SEMISELECTIVE COIDEALS

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Все счастливые семьи похожи друг на друга,
каждая несчастливая семья несчастлива по своему.[1]
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In this note we give an answer to the following problem of Todorcevic: Find out the combinatorial essence behind the fact that the family $\mathcal{H}$ of the ground-model infinite sets of integers in a Perfect-set forcing extension has the property that for any Borel $f\colon[\mathbb{N}]^\omega \to \{0,1\}$ there exists an $A \in \mathcal{H}$ such that f is constant on $[A]^\omega$ (see [7], [13]). In other words, one needs to capture the combinatorial properties of the family $\mathcal{H}$ of ground-model subsets of $\mathbb{N}$ which assure that it diagonalizes all Borel partitions. It turns out that the notion which results from our analysis of this problem is a bit more optimal than the older notion of a "happy family" (or selective coideal) introduced by A.R.D. Mathias [16] long ago in order to extend the well-known theorems of Galvin–Prikry [6] and Silver [25] (see theorems 3.1 and 4.1 below). We should remark that these Mathias-style extensions can indeed be as useful in the applications as the original partition theorems. For example, one such application (where the original partition theorem of Galvin–Prikry and Silver does not seem to fit) was recently found by Todorcevic ([28]) in order to supply a new proof of the famous Bourgain–Fremlin–Talagrand theorem ([2]). Other applications can be found in the so-called parametrized partition calculus (see e.g. [17], [19], [29], [38]). One can also use these Mathias-style extensions of the Galvin–Prikry and Silver theorems to give a new proof of the well-known perfect-tree theorem of J. Stern ([39], see also [38, §C]).

We have organized our paper so that it can be read by two kinds of mathematicians, those unfamiliar with the forcing technique and the others. In §1 (on which the latter sections do not depend and which can be skipped by readers not familiar with forcing) we prove that Mathias' notion of selective coideal is too strong to be preserved by the Perfect-set forcing. In §2 we introduce the notion of a semiselective coideal and compare it with the stronger notion of a selective coideal. Sections §3 and §4 are independent from each other, but they both rely on §2. In §3 we use topological methods to prove that a coideal is semiselective exactly when the abstract Baire property and the Ramsey property, naturally associated with it, coincide (see Theorem 3.1 below). In §4 we prove that a coideal $\mathcal{H}$ is semiselective

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[1] All happy families resemble one another, each unhappy family is unhappy in its own way.

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iff the Mathias poset $\mathcal{M}_{\mathcal{H}}$ associated with $\mathcal{H}$ has the Prikry property iff $\mathcal{M}_{\mathcal{H}}$ has the Mathias property (i.e. subsets of a generic set are generic). We also show that semiselective coideals are preserved by the Perfect-set forcing and describe a situation in which every semiselective coideal diagonalizes all definable sets.

We denote finite sets of positive integers by $s, t, u, \ldots$ and infinite sets of positive integers by $A, B, C, \ldots$. Symbols $[\mathbb{N}]^{<\omega}$ (resp. $[\mathbb{N}]^{\omega}$) denote families of all finite (resp. infinite) sets of positive integers. The following notation will be useful:

$$A/s = A \setminus \{1, \ldots, \max s\},$$
$$A/n = A \setminus \{1, \ldots, n\}.$$

The symbol $s \sqsubseteq t$ means that s is an initial segment of t. We consider two topologies on $[\mathbb{N}]^{\omega}$, the separable metric given by basic open sets

$$[s] = \{B \subseteq \mathbb{N} \colon s \sqsubseteq B\}$$

and the exponential whose basic open sets are

$$[s, A] = \{B \subseteq \mathbb{N} \colon s \sqsubseteq B \quad \text{and} \quad B \subseteq s \cup A\}.$$

A family $\mathcal{H}$ of sets of integers is a *coideal* if its complement $\mathcal{P}(\mathbb{N}) \setminus \mathcal{H}$ is an ideal; in other words, coideal is a family which satisfies the following two axioms:

(I0) $F_1 \supseteq F_2$ and $F_2 \in \mathcal{H}$ implies $F_1 \in \mathcal{H}$.

(I1) If $F_1 \cup F_2$ is in $\mathcal{H}$ then either F_1 is in $\mathcal{H}$ or F_2 is in $\mathcal{H}$.

For an $A \in \mathcal{H}$ we define

$$\mathcal{H} \restriction A = \{B \subseteq A \colon B \in \mathcal{H}\}.$$

A set $\mathcal{X} \subseteq [\mathbb{N}]^{\omega}$ is *$\mathcal{H}$-Ramsey* iff for every $[s, A]$ such that $A \in \mathcal{H}$ there is $B \in \mathcal{H} \restriction A$ such that $[s, B]$ is either included in or disjoint from $\mathcal{X}$. It is *$\mathcal{H}$-Ramsey null* if it is $\mathcal{H}$-Ramsey but the set $[s, B]$ as above is always disjoint from it. The algebra of $\mathcal{H}$-Ramsey sets and its ideal of $\mathcal{H}$-Ramsey null sets are denoted by $\mathcal{R}(\mathcal{H})$ and $\mathcal{R}_0(\mathcal{H})$ respectively. A set $\mathcal{X} \subseteq \mathbb{R} \times [\mathbb{N}]^{\omega}$ is *perfectly $\mathcal{H}$-Ramsey* iff for every perfect $P \subseteq \mathbb{R}$ and every $[s, A]$ such that $A \in \mathcal{H}$ there is a perfect $Q \subseteq P$ and a $B \in \mathcal{H} \restriction A$ such that $Q \times [s, B]$ is either included in or disjoint from $\mathcal{X}$. It is *perfectly $\mathcal{H}$-Ramsey null* if it is $\mathcal{H}$-Ramsey but the set $Q \times [s, B]$ as above is always disjoint from it. The algebra of perfectly $\mathcal{H}$-Ramsey sets and its ideal of perfectly $\mathcal{H}$-Ramsey null sets are denoted by $\mathcal{PR}(\mathcal{H})$ and $\mathcal{PR}_0(\mathcal{H})$ respectively. Using these notions one can state Todorcevic's question avoiding the terminology of forcing (see [19], [17]):

Which combinatorial properties of the family $[\mathbb{N}]^{\omega}$ of all infinite sets of integers are responsible for the fact that all Borel sets are perfectly $[\mathbb{N}]^{\omega}$-Ramsey?

We will always assume that ideals are proper and that they include the Fréchet ideal $[\mathbb{N}]^{<\omega}$, i.e. that coideals are closed under finite changes and differ from $\mathcal{P}(\mathbb{N})$. With this convention, every coideal can be naturally considered as a poset ordered by almost inclusion, $\subseteq^*$. So we will adopt the terminology of forcing and talk about *dense open subsets*, *incompatible elements*, *maximal antichains*, etc., of $\mathcal{H}$, always referring to the poset $\langle \mathcal{H}, \subseteq^* \rangle$. The same applies to the poset $\mathcal{P}$ of all perfect subsets of reals ordered by the inclusion – this is the so-called *Perfect-set* (or *Sacks*) forcing – and to the poset $\mathcal{P} \times \mathcal{H}$ with the product ordering.

Let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ be an infinite decreasing sequence of sets of integers. Then a set F_∞ is a *diagonalization* of this sequence if $F_\infty/n \subseteq F_n$ for all $n \in F_\infty$.

By $A^{<\omega}$ we denote the set of all finite sequences of elements of A. A family of nonempty perfect sets $\{P_s : s \in \{0,1\}^{<\omega}\}$ is a *fusion sequence* if for every s sets $P_{s^\frown 0}$ and $P_{s^\frown 1}$ are disjoint perfect subsets of P_s, and the diameter of P_s converges to 0 as the length of s approaches infinity. The perfect set

$$\bigcup_{f\in\{0,1\}^\omega} \bigcap_{n\in\omega} P_{f\restriction n} = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} P_s$$

is the *fusion* of this family. The $\mathcal{A}$-*operation* is defined for a family $\{F_s : s \in \mathbb{N}^{<\omega}\}$ of sets of reals by

$$\mathcal{A}(F_s : s \in \mathbb{N}^{<\omega}) = \bigcup_{f\in\mathbb{N}^\mathbb{N}} \bigcap_{n\in\mathbb{N}} F_{f\restriction n}.$$

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1. PRESERVING SELECTIVITY

Recall the following well-known notion introduced by Mathias in [16].

DEFINITION 1.1 *A coideal $\mathcal{H} \subseteq [\mathbb{N}]^\omega$ is selective if every decreasing sequence in $\mathcal{H}$ has a diagonalization in $\mathcal{H}$.*

We note that in [16] selective coideals were called *happy families*. It is easy to see that $[\mathbb{N}]^\omega$ is an example of a selective coideal but there are less obvious examples (see [16], [28]). One might be led to expect that selectivity is the notion we seek by the following well-known results.

THEOREM 1.2 (Mathias) Analytic sets are $\mathcal{H}$-Ramsey for every selective coideal $\mathcal{H}$. □

THEOREM 1.3 If $\mathcal{U}$ is a selective ultrafilter then in a Perfect-set forcing extension it still generates a selective ultrafilter. □

For the first, see [16; Theorem 4.3]. In an unpublished manuscript of Mathias [42] one finds an interesting history of the second result which we reproduce here following the suggestion of the referee. In the course of a lecture at Berkeley in 1967 Harvey Friedman announced that in an extension by one Sacks real, every infinite subset of $\mathbb{N}$ contains or is disjoint from one in the ground model, and sketched a proof. In discussion after the lecture, Prikry showed that Friedman's proof was defective and Mathias found a correct one. Around 1971 Kunen observed in a letter to Mathias that the latter's proof of Friedman's statement would establish that a selective ultrafilter in the ground model generated one in an extension by one Sacks real. In reply to Kunen's letter, Mathias remarked that Kunen's observation would imply that where a is Sacks-generic over V and b is $\mathcal{P}_\mathcal{U}$-generic (see §4) over V for a selective ultrafilter $\mathcal{U}$ in V, then a and b are automatically mutually generic. Later on, Miller ([17]) found this curious fact's elegant consequence: All Borel sets are perfectly Ramsey. In his letter Kunen also claimed, but later retracted, the corresponding preservation result for arbitrary Sacks extensions (see [40; p. 93]). Later work (see [1], [7], [26]) established the truth of that claim and its connection to a deep combinatorial fact known under the name of Halpern–Läuchli theorem (see Remark 4.1 below, also [13], [26; §6]).

It is not difficult to see that the standard proof of Theorem 1.3 gives that in the Perfect-set forcing extension a selective coideal still generates a coideal (see Corollary 4.2 below). However, the following lemma shows that selectivity is destroyed whenever the coideal $\mathcal{H}$ has infinite cellularity.

LEMMA 1.4 If $\mathcal{H}$ is a selective coideal which has an infinite antichain then in the Perfect-set forcing extension it does not generate a selective coideal.

Proof Assume that $\mathcal{H}$ has an infinite antichain $\{X_n\}$. We can further assume that X_n's are pairwise disjoint (by replacing X_n with $\bar{X}_n = X_n \setminus \bigcup_{i=1}^{n} X_i$), and that it is bijectively enumerated as B_t ($t \in \{0,1\}^{<\omega}$). So we can recursively embed the binary tree $\langle\{0,1\}^{<\omega}, \sqsubset\rangle$ into $\langle\mathcal{H}, \subset\rangle$: let

$$A_s = \bigcup_{t \sqsupseteq s} B_t, \qquad \text{for } s \in \{0,1\}^{<\omega}.$$

Then we have the following:

(1) $A_t \supseteq A_s$ for all $t \sqsubseteq s$,
(2) $A_t \cap A_s = \emptyset$ for all $t \neq s$ of the same length.

Let r be a new real in a forcing extension of V, and in the extension let $\{F_n\}$ be the following sequence (recall that $r \restriction n$ is the sequence of first n digits of r):

$$F_n = A_{r \restriction n}.$$

Then this is a decreasing sequence of elements of $\mathcal{H}$. We claim that no ground-model infinite subset of ω diagonalizes it, moreover

CLAIM If $X \subseteq \omega$ is infinite and almost included in all F_n's, then $r \in V[X]$.

Proof Let

$$T = \{t \in \{0,1\}^{<\omega} : X \cap A_t \text{ is infinite}\}.$$

This set is obviously downwards closed. It is infinite, because all initial segments of r belong to it. We claim that r is the unique infinite branch of T: if $x \in \{0,1\}^{\omega}$ is distinct from r and all its initial segments are in T, look at maximal $t \in T$ such that both x and r extend it. Then both $X \cap A_{t^\frown 0}$ and $X \cap A_{t^\frown 1}$ are infinite, but this contradicts to the choice of X. So r is definable from X, as required. $\square$

So in $V[r]$ we have a sequence $\{F_n\}$ in $\mathcal{H}$ with no diagonalization in $\mathcal{H}$, and therefore $\mathcal{H}$ is destructible. $\square$

We say that a family of sets of integers *generates a coideal* if its upwards closure is a coideal.

THEOREM 1.5 A selective coideal is preserved by the Perfect-set forcing iff it is generated by finitely many selective ultrafilters.

Proof Assume $\mathcal{H}$ is a selective coideal which in a Perfect-set forcing extension generates a selective coideal. Let $\mathcal{A} \subseteq \mathcal{H}$ be a maximal family such that for all $A \neq B$ in $\mathcal{A}$:

(1) $A \cap B$ is finite, and

(2) $\mathcal{H} \restriction A$ has the f.i.p. (finite intersection property).

By Lemma 1.1, the family $\mathcal{A}$ is finite, say $\mathcal{A} = \{A_1, \dots, A_n\}$. Then for every $C \in \mathcal{H}$ there is $A \in \mathcal{A}$ such that $A \cap C \in \mathcal{H}$, and $\mathcal{U}_i = \mathcal{H} \restriction A_i$ is a selective ultrafilter on A_i for $i = 1, \dots n$. Therefore $\mathcal{H}$ is generated by $\bigcup_{i=1}^n \mathcal{U}_i$, as required.

Let $\mathcal{H} = \bigcup_{i=1}^n \mathcal{U}_i$, where $\mathcal{U}_i$'s are distinct selective ultrafilters. By Theorem 1.2 in a Perfect-set forcing extension all $\mathcal{U}_i$'s remain selective. To see that $\mathcal{H}$ itself is preserved, it is enough to see that the condition from Definition 1.1 is preserved "locally", i.e. that it remains true on a dense subset of $\mathcal{H}$. But this obviously reduces to the fact that each $\mathcal{U}_i$ is selective in an extension. $\square$

2. SEMISELECTIVITY

If $\mathcal{D}_n$ is a sequence of families of sets of integers, then a set F_∞ is a *diagonalization* of it if F_∞/n is in $\mathcal{D}_n$ for all $n \in F_\infty$. (In other words, if there are sets $F_n \in \mathcal{D}_n$ such that F_∞ is their diagonalization.)

DEFINITION 2.1 *A coideal $\mathcal{H} \subseteq [\mathbb{N}]^\omega$ is semiselective if for every sequence $\{\mathcal{D}_n\}$ of dense open subsets of $\mathcal{H}$ the family of all diagonalizations of this sequence is dense in $\mathcal{H}$.*

Thus, semiselectivity is a weakening of selectivity where we require σ-distributivity instead of σ-closedness. Note that in [11] Kunen uses the term semiselective for a weakening of selectivity different from ours – he retains σ-closedness but relaxes the Q^+ property (for definitions see the paragraph before Theorem 2.3 below).

THEOREM 2.2 Let $\mathcal{H}$ be a semiselective coideal. **a)** The family of all sets homogeneous for some partition of $[\mathbb{N}]^2$ into two pieces is $\mathcal{H}$-Ramsey. **b)** Analytic sets are $\mathcal{H}$-Ramsey and perfectly $\mathcal{H}$-Ramsey. **c)** The algebras of $\mathcal{H}$-Ramsey and perfectly $\mathcal{H}$-Ramsey sets are σ-algebras closed under the $\mathcal{A}$-operation.

Proof **a)** Let $[\mathbb{N}]^2 = K_0 \cup K_1$ be a partition. Define

$$\mathcal{D}_n = \{A \in \mathcal{H} \colon \{n\} \times A \subseteq K_0 \text{ or } \{n\} \times A \subseteq K_1\}.$$

Then each $\mathcal{D}_n$ is a dense subset of $\mathcal{H}$, so we can pick a set $A \in \mathcal{H}$ such that $A/n \in \mathcal{D}_n$ for all n; by definitions, this means that A is *min–homogeneous*, i.e. that there is a sequence i_n $(n \in A)$ such that $\{n, m\} \in K_{i_n}$ for all $n < m$ in A. Let $A_0 = \{n \in A \colon i_n = 0\}$ and $A_1 = \{n \in A \colon i_m = 1\}$; these two sets are homogeneous and one of them is in $\mathcal{H}$. $\square$

Parts **b)** and **c)** will be proved both in sections §3 and §4, by using different methods, but we will first prove special cases of **b)** and **c)** now to illustrate typical applications of semiselective coideals. Applications of the following theorem (i.e. its version when $\mathcal{H} = [\mathbb{N}]^\omega$), are found in several areas of mathematics, like better quasi-ordering theory ([18]), partition calculus ([6]), convergence theory in function spaces ([23]) and Borel graph theory ([10]). Recall that a family $\mathcal{F}$ of finite sets is a *barrier on A* iff every infinite subset of A has an initial part in $\mathcal{F}$ and $\mathcal{F}$ is an antichain in the $\sqsubseteq$ ordering. For a set $B \subseteq A$ let $\mathcal{F} \restriction B$ denote the family $\mathcal{F} \cap [B]^{<\omega}$. The following theorem was proved by Nash-Williams and Galvin in the case when $\mathcal{H} = [\mathbb{N}]^\omega$, and by Mathias in the case when $\mathcal{H}$ is selective.

THEOREM 2.3 (Semiselective Nash-Williams; Galvin Lemma) Let $\mathcal{H}$ be a semiselective coideal. If $\mathcal{F}$ is a family of finite subsets of $\mathbb{N}$, then there is a $B \in \mathcal{H}$ such that $\mathcal{F} \restriction B$ is either empty or it includes a barrier.

Proof We say $A \in \mathcal{H}$ ***accepts*** s if every element of $[s, A]$ has an initial segment in $\mathcal{F}$, A ***rejects*** s if no element of $[s, A] \cap \mathcal{H}$ accepts s, and A ***decides for*** s if A either rejects or accepts s. The following statements are obvious:

(1) If A accepts (rejects) s then every B in $\mathcal{H} \restriction A$ accepts (rejects) s.
(2) For every pair s, A there is a B in $\mathcal{H} \restriction A$ which either accepts or rejects s.
(3) If A accepts s then A accepts $s \cup \{n\}$ for all n in A/s.

An analogue of (3) for "rejects" is easily seen to be false – A can reject s while accepting $s \cup \{\bar{n}\}$ for some $\bar{n} \in A/s$ (e.g. if $\mathcal{F} = \{t : \bar{n} \in t\}$).

CLAIM 1 If A rejects s, then the set of $n \in A$ such that A accepts $s \cup \{n\}$ is not in $\mathcal{H}$.

Proof Suppose the contrary, that the set B of all such n is in $\mathcal{H}$. Then B accepts s, contrary to our assumption and (1). □

CLAIM 2 There is a B in $\mathcal{H}$ which decides for each of its finite subsets.

Proof Let

$$\mathcal{D}_s = \{C \in \mathcal{H} : C \text{ decides for } s\}.$$

Then sets $\mathcal{D}_n = \bigcap_{\max s \leq n} \mathcal{D}_s$ are dense open in $\mathcal{H}$, so if B is their diagonalization then by (2) above it decides for all of its finite subsets. □

Let B be as in Claim 2 above. If B accepts $\emptyset$ then $\mathcal{F} \restriction B$ includes a barrier, so we can assume B rejects $\emptyset$. For $s \subseteq B$ such that B rejects it let

$$\mathcal{E}_s = \{C \in \mathcal{H} \restriction B : C \text{ rejects } s \cup \{n\} \text{ for all } n \in C/s\},$$

and let $\mathcal{E}_s = \mathcal{H} \restriction B$ otherwise. By Claim 1 sets $\mathcal{E}_s$ are dense below B, so let $C \in \mathcal{H} \restriction B$ be a diagonalization of $\{\mathcal{E}_s\}$. Then it follows (by induction on the length of s) that C rejects all of its finite subsets; in particular every $s \subseteq C$ is not in $\mathcal{F}$, so $\mathcal{F} \restriction C$ is empty. This finishes the proof. □

We will now prove a parametrized version of Theorem 2.2. For an $x \in \{0,1\}^\omega$ and $k \in \mathbb{N}$ by $x \restriction k$ we denote the sequence of first k digits of x. For a perfect set $P \subseteq \{0,1\}^\omega$ by T_P we denote the set of all $x \restriction k$ for $x \in P$ and $k \in \mathbb{N}$, and by $T_P \restriction k$ we denote the set of all $u \in T_P$ of length k; if $u \in T_P \restriction k$ then we write $|u| = k$. The space $\{0,1\}^\omega$ is considered with the metric $d(x,y) = 1/(\Delta(x,y)+1)$, where $\Delta(x,y)$ is the minimal k such that $x \restriction k \neq y \restriction k$.

THEOREM 2.4 Let $\mathcal{H}$ be a semiselective coideal. For every $\mathcal{F} \subseteq \{0,1\}^{<\omega} \times [\mathbb{N}]^{<\omega}$ there is a perfect $P \subseteq \{0,1\}^\omega$ and $A \in \mathcal{H}$ so that one of the following two alternatives applies:

(a) For all $x \in P$ and all infinite $B \subseteq A$ there are integers k, l such that the pair $\langle x \restriction k, B \cap \{1, \ldots, l\} \rangle$ is in $\mathcal{F}$, or
(b) The set $T_P \times [A]^{<\omega}$ is disjoint from $\mathcal{F}$.

LEMMA 2.5 If $\mathcal{D}_n$ ($n \in \mathbb{N}$) are dense open subsets of $\mathcal{P} \times \mathcal{H}$ then the set

$$\mathcal{D}_\infty = \{\langle P, A \rangle : \langle P \cap [u], A/n \rangle \in \mathcal{D}_n \text{ for all } n \in A \text{ and } u \in T_P \restriction n\}$$

is dense open in $\mathcal{P} \times \mathcal{H}$.

Proof For a set $\mathcal{E} \subseteq \mathcal{P} \times \mathcal{H}$ let $\pi_2(\mathcal{E})$ denote the set of all $A \in \mathcal{H}$ such that $\langle P, A \rangle$ is in $\mathcal{E}$ for some $P \in \mathcal{P}$. Note that $\pi_2(\mathcal{D}_n)$ is a dense open subset of $\mathcal{H}$ for all n. Find a maximal antichain $\mathcal{A}_1 \subseteq \pi_2(\mathcal{D}_1)$ and for each $A \in \mathcal{A}_1$ fix a perfect set $P = P_{\langle\rangle}(A)$ such that $\langle P, A/1 \rangle$ is in $\mathcal{D}_1$. Then let $\mathcal{D}_2'$ be the family of all $\langle Q, B \rangle \in \mathcal{D}_2$ such that $B \subseteq A$ for some $A \in \mathcal{A}_1$ and $Q \subseteq P(A)$. Note that this $A = A(B)$ is unique, if it exists. Find a maximal antichain $\mathcal{A}_2$ included in $\pi_2(\mathcal{D}_2')$, and for each $B \in \mathcal{A}_2$ find perfect sets $P_{\langle 0 \rangle}(B)$, $P_{\langle 1 \rangle}(B)$ of diameter $\leq 1/2$ included in $P_{\langle\rangle}(A(B))$. Proceeding in this way, we define maximal antichains $\mathcal{A}_n$ in $\mathcal{H}$ and for each $A \in \mathcal{A}_n$ disjoint perfect sets $P_s(A)$ $(s \in \{0,1\}^{n-1})$ such that:

(1) $\langle P_s(A), A/n \rangle$ is in $\mathcal{D}_n$,
(2) the diameter of $P_s(A)$ is at most $1/n$, and
(3) if $A \subseteq B$ for $A \in \mathcal{A}_n$ and $B \in \mathcal{A}_m$ $(n > m)$ then $P_s(A) \subseteq P_t(B)$ for all $s \sqsupseteq t$.

Let

$$\mathcal{E}_n = \{C \in \mathcal{H} \colon C \text{ is almost included in some member of } \mathcal{A}_n\}.$$

Sets $\mathcal{E}_n$ are dense open in $\mathcal{H}$, so they have a diagonalization A_∞ in $\mathcal{H}$. Then for each n there is $A_n \in \mathcal{A}_n$ such that A_∞ diagonalizes the sequence $\{A_n\}$, and sets $P_s^n = P_s(A_n)$ form a fusion sequence such that $T_{P_s^n} \restriction n$ has only one element. Let P_∞ be the fusion of $\{P_s^n\}$; then for each $n \in A$ and $u \in T_P \restriction n$ we have t such that

$$\langle P \cap [u], A/n \rangle \subseteq \langle P_t^n, A_n \rangle$$

so the pair $\langle P_\infty, A_\infty \rangle$ is in $\mathcal{D}_\infty$. $\square$

Proof (Theorem 2.5) For a pair $\langle u, s \rangle \in \{0,1\}^{<\omega}$, perfect $Q \subseteq \{0,1\}^\omega$, and $A \in \mathcal{H}$ we say $\langle Q, A \rangle$ *accepts* $\langle u, s \rangle$ if $u \in T_Q$ and for every $x \in Q \cap [u]$ and $B \in [s, A]$ there are integers k, l such that $\langle x \restriction k, B \cap \{1, \ldots, l\} \rangle \in \mathcal{F}$. We say $\langle Q, A \rangle$ *rejects* $\langle u, s \rangle$ if $\langle P, B \rangle$ does not accept $\langle u, s \rangle$ for all perfect $P \subseteq Q$ and $B \in \mathcal{H} \restriction A$, and that $\langle Q, A \rangle$ *decides for* $\langle u, s \rangle$ if it either accepts or rejects it. The following statements are obvious:

(1) If $\langle Q, A \rangle$ accepts (rejects) $\langle u, s \rangle$, then every $\langle P, B \rangle$ such that P is a perfect subset of $Q \cap [u]$ and $B \in \mathcal{H} \restriction A$ also accepts (rejects) $\langle u, s \rangle$.
(2) For all $\langle u, s \rangle$, A and Q there is a perfect $P \subseteq Q \cap [s]$ and a $B \in \mathcal{H} \restriction A$ such that $\langle P, B \rangle$ decides for $\langle u, s \rangle$.
(3) If $\langle Q, A \rangle$ accepts $\langle u, s \rangle$ then it also accepts $\langle u, s \cup \{n\} \rangle$ for all $n \in A/s$.
(4) If $\langle Q, A \rangle$ rejects $\langle u, s \rangle$ then the set of all $n \in A/s$ such that $\langle Q, A \rangle$ accepts $\langle u, s \cup \{n\} \rangle$ is not in $\mathcal{H}$.
(5) If $\langle Q, A \rangle$ accepts (rejects) $\langle u, s \rangle$ then it also accepts (rejects) $\langle v, s \rangle$ for all $v \sqsupseteq u$ in T_Q.

Claim There is a pair $\langle Q, A \rangle$ which decides for every $\langle u, s \rangle \in T_Q \times [A]^{<\omega}$ such that $\max(s) \leq |u|$.

Proof Let

$$\mathcal{D}_n = \{\langle Q, A \rangle \colon \langle Q, A \rangle \text{ decides for all } \langle u, s \rangle \in T_Q \times [A]^{<\omega} \text{ such that } \max(s) \leq |u| = n.\}$$

Then (2) implies $\mathcal{D}_n$ is a dense open subset of $\mathcal{P} \times \mathcal{H}$, so let $\langle Q, A \rangle$ be as guaranteed by Lemma 2.1. Then $\langle Q, A \rangle$ is as required, because $\langle P, A \rangle$ decides for $\langle u, s \rangle$ iff $\langle P \cap [u], A \rangle$ does. $\square$

Let $\langle Q, A\rangle$ be as in Claim. If it accepts $\langle\langle\rangle,\emptyset\rangle$, then it satisfies (a), so we can assume it rejects $\langle\langle\rangle,\emptyset\rangle$. Define sets $\mathcal{E}_{u,s}$ for $\langle u,s\rangle \in T_Q \times [A]^{<\omega}$ as follows:

$$\mathcal{E}_{u,s} = \{\langle P,B\rangle \in \mathcal{P}\times\mathcal{H}\restriction A \colon P \subseteq Q\cap[u] \text{ and } \langle P, B/n\rangle \text{ rejects } \langle u, s\cup\{n\}\rangle\}$$

if $\langle Q,A\rangle$ rejects $\langle u,s\rangle$ and $\mathcal{E}_{u,s} = \mathcal{P}\times\mathcal{H}$ otherwise. By (4) sets

$$\mathcal{E}_n = \bigcap_{\substack{u\in T_Q\restriction n\\ \max s\leq n}} \mathcal{E}_{u,s}$$

are dense open in $\mathcal{H}\restriction A$, so let $\langle P,B\rangle$ be as guaranteed by Lemma 2.1 We claim that $\langle P,B\rangle$ satisfies (b). Assume the contrary, and let $\langle u,s\rangle \in \mathcal{F}\cap(T_P\times[B]^{<\omega})$ be such that $\max(s)$ is minimal (taking $\max(\emptyset)=0$). Then $\langle P\cap[u],B\rangle$ accepts $\langle\langle\rangle,s\rangle$, so in particular $s\neq\emptyset$. Let $n=\max(s)$ and $t = s\setminus\{n\}$. Then if $v\sqsubseteq u$ is of length $\max(t)$, $\langle P,B\rangle$ rejects $\langle v,t\rangle$, therefore $\langle P\cap[u],B\rangle$ is not in $\mathcal{E}_{v,t}$ – a contradiction. □

Now we will investigate the relationship between selective and semiselective coideals. We say that a coideal $\mathcal{H}$ is *Ramsey* iff it diagonalizes all two-dimensional partitions of its elements, and that it has the Q^+*-property* if for every $A\in\mathcal{H}$ and every partition of A into finite sets there is a $B\in\mathcal{H}\restriction A$ which intersects every piece of the partition in at most one point. It follows from the definition that $\mathcal{H}$ is selective if and only if it is σ-closed and has the Q^+-property, and that $\mathcal{H}$ is semiselective if and only if it is σ-distributive and has the Q^+-property. Also, Ramsey coideals always have the Q^+-property. [For every partition of $\mathbb{N}$ into disjoint pieces there is a partition of $[\mathbb{N}]^2$ such that every homogeneous set is either included in one piece or it intersects each piece in at most one point.] So we have the following implications:

selective ⇒ semiselective ⇒ Ramsey

Examples 2.1 and 2.2 below demonstrate that none of these arrows is reversible (and it is well-known that in general Q^+-property does not imply Ramseyness). On the other hand, by [16; Theorem 0.10], an ultrafilter $\mathcal{U}$ is selective iff it is Ramsey iff every partition of $[\mathbb{N}]^2$ into two pieces has a homogeneous set in $\mathcal{U}$, so we have:

THEOREM 2.6 An ultrafilter is Ramsey iff it is selective iff it is semiselective. □

EXAMPLE 2.7 A semiselective coideal which is not selective. Our working copy of $\mathbb{N}$ will be $\mathbb{N}\times\mathbb{N}$. Let $\mathcal{I}$ be the ideal of all subsets of $\mathbb{N}\times\mathbb{N}$ having finite intersections with all vertical sections $\{n\}\times\mathbb{N}$. We claim the coideal $\mathcal{H} = \mathcal{P}(\mathbb{N})\setminus\mathcal{I}$ is semiselective, yet not selective. To see that $\mathcal{H}$ is not selective, consider the sets

$$A_n = [n,\infty)\times\mathbb{N}.$$

All A_n's are in $\mathcal{H}$, but every diagonalization of the sequence $\{A_n\}$ has to be in $\mathcal{I}$. On the other hand, $\mathcal{I}$ is a *nowhere dense ideal*, i.e. every $A\in\mathcal{H}$ has an infinite subset B such that $\mathcal{I}\restriction B$ equals the Fréchet ideal; therefore $\mathcal{I}$ is semiselective.

EXAMPLE 2.8 A Ramsey coideal which is not semiselective. A generating family for this coideal is $\{A_u : u \text{ is a finite sequence of reals}\}$ such that $A_{\langle\rangle} = \mathbb{N}$ and

> $A_{u^\frown\xi}$ ($\xi\in\mathbb{R}$) is an almost disjoint family of subsets of A_u and for every analytic $\mathcal{A}\subseteq[A_u]^\omega$ there is ξ so that $[A_{u^\frown\xi}]^\omega$ is either disjoint from or included in $\mathcal{A}$.

Such family is constructed by the recursion as follows: Let $\mathcal{A}_\xi$ ($\xi \in \mathbb{R}$) be an enumeration of all analytic subsets of $[\mathbb{N}]^\omega$. If A_u is chosen, pick an almost disjoint family B_ξ ($\xi \in \mathbb{R}$) of subsets of A_u. Using Silver's theorem for every $\xi \in \mathbb{R}$ find $A_{u^\frown\xi} \subseteq B_\xi$ which is either disjoint from or included in $\mathcal{A}_\xi$. By the construction, the family $\{A_u\}$ generates a Ramsey coideal. Dense open sets $\mathcal{D}_n$ generated by $\{A_u : |u| = n\}$ verify that this coideal is not σ-distributive.

Example 2.2 can be modified to obtain semiselective coideals with special properties; see e.g. Example 4.1.

3. THE ABSTRACT BAIRE PROPERTY

In [19] Pawlikowski used the work of Marczewski ([35]) and Morgan ([36]) on the abstract Baire property to analyze the algebra of perfectly $[\mathbb{N}]^\omega$-Ramsey sets. In this section we will apply his methods in the context of semiselective coideals. We should note that [19] originated in a challenge to find elementary proofs for the results in [17] about perfectly Ramsey sets.

Throughout this section we assume $\mathcal{H}$ is a semiselective coideal, and $\mathcal{P}$ will denote the family of all perfect (nonempty) subsets of a Cantor cube $\{0,1\}^\omega$. For a family of sets of integers $\mathcal{A}$ we define a family $\mathrm{Exp}(\mathcal{A})$ by

$$\mathrm{Exp}(\mathcal{A}) = \{[s, A] : A \in \mathcal{A}\}.$$

We always assume $\mathbb{N}$ is in $\mathcal{A}$ so that $\mathrm{Exp}(\mathcal{A})$ extends the standard basis for the separable metric topology on $[\mathbb{N}]^\omega$. The family $\mathrm{Exp}(\mathcal{H})$ for a coideal $\mathcal{H}$ is rarely a basis for some topology, as the following easy fact shows.

FACT 3.1 $\mathrm{Exp}(\mathcal{A})$ is a basis for the topology if and only if for all $A, B \in \mathcal{A}$ which are not almost disjoint there is a $C \in \mathcal{A}$ included in their intersection. □

DEFINITION 3.1 *Let $\mathcal{B}$ be a family of subsets of some set Y. A set $\mathcal{X} \subseteq Y$*

- *is $\mathcal{B}$-open if it is equal to the union of some subfamily of $\mathcal{B}$,*
- *is $\mathcal{B}$-nowhere dense if every U in $\mathcal{B}$ has a subset in $\mathcal{B}$ disjoint from $\mathcal{X}$,*
- *has the abstract $\mathcal{B}$-Baire property if every $U \in \mathcal{B}$ has a subset in $\mathcal{B}$ which is either included in or disjoint from $\mathcal{X}$,*
- *has the $\mathcal{B}$-Baire property if there is a $\mathcal{B}$-open set $\mathcal{O}$ such that $\mathcal{X} \Delta \mathcal{O}$ is $\mathcal{B}$-nowhere dense.*

It is easy to see that the $\mathcal{H}$-Ramsey property implies the abstract $\mathrm{Exp}(\mathcal{H})$-Baire property, which in turn implies the $\mathrm{Exp}(\mathcal{H})$-Baire property. We will see that in the case when $\mathcal{H}$ is semiselective we can say more. By $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$ denote the family of all subsets of $\mathbb{R} \times [\mathbb{N}]^\omega$ of the form $P \times [s, A]$ for a perfect P and $[s, A] \in \mathrm{Exp}(\mathcal{H})$.

In the case when $\mathcal{H} = [\mathbb{N}]^\omega$, (b)–(d) of the following theorem were proved by Ellentuck in [3], (e)–(g) were proved by Pawlikowski in [19], and in the case when $\mathcal{H}$ is a selective ultrafilter (b)–(d) were proved by Louveau in [14].

THEOREM 3.2 Let $\mathcal{H}$ be a coideal. Then the following are equivalent:

(a) $\mathcal{H}$ is semiselective.
(b) The $\mathcal{H}$-Ramsey sets are exactly the sets with the abstract $\mathrm{Exp}(\mathcal{H})$-Baire property.
(c) All $\mathrm{Exp}(\mathcal{H})$-nowhere dense sets are $\mathcal{H}$-Ramsey.

(d) The following three families of subsets of $[\mathbb{N}]^\omega$ coincide:
- $\mathcal{H}$-Ramsey null sets,
- $\mathrm{Exp}(\mathcal{H})$-nowhere dense sets, and
- $\mathrm{Exp}(\mathcal{H})$-meager sets.

(e) The perfectly $\mathcal{H}$-Ramsey sets are exactly the sets with the abstract $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$-Baire property.

(f) All $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$-nowhere dense sets are perfectly $\mathcal{H}$-Ramsey.

(g) The following three families of subsets of $[\mathbb{N}]^\omega$ coincide:
- perfectly $\mathcal{H}$-Ramsey null sets,
- $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$-nowhere dense sets, and
- $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$-meager sets.

Lemma 3.3 If $\mathcal{H}$ is semiselective, then the ideal $\mathcal{R}_0(\mathcal{H})$ is a σ-ideal and the algebra $\mathcal{R}(\mathcal{H})$ is a σ-algebra.

Proof Let $\mathcal{A}_n$ be a sequence of $\mathcal{H}$-Ramsey null sets and fix $[t, B]$ for $B \in \mathcal{H}$; we can assume $t = \emptyset$. For $s \in [\mathbb{N}]^{<\omega}$ define dense open subsets of $\mathcal{H}$

$$\mathcal{D}_s = \{A \in \mathcal{H} \colon [s, A] \cap \mathcal{A}_n = \emptyset \text{ for all } n \leq |s|\}.$$

If C is a diagonalization of the sequence $\mathcal{D}_s$ (i.e. a diagonalization of the sequence $\mathcal{D}_n = \bigcap_{\max s \leq n} \mathcal{D}_n$) then $[\emptyset, C]$ is disjoint from $\bigcup \mathcal{A}_n$.

Let $\mathcal{A}_n$ be a sequence of $\mathcal{H}$-Ramsey sets and fix $[t, B]$ for $B \in \mathcal{H}$. Then either there is $C \in \mathcal{H} \upharpoonright B$ such that $[t, C]$ is included in some $\mathcal{A}_n$ or a proof similar to the above gives $C \in \mathcal{H} \upharpoonright B$ such that $[t, C]$ is disjoint from $\bigcup \mathcal{A}_n$. □

Lemma 3.4 If $\mathcal{H}$ is semiselective, then the ideal $\mathcal{PR}_0(\mathcal{H})$ is a σ-ideal and the algebra $\mathcal{PR}(\mathcal{H})$ is a σ-algebra.

Proof Let $\mathcal{A}_n$ be a sequence of perfectly $\mathcal{H}$-Ramsey null sets and fix $\langle P, [t, B]\rangle$ in $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$. We can assume $t = \emptyset$. For $s \in [\mathbb{N}]^{<\omega}$ and $u \in T_P$ so that $\max(s) \leq |u| = m$ define a dense open subset of $\mathcal{P} \times \mathcal{H}$

$$\mathcal{D}_s = \{\langle Q, A\rangle \colon \ A \in \mathcal{H} \quad \text{and} \quad Q \times [s, A] \cap \mathcal{A}_n = \emptyset \text{ for all } n \leq m\}$$

then an application of Lemma 2.1 gives us $\langle Q, A\rangle \in \mathcal{D}_\infty$ so that the set $Q \times [t, A]$ avoids $\bigcup \mathcal{A}_n$.

Let $\mathcal{A}_n$ be a sequence of perfectly $\mathcal{H}$-Ramsey sets and fix $[t, B] \in \mathrm{Exp}(\mathcal{H})$ and $P \in \mathcal{P}$. Then either there are $C \in \mathcal{H} \upharpoonright B$ and a perfect $Q \subseteq P$ such that $Q \times [t, C]$ is included in some $\mathcal{A}_n$ or a proof similar to the above gives $C \in \mathcal{H} \upharpoonright B$ and a perfect $Q \subseteq P$ such that $Q \times [t, C]$ is disjoint from $\bigcup \mathcal{A}_n$. □

Lemma 3.5 If $\mathcal{H}$ is semiselective, then the $\mathcal{H}$-Ramsey property is equivalent to the abstract $\mathrm{Exp}(\mathcal{H})$-Baire property.

Proof Let $\mathcal{O}$ be a set with the abstract $\mathrm{Exp}(\mathcal{H})$-Baire property. Fix $[t, B]$ such that $B \in \mathcal{H}$. Assuming $t = \emptyset$ makes this proof only notationally simpler, therefore we will do so. For $s \in [\mathbb{N}]^{<\omega}$ let

$$\begin{aligned}\mathcal{D}_s = &\{B \in \mathcal{H} \colon [s, B] \subseteq \mathcal{O} \quad \text{or} \quad [s, B] \cap \mathcal{O} = \emptyset \\ &\text{or } [s, C] \text{ is neither included in nor disjoint from } \mathcal{O} \text{ for all } C \in \mathcal{H} \upharpoonright B\}.\end{aligned}$$

Then each $\mathcal{D}_s$ is dense open so let $C \in \mathcal{H} \upharpoonright B$ be such that $C/s \in \mathcal{D}_s$ for all $s \subseteq C$. Let $\mathcal{F}_0$ be the set of all s such that $[s, C] \subseteq \mathcal{O}$ and let $\mathcal{F}_1$ be the set of all s such that $[s, C] \cap \mathcal{O} = \emptyset$. If $\mathcal{F}_0 \upharpoonright C$ ($\mathcal{F}_1 \upharpoonright C$) includes a barrier on C, then $[\emptyset, C] \subseteq \mathcal{O}$ ($[\emptyset, C] \cap \mathcal{O} = \emptyset$, resp.), so by Theorem 2.2 we can assume $[C]^{<\omega}$ is disjoint from $\mathcal{F}_0 \cup \mathcal{F}_1$. But by the abstract Baire property of $\mathcal{O}$ there is an $s \in [C]^{<\omega}$ and $D \in \mathcal{H} \upharpoonright C$ so that $[s, D] \subseteq \mathcal{O}$ or $[s, D] \cap \mathcal{O} = \emptyset$, so $s \in \mathcal{F}_1 \cup \mathcal{F}_1$ – a contradiction. Therefore $\mathcal{O}$ is $\mathcal{H}$-Ramsey. The other direction is trivial. □

LEMMA 3.6 If $\mathcal{H}$ is semiselective, then the perfectly $\mathcal{H}$-Ramsey sets are exactly the sets with the abstract $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$-Baire property.

Proof Assume $\mathcal{O}$ has the $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$-Baire property. Fix $[t, A] \in \mathrm{Exp}(\mathcal{H})$ and $P \in \mathcal{P}$; without a loss of generality $t = \emptyset$. For $s \in [\mathbb{N}]^{<\omega}$ and $u \in T_P$ let

$$\begin{aligned}\mathcal{D}_{u,s} =\{&\langle Q, B\rangle : B \in \mathcal{H},\ Q \subseteq P \text{ is perfect, } Q \times [s, B] \subseteq \mathcal{O} \quad\text{or}\quad Q \times [s, B] \cap \mathcal{O} = \emptyset\\ &\text{or } R \times [s, C] \text{ is neither included in nor disjoint from } \mathcal{O}\\ &\text{for all } C \in \mathcal{H} \upharpoonright B \text{ and perfect } R \subseteq Q\}.\end{aligned}$$

$$\mathcal{D}_n = \bigcap_{\substack{u \in T_P \upharpoonright n \\ \max s \leq n}} \mathcal{D}_{u,s}$$

By Lemma 2.1 let $C \in \mathcal{H} \upharpoonright B$ and a perfect $R \subseteq P$ be such that $\langle R \cap [u], C/s\rangle \in \mathcal{D}_n$ for all $u \in T_R \upharpoonright n$ and $s \subseteq \{1, \ldots, n\}$. Let

$$\begin{aligned}\mathcal{F}_0 =&\{\langle u, s\rangle \in T_R \times [C]^{<\omega} : (P \cap [u]) \times [s, C] \subseteq \mathcal{O}\}\\ \mathcal{F}_1 =&\{\langle u, s\rangle \in T_R \times [C]^{<\omega} : (P \cap [u]) \times [s, C] \cap \mathcal{O} = \emptyset\}.\end{aligned}$$

If $\langle P, C\rangle$ and $\mathcal{F}_0$ ($\mathcal{F}_1$) satisfy (a) of Theorem 2.3 (with A replaced by C) then $P \times [\emptyset, C] \subseteq \mathcal{O}$ ($P \times [\emptyset, C] \cap \mathcal{O} = \emptyset$, resp.), so we can assume (b) of this theorem applies, and $\mathcal{F}_0 \cup \mathcal{F}_1$ is disjoint from $T_P \times [C]^{<\omega}$. But by the abstract $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$-Baire property of $\mathcal{O}$ this is impossible, therefore $\mathcal{O}$ is perfectly $\mathcal{H}$-Ramsey. The other direction follows by the definitions. □

LEMMA 3.7 If $\mathcal{H}$ is not semiselective, then there is an $\mathrm{Exp}(\mathcal{H})$-nowhere dense set $\mathcal{O}$ which is not $\mathcal{H}$-Ramsey.

Proof Let $\mathcal{A}_n$ be a sequence of maximal antichains in $\mathcal{H}$ with no diagonalization in $\mathcal{H}$. Let $\mathcal{O}$ be the family of all diagonalizations of this sequence; it is not $\mathcal{H}$-Ramsey because it is disjoint from $\mathcal{H}$, yet every element of $\mathcal{H}$ has an infinite subset in $\mathcal{O}$. On the other hand, $\mathcal{O}$ is $\mathrm{Exp}(\mathcal{H})$-nowhere dense: For any $[t, A] \in \mathrm{Exp}(\mathcal{H})$ we can find $n \in A$ such that A/n is not in $\mathcal{A}_n$. Pick $B \in \mathcal{A}_n$ so that $B \cap A \in \mathcal{H}$ and $m > n$ in $A \setminus B$; extend t to s so that $n, m \in s$. Then $[s, A \cap B]$ is disjoint from $\mathcal{O}$. □

Proof (Theorem 3.7) We have proved (a)⇒(b),(a)⇒(e) and (c)⇒(a), and the implications (d)⇒(c), (g)⇒(d), (f)⇒(c), (e)⇒(b), are obvious.

(b)⇒(c) If a set is both $\mathrm{Exp}(\mathcal{H})$-nowhere dense and $\mathcal{H}$-Ramsey, then it has to be $\mathcal{H}$-Ramsey null.

(e)⇒(f) Similar to (b)⇒(c) above.

(a)⇒(d) If $\mathcal{H}$ is semiselective, then by Lemma 1 $\mathcal{H}$-Ramsey null sets form a σ-ideal, and by (c) all $\mathrm{Exp}(\mathcal{H})$-nowhere dense sets are $\mathcal{H}$-Ramsey; therefore $\mathrm{Exp}(\mathcal{H})$-nowhere dense sets form a σ-ideal, and they coincide with $\mathrm{Exp}(\mathcal{H})$-meager sets.

(a)⇒(g) This proof is analogous to the proof of (a)⇒(d) above. □

To prove Theorem 2.1, we will make use of the following classical result whose proof can be found e.g. in [30], [31] or [28]. If $\mathcal{B}$ is a σ-algebra of subsets of $[\mathbb{N}]^\omega$ and $\mathcal{B}_0$ is its σ-ideal, we say that a pair $\langle \mathcal{B}, \mathcal{B}_0 \rangle$ is a *Marczewski pair* if for every $\mathcal{X} \subseteq [\mathbb{N}]^\omega$ there is $\Phi(\mathcal{X}) \supseteq \mathcal{X}$ in $\mathcal{B}$ such that $\mathcal{Y} \subseteq \Phi(\mathcal{X}) \setminus \mathcal{X}$ is in $\mathcal{B}$ iff it is in $\mathcal{B}_0$.

THEOREM 3.8 (Marczewski) An algebra $\mathcal{B}$ which together with one of its σ-ideals forms a Marczewski pair is closed under the $\mathcal{A}$-operation. □

Proof (Theorem 2.1**b)**) By Theorem 2.2 open sets are $\mathcal{H}$-Ramsey, so it suffices to prove that $\mathcal{R}(\mathcal{H})$ is closed under the $\mathcal{A}$-operation. The following is a version of the proof given by Todorcevic in [28, Section 9] for the case when $\mathcal{H}$ is selective. The first step is to prove that the Continuum Hypothesis, CH, implies $\langle \mathcal{R}(\mathcal{H}), \mathcal{R}_0(\mathcal{H}) \rangle$ is a Marczewski pair. The second step makes use of the following well-known result of Platek ([34]):

THEOREM 3.9 The use of the Continuum Hypothesis can be eliminated from the proof of any statement involving only quantification over the reals and possibly some fixed set of reals as a predicate. □

CLAIM If CH holds and $\mathcal{H}$ is a semiselective coideal, then $\langle \mathcal{R}(\mathcal{H}), \mathcal{R}_0(\mathcal{H}) \rangle$ is a Marczewski pair.

Proof First observe that Lemma 3.9 implies the following:

(M) If $[s_n, A_n]$ $(n \in \mathbb{N})$ and $[t, B]$ are in $\mathrm{Exp}(\mathcal{H})$ and the set $[t, B] \setminus \bigcup_{n \in \mathbb{N}} [s_n, A_n]$ is $\mathrm{Exp}(\mathcal{H})$-nowhere dense, then for some n the set $[t, B] \cap [s_n, A_n]$ is not $\mathrm{Exp}(\mathcal{H})$-nowhere dense.

Since CH implies that size of $\mathrm{Exp}(\mathcal{H})$ equals $\aleph_1$, (M) says that, in the terminology of [19], $\mathrm{Exp}(\mathcal{H})$ is *M-like*. Therefore, by Lemma 3.1 and [19, §2] the pair $\langle \mathcal{R}(\mathcal{H}), \mathcal{R}_0(\mathcal{H}) \rangle$ is a Marczewski pair. □

Now fix $\mathcal{H}$-Ramsey sets F_s $(s \in \mathbb{N}^{<\omega})$. Statement "$\mathcal{A}(F_s)$ is not $\mathcal{H}$-Ramsey" is false under CH by Claim and Marczewski's theorem, and it is of the form to which Platek's theorem applies. Therefore the set $\mathcal{A}(F_s)$ is $\mathcal{H}$-Ramsey, as required. □

REMARK 3.1 A natural question is whether one can prove $\langle \mathcal{R}(\mathcal{H}), \mathcal{R}_0(\mathcal{H}) \rangle$ is a Marczewski pair without the use of an additional set-theoretic axiom such as CH. The answer to this question is negative: There can be a selective coideal without this property. More precisely, Alan Dow observed ([33]) that under some additional Set-Theoretic axiom (see [37, §8]) there is a closed subset X of the remainder of the Čech–Stone compactification of the integers, $\mathbb{N}^*$, and two relatively open disjoint subsets U, V of X such that whenever U' and V' are open subsets of $\mathbb{N}^*$ such that $U' \cap X = U$ and $V' \cap X = V$, then U' and V' are not disjoint. This X can moreover be chosen so that the corresponding coideal $\mathcal{H} = \bigcup X$ is selective, and it is not difficult to use U and V as above to construct a subset of $\mathcal{P}(\mathbb{N})$ witnessing that $\langle \mathcal{R}(\mathcal{H}), \mathcal{R}_0(\mathcal{H}) \rangle$ is not a Marczewski pair.

Now we turn to the perfectly $\mathcal{H}$-Ramsey part of Theorem 2.1.

Proof (Theorem 2.1.**c)**) In Theorem 2.3 we proved that open sets are perfectly $\mathcal{H}$-Ramsey, so it remains to prove $\mathcal{PR}(\mathcal{H})$ is closed under the $\mathcal{A}$-operation. The scenario is analogous to one used in the proof of the part **b)** above.

CLAIM If CH holds and $\mathcal{H}$ is a semiselective coideal, then $\langle \mathcal{PR}(\mathcal{H}), \mathcal{PR}_0(\mathcal{H}) \rangle$ is a Marczewski pair.

Proof Using Pawlikowski's terminology introduced before, $\mathcal{P}$ is M-like (by [19, Lemma 28]) so since $\mathrm{Exp}(\mathcal{H})$ is M-like and the sizes of $\mathcal{P}$ and $\mathrm{Exp}(\mathcal{H})$ are equal, $\mathcal{P} \times \mathrm{Exp}(\mathcal{H})$ is M-like as well ([19, Lemma 2.7]) therefore by Lemma 3.2 and [19, §2] the conclusion follows. □

By Marczewski's theorem and Claim, CH implies the desired conclusion. By Platek's theorem, the use of CH can be eliminated from this proof. □

Note that the $\mathrm{Exp}(\mathcal{H})$-Baire property implies the abstract $\mathrm{Exp}(\mathcal{H})$-Baire property only when $\mathcal{H}$ is an ultrafilter or when its dual ideal is *nowhere dense* – i.e. for every $A \in \mathcal{H}$ there is a $B \in \mathcal{H} \restriction A$ such that $[B]^\omega$ is included in $\mathcal{H}$. This is because if there are disjoint A and B in $\mathcal{H}$ such that for every $C \in \mathcal{H} \restriction B$ the set $[C]^\omega$ is not included in $\mathcal{H}$ then the set

$$\mathcal{O} = \bigcup \{[A \cup C]^\omega : C \subseteq B \quad \text{and} \quad C \notin \mathcal{H}\}$$

is $\mathrm{Exp}(\mathcal{H})$-open, yet not $\mathcal{H}$-Ramsey, since for all $C \in \mathcal{H} \restriction B$ the set $[\emptyset, C]$ is neither disjoint from nor included in $\mathcal{O}$.

We will now characterize those $\mathcal{H}$ for which there is a family $\mathcal{H}' \subseteq \mathcal{H}$ such that the $\mathrm{Exp}(\mathcal{H}')$-Baire property is equivalent to the $\mathcal{H}$-Ramsey property. Let $\mathcal{H}$ be a semiselective coideal. Consider the following statements:

(E1) $\mathcal{H}$ is *Ellentuck*: There is a dense $\mathcal{H}' \subseteq \mathcal{H}$ such that $\mathrm{Exp}(\mathcal{H}')$ forms a basis for a topology.
(E2) There is a dense $\mathcal{H}' \subseteq \mathcal{H}$ such that the $\mathcal{H}$-Ramsey property is equivalent to the $\mathrm{Exp}(\mathcal{H}')$-Baire property.
(E3) There is a dense $\mathcal{H}' \subseteq \mathcal{H}$ such that all $A, B \in \mathcal{H}'$ are either almost disjoint or their intersection is in $\mathcal{H}$.

THEOREM 3.10 The statements (E1)–(E3) are all equivalent.

Proof (E3) ⇔ (E1) By the Fact 1 above, the family $\mathcal{H}' \cup \{\mathbb{N}\}$ verifies (E1) iff the family $\mathcal{H}'$ verifies (E3).

(E3)⇒(E2) Let $\mathcal{H}'$ be as in (E3); we claim that it verifies (E2) as well. Obviously all $\mathcal{H}$-Ramsey sets have the $\mathrm{Exp}(\mathcal{H}')$-Baire property, so we can concentrate on proving the other direction.

CLAIM 1 $\mathrm{Exp}(\mathcal{H}')$-open sets are $\mathcal{H}$-Ramsey.

Proof Let $\mathcal{O}$ be an $\mathrm{Exp}(\mathcal{H}')$-open set and fix $[t, B]$ for $B \in \mathcal{H}$. For $s \sqsupseteq t$ define a dense open subset of $\mathcal{H}$ by

$$\mathcal{D}_s = \{A \in \mathcal{H} \colon [s, A] \subseteq \mathcal{O} \text{ or } [s, C] \nsubseteq \mathcal{O} \text{ for all } C \in \mathcal{H} \restriction A\}.$$

By the semiselectiveness, there is a $C \in \mathcal{H}' \restriction B$ such that $C \in \mathcal{D}_s$ for all $s \sqsupseteq t$ included in C. We claim that $\mathcal{O}$ is relatively open on $[t, C]$ even in the standard metric topology, more precisely if we let $\mathcal{F} = \{s \colon [s, C] \subseteq \mathcal{O}\}$ then $\mathcal{O} \cap [t, C] = \bigcup_{s \in \mathcal{F}} [s] \cap [t, C]$. Only "$\subseteq$" requires a proof, so pick $D \in \mathcal{O} \cap [s, C]$. Then there is a $[u, D'] \subseteq \mathcal{O}$ such that $D' \in \mathcal{H}'$ and $D \in [u, D']$; but by our assumption on $\mathcal{H}'$ the set $C \cap D'$ is in $\mathcal{H}$ and therefore $u \in \mathcal{F}$, so $D \in \bigcup_{s \in \mathcal{F}} [s]$, as required.

Since by Theorem 2.1 open sets in the metric topology are $\mathcal{H}$-Ramsey, this ends the proof. □

Let $\mathcal{A}$ be a set with the $\mathrm{Exp}(\mathcal{H}')$-Baire property. Then there is an $\mathrm{Exp}(\mathcal{H}')$-open set $\mathcal{O}$ such that $\mathcal{A}\Delta\mathcal{O}$ is $\mathrm{Exp}(\mathcal{H}')$-meager; but then $\mathcal{A}\Delta\mathcal{O}$ is $\mathrm{Exp}(\mathcal{H}')$-nowhere dense and the $\mathcal{H}$-Ramsey property of $\mathcal{A}$ reduces to the $\mathcal{H}$-Ramsey property of $\mathcal{O}$, i.e. to Claim 1.

(E2)⇒(E3) Assume $\mathcal{H}'$ satisfies (E2) and enumerate $\mathcal{H}'$ as $\{A_\xi\}_{\xi<\lambda}$. Then recursively pick a sequence B_ξ $(\xi < \lambda)$ such that for all $\xi < \eta < \lambda$ we have

(1) $B_\xi \in \mathcal{H}'$,
(2) $B_\xi \subseteq A_\xi$,
(3) $B_\eta \subseteq^* B_\xi$ or B_η and B_ξ are almost disjoint.

Set B_η is constructed by applying (E2) to sets $\mathcal{O} = \bigcup_{\xi<\eta}[B_\xi]^\omega$ and $[\emptyset, A_\eta]$. Then $\mathcal{H}'' = \{B_\xi\}_{\xi<\lambda}$ satisfies (E3). □

COROLLARY 3.11 Every selective ultrafilter is Ellentuck. □

We can say slightly more, namely

LEMMA 3.12 If in $\mathcal{H}$ there are no uncountable antichains then $\mathcal{H}$ is Ellentuck.

Proof In the case when $\mathcal{H} \restriction A$ does not have the f.i.p. for all $A \in \mathcal{H}$ the proof proceeds by transfinite induction and the construction of $\mathcal{H}'$ as in (E1) is very similar to that of Example 2.2. To prove Lemma in the general case, let

$$\mathcal{D}_0 = \{A \in \mathcal{H}\colon \mathcal{H} \restriction A \text{ has the f.i.p.}\}.$$
$$\mathcal{D}_1 = \{A \in \mathcal{H}\colon \text{ no set in } \mathcal{H} \restriction A \text{ is in } \mathcal{D}_0\}.$$

Then $\mathcal{D}_0 \cup \mathcal{D}_1$ is dense open in $\mathcal{H}$, so we can pick an almost disjoint family A_n $(n < \omega)$ of sets in $\mathcal{H}$ so that each A_n is in $\mathcal{D}_0 \cup \mathcal{D}_1$. Then by the above and Corollary 3.1 there is a dense $\mathcal{H}'_n$ in $\mathcal{H} \restriction A_n$ for each n which satisfies (c); let $\mathcal{H}'$ be the union of all $\mathcal{H}'_n$'s. it is easy to check that $\mathcal{H}'$ is as required. □

LEMMA 3.13 The Continuum Hypothesis implies that every semiselective coideal $\mathcal{H}$ whose complement is a P-ideal is Ellentuck.

Proof This follows by [8; Theorem 2]. □

4. SEMISELECTIVITY AND FORCING

The reader is referred to [12] for the general theory of forcing, and to [32] for Perfect-set (or Sacks) forcing. Recall that if $\mathcal{H}$ is a selective coideal, then forcing by $\langle \mathcal{H}, \subseteq^* \rangle$ adjoins no new reals to the universe and its generic filter is a selective ultrafilter. We will prove that semiselective coideals still retain this desirable property. From now on, $\mathcal{H}$ will always denote a semiselective coideal.

LEMMA 4.1 The following are equivalent for a family $\mathcal{H}$ of sets of integers:

(a) $\mathcal{H}$ generates a semiselective coideal.
(b) Forcing by $\langle \mathcal{H}, \subseteq^* \rangle$ adjoins no new reals and its generic filter is a selective ultrafilter.

Proof (⇒) By our definition, the poset $\langle \mathcal{H}, \subseteq^* \rangle$ is σ-distributive, so it adds no new reals. Let $\dot{\mathcal{U}}$ be a name for a generic subset of $\mathcal{H}$; by genericity it is an ultrafilter, and the semiselective version of Nash-Williams; Galvin Lemma implies that every partition of $[\mathbb{N}]^2$ into two pieces has a homogeneous set in $\dot{\mathcal{U}}$. By Theorem 2.4, $\dot{\mathcal{U}}$ is selective.

($\Leftarrow$) If $\mathcal{H}$ does not generate a semiselective coideal, ten there is a sequence $\{\mathcal{A}_n\}$ of maximal antichains in $\langle \mathcal{H}, \subseteq^* \rangle$ with no diagonalization in $\mathcal{H}$. In the forcing extension by $\langle \mathcal{H}, \subseteq^* \rangle$ let A_n be the unique element of $\mathcal{A}_n \cap \dot{\mathcal{U}}$; then the sequence $\{A_n\}$ has no diagonalization in $\dot{\mathcal{U}}$, and therefore $\dot{\mathcal{U}}$ is not selective. $\square$

The $\mathcal{H}$-Ramsey part of Theorem 2.1.**b)** and **c)** easily follows from Lemma 4.1 by an absoluteness argument (see [16; §4]), but we choose to give a more informative proof here. We need some definitions first.

DEFINITION 4.2 *If $\mathcal{H}$ is a coideal then let $\mathcal{M}_{\mathcal{H}}$ be the Mathias poset associated to $\mathcal{H}$. Conditions are pairs $\langle s, A \rangle$ such that* $\max s < \min A$ *and $A \in \mathcal{H}$. The ordering is defined by*

$$\langle s, A \rangle \leq \langle t, B \rangle \quad \text{iff} \quad s \sqsupseteq t, \quad A \subseteq B \quad \text{and} \quad s \setminus t \subseteq B.$$

If $\mathcal{H}$ is an ultrafilter poset $\mathcal{M}_{\mathcal{H}}$ is denoted by $\mathcal{P}_{\mathcal{H}}$ and called the Prikry poset.

Poset $\mathcal{M}_{\mathcal{H}}$ has the Prikry property if for every sentence of the forcing language ϕ and every condition $\langle s, A \rangle$ there is a $B \in \mathcal{H} \restriction A$ such that $\langle s, B \rangle$ decides ϕ. It has the Mathias property if every subset of an $\mathcal{M}_{\mathcal{H}}$-generic set is $\mathcal{M}_{\mathcal{H}}$-generic.

Let $\dot{\mathcal{U}}$ be a canonical name for the $\langle \mathcal{H}, \subseteq^* \rangle$-generic ultrafilter. Then it is a routine to check that the poset $\mathcal{M}_{\mathcal{H}}$ is equivalent to the two-step iteration of posets $\langle \mathcal{H}, \subseteq^* \rangle$ and $\mathcal{P}_{\dot{\mathcal{U}}}$, hence it is of the form σ-distributive$*$ccc.

In the case when $\mathcal{H}$ is a normal ultrafilter on a measurable cardinal, ($\mathcal{M}2$) of the following theorem was introduced and proved by Prikry in [20] and ($\mathcal{M}3$) was introduced and proved by Mathias in [15], while ($\mathcal{M}2$) and ($\mathcal{M}3$) for a selective coideal on $\mathbb{N}$ were proved by Mathias in [16]. As far as we know, even the equivalence of ($\mathcal{M}2$) and ($\mathcal{M}3$) below is new.

THEOREM 4.3 For a coideal $\mathcal{H}$ the following are equivalent:

($\mathcal{M}1$) $\mathcal{H}$ is semiselective.
($\mathcal{M}2$) $\mathcal{M}_{\mathcal{H}}$ has the Prikry property.
($\mathcal{M}3$) $\mathcal{M}_{\mathcal{H}}$ has the Mathias property.

Proof ($\mathcal{M}1$)$\Rightarrow$($\mathcal{M}2$) Assume $\mathcal{H}$ is semiselective, and fix a sentence of the forcing language ϕ and a condition $\langle s, A \rangle$ in $\mathcal{M}_{\mathcal{H}}$. For $t \sqsupseteq s$ let $\mathcal{D}_t$ be the set of all $B \in \mathcal{H} \restriction A$ such that either $\langle t, B \rangle$ decides ϕ or $\langle t, C \rangle$ does not decide ϕ for all $C \in \mathcal{H} \restriction B$. Sets $\mathcal{D}_t$ are dense open in $\langle \mathcal{H}, \subseteq^* \rangle$, so let $B \in \mathcal{H} \restriction A$ be such that $B/n \in \mathcal{D}_t$ for all $n \in B$ and all $t \subseteq \{1, \dots, n\}$. Let

$$\mathcal{F}_1 = \{t \sqsupseteq s : \langle t, B \rangle \text{ forces } \phi\}$$
$$\mathcal{F}_2 = \{t \sqsupseteq s : \langle t, B \rangle \text{ forces } \neg\phi\}.$$

Then let $C \in \mathcal{H} \restriction B$ be as guaranteed by the semiselective Nash-Williams; Galvin Lemma as applied to $\mathcal{F}_1$ and to $\mathcal{F}_2$. We claim that $\langle s, C \rangle$ decides ϕ. To prove this, it suffices to check that if $\langle t_1, C_1 \rangle$ and $\langle t_2, C_2 \rangle$ are different extensions of $\langle s, C \rangle$ deciding ϕ, then they both decide it in the same way. So assume the contrary, that $\langle t_1, C_1 \rangle$ forces ϕ and $\langle t_2, C_2 \rangle$ forces its negation. Then $t_1 \in \mathcal{F}_1$ and $t_2 \in \mathcal{F}_2$, and both are finite subsets of C. This together with the choice of C means that every infinite subset of C has an initial segment both in $\mathcal{F}_1$ and in $\mathcal{F}_2$; so we can pick two compatible conditions one of which forces ϕ and the other forces its negation – a contradiction. So $\langle s, C\rangle$ does decide ϕ.

$(\mathcal{M}1)\Rightarrow(\mathcal{M}3)$ Assume $\mathcal{H}$ is semiselective. In Lemma 4.2 below we prove a statement which immediately implies $(\mathcal{M}3)$. For $X \subseteq \mathbb{N}$ let

$$G(X) = \{A \in \mathcal{H} \colon X \subseteq^* A\}$$
$$G_{\mathcal{M}}(X) = \{\langle s, A\rangle \in \mathcal{M}_{\mathcal{H}} \colon X \in [s, A]\}.$$

Lemma 4.4 The filter $G(X)$ is $\mathcal{H}$-generic if and only if the filter $G_{\mathcal{M}}(X)$ is $\mathcal{M}_{\mathcal{H}}$-generic (over a model of a large enough part of ZFC, of course).

Proof $(\Rightarrow)$ Follows from the decomposition of Mathias forcing $\mathcal{M}_{\mathcal{H}}$ as $\mathcal{H} * \mathcal{P}_{\dot{\mathcal{U}}}$.

$(\Leftarrow)$ For a dense open subset $\mathcal{E}$ of $\mathcal{M}_{\mathcal{H}}$ let $\mathcal{D}(\mathcal{E})$ be the family of all $A \in \mathcal{H}$ such that the family

$$\mathcal{F} = \{s \colon \langle s, A\rangle \in \mathcal{E}\}$$

includes a barrier on A.

Claim If $\mathcal{E}$ is dense open in $\mathcal{M}_{\mathcal{H}}$ then $\mathcal{D}(\mathcal{E})$ is dense open in $\langle \mathcal{H}, \subseteq^*\rangle$.

Proof Pick $\langle s, A\rangle \in \mathcal{H}$, and let $\mathcal{D}_t$ (for $t \sqsupseteq s$) be the set of all $B \in \mathcal{H} \restriction A$ such that either $\langle t, B\rangle$ is in $\mathcal{E}$ or $\langle t, C\rangle$ is not in $\mathcal{E}$ for all $C \in \mathcal{H} \restriction B$. Sets $\mathcal{D}_t$ are dense open in $\langle \mathcal{H} \restriction A, \subseteq^*\rangle$, so let $B \in \mathcal{H} \restriction A$ be their diagonalization. By the semiselective Nash-Williams; Galvin Lemma applied to the family

$$\mathcal{G} = \{s \colon \langle s, B\rangle \in \mathcal{E}\}$$

get $C \in \mathcal{H} \restriction B$ such that either C is in $\mathcal{D}(\mathcal{E})$ or no finite subset of C is in $\mathcal{G}$. If the second possibility applies, then we find $\langle t, D\rangle \in \mathcal{E}$ which extends $\langle \emptyset, C\rangle$. Let $n = \min C/t$ and $D \in \mathcal{D}_t$ be such that $B/n \subseteq D$; then we must have $\langle t, B\rangle \in \mathcal{E}$ and therefore $t \in \mathcal{G}$ – a contradiction. So the first possibility applies here and C is in $\mathcal{D}(\mathcal{E})$. □

If $G(X)$ is $\mathcal{H}$-generic, then by Claim for every dense open subset $\mathcal{E}$ of $\mathcal{M}_{\mathcal{H}}$ there is a $B \in \mathcal{D}(\mathcal{E}) \cap G(X)$ such that $X \subseteq A$. The family $\mathcal{F}$ defined as above remains a barrier in any forcing extension (because well-foundedness is absolute) so there is a $t \sqsubset X$ such that $\langle t, A\rangle$ is in $\mathcal{E}$. But $\langle t, A\rangle$ is in $G_{\mathcal{M}}(X)$, so $G_{\mathcal{M}}(X)$ is generic. □

$\neg(\mathcal{M}1) \Rightarrow \neg(\mathcal{M}2), \neg(M3)$. Assume $\mathcal{H}$ is not semiselective, and let $\mathcal{A}_n$ be a sequence of maximal antichains in $\langle \mathcal{H}, \subseteq^*\rangle$ such that no element of $\mathcal{H}$ is a diagonalization of this sequence. Let $\dot{X}$ be a canonical name for an $\mathcal{M}_{\mathcal{H}}$-generic subset of $\mathbb{N}$, and let $\dot{A}_n$ be a name for the (unique) element of $\mathcal{A}_n$ such that $\dot{X} \subseteq^* \dot{A}_n$.

To see that $(\mathcal{M}3)$ fails, we claim that $\dot{X}$ is forced not to be a diagonalization of this sequence. Suppose the contrary, that $\langle t, B\rangle$ forces

$$\dot{X}/n \subseteq \dot{A}_n \text{ for all } n \in \dot{X}.$$

Pick $n \in B/t$ such that B/n is not included in any element of $\mathcal{A}_n$, so there is $A_n \in \mathcal{A}_n$ such that $B \cap A_n \in \mathcal{H}$ and there is $m \in A/n \setminus A_n$. Then the condition

$\langle t \cup \{n, m\}, B \cap A_n \rangle$ forces that $\dot{A}_n = A_n$ and that $m \in \dot{X}/n \setminus A_n$ – a contradiction. Therefore $\dot{X}$ is forced not to be a diagonalization of $\{\dot{A}_n\}$. But $\dot{X}$ has an infinite subset Y which is a diagonalization of $\{\dot{A}_n\}$, and the latter set can not be $\mathcal{M}_h$-generic.

To see that ($\mathcal{M}2$) fails, assume further that $\mathcal{A}_{n+1}$ *refines* $\mathcal{A}_n$ for all n; namely, that every element of $\mathcal{A}_{n+1}$ is included in a unique element of $\mathcal{A}_n$. Let ϕ be the statement:

(ϕ) If $\dot{n}$ is the least $n \in \dot{X}$ such that $\dot{X}/n$ is not included in $\dot{A}_n$, then the size $\dot{k}$ of the set $\dot{X} \cap \{1, \ldots, n-1\}$ is an even number.

Since $\dot{X}$ does not diagonalize the sequence $\mathcal{A}_n$, $\dot{n}$ is well-defined. Assume $\mathcal{M}_\mathcal{H}$ has the Prikry property, so let $\langle \emptyset, A \rangle$ be a condition which decides ϕ. Find $n_1 < m_1 < n_2 < m_2$ as follows: n_1 is the minimal $n \in A$ such that A/n is not included in any member of $\mathcal{A}_n$. Let $A_1 \in \mathcal{A}_n$ be such that $A \cap A_1$ is in $\mathcal{H}$, and find $m_1 > n_1$ in $A \setminus A_1$. The set $A' = A \cap A_1 \setminus \{1, \ldots, m_1\}$ does not diagonalize $\mathcal{A}_n$, so let n_2 be the minimal $n \in A'$ such that A'/n is not included in any member of $\mathcal{A}_n$. Pick $A_2 \in \mathcal{A}_n$ and let $m_2 > n_2$ in $A' \setminus A_2$. Let $A'' = A' \cap A_2 \setminus \{1, \ldots, m_2\}$. Then $\langle \{n_1, m_1\}, A' \rangle$ decides $\dot{n} = n_1$ and $\dot{k} = 0$ (so ϕ is forced), while $\langle \{n_1, n_2, m_2\}, A'' \rangle$ decides $\dot{n} = n_2$ and $\dot{k} = 1$ (so $\neg\phi$ is forced). □

Remark The importance of the above theorem is twofold – on the one hand, it gives us a new forcing notion with nice properties which is not of the usual form σ-closed$*$ccc, and which even need not be proper – see Example 4.1 below. On the other hand, it gives us an alternative definition of semiselectivity and shows that this is a very natural notion. The following fact is also worth mentioning

Corollary 4.5 If $\mathcal{M}_\mathcal{H}$ has the Prikry property and/or the Mathias property then in the Perfect-set forcing extension it retains this property.

Proof This follows by Theorem 4.5 and by the fact that Perfect-set forcing preserves semiselectivity (Theorem 4.2 below). □

Proof (Theorem 2.1., $\mathcal{H}$-Ramsey part)

b) Let $\mathcal{A}$ be an analytic set and let $[t, C]$ be such that $C \in \mathcal{H}$; we can, and will, assume $t = \emptyset$ and $C = \mathbb{N}$. There is a tree $T \subseteq [\mathbb{N}]^{<\omega} \times \mathbb{N}^{<\omega}$ such that B is in $\mathcal{A}$ iff the tree

$$T_B = \{s \in \mathbb{N}^{<\omega} : \langle B \cap |s|, s \rangle \in T\}$$

has an infinite branch. Find $A \in \mathcal{H}$ such that the condition $\langle \emptyset, A \rangle$ decides whether $T_{\dot{X}}$ ($\dot{X}$ is a name for an $\mathcal{M}_\mathcal{H}$-generic set) has an infinite branch or not. So in the first case let $\dot{f}$ be a name for the leftmost branch of $T_{\dot{X}}$ and in the second case let $\dot{H}$ be a name for the rank mapping witnessing that $T_{\dot{X}}$ is well-founded, i.e. an order-reversing mapping from $T_{\dot{X}}$ into ordinals. Since the tree T is countable, there is a a countable sequence of dense open subsets $\{\mathcal{E}_n\}$ of $\mathcal{M}_\mathcal{H}$ such that if $G = G_\mathcal{M}(X)$ intersects all of them then $\mathrm{val}_G(\dot{f})$ ($\mathrm{val}_G(\dot{H})$, resp.) is an infinite branch through T_X (is a rank mapping for T_X, resp.). It remains to find $X \in \mathcal{H}$ such that $G_\mathcal{M}(X)$ is sufficiently generic in the above sense. But by Lemma 4.5 it suffices that $G(X)$ intersects dense open subsets $\mathcal{D}(\mathcal{E}_n)$ of $\mathcal{H}$, and therefore by the semiselectivity there is $X \in \mathcal{H}$ such that $G_\mathcal{M}(X)$ intersects all $\mathcal{E}_n$'s. Fix such an X.

If $\langle \emptyset, A \rangle$ forces $T_{\dot{X}}$ has an infinite branch, then the $G_\mathcal{M}(X)$-interpretation f of $\dot{f}$ in $G_\mathcal{M}(X)$ is an infinite branch of T_X, so $X \in \mathcal{A}$. If $\langle \emptyset, A \rangle$ forces $T_{\dot{X}}$ is well-founded, then the $G_{\mathcal{M}}(X)$-interpretation H of $\dot{H}$ witnesses T_X is well-founded, so $X \notin \mathcal{A}$. Moreover, by the Mathias property the same applies to all infinite subsets of X, and therefore $[X]^\omega$ is either included in or disjoint from $\mathcal{A}$.

c) Assume $\langle F_s : s \in \mathbb{N}^{<\omega}\rangle$ are $\mathcal{H}$-Ramsey sets and $[t, A]$ is such that $A \in \mathcal{H}$. We need to provide $B \in \mathcal{H} \restriction A$ such that $[t, B]$ is either included in or disjoint from $\mathcal{A}(F_s)$, and it is easy to see that we can assume $t = \emptyset$ and $A = \mathbb{N}$. For $n \in \mathbb{N}$ and $s \in \mathbb{N}^{<\omega}$ let

$$\begin{aligned}\mathcal{D}_{n,s} &= \{B \in \mathcal{H} : \forall u \subseteq \{1, \dots, n\}[u, B] \subseteq F_s \text{ or } [u, B] \cap F_s = \emptyset\}\\ \mathcal{E}_n &= \bigcap_{|s|\leq n} \mathcal{D}_{n,s}.\end{aligned}$$

Since each F_s is $\mathcal{H}$-Ramsey, sets $\mathcal{E}_n$ are dense open in $\mathcal{H}$, so they have a diagonalization B in $\mathcal{H}$. Then F_s is clopen when restricted to $[B]^\omega$ for all s, so $\mathcal{A}(F_s)$ is analytic when restricted to $[B]^\omega$ and by part **b)** there is a $C \in \mathcal{H} \restriction B$ such that $[C]^\omega$ is either included in or disjoint from $\mathcal{A}(F_s)$. □

Now we turn to the forcing proof of perfectly $\mathcal{H}$-Ramsey part of Theorem 2.1. This reduces to proving that semiselective coideals generate semiselective coideals in the Perfect-set forcing extension. For definitions see [13] or [26].

THEOREM 4.6 If $\mathcal{H}$ is a semiselective coideal then in the Perfect-set forcing extension it generates a semiselective coideal.

Proof By Lemma 4.1, it suffices to prove that the iteration Sacks$*\langle \mathcal{H}, \subseteq^*\rangle$ forces $\dot{\mathcal{U}}$ is selective. But this iteration can be considered as the product of $\langle \mathcal{H}, \subseteq^*\rangle$ and Sacks because the definition of $\langle \mathcal{H}, \subseteq^*\rangle$ is not changed by the Sacks forcing, and therefore the conclusion follows from the well-known fact that Sacks forcing preserves selective ultrafilters (see Theorem 1.1). □

COROLLARY 4.7 In the Perfect-set forcing extension the family of ground-model sets of integers diagonalizes all Borel partitions.

Proof Clearly, the coideal $[\mathbb{N}]^\omega$ is selective, so by Theorem 4.7 it remains semiselective in an extension, and therefore by Theorem 2.1.**b)** the conclusion follows. □

REMARK 4.1 All of the above results are true in the case of adding any number of side-by-side Sacks reals with countable supports; this can be proved by using the above methods, or by the direct fusion argument using the infinitary version of the Halpern–Laüchli theorem proved by Laver ([13]), or more precisely, its semiselective version that we will now formulate. If $\{T_n\}$ is a sequence of perfect trees and $A \subseteq \mathbb{N}$ then by $\bigotimes_{n=1}^{\infty} T_n \restriction A$ we denote the set $\bigcup_{i\in A} \prod_{n=1}^{\infty} T_n(i)$ (here $T(i)$ denotes the i-th level of the tree T). For a coideal $\mathcal{H}$ consider the following statement:

$\mathrm{HL}_\omega(\mathcal{H})$ For every $A \in \mathcal{H}$, sequence of perfect trees T^n, and $f : \bigotimes_{n=1}^{\infty} T^n \to \{0, 1\}$ there is a $B \in \mathcal{H} \restriction A$ and perfect subtrees $P^n \subseteq T^n$ such that f assumes a constant value on $\bigotimes_{n=1}^{\infty} P^n \restriction B$.

Then $\mathrm{HL}_\omega(\mathcal{H})$ for some coideal $\mathcal{H}$ is equivalent to the following:

> $\mathcal{H}$ generates a coideal in any forcing extension by a countable support product of Perfect-set forcing.

[An explanation of this equivalence in the case when $\mathcal{H} = [\mathbb{N}]^\omega$ is given in [13], where (1) and (2) correspond respectively to $\mathrm{HL}_\omega(\mathcal{H})$ and the above preservation statement.] We note that $\mathrm{HL}_\omega(\mathcal{H})$ is true whenever $\mathcal{H}$ is semiselective by Theorem 6.8 of [26], where $\mathrm{HL}_\omega(\mathcal{H})$ for a selective ultrafilter $\mathcal{H}$ was proved.

Let us recall some definitions before we turn to the proof of the remaining part of Theorem 2.1. To every analytic subset $\mathcal{A}$ of $\{0,1\}^\omega \times [\mathbb{N}]^\omega$ we associate the tree $T_\mathcal{A}$ on $\{0,1\}^{<\omega} \times [\mathbb{N}]^{<\omega} \times \mathbb{N}^{<\omega}$ such that $\mathcal{A}$ equals the set of all $\langle x, B\rangle$ for which the tree

$$T_{x,B} = \{s \in \mathbb{N}^{<\omega} : \langle x \restriction |s|, B \cap \{1,\dots,|s|\}, s\rangle \in T\}$$

has an infinite branch. This gives a name for the set $\mathcal{A}$ in a forcing extension. Note that a pair $\langle y, A\rangle$ is not in $\mathcal{A}$ if and only if tree $T_{y,B}$ is well-founded, if and only if there are a countable ordinal α and a strictly decreasing mapping H of $T_{y,A}$ into α. The minimal such α is called the *rank* of $T_{y,a}$.

Proof (Theorem 2.1, perfectly $\mathcal{H}$-Ramsey part)

b) Fix an analytic set $\mathcal{A} \subseteq \{0,1\}^\omega \times [\mathbb{N}]^\omega$, a perfect $P \subseteq \{0,1\}^\omega$, and $[t, A]$ $(A \in \mathcal{H})$. We have to find a perfect $Q \subseteq P$ and $B \in \mathcal{H} \restriction A$ such that $Q \times [s, B]$ is either included in or disjoint from $\mathcal{A}$. We can, and will, assume $P = \{0,1\}^\omega$, $t = \emptyset$ and $A = \mathbb{N}$. Let $\dot{y}$ $(\dot{X})$ be the canonical name for a Sacks (Mathias, resp.) -generic real. By Theorem 4.2 and the already proved $\mathcal{H}$-Ramsey part of Theorem 2.1, there is a perfect Q and $B \in \mathcal{H}$ such that P forces set $\{\dot{y}\} \times [B]^\omega$ is either included in or disjoint from $\mathcal{A}$. Like in the proof of $\mathcal{H}$-Ramsey part of this theorem above, let T be the tree on $\{0,1\}^{<\omega} \times [\mathbb{N}]^{<\omega} \times \mathbb{N}^{<\omega}$ such that

$$\mathcal{A} = \{\langle y, X\rangle : T_{y,X} \text{ has an infinite branch}\}.$$

There are countably many dense open subsets $\{\mathcal{D}_n\}$ of the product Sacks$\times\mathcal{M}_\mathcal{H}$ relevant to the decision of the statement "$\langle \dot{y}, \dot{X}\rangle \in \mathcal{A}$", namely these are the sets deciding the infinite branch of the tree $T_{\dot{y},\dot{X}}$ or a strictly decreasing mapping of this tree into some countable ordinal. By the preservation of semiselective coideals by Sacks forcing (Theorem 4.2) and Lemma 4.2, we can translate $\mathcal{D}_n$'s to dense open subsets $\mathcal{E}_n$ of the poset Sacks$\times\mathcal{H}$, so that $\langle \dot{y}, \dot{X}\rangle$ is $\langle \mathcal{D}_n$,Sacks$\times\mathcal{M}_\mathcal{H}\rangle$-generic if and only if $\langle \dot{y}, G(\dot{X})\rangle$ is $\langle \mathcal{E}_n$,Sacks$\times\mathcal{H}\rangle$-generic. By Lemma 2.1, there are a perfect $R \subseteq Q$ and $C \in \mathcal{H} \restriction B$ such that for all $n \in C$, all $u \in T_R \restriction n$ we have

$$\langle R \cap [u], C/n\rangle \in \bigcap_{i=1}^{n} \mathcal{E}_n,$$

and therefore $\langle y, D\rangle$ is Sacks$\times\mathcal{M}_\mathcal{H}$-generic for all $y \in R$ and all $D \subseteq C$. In particular, either $R \times [C]^\omega \subseteq \mathcal{A}$ or $R \times [C]^\omega \cap \mathcal{A} = \emptyset$, depending on what $\langle Q, \langle \emptyset, B\rangle\rangle$ forces.

c) Fix perfectly $\mathcal{H}$-Ramsey sets $F_s \subseteq \{0,1\}^\omega \times [\mathbb{N}]^\omega$ $(s \in \mathbb{N}^{<\omega})$, a perfect $P \subseteq \{0,1\}^\omega$, and $[t, A]$ such that $A \in \mathcal{H}$. We have to find a perfect $Q \subseteq P$ and a $B \in \mathcal{H} \restriction A$ such that $Q \times [t, B]$ is either included in or disjoint from $\mathcal{A}(F_s)$. We can, and will, assume $P = \{0,1\}^\omega$, $s = \emptyset$ and $A = \mathbb{N}$. For a positive integer n let

$$\begin{aligned}\mathcal{D}_n = \{\langle P, B\rangle :\ & P \text{ perfect, } B \in \mathcal{H}, \text{ and for all } u \subseteq \{1,\dots,n\} \text{ and all } s \in \mathbb{N}^n \\ & \text{either } P \times [u, A] \subseteq F_s \text{ or either } P \times [u, A] \cap F_s = \emptyset\}\end{aligned}$$

Then by Lemma 2.1 there are a perfect R and $C \in \mathcal{H}$ such that for all $n \in C$, $t \in T_R \restriction n$, $u \subseteq \{1,\dots,n\}$ and $s \in \mathbb{N}^n$ set

$$(R \cap [t]) \times [u, C]$$

is either disjoint from or included in F_s, namely F_s is relatively clopen when restricted to $R \times [C]^\omega$. Therefore the set $\mathcal{A}(F_s)$ is analytic when restricted to the same set, and the desired conclusion follows from part **b)** of this theorem. □

Now we describe a situation in which all definable sets are $\mathcal{H}$-Ramsey for every semiselective $\mathcal{H}$. In this section we will use the word "definable" as a synonym for "an element of $L(\mathbb{R})$". For the other definitions see [9].

THEOREM 4.8 If there is a supercompact cardinal and $\mathcal{H}$ is a semiselective coideal, then all definable sets are $\mathcal{H}$-Ramsey and perfectly $\mathcal{H}$-Ramsey.

Shelah and Woodin ([24]) proved that if there is a supercompact cardinal then all definable sets are Ramsey. Using a version of their result, Todorcevic proved the following

THEOREM 4.9 If there is a supercompact cardinal, then every selective ultrafilter is $[\mathbb{N}]^\omega$-generic over $L(\mathbb{R})$.

In other words, all definable sets are $\mathcal{U}$-Ramsey for every selective ultrafilter $\mathcal{U}$. This unpublished result appears in [27] and since we shall need this result in order to supply a proof of Theorem 4.3, we include its proof with Todorcevic's kind permission. To state the lemma of Shelah and Woodin on which the proof of Theorem 4.4 is based we need the following:

DEFINITION 4.10 *Let M be a countable elementary submodel of some structure of the form H_θ which contains a poset $\mathcal{P}$ and a $\mathcal{P}$-name $\dot{r}$ for a real. Then we say that M is $(L(\mathbb{R}), \mathcal{P})$-correct if for every $(M, \mathcal{P})$-generic filter $G \subseteq \mathcal{P} \cap M$ and every formula $\phi(x, \vec{p})$ with parameter $\vec{p}$ in M the formula $\phi(\mathrm{val}_G(\dot{r}), \vec{p})$ is true in $L(\mathbb{R})$ if and only if there is a condition in G which forces this. We shall say that truth in $L(\mathbb{R})$ is unchangeable by forcing if the following condition is satisfied:*

(*) *For every poset $\mathcal{P}$ there is a large enough θ so that there are stationarily many countable elementary submodels M of H_θ which are $(L(\mathbb{R}), \mathcal{P})$-correct.*

LEMMA 4.11 ([4], [24], [5]) If there is a supercompact cardinal, then truth in $L(\mathbb{R})$ is unchangeable by forcing. □

We will also need the following lemma.

LEMMA 4.12 Assume that truth in $L(\mathbb{R})$ is unchangeable by forcing. If X is a ccc Baire space and $f: X \to \mathbb{R}$ is continuous then the f-inverse image of any set of reals from $L(\mathbb{R})$ has the property of Baire.

Proof Fix a set of reals A from $L(\mathbb{R})$ and θ which verifies $(*)$ for $\mathcal{P} = \mathrm{ro}(X)$. Every set of reals from $L(\mathbb{R})$ is determined by a formula $\phi(\cdot,\cdot)$ and a finite sequence of parameters $\vec{p}$ which are either reals or ordinals in the sense that x belongs to A if and only if $\phi(x, \vec{p})$ is true in $L(\mathbb{R})$ (see [9]). So let $\phi(\cdot,\cdot)$ and $\vec{p}$ be such that

$$A = \{x \in \mathbb{R} \colon \phi(x, \vec{p}) \text{ is true in } L(\mathbb{R})\}.$$

If $\dot{G}$ is a name for a $\mathcal{P}$-generic filter, then let $\dot{r}$ be the $\mathcal{P}$-name for a real such that

$$\dot{r} \in [\check{s}] \quad \text{iff} \quad f^{-1}[\check{s}] \in \dot{G},$$

for all basic intervals s. Let M be an $(L(\mathbb{R}), \mathcal{P})$-correct countable elementary submodel of H_θ and such that everything relevant is in M. Let $\boldsymbol{B}$ be the Boolean value of the formula of the $\mathcal{P}$-forcing language which says that $\phi(\dot{r}, \vec{p})$ is true in $L(\mathbb{R})$. Then $\boldsymbol{B}$ is a regular open subset of X.

Claim $f^{-1}(A)\Delta\boldsymbol{B}$ is a meager subset of X.

Proof Suppose that it is not; say, $f^{-1}(A) \setminus \boldsymbol{B}$ is nonmeager. So we can pick

$$x \in (f^{-1}(A) \setminus \boldsymbol{B}) \cap \bigcap \{U : U \text{ is a dense open subset of } X \text{ and } U \in M\}.$$

Since X is a ccc space it is easily seen that $G = \{U \in \mathcal{P} : x \in U\}$ is a $(\mathcal{P}, M)$-generic filter. Note also that $\mathrm{val}_G(\dot{r}) = f(x)$ is an element of A, so the formula $\phi(\mathrm{val}_G(\dot{r}), \vec{p})$ is true in $L(\mathbb{R})$. This contradicts the assumption that M is $(L(\mathbb{R}), \mathcal{P})$-correct, since clearly $\boldsymbol{B}$ is not in G. The assumption that $\boldsymbol{B} \setminus f^{-1}(A)$ is nonmeager would lead to a similar contradiction. □

Since $\boldsymbol{B}$ is a regular open subset of X, this proves Lemma 4.5. □

Proof (Theorem 4.4) Let $\mathcal{U}$ be a selective ultrafilter and let X be the topological space $[\mathbb{N}]^\omega$ whose basis is the family $\mathrm{Exp}(\mathcal{U})$ as defined in §3. Then by Corollary 3.1 this is a topology. By Theorem 3.3 X is a Baire space and subsets of X with the Baire property are exactly the $\mathcal{U}$-Ramsey sets. Moreover X is a ccc space because $\mathcal{P}_\mathcal{U}$ is a ccc poset. Let $f: X \to [\mathbb{N}]^\omega$ be the identity mapping; then f is continuous if $[\mathbb{N}]^\omega$ is considered with its separable metric topology, and by Lemma 4.12 every set of reals $\mathcal{X}$ in $L(\mathbb{R})$ is $\mathcal{U}$-Ramsey. So if such an $\mathcal{X}$ is, moreover, dense in $\langle [\mathbb{N}]^\omega, \subseteq^* \rangle$, then there is $A \in \mathcal{U}$ such that $[A]^\omega \subseteq \mathcal{X}$. Therefore $\mathcal{U}$ is $(L(\mathbb{R}), [\mathbb{N}]^\omega)$-generic. □

Proof (Theorem 4.3) Let $f: [\mathbb{N}]^\omega \to \{0, 1\}$ be definable. Force with $\langle \mathcal{H}, \subseteq^* \rangle$, and let $\mathcal{U}$ be a generic selective ultrafilter. Forcing with a small poset preserves supercompactness, so by Theorem 4.4 ultrafilter $\mathcal{U}$ is generic over $L(\mathbb{R})$. By the same result, the family of all $A \subseteq \mathbb{N}$ such that f is constant on A is a dense open subset of $[\mathbb{N}]^\omega$, so $\mathcal{U}$ intersects it; therefore $\mathcal{H}$ intersects it as well, and there is an $A \in \mathcal{H}$ such that f is constant on A. Since there are no new reals in an extension, A is in the ground model and this proves the $\mathcal{H}$-Ramsey part of our theorem. The perfectly $\mathcal{H}$-Ramsey part follows from the fact that semiselectivity is preserved in the Perfect-set forcing extension, and the unchangeability of theory of $L(\mathbb{R})$ by Sacks forcing. □

We finish this section with an example that has been promised after the proof of Theorem 4.1 above.

Example 4.13 A semiselective coideal $\mathcal{H}$ such that the poset $\mathcal{M}_\mathcal{H}$ is not proper. For a stationary, costationary subset S of ω_1 let $\mathcal{J}(S)$ be Jensen's poset of all countable closed subsets of S ordered by the end-extension (see [12, p. 250]). It is σ-distributive yet it destroys the stationarity of $\omega_1 \setminus S$ and therefore it is not proper. Observe that $\mathcal{J}(S)$ is a σ-distributive tree of height ω_1 with no uncountable branches such that every one of its nodes has continuum many incomparable successors. By an extension (literally) of the construction of Example 2.2, for every such tree T there is a Ramsey coideal $\mathcal{H}(T)$ such that the regular open algebras of $\mathcal{H}(T)$ and T are isomorphic; in particular, $\mathcal{H}$ is σ-distributive. But every σ-distributive Ramsey coideal is semiselective, so this finishes the construction.

5. CONCLUDING REMARKS

We would like to point out some analogies between §3 and §4. Note that the fact that Exp($\mathcal{H}$)-Baire property implies the $\mathcal{H}$-Ramsey property roughly corresponds to the poset $\mathcal{M}_{\mathcal{H}}$ having the Prikry property, while the fact that Exp($\mathcal{H}$)-nowhere dense sets are $\mathcal{H}$-Ramsey null roughly corresponds to $\mathcal{M}_{\mathcal{H}}$ having the Mathias property. "Roughly" because e.g. for a condition $[t, B]$ to decide the statement "$\dot{X} \in \mathcal{A}$" (where $\dot{X}$ is a name for an $\mathcal{M}_{\mathcal{H}}$-generic and $\mathcal{A}$ is an analytic set) it is not necessary that $[t, B] \subseteq \mathcal{A}$ or $[t, B] \cap \mathcal{A} = \emptyset$. An alternative approach was taken by Matet [41], who has studied $\mathcal{H}$-Ramsey property for a selective coideal $\mathcal{H}$ using games.

Many partition theorems, like canonical ones of Pudlak–Rödl [22], Mathias [16; §6] and Prömel–Voigt [21] have their semiselective versions. Functions to which these results apply are exactly the functions which have the *abstract* Exp($\mathcal{H}$)*-Baire property*, i.e. such that the inverse images of open sets have the abstract Exp($\mathcal{H}$)-Baire property (see §3). Their proofs rely on the following strengthening of Theorem 3.3.(b) which is of the independent interest as an analogue of a classical theorem of Baire (see [30], also [16; 6.2]).

Lemma 5.1 If $\mathcal{H}$ is a semiselective coideal and $f : [\mathbb{N}]^\omega \to \mathbb{R}$ has the abstract Exp($\mathcal{H}$)-Baire property, then there is $A \in \mathcal{H}$ such that $f \restriction [A]^\omega$ is continuous. □

The proof of the semiselective version of a partition theorem can be obtained either by mimicking the original proof, from the theorem itself (like e.g. in [26, Theorem 6.8]) or by using forcing arguments. An explanation of the phenomenon of the existence of (semi)selective versions of partition theorems lies in Theorems 4.3 and 4.4 and the fact that partition theorems can usually be formulated as the statements of $L(\mathbb{R})$.

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