Selected applications of logic to classification problem for C*-algebras[a]

Ilijas Farah

Department of Mathematics and Statistics
York University, 4700 Keele Street
North York, Ontario, Canada, M3J 1P3
and Matematicki Institut, Kneza Mihaila 34, Belgrade, Serbia
http://www.math.yorku.ca/∼ifarah
ifarah@yorku.ca

Basics of Elliott's classification program are outlined and juxtaposed with the abstract classification theory from descriptive set theory. Some preliminary estimates on the complexity of the isomorphism relation of separable C*-algebras are given.

Contents

[a]Partially supported by NSERC

0. Introduction

In recent years we have witnessed a number of applications of set theory to operator algebras. In the present notes I will focus on one very specific (yet particularly exciting) aspect of this development, applications of descriptive set theory (more precisely, theory of abstract classification) to Elliott's classification program for nuclear C*-algebras. These lecture notes contain a plenty of exercises, many (but not all) of them fairly straightforward. They are meant as a bridge towards more advanced literature on classification of C*-algebras, a subject with an abundance of excellent literature. Hints to exercises often refer to the material covered at a later point in the notes and thus provide additional motivation for the introduced notions. The initial parts of the lecture notes, and §2 in particular, are very sketchy and are meant to outline the main ideas and provide a guide to the literature rather than to serve as a textbook.

The basic premise is that every classical classification program in mathematics deals with analytic equivalence relations on Polish spaces. Moreover, classification invariants are usually coded by elements of a Polish space and the computation of invariants is given by a Borel-measurable function.

I shall always start counting at zero. Therefore $i < n$ means that i assumes n distinct values: $0, 1, \dots, n-1$. Also, an $a \in M_n(\mathbb{C})$ is identified with its matrix entries a_{ij} for $i < n$, $j < n$.

Suggested references.

For functional analysis, [64]. For the theory of C*-algebras, [62] and [12], and [68] for their K-theory. General theory of operator algebras is excellently surveyed in [4] and standard reference for the Elliott program as of 2002 is [70]. Very detailed notes on topics on operator theory starting with simple (partial isometries) to very advanced can be found at [76]. Classical reference for classical descriptive set theory is [48]. Somewhat dated surveys of some other applications of set theory to C*-algebras are given in [84] and [34]. Papers [33] and [32] contain more details and proofs of the results presented in §7 and §8.2. Survey of the latest developments in Elliott's program will be available shortly in [88].

Acknowledgments.

These notes are based on two series of lectures given in 2012. At the Luminy Young Set Theory workshop in April 2012 I covered sections dealing with general theory of C*-algebras and Ellliott's classification of AF algebras by K-theoretic invariants. At Asian Initiative for Infinity course in addition I covered this in more detail, and presented the abstract classification viewpoint of Elliott's program. I am indebted to the organizers of both of these meetings for inviting me. I would like to thank to Asger Törnquist for kindly permitting me to use his writeup of Lemma 8.6, taken

from the appendix to the original version of [33]. Finally, I would like to thank John Campbell, Boris Kadets, Vladislav Kalashnyk and Jiewon Park for noticing several mistakes in an earlier version of these notes.

1. Operators on Hilbert spaces

We begin with a review of the basic properties of operators on a Hilbert space. Throughout we let H denote a complex infinite-dimensional separable Hilbert space, and we let (e_n) be an orthonormal basis for H (see Example 1.1). For $\xi, \eta \in H$, we denote their inner product by $(\xi|\eta)$. We recall that

$$(\eta|\xi) = \overline{(\xi|\eta)}$$

and the norm defined by

$$\|\xi\| = \sqrt{(\xi|\xi)}.$$

The Cauchy–Schwartz inequality says that

$$|(\xi|\eta)| \leq \|\xi\|\|\eta\|.$$

Example 1.1. The space

$$\ell^2(\mathbb{N}) = \left\{(\alpha_k)_{k\in\mathbb{N}} : \alpha_k \in \mathbb{C}, \|\alpha\|^2 = \sum |\alpha_k|^2 < \infty\right\}$$

(sometimes denoted simply by ℓ^2) is a Hilbert space under the inner product $(\alpha|\beta) = \sum \alpha_k \overline{\beta_k}$. If we define $e^n \in \ell^2(\mathbb{N})$ by $e^n_k = \delta_{nk}$ (the Kronecker's δ), then (e^n) is an orthonormal basis for ℓ^2. For any $\alpha \in \ell^2$, $\alpha = \sum \alpha_n e^n$.

Any Hilbert space has an orthonormal basis, and this can be used to prove that all separable infinite-dimensional Hilbert spaces are isomorphic. Moreover, any two infinite-dimensional Hilbert spaces with the same density character (the minimal cardinality of a dense subset) are isomorphic.

Example 1.2. If (X, μ) is a measure space,

$$L^2(X,\mu) = \left\{f : X \to \mathbb{C} \text{ measurable} : \int |f|^2 d\mu < \infty\right\} / \{f : f = 0 \text{ a.e.}\}$$

is a Hilbert space under the inner product $(f|g) = \int f\bar{g}d\mu$ and with the norm defined by $\|f\|^2 = \int |f|^2 d\mu$.

We will let $a, b, \ldots$ denote linear operators $H \to H$. We recall that

$$\|a\| = \sup\{\|a\xi\| : \xi \in H, \|\xi\| = 1\}.$$

If $\|a\| < \infty$, we say a is *bounded*. An operator is bounded if and only if it is continuous. We denote the algebra of all bounded operators on H by $\mathcal{B}(H)$ (some authors use $L(H)$), and throughout the paper all of our operators will be bounded. We define the *adjoint* a^* of a to be the unique operator satisfying

$$(a\xi|\eta) = (\xi|a^*\eta)$$

for all $\xi, \eta \in H$. Note that since an element of H is determined by its inner products with all other elements of H (e.g., take an orthonormal basis), an operator a is determined by the values of $(a\xi|\eta)$ for all ξ, η or even by the values $(ae_m|e_n)$ for m and n in $\mathbb{N}$.

Lemma 1.3. *For all a, b in $\mathcal{B}(H)$ we have*

(1) $(a^*)^* = a$
(2) $(ab)^* = b^*a^*$
(3) $\|a\| = \|a^*\|$
(4) $\|ab\| \leq \|a\| \cdot \|b\|$
(5) $\|a^*a\| = \|a\|^2$

Proof. These are all easy calculations. For example, for (5), for $\|\xi\| = 1$,

$$\|a\xi\|^2 = (a\xi|a\xi) = (\xi|a^*a\xi) \leq \|\xi\| \cdot \|a^*a\xi\| \leq \|a^*a\|,$$

the first inequality holding by Cauchy–Schwartz. Taking the sup over all ξ, we obtain $\|a\|^2 \leq \|a^*a\|$. Conversely,

$$\|a^*a\| \leq \|a^*\|\|a\| = \|a\|^2$$

by (3) and (4). □

Entries (1)–(3) state that $\mathcal{B}(H)$ is a *Banach *-algebra* (or a *Banach algebra with involution **) and (5) is sometimes called the *C*-equality*.

1.1. Subspaces and subalgebras of $\mathcal{B}(H)$

Before focusing on C*-algebras, I shall list some other structures based on $\mathcal{B}(H)$. Each one of these categories is replete with possible applications of logic.

1.1.1. *Operator space* is a closed linear subspace of $\mathcal{B}(H)$. In addition to its Banach space structure, an operator space X is considered with the Banach space structure on $M_n(X)$, $n \times n$ matrices of elements of X identified with operators from H^n to H^n with respect to the operator norm. Morphisms in this category are *completely bounded* maps—linear maps whose canonical extension to $M_n(X)$ is bounded for all n. The standard reference on operator spaces is [17].

1.1.2. *Operator system* is an operator space that is in addition closed under the adjoint. Morphisms are *completely positive* maps—linear maps that preserve positivity of operators and matrices. They are indispensable in the theory of C*-algebras, and more on this can be found in [63] and [5].

1.1.3. *Concrete C*-algebras* are in addition closed under multiplication of operators. Morphisms are *-homomorphisms—maps that preserve all algebraic structure (+, $\cdot$ and *). Remarkably, such maps are automatically completely positive and completely contractive.

We import the terminology from §1.5.1 wholesale and talk about operators in a C*-algebra that are normal, self-adjoint, positive, projections, etc. This will also apply to abstract C*-algebras once they are introduced in Definition 2.2.

1.1.4. *Non-self-adjoint subalgebras* are norm-closed and closed under + and $\cdot$, but not necessarily closed under *. See [66] for more on this subject.

1.1.5. *Von Neumann algebras* are C*-algebras that are closed in weak operator topology. An excellent introductory source is [46].

Theories of these categories have many connections between each other. For example, the study of tensor products of C*-algebras and their finite-dimensional approximation is deeply steeped in the study of operator systems. In the present lecture notes I shall focus on C*-algebras.

1.2. Exercises

1.2.1. Prove that for all $\xi, \eta \in H$ the following so-called *polarization identity* holds (note that $i = \sqrt{-1}$)

$$(\xi|\eta) = \frac{1}{4}\sum_{k=0}^{3} i^k(\xi + i^k\eta|\xi + i^k\eta).$$

The right-hand side is a linear combination of norms of vectors $\xi + i^k\eta$ for $0 \leq k \leq 3$. Consequentially, the scalar product on H is uniquely determined by its Hilbert space norm.

The *dimension* of a Hilbert space is the least cardinality of an orthonormal basis.

1.2.2. All orthonormal bases in a fixed Hilbert space have the same cardinality. Two complex Hilbert spaces are isomorphic if and only if they have the same dimension.

1.2.3. Prove that a Hilbert space is separable if and only if it has a finite or countable orthonormal basis.

1.2.4 (Inner automorphisms)**.** Assume u is a linear isometry from H onto itself. Prove that $\mathrm{Ad}\, u\colon \mathcal{B}(H) \to \mathcal{B}(H)$ defined by

$$(\mathrm{Ad}\, u)a = uau^*$$

is an automorphism of C*-algebra $\mathcal{B}(H)$.

1.2.5. If $u\colon H_1 \to H_2$ is an isomorphism between Hilbert spaces, then

$$(\mathrm{Ad}\, u)a = uau^*$$

is an isomorphism between $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$. The operator $\mathrm{Ad}\, u(a)$ is just a with its domain and range identified with H_2 via u.

1.2.6. An operator u in a unital C*-algebra is a *unitary* if $uu^* = u^*u = 1$. Prove that $\mathrm{Ad}\, u$ is an automorphism of A.

1.2.7. Prove that all automorphisms of $\mathcal{B}(H)$ are inner.

(Hint 1: As a warmup prove that all automorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})$ are "trivial." You first have to replace the quotation marks with an appropriate definition of "trivial."

Hint 2 (for a better proof): Fix a unit vector ξ and let η be a unit vector such that $\Phi(p_\xi) = p_\eta$ (here p_ξ denotes the projection to the subspace spanned by ξ). With $v_{\xi,\zeta}$ denoting the rank 1 linear map sending ξ to ζ, let $u(\alpha) = \beta$, where β is the unique vector such that $\Phi(v_{\xi,\alpha}) = v_{\eta,\beta}$.)

A **-polynomial* is a term in the language $\{+, \cdot, *\}$. If $P(x)$ is a *-polynomial and a is an operator then $P(a)$ is naturally defined (again, logic for metric structures provides the right setting for this; see [29]). For a set of operators X in $\mathcal{B}(H)$ or in a C*-algebra A let $C^*(X)$ denote the C*-algebra generated by X. If $X = \{x_1, \dots, x_n\}$ is finite we may write $C^*(x_1, \dots, x_n)$ instead of $C^*(X)$.

1.2.8. Prove that $C^*(X)$ is the norm-closure of $\{P(\bar{a}) : P(\bar{x})$ is a *-polynomial in n (non-commuting) variables with complex coefficients and $\bar{a}$ is an n-tuple in X for $n \in \mathbb{N}\}$.

1.2.9. Prove that $C^*(a)$ is abelian if and only if a is normal.

An *ideal* in a C*-algebra is a two-sided, norm closed, self-adjoint ideal (there is some redundancy in this definition).

1.2.10. Prove that an ideal generated by an element a of a C*-algebra is equal to the closure of the linear span of $\{bac : b, c \in A\}$

1.3. Spectrum and spectral radius

We start with a useful lemma.

Lemma 1.4. *If* $\|a\| < 1$ *then* $1 - a$ *is invertible in* $\mathcal{B}(H)$.

Proof. The series $b = \sum_{n=0}^{\infty} a^n$ is convergent and hence in $\mathcal{B}(H)$. By considering partial sums one sees that $(1 - a)b = b(1 - a) = 1$. □

The *spectrum* of an operator a in $\mathcal{B}(H)$ is

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda I \text{ is not invertible}\}.$$

The spectrum of a bounded linear operator is always a compact subset of $\mathbb{C}$ (Exercise 1.4.4), and it is moreover always nonempty (the latter fact follows from a clever use of Liouville's theorem; see e.g., [64, Theorem 4.1.13]). Also, for a normal operator a we have that $\|a\| = r(a)$ (Exercise 1.4.8), where $r(a)$ is the *spectral radius* of a defined as follows:

$$r(a) = \max\{|\lambda| : \lambda \in \mathrm{sp}(a)\}.$$

1.4. Exercises

1.4.1. Prove that the spectrum of a finite-dimensional matrix is equal to the set of its eigenvalues.

1.4.2. Prove that the invertible elements form an open subset of $\mathcal{B}(H)$.

(Hint: If $ab = 1 = ba$, find an $\varepsilon > 0$ such that $\|b - c\| < \varepsilon$ implies $\|ac - 1\| < 1$ and $\|ca - 1\| < 1$. Then apply Lemma 1.4.)

1.4.3. Prove that $\mathrm{sp}(a) \subseteq \{\lambda : |\lambda| \leq \|a\|\}$ for all a.

1.4.4. Prove that $\mathrm{sp}(a)$ is compact for all a.

(Hint: Use Exercise 1.4.2 and Exercise 1.4.3.)

1.4.5. Prove that $\|a\| \geq r(a)$ for all a.

1.4.6. Find an example of a nonzero a such that $\mathrm{sp}(a) = \{0\}$, and hence $\|a\| > r(a)$.

(Hint: A well-chosen 2×2 matrix would do.)

1.4.7. Prove that $\lim_n \|a^n\|^{1/n}$ exists and is equal to $r(a)$.
(Hint: See [64, Lemma 4.1.13].)

1.4.8. Assume a is normal. Prove $\|a\| = r(a)$.
(Hint: First prove that $\|a^{2^n}\| = \|a\|^{2^n}$ and then use Exercise 1.4.7. Prove this equality in the self-adjoint case first, using the C*-equality.)

1.4.9. Show that if $\lambda \notin \mathrm{sp}(a)$ then $\|(a - \lambda \cdot 1)^{-1}\| \leq 1/\operatorname{dist}(\lambda, \mathrm{sp}(a))$.

1.4.10. Find a normal operator on a Hilbert space that has no eigenvectors.
(Hint: Example 1.5.)

1.5. Normal operators and the spectral theorem

In this section we introduce some distinguished classes of operators in $\mathcal{B}(H)$, such as normal and self-adjoint operators (cf. §1.5.1).

Example 1.5. Assume (X, μ) is a probability measure space. If $H_0 = L^2(X, \mu)$ and $f\colon X \to \mathbb{C}$ is bounded and measurable, then

$$H_0 \ni g \overset{m_f}{\longmapsto} fg \in H_0$$

is a bounded linear operator. We have $\|m_f\| = \|f\|_\infty$ and

$$m_f^* = m_{\bar{f}}.$$

Hence $m_f^* m_f = m_f m_f^* = m_{|f|^2}$. We call operators of this form *multiplication operators.*

Recall that an operator a is *normal* if $aa^* = a^*a$. Clearly, all multiplication operators are normal. When H is a complex Hilbert space, normal operators have a nice structure theory. It is summarized in the following theorem, stated a bit prematurely since its proof involves Proposition 2.5.

Theorem 1.6 (Spectral Theorem)**.** *If a is a normal operator then there is a probability measure space (X, μ), a measurable function f on X, and a Hilbert space isomorphism $\Phi\colon L^2(X, \mu) \to H$ such that $\Phi a \Phi^{-1} = m_f$.*

Proof. For a proof see e.g., [2, Theorem 2.4.5]. □

Therefore every normal operator is a multiplication operator for some identification of H with an L^2 space. Conversely, every multiplication operator is clearly normal. If X is discrete and μ is the counting measure, the characteristic functions of the points of X form an orthonormal basis for $L^2(X, \mu)$ and the spectral theorem says that a is diagonalized by this basis. In general, the spectral theorem says that normal operators are "measurably diagonalizable".

An operator a is *self-adjoint* if $a = a^*$. Self-adjoint operators are obviously normal. For any $b \in \mathcal{B}(H)$, the "real" and "imaginary" parts of b, defined by $b_0 = (b + b^*)/2$ and $b_1 = (b - b^*)/2i$, are self-adjoint and satisfy $b = b_0 + ib_1$. Thus any operator is a linear combination of two self-adjoint operators. It is easy to check that an operator is normal if and only if its real and imaginary parts commute, so the normal operators are exactly the linear combinations of commuting self-adjoint operators.

1.5.1. *Types of bounded operators* The following distinguished classes of operators will play an important role.

(1) a is *normal* if $aa^* = a^*a$,
(2) a is *self-adjoint* (or *Hermitian*) if $a = a^*$,
(3) a is *positive* if $a = b^*b$ for some $b \in H$,
(4) a is a *projection* if $a^2 = a^* = a$,
(5) a is *positive* (or $a \geq 0$) if $a = b^*b$ for some b,
(6) a is *unitary* if $aa^* = a^*a = 1$,
(7) a is an *isometry* if aa^* is a projection and $a^*a = 1$.
(8) a is *partial isometry* if both aa^* and a^*a are projections.

Note that a positive element is automatically self-adjoint. For self-adjoint elements a and b write $a \leq b$ if $b - a$ is positive.

Any complex number z can be written as $z = re^{i\theta}$ for $r \geq 0$ and $|e^{i\theta}| = 1$. Considering $\mathbb{C}$ as the set of operators on a one-dimensional Hilbert space, there is an analogue of this on an arbitrary Hilbert space.

Theorem 1.7 (Polar Decomposition). *Any $a \in \mathcal{B}(H)$ can be written as $a = bv$ where b is positive and v is a partial isometry. Moreover, v can be chosen so that* $\ker(v) = \ker(a)$. *This additional requirement makes v unique.*

Proof. See [64], but apply it to a^* instead. □

1.6. Exercises

1.6.1. The real and imaginary parts of a multiplication operator m_f are $m_{\Re f}$ and $m_{\Im f}$. A multiplication operator m_f is self-adjoint if and only if f is real (a.e.). By the spectral theorem, all self-adjoint operators are of this form up to the unitary equivalence.

1.6.2. Two projections p and q commute if and only if pq is a projection.

1.6.3. For projections p and q write $p \leq q$ if $pq = p$ (equivalently, $qp = p$). Prove the following.

(1) $p \leq q$ implies that p and q commute.
(2) This is a partial ordering on any set of projections.
(3) $p \leq q$ if and only if the self-adjoint operator $q - p$ is positive.
(4) $p \leq q$ and $\|p - q\| < 1$ implies $p = q$.

1.6.4. If p and q are projections such that $pq = q$ and $(1-p)(1-q) = 1-q$ then $p = q$.

1.6.5. Every self-adjoint unitary is of the form $u = 1-2p$ for a projection p.

2. Preliminaries on C*-algebras

2.1. Positivity, states and the GNS theorem

Let X be a locally compact Hausdorff space. Recall that $C_0(X)$ denotes the space of continuous complex-valued functions on X such that for every $\varepsilon > 0$ the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact. It is considered as a Banach algebra with respect to $+$, $\cdot$ and adjoint defined as pointwise conjugation. If X is compact then we write $C(X)$. By the following remarkable result, these algebras are exactly the abstract abelian C*-algebras.

Theorem 2.1 (Gelfand–Naimark)**.** *Every abelian C*-algebra is isomorphic to $C_0(X)$ for a unique locally compact Hausdorff space X. The algebra is unital if and only if X is compact.* □

Space X is equal to the space of characters on A (see Exercise 2.2.2). A proof of this theorem can be found in e.g., [64] or [2]. In fact, the Gelfand–Naimark theorem is functorial: the category of abelian C*-algebras is contravariantly isomorphic to the category of locally compact Hausdorff spaces (cf. Exercise 2.2.6).

Definition 2.2. An *abstract* C*-algebra is a Banach algebra satisfying the conclusion of Lemma 1.3.

Concrete C*-algebras were introduced in §1.1.3. By the definition every concrete C*-algebra is an abstract C*-algebra. The fact that the converse is true can hardly be overestimated.

Theorem 2.3 (Gelfand–Naimark–Segal)**.** *Every abstract C*-algebra A is isomorphic to a concrete C*-algebra.* □

From now on we will usually refer to abstract C*-algebras as C*-algebras. An occasional concrete C*-algebra will also be treated in the abstract way, independently from its actual representation on $\mathcal{B}(H)$.

To a logician, Theorem 2.3 states that C*-algebras form an axiomatizable class in an appropriately chosen logic. This fact was made precise and taken advantage of in [28] and [29].

2.2. Exercises

2.2.1. Check the easy direction of Theorem 2.3.

In the following three exercises we define the Gelfand transform and give a (very rough!) outline of the proof of Gelfand–Naimark theorem. *Character* of a C*-algebra A is a *-homomorphism $\phi\colon A \to \mathbb{C}$. Let $\hat{A}$ denote the set of characters of A, and note that $\hat{A} = \{0\}$ for many C*-algebras A (e.g., all matrix algebras).

2.2.2. Prove that every character is continuous and has norm ≤ 1, and that $\hat{A}$ is a weak*-compact subset of A^*.

(Hint: For the first part one needs to check that the kernel is closed. For this apply Lemma 1.4. In the second part, by Alaoglu's theorem (the unit ball of A^* is weak*-compact) one only needs to check that the characters form a closed subset of the unit ball of A^*.)

2.2.3. If A is an abelian C*-algebra then $\hat{A}$ is equal to the set of pure states of A.

(Hint: This is a consequence of the Riesz Representation Theorem. See Exercise 3.17.)

2.2.4. If $X \subseteq \mathbb{C}$ is compact, then $C(X) \cong C^*(\iota_X, 1)$, where ι_X is the identity function on X and 1 is the constantly one function.

(Hint: Stone–Weierstrass.)

2.2.5. If A is a C*-algebra define the map $\Gamma\colon A \to C(\hat{A})$ by

$$\Gamma(a)(\phi) = \phi(a).$$

Show that Γ is a *-homomomorphism. If A is moreover abelian, show that Γ is an isometric isomorphism.

(Hint: For the second part, combine Exercise 2.2.3 and Exercise 1.4.8.)

2.2.6. Assume X and Y are compact Hausdorff spaces and $\Phi\colon C(X) \to C(Y)$ is a *-homomorphism.

(1) Prove that there exists a unique continuous $f\colon Y \to X$ such that $\Phi(a) = a \circ f$ for all $a \in C(X)$.
(2) Prove that Φ is a surjection if and only if f is an injection.
(3) Prove that Φ is an injection if and only if f is a surjection.
(4) Prove that for every $f\colon Y \to X$ there exists a unique $\Phi\colon C(X) \to C(Y)$ such that (1)–(3) above hold.

Let A be a C*-algebra. A continuous linear functional $\phi : A \to \mathbb{C}$ is *positive* if $\phi(a) \geq 0$ for all positive $a \in A$. It is a *state* if it is positive and of norm 1. We denote the space of all states on A by $\mathbb{S}(A)$.

2.2.7. If $\xi \in H$ is a unit vector, define a functional ω_ξ on $\mathcal{B}(H)$ by

$$\omega_\xi(a) = (a\xi|\xi).$$

Then $\omega_\xi(a) \geq 0$ for a positive a and $\omega_\xi(I) = 1$; hence it is a state. We call a state of this form a *vector state*.

2.2.8. If A is unital then $\mathbb{S}(A) = \{\phi \in A^* : \|\phi\| = 1 = \phi(1)\}$.

2.2.9. Prove that if A is a unital subalgebra of B then all states of A extend to states of B.

(Hint: Exercise 2.2.8.)

In the following $\mathfrak{X}^*$ denotes the Banach space dual of Banach space $\mathfrak{X}$.

2.2.10. Assume $\Phi\colon A \to B$ is a unital *-homomorphism. Define $\Phi^*\colon B^* \to A^*$ via $\Phi^*(\psi) = \psi\colon \Phi$.

(1) Prove that Φ^* maps $\mathbb{S}(B)$ into $\mathbb{S}(A)$.
(2) Prove that Φ^* maps $T(B)$ into $T(A)$ (the definition of a trace and$T(A)$ is given in §3.6).
(3) Prove that Φ^* is injective if and only if Φ is surjective.
(4) Prove that Φ^* is surjective if and only if Φ is injective.

2.3. Continuous functional calclus

We are about to introduce one of the key tools in the theory of C*-algebras building on Gelfand–Naimark theorem in Proposition 2.5 below. Following §1.3, a spectrum $\mathrm{sp}_A(a)$ of an element a of an arbitrary unital C*-algebra A can be defined as

$$\mathrm{sp}_A(a) = \{\lambda \in \mathbb{C} | a - \lambda \cdot 1 \text{ is not invertible in } A\}.$$

The reason one this notation is not in standard usage is contained in Lemma 2.6 below. Let us first prove its special case.

Lemma 2.4. *Let a be a normal element in a unital C*-algebra A and let $B = C^*(a, I)$ (the algebra generated by a and the identity). Then* $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$.

Proof. Since B is a subalgebra of A, we have that $\mathrm{sp}_B(a) \supseteq \mathrm{sp}_A(a)$. In order to show the converse inclusion, we need to show that an operator $b \in B$ that is not invertible in B is not invertible in A.

Assume the contrary, and fix a $b \in B$ which is not invertible in B but has an inverse d in A. By Theorem 2.1, B is isomorphic to $C(X)$ for a compact Hausdorff space X, and we can identify b with a function f on X. Since f is not invertible, we have $f(x) = 0$ for some $x \in X$. Fix $\varepsilon > 0$ pick an open neighbourhood U of x such that $|f(y)| < \varepsilon$ for all $y \in U$. Now let $g \in C(X)$ be a continuous function such that $g(x) = 1$, $0 \leq g(y) \leq 1$ for all y and $g(z) = 0$ for $z \notin U$. Let $c \in B$ correspond to g.

Then $\|cbd\| = \|c\| = 1$. On the other hand, $\|cb\| = \max_{y \in X} |g(y)f(y)\| \leq \varepsilon$. Therefore, $\|d\| \geq 1/\varepsilon$. Since ε was arbitrarily small and did not depend on d, this is a contradiction. □

Proposition 2.5. *If $a \in \mathcal{B}(H)$ is normal then $C(\mathrm{sp}(a)) \cong C^*(a, I)$. The isomorphism sends function $f \in C(\mathrm{sp}(a))$ to $f(a)$, where $f(a)$ is defined naturally in case when f is a *-polynomial.*

Proof. By Lemma 2.4 it suffices to prove that $C^*(a, I)$ is isomorphic to $C(\mathrm{sp}_0(a))$, where $\mathrm{sp}_0(a)$ denotes the spectrum of a as defined in $C^*(a, I)$. Let X be a compact Hausdorff space such that $C^*(a, I) \cong C(X)$, as guaranteed by Gelfand–Naimark theorem. For any $\lambda \in \mathrm{sp}(a)$, $a - \lambda \cdot 1$ is not invertible so there exists $\phi_\lambda \in X$ such that $\phi_\lambda(a - \lambda \cdot 1) = 0$, or $\phi_\lambda(a) = \lambda$. Conversely, if there is $\phi \in X$ such that $\phi(a) = \lambda$, then $\phi(a - \lambda \cdot 1) = 0$ so $\lambda \in \mathrm{sp}(a)$. Since any nonzero homomorphism to $\mathbb{C}$ is unital, an element $\phi \in X$ is determined entirely by $\phi(a)$. Since X has the weak* topology, $\phi \mapsto \phi(a)$ is thus a continuous bijection from X to $\mathrm{sp}(a)$, which is a homeomorphism since X is compact. □

Lemma 2.6. *Suppose A is a unital subalgebra of B and $a \in A$ is normal. Then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$, where $\mathrm{sp}_A(a)$ and $\mathrm{sp}_B(a)$ denote the spectra of a as an element of A and B, respectively.*

Proof. Since an element invertible in the smaller algebra is clearly invertible in the larger algebra, we have that $\mathrm{sp}_B(a) \subseteq \mathrm{sp}_A(a)$ and we only need to check that $\mathrm{sp}_A(a) \subseteq \mathrm{sp}_B(a)$. Pick $\lambda \in \mathrm{sp}_A(a)$. We need to prove that $a - \lambda \cdot 1$ is not invertible in B. Assume the contrary and let b be the inverse of $a - \lambda \cdot 1$. Fix $\varepsilon > 0$ and let $U \subseteq \mathrm{sp}_A(a)$ be the open ball around λ od radius ε. Let $g \in C(\mathrm{sp}_A(a))$ be a function supported by U such that $\|g\| = 1$. Then $g = b(a - \lambda \cdot 1)g$, hence $\|b(a - \lambda \cdot 1)g\| = 1$. On the other hand, $(a - \lambda \cdot 1)g = f \in C(\mathrm{sp}_A(a))$ so that f vanishes outside of U and $\|f(x)\| < \varepsilon$ for $x \in U$, hence $\|(a - \lambda \cdot 1)g\| < \varepsilon$. Thus $\|b\| > 1/\varepsilon$ for every $\varepsilon > 0$, a contradiction. □

Note that the isomorphism defined in Proposition 2.5 is canonical and maps a to the identity function on $\mathrm{sp}(a)$. It follows that for any polynomial p, the isomorphism maps $p(a)$ to the function $z \mapsto p(z)$. More generally, for any continuous function $f : \mathrm{sp}(a) \to \mathbb{C}$, we can then define $f(a) \in C^*(a, I)$ as the preimage of f under the isomorphism. For example, we can define $|a|$ and if a is self-adjoint then it can be written as a difference of two positive operators as

$$a = \frac{|a| + a}{2} - \frac{|a| - a}{2}.$$

If $a \geq 0$, then we can also define $\sqrt{a}$. Lemma 2.7 is an another application of the "continuous functional calculus" of Corollary 2.5. A remark about terminology is in order. It is customary among C*-algebraists to call 1-Lipshitz maps *contractions*. Recall that a map Φ is *Lipshitz* if $d(\Phi(x), \Phi(y)) \leq d(x, y)$ for all x and y. Although this terminology makes various fixed-point theorems false[b], I shall use it in order to be compatible with the standard terminology.

Lemma 2.7. *Any *-homomorphism $\Phi : A \to B$ between C*-algebras is a contraction (in particular, it is continuous). Therefore, any (algebraic) isomorphism between C*-algebras is an isometry.*

Proof. By passing to the unitizations, we may assume A and B are unital and Φ is unital as well (i.e., $\Phi(I_A) = 1_B$).

Note that for any $a \in A$, $\mathrm{sp}(\Phi(a)) \subseteq \mathrm{sp}(a)$ (by the definition of the spectrum). Thus for a normal using Exercise 1.4.8 we have

$$\begin{aligned}\|a\| &= \sup\{|\lambda| : \lambda \in \sigma(a)\} \\ &\geq \sup\{|\lambda| : \lambda \in \sigma(\Phi(a))\} \\ &= \|\Phi(a)\|.\end{aligned}$$

[b]outside of the theory of operator algebras, contractions are usually required to strictly decrease the distance

For general a, aa^* is normal so by the C*-equality we have

$$\|a\| = \sqrt{\|aa^*\|} \geq \sqrt{\|\Phi(aa^*)\|} = \|\Phi(a)\|,$$

concluding the proof. □

For subsets K and L of a metric space (X, d) and $\varepsilon > 0$ we write $K \subseteq_\varepsilon L$ if and only if $\inf_{y \in L} d(x, y) \leq \varepsilon$ for all $x \in K$. Note that X and d are implicit in this notation.

Lemma 2.8. *Assume a and b are normal and $\|a - b\| \leq \varepsilon$. Then $\mathrm{sp}(a) \subseteq_\varepsilon \mathrm{sp}(b)$ and $\mathrm{sp}(b) \subseteq_e \mathrm{sp}(a)$.*

Proof. It suffices to prove that for an arbitrary $\lambda \in \mathbb{C}$ such that $\mathrm{dist}(\lambda, \mathrm{sp}(a)) > \varepsilon$ we have $\lambda \notin \mathrm{sp}(b)$.

Fix such λ and let $c = (a - \lambda \cdot 1)^{-1}$. By Exercise 1.4.9 we have that $\|c\| < 1/\varepsilon$. Then $c(b - \lambda \cdot 1) = c(a - \lambda \cdot 1) - c(a - b) = 1 - c(a - b)$. The right-hand side is invertible by Lemma 1.4, and therefore $b - \lambda \cdot 1$ is invertible as well. □

The following slightly amusing remark can be safely ignored. Consider $K(\mathbb{C})$, the space of compact subsets of $\mathbb{C}$, as a Polish space with respect to the Hausdorff distance

$$d_H(K, L) = \max\{\inf\{\varepsilon : K \subseteq_\varepsilon L \text{ and } L \subseteq_e K\}.$$

Lemma 2.8 states that the map $a \mapsto \mathrm{sp}(a)$ from normal operators in A into $K(\mathbb{C})$ is a contraction (cf. the paragraph before Lemma 2.7).

2.4. Exercises

2.4.1. Prove the following are equivalent for all $a \in A$.

(1) $a = b^*b$ for some $b \in A$.
(2) a is normal and $\mathrm{sp}(a) \subseteq [0, \infty)$.

2.4.2. Assume a is a normal operator. Characterize a being self-adjoint, projection, positive, unitary, in terms of the spectrum of a.

2.4.3. A multiplication operator m_f is invertible if and only if there is some $\varepsilon > 0$ such that $|f| > \varepsilon$ (a.e.). Thus since $m_f - \lambda I = m_{f-\lambda}$, $\mathrm{sp}(m_f)$ is the essential range of f (the set of $\lambda \in \mathbb{C}$ such that for every neighborhood U of λ, $f^{-1}(U)$ has positive measure).

2.4.4. If f is a continuous function on $\mathrm{sp}(a)$, prove that $\mathrm{sp}(f(a)) = \{f(\lambda) : \lambda \in \mathrm{sp}(a)\}$.

2.4.5. Assume a is normal. Show that $C^*(a) \cong C_0(\mathrm{sp}(a) \setminus \{0\})$.

2.4.6. Show that Theorem 1.6 is a corollary of Proposition 2.5.

2.5. Constructions of C*-algebras

In many of the constructions of C*-algebras given below the fact that the algebraic structure determines the norm considerably simplifies the discussion.

2.5.1. *Unitization* Every C*-algebra A can be embedded in a unital C*-algebra $\tilde{A}$ in a minimal way as follows. On $A \times \mathbb{C}$ define the operations as follows: $(a, \lambda)(b, \xi) = (ab + \lambda b + \xi a, \lambda\xi)$, $(a, \lambda)^* = (a^*, \bar{\lambda})$ and $\|(a, \lambda)\| = \sup_{\|b\| \leq 1} \|ab + \lambda b\|$ and check that this is still a C*-algebra.

A straightforward calculation shows that $(0, 1)$ is the unit of $\tilde{A}$ and that $A \ni a \mapsto (a, 0) \in \tilde{A}$ is an isomorphic embedding of A into $\tilde{A}$.

2.5.2. *Direct sums.* Given C*-algebras A and B we define their direct sum $A \oplus B$ to be the set of all pairs (a, b) with the pointwise defined operations and norm defined by $\|(a, b)\| = \max\{\|a\|, \|b\|\}$. This is easily seen to be an abstract C*-algebra. If A and B are given with concrete representations on spaces H and K, respectively, then it is equally easy to represent $A \oplus B$ on $H \oplus K$ as block-matrices

$$(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

As is customary we write $a + b$ instead of (a, b). One similarly defines a direct sum of any finite number of C*-algebras. Given an infinite family of C*-algebras A_i, for $i \in I$, the direct sum is defined as

$$\bigoplus_{i \in I} A_i = \{(a_i : i \in I) : a_i \in A_i \text{ for all } i, \text{ and } \{i : \|a_i\| > 1/n\} \text{ is finite for all } n\}.$$

Operations are defined pointwise and the norm is the supremum norm. It is again easy to see that this is a C*-algebra.

2.5.3. *Direct products* Finite direct products coincide with finite sums. Given an infinite family of C*-algebras A_i, for $i \in I$, one defines

$$\prod_{i \in I} A_i = \{(a_i : i \in I) : \sup_i \|a_i\| < \infty\}.$$

With operations and norm defined as in §2.5.2 this is a C*-algebra that has $\bigoplus_{i \in I} A_i$ as an ideal. One can prove that $\prod_{i<I} A_i$ is nonseparable unless all but finitely many A_i are isomorphic to $\mathbb{C}$.

2.5.4. *Direct limits (also called inductive limits)* Let A_i, for $i \in \Lambda$ be family of C*-algebras indexed by a directed set. Also assume that for $i < j'$ we have a *-homomorphism $F_{ij} \colon A_i \to A_j$ and that these *-homomorphisms commute. Then A_i together with F_{ij} form a *directed system* of C*-algebras.

Then the *direct limit* of this system, $A = \lim_i A_i$, is defined as follows. Consider the set of all $(a_i : i \geq j)$ such that $j \in \Lambda$, $a_j \in A_j$ and $a_i = F_{ji}(a_j)$ for all $i \geq j$. This limit comes equipped with canonical *-homomorphisms $F_i \colon A_i \to A$ for all i which commute with all F_{ij}.

One should keep in mind that this is an abuse of notation, since the direct limit depends on connecting maps F_{ij} as well as the algebras A_i (see the examples in §4.4).

2.5.5. *Matrix algebra over A* Given a C*-algebra A and $n \in \mathbb{N}$ we define C*-algebra $M_n(A)$ as follows. Its elements are $n \times n$ matrices over A. The algebraic operations are defined to be the usual matrix operations. In order to define norm fix a faithful representation of A on a Hilbert space H. Now interpret each $a \in M_n(A)$ naturally as an operator on the direct sum of n copies of H. We equip $M_n(A)$ with the corresponding operator norm.

By the automatic continuity (Lemma 2.7) the norm on $M_n(A)$ is canonical. However, it is notoriously nontrivial to compute. For example, a deceivingly simple Anderson's paving conjecture is equivalent to the positive solution to the central Kadison–Singer problem on extensions of pure states.

2.5.6. *Stabilization* Story goes that in the olden days, whenever encountered with a non-unital C*-algebra one would immediately unitize it. Nowadays, whenever encountered with a unital C*-algebra one stabilizes it and hence turns it into a non-unital C*-algebra. The motivation for this behaviour will become apparent in §5.

Given A, define a direct limit as follows. Let A_n be $M_n(A)$ and let $F_{n,n+1} \colon A_n \to A_{n+1}$ be given by adding the $n+1$-st zero row and zero column to a, or in block-matrix notation $F_n(a) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$.

Then maps $F_{n,n+1}$ for $n \in \mathbb{N}$ define a commuting system of non-unital *-homomorphisms. The direct limit is a non-unital algebra called the *stabilization* of A. This algebra is just a special case of minimal C*-algebraic tensor product defined in §2.5.7 below. More precisely, $M_n(A)$ is (isomorphic to) $A \otimes M_n(\mathbb{C})$ and the stabilization of A is (isomorphic to) $A \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the algebra of compact operators on an infinite-dimensional, separable, complex Hilbert space.

For future use in §5 we record a bit of notation. By $M_\infty(A)$ we denote $\bigcup_n M_n(A)$, with the connecting maps as defined above.

A C*-algebra is A *stable* if it is isomorphic to $A \otimes \mathcal{K}$. Since $\mathcal{K}$ itself is stable, A is stable if and only if it is isomorphic to $B \otimes \mathcal{K}$ for some B.

2.5.7. *Minimal tensor product* An *algebraic tensor product* of C*-algebras A and B is defined as a quotient of the linear span of elementary tensors $a \otimes b$. It is customary to denote this algebraic tensor product by $A \odot B$. On this complex *-algebra one wants to define a norm satisfying axioms listed in Lemma 1.3 and take the completion. By the GNS theorem (Theorem 2.3) such completion is a C*-algebra. It turns out that in some cases there is no unique C*-norm on $A \odot B$; for example, this is the case with $\mathcal{B}(H) \odot \mathcal{B}(H)$. This is even more remarkable in light of the fact that the tensor product of Hilbert spaces H and K is uniquely defined: If (e_ξ) is an orthonormal basis of H and (f_η) is an orthonormal basis of K then $e_\xi \otimes f_\eta$ is an orthonormal basis of $H \otimes K$.

I shall cut the corners and only describe construction of the so-called *minimal* C*-algebraic tensor product, without even explaining why is it minimal. As a matter of fact, I shall not even prove that it is uniquely defined (this requires showing a true, albeit not obvious, fact that $A \otimes B$ does not depend on the choice of representations of A and B).

Assume A and B are unital C*-algebras. By the GNS theorem (Theorem 2.3) we can fix *-isomorphisms $\Phi\colon A \to \mathcal{B}(H)$ and $\Psi\colon B \to \mathcal{B}(K)$. Without a loss of generality, we may assume these *-homomorphisms are unital. We can canonically identify $\mathcal{B}(H)$ with a subalgebra of $\mathcal{B}(H \otimes K)$, by sending each a to the operator such that $a(e \otimes f) = a(e) \otimes f$ for all $e \in H$ and $f \in K$. Similarly we identify $\mathcal{B}(K)$ with a subalgebra of $\mathcal{B}(H \otimes K)$. This defines representations of A and B on $\mathcal{B}(H \otimes K)$, and we identify A and B with their respective images. Then $ab = ba$ for all $a \in A$ and $b \in B$ and we define $A \otimes B$ to be the C*-algebra generated by A and B. This product is sometimes denoted $A \otimes_{\min} B$.

One can similarly define a tensor product of a family (finite or infinite) of C*-algebras, $\bigotimes_{i \in \mathcal{I}} A_i$. If $\mathcal{I}$ is infinite then one needs to assume that all but finitely many of A_i are unital and let $\bigodot_{i \in \mathcal{I}} A_i$ be the span of the set of all elementary tensors $\bigotimes_{i \in \mathcal{I}} a_i$ where $a_i \in A_i$ and $a_i = I_A$ for all but finitely many i.

The assumption that A is unital was needed in order to have an isomorphic copy of B, $1 \otimes B$, inside $A \otimes B$. The unitality of B is used in the analogous way.

Definition 2.9. C*-algebra A is *nuclear* if for every C*-algebra B there is a unique C*-algebra norm on $A \odot B$.

It is not difficult to check that all finite-dimensional C*-algebras are nuclear and that the class of nuclear algebras is closed under taking tensor products and direct limits. Also, all abelian C*-algebras are nuclear. Therefore, all algebras (except $\mathcal{B}(H)$) considered in these notes are nuclear and all tensor product norms used here will be uniquely determined.

The theory of tensor products of C*-algebras is full of surprises and two of the most important classes of C*-algebras, nuclear and exact algebras, are defined by their behaviour with respect to the tensor products. This exciting subject is beyond the scope of the present paper and the reader may want to consult [5] for more details. See also Exercise 2.6.5 and 2.6.6.

2.5.8. *Continuous fields of C*-algebras* Given a compact space X and a C*-algebra A, let $C(X, A)$ denote the algebra of all continuous functions $f : X \to A$. The operations are given pointwise and the norm is the supremum norm, $\|f\| = \sup_{x \in X} \|f(x)\|$.

One can vary this definition by restricting the range of functions f to obtain more general C*-algebras.

2.5.9. *Corners* This is a special case of a hereditary subalgebra. Given a C*-algebra A and a projection $p \in A$, we can consider the subalgebra $pAp = \{pap : a \in A\}$. This is a unital C*-algebra, although it is typically not a unital subalgebra of A, even if A has a unit.

2.5.10. *. . . and so on* Some important constructions of C*-algebras, such as maximal tensor products, group C*-algebras (both full and reduced), multiplier algebras and coronas will not be used in these notes.

2.6. Exercises

2.6.1. Let X be a locally compact, non-compact, Hausdorff space. By Gelfand–Naimark theorem, the unitization of $C_0(X)$ is isomorphic to $C(Y)$ for some compact Hausdorff space Y. What is the relation between X and Y?

2.6.2. Prove that a direct product of infinitely many C*-algebras is non-separable unless all but finitely many of them are isomorphic to $\mathbb{C}$.

2.6.3. Assume $A = \lim_i A_i$ is unital. Prove that there is $i_0 \in \lambda$ such that for all $i_0 < i$ algebra A_i is unital and for all $i_0 < i < j$ the map F_{ij} is unital.

2.6.4. Check that $M_n(M_k(\mathbb{C}))$ is isomorphic to $M_{nk}(\mathbb{C})$.

2.6.5. Prove that $M_n(A)$ is algebraically isomorphic to $M_n(\mathbb{C}) \otimes A$ and that it carries a uniquely defined C*-norm.

(Hint: To prove uniqueness of the norm use Lemma 2.7.)

2.6.6. Show that the CAR algebra can be identified with the unital direct limit of algebras $A_n = \bigotimes_n M_2(\mathbb{C})$, which in turn can be identified with the infinite tensor product $\bigotimes_{\mathbb{N}} M_2(\mathbb{C})$. (Just like in the case of direct limits, C*-algebraic tensor product is the norm-completion of the algebraic tensor product.)

2.6.7 (Matrix units I)**.** Prove that $M_n(\mathbb{C})$ is the unique C*-algebra generated by elements $(e_{ij})_{i<n,j<n}$ with the following properties.

(1) Each e_{ii}, for $i < n$, is a projection.
(2) $\sum_{i<n} e_{ii} = 1$.
(3) $e_{ij}e_{kl} = \delta_{jk}e_{il}$, where $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq 0$ (Kronecker's delta),
(4) $e_{ij}^* = e_{ji}$.

2.6.8 (Matrix units II)**.** Prove that $M_n(\mathbb{C})$ is generated (as a C*-algebra) by matrix units e_{1j}, for $j < n$, as in Exercise 2.6.7.

2.6.9. Prove that an element a of $C(X, A)$ is normal (self-adjoint, unitary, projection,...) if and only if $a(x)$ is normal (self-adjoint, unitary, projection,...) for all $x \in X$.

The above exercise makes connection between projections in $C(X, M_n(\mathbb{C}))$ and vector bundles over X (see §6.0.1).

2.6.10. Prove that $C(X) \otimes C(Y) \cong C(X \times Y)$.

2.6.11. Prove that $M_m(\mathbb{C}) \otimes M_n(\mathbb{C}) \cong M_m(M_n(\mathbb{C})) \cong M_{mn}(\mathbb{C})$.
(Hint: This is Exercise 2.6.4.)

2.6.12. Assume there is a unital *-homomorphism $\Phi\colon M_k(\mathbb{C}) \to A$ and let $p \in A$ be an image of a minimal projection in A. Prove that $A \cong M_k(pAp)$.
(Hint: Exercise 2.6.5.)

2.6.13. With the notation used in §2.5.7, prove that $\mathcal{B}(H) \otimes \mathcal{B}(H)$ is a proper subset of $\mathcal{B}(H \otimes H)$ if H is infinite-dimensional.

(Hint: Let e_n, for $n \in \mathbb{N}$, be an orthonormal sequence in H. Let p be the orthogonal projection to the closed linear span of $\{e_n \otimes e_n : n \in \mathbb{N}\}$.

This operator is in $\mathcal{B}(H \otimes H)$ but it cannot be approximated by finite linear combinations of elementary tensors.)

2.6.14. Prove that the stabilization of A (§2.5.6) is isomorphic to $A \otimes \mathcal{K}$.

2.6.15. Prove that $\mathcal{K} \cong \mathcal{K} \otimes \mathcal{K}$.

3. Local theory of C*-algebras

3.1. Polar decomposition

I now collect several somewhat technical (yet illuminating) results on the local structure of C*-algebras. By polar decomposition theorem (Theorem 1.7) every operator a in $\mathcal{B}(H)$ can be represented as a product of a positive element and a partial isometry, $a = |a|v$. The positive part of a is given by formula $(aa^*)^{1/2}$ and therefore belongs to $C^*(a)$. In Exercise 3.3.2 we shall see that the partial isometry v need not belong to $C^*(a)$. We now discuss to what extent this representation, analogous to $z = \rho e^{i\theta}$ for complex numbers, works in arbitrary C*-algebras.

Lemma 3.1. *If b is invertible, then $b = cu$ for a unitary u and a positive c. Also, $c = (bb^*)^{1/2}$.*

Proof. Since bb^* is positive, $c = (bb^*)^{1/2}$ is well-defined by continuous functional calculus. Also, since b is invertible, so are b^*, bb^* and c. Then $u = c^{-1}b$ satisfies

$$uu^* = (bb^*)^{-1/2}bb^*(bb^*)^{-1/2} = 1$$

and

$$u^*u = b^*(bb^*)^{-1/2}(bb^*)^{-1/2}b^* = 1,$$

and is therefore a unitary. □

Lemma 3.2. *If 0 is not an accumulation point of $\mathrm{sp}((aa^*)^{1/2})$ then there is a partial isometry v in $C^*(a)$ such that $a = |a|v$.*

Proof. For a moment consider $C^*(a)$ as a concrete C*-algebra on some Hilbert space and let $a = |a|v$ be the polar decomposition of a. We have that $b = (aa^*)^{1/2}$ is in $C^*(a)$ and it remains to prove that $v \in C^*(a)$.

Let $f\colon \mathrm{sp}(b) \to \mathbb{C}$ be defined by $f(0) = 0$ and $f(t) = 1/t$ if $t \neq 0$. By our assumption f is continuous on $\mathrm{sp}(b)$ and therefore by the continuous functional calculus we have $f(b) \in C^*(a)$. Since $f(b)$ and b commute, $f(b)b$ is a self-adjoint element whose spectrum is included in $\{0, 1\}$ and it is therefore a projection. Denote this projection by p and note that $p \in C^*(a)$. Since $\ker(p) = \ker((aa^*)^{1/2}) = \ker(a^*)$ we have $pv = v$. Finally, $pv = f(b)|a|v = f(b)a$ and therefore $pv \in C^*(a)$. □

3.2. Stability

If b is self-adjoint and $\|a - b\| < \varepsilon$ then $c = (a + a^*)/2$ is self-adjoint, belongs to $C^*(a)$, and satisfies $\|c-b\| < \varepsilon$. This is an instance of the stability phenomenon: if an operator b belongs to a distinguished class of operators and $\|a-b\|$ is small, then there is $c \in C^*(a)$ in the same distinguished class as b such that $\|c - b\|$ is small.

Lemma 3.3. *Assume p is a projection and $\|a-p\| < \varepsilon$ with $\varepsilon < 1/2$. Then there is a projection $q \in C^*(a)$ such that $\|p - q\| < 2\varepsilon$.*

Proof. We may assume $p \neq 0$ (otherwise take $q = 0$). By replacing a with $(a + a^*)/2$ we may assume a is self-adjoint. By Lemma 2.8 we have that $\mathrm{sp}(a) \subseteq (-\varepsilon, \varepsilon) \cup (1 - \varepsilon, 1 + \varepsilon)$. Since $\varepsilon < 1/2$ the function f on $\mathrm{sp}(a)$ that sends $(-\varepsilon, \varepsilon)$ to 0 and $(1 - \varepsilon, 1 + \varepsilon)$ to 1 is well-defined and continuous. By continuous functional calculus we have $q = f(a) \in C^*(a)$ such that $\mathrm{sp}(q) = \{0, 1\}$. Therefore q is a projection. A straightforward computation (Exercise 3.3.1) shows that $\|q - a\| < \varepsilon$ and therefore q is as required. □

A straightforward modification of the proof of Lemma 3.3 gives the following.

Lemma 3.4. *Let F be a finite subset of $\mathbb{C}$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(F, \varepsilon) > 0$ with the following property. If b is normal and such that $\mathrm{sp}(b) = F$ and $\|a - b\| < \delta$, then there exists a normal $c \in C^*(a)$ such that $\|c - b\| < \varepsilon$ and $\mathrm{sp}(c) = F$.* □

Although partial isometries are not necessarily normal, a result similar to the above still applies.

Lemma 3.5. *For every $\varepsilon > 0$ there exists $\delta > 0$ with the following property. Assume v is a partial isometry and $\|a - v\| < \delta$. Then there is a partial isometry $w \in C^*(a)$ such that $\|v - w\| < \varepsilon$. Moreover, if p and q are projections such that $\|vv^* - p\| < \delta$ and $\|v^*v - q\| < \delta$ then we can choose partial isometry $w \in C^*(a, p, q)$ so that $ww^* = p$ and $w^*w = q$.*

Proof. Fix $\delta < \min(e^2/2, 1/12)$. If $\|a - v\| < \delta$ then $\|aa^* - vv^*\| \leq \|a(a^* - v^*)\| + \|(a - v)v^*\| \leq \delta\|a\| + \delta \leq 3\delta$ since $\|a\| \leq 1 + \delta$. By Lemma 2.8 this implies $\mathrm{sp}(aa^*) \subseteq_{3\delta} \{0, 1\}$ and therefore Exercise 2.4.4 implies $\mathrm{sp}((aa^*)^{1/2}) \subseteq_{\sqrt{3\delta}} \{0, 1\}$. Thus $\mathrm{sp}((aa^*)^{1/2})$ is included in two short intervals centered at 0 and 1. Let $f \in C((\mathrm{sp}(aa^*)^{1/2})$ be the function that maps the first interval to 0 the second to 1. Then $f((aa^*)^{1/2})$ is a self-adjoint element whose spectrum is $\{0,1\}$, and it is therefore a projection. It is also within $\sqrt{3\delta}$ of $(aa^*)^{1/2}$ and it therefore satisfies the assumptions of Lemma 3.2. Hence there is a partial isometry $w \in C^*(a)$ such that $a = (aa^*)^{1/2}w$.

For the moreover part, pick δ small enough so that pvq satisfies the assumptions of the first part of the lemma. Applying it to pvw we obtain $w \in C^*(a,p,q)$ such that $pwq = w$, $\|ww^* - p\| < 1$ and $\|w^*w - q\| < 1$. By Exercise 1.6.3, we have $ww^* = p$ and $w^*w = q$. □

The last few lemmas are concerned with stability of classes of operators in C*-algebras. For more information on this exciting subject the reader can consult the excellent [56].

The following lemma will be important in the analysis of the structure of UHF algebras. We write $A_1 = \{a \in A : \|a\| \leq 1\}$.

Lemma 3.6. *For every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists $\delta > 0$ with the following property. If A and B are unital subalgebras of C such that A is isomorphic to $M_n(\mathbb{C})$ and $A_1 \subseteq_\delta B$, then there exists a unitary $u \in C$ such that $uAu^* \subseteq B$ and $\|1 - u\| \leq \varepsilon$.*

Proof. I shall try to avoid the computation of δ (it is given in [12] and in [39]). Assume $\delta_1 > 0$ is very small. Consider $a \in A$ defined by $a = \mathrm{diag}(1,2,3,\dots,n)$. Then $a = \sum_{j<n} jp_j$, where p_j for $j < n$ is a projection in A and $\sum_{j<n} p_j = 1$. By Lemma 3.4 there exists $b \in B$ such that $\|b - a\| < \delta_2$ and b is self-adjoint with $\mathrm{sp}(b) = \{1,2,\dots,n\}$. Then $b = \sum_{j<n} jq_j$, where q_j for $j < n$ is a projection and $\sum_{j<n} q_j = 1$. We have $\|p_j - q_j\| < \delta_2$ for all j. As in Exercise 2.6.8, let e_{1j} for $j < n$ be partial isometries[c] generating A. Then $e_{1j}^* e_{1j} = p_j$ for $j < n$ and $e_{1j} = p_1 e_{1j} p_j$ for $j < n$.

By choosing δ small enough so that Lemma 3.5 can be applied to each e_{1j}, we find partial isometries e_{1j} for $1 \leq j \leq n$ in B as required. Then define $e_{ij} = e_{1i}^* e_{1j}$ for all i and j, note that $p_i = e_{ii}$ and that e_{ij} are matrix units as required. □

The following lemmas will be important in the analysis of Murray-von Neumann equivalence of projections (see §3.4).

[c]projections are partial isometries, too

Lemma 3.7. *If a is self-adjoint and there exist x and a self-adjoint b such that $xa = bx$ then x^*x commutes with a.*

If x is moreover invertible then b is unitarily equivalent to a, i.e., there exists unitary $u \in A$ such that $b = uau^*$.

Proof. By the self-adjointness of aa nd b we have $ax^* = x^*b$. Therefore

$$x^*xa = x^*bx = ax^*x$$

as required.

Now assume x is invertible. By Lemma 3.1 we have $x = u|x|$ for some $u \in A$. By the first part, x^*x commutes with a. Since $|x| = (x^*x)^{1/2}$ belongs to the C*-algebra generated by x^*x, it also commutes with a. We therefore have $b = xax^{-1} = u|x|a|x|^{-1}u^* = uau^*$. □

We state an immediate consequence of the above lemma for future reference.

Lemma 3.8. *If a and b are self-adjoint and $b = xax^{-1}$ for an invertible x then $b = uau^*$ for a unitary u.* □

3.3. Exercises

3.3.1. Assume a is normal and f and g are continuous functions on $\mathrm{sp}(a)$. Then $\|f(a) - g(a)\| = \|f - g\|_\infty$.

3.3.2. Find an operator a such that $C^*(a)$ does not contain partial isometry v such that $a = |a|v$.

(Hint: Choose a to be compact but of infinite rank.)

3.3.3. Show that invertible elements form an open set in a unital C*-algebra. (Hint: Use the proof of Lemma 1.4 to show that b invertible and $\|b - c\| < \|b\|$ implies c is invertible.)

3.3.4. Prove that a is invertible if and only if a^* is invertible if and only if aa^* invertible if and only if $|a|$ is invertible.

3.3.5. Prove Lemma 3.4 and express δ in terms of ε and F.

3.4. Murray–von Neumann equivalence of projections

Murray–von Neumann equivalence of projections is a noncommutative analogue of equinumerosity relation of sets and also a continuous variant of the dimension of a closed subspace of the Hilbert space (see Example 3.9 (1)). It was introduced by Murray and von Neumann in their seminal series of papers 'Rings of operators' (an old name for von Neumann algebras) where it played a fundamental role in type classification (see e.g., [46]). While C*-algebras are not nearly as well-behaved as von Neumann algebras Murray–von Neumann equivalence of projections is a useful tool in classification problem for some well-behaved classes of C*-algebras.

Assume A is a C*-algebra. Two projections p and q in A are *Murray-von Neumann equivalent* if there exists $v \in A$ such that

$$vv^* = p \text{ and } v^*v = q. \tag{MvN}$$

In this case we write $p \sim q$ and keep in mind that the relation depends on the ambient algebra A.

Note that a witness v of Murray-von Neumann equivalence is necessarily a partial isometry.

In some C*-algebras $p \sim q$ is strictly weaker than the requirement that p and q are conjugate by a unitary (see Example 3.9 (1)).

Example 3.9. (1) If $A = \mathcal{B}(H)$ then $p \sim q$ if and only if the range of p and the range of q have the same dimension, where the dimension of a closed subspace of the Hilbert space is the minimal cardinality of an orthonormal basis. This is an immediate consequence of the fact that two complex Hilbert spaces with the same dimension are linearly isometric.

(2) A special case of (1) is $A = M_n(\mathbb{C})$, where two projections are Murray-von Neumann equivalent if and only if they have the same rank. This extends to the algebra $\mathcal{K}$ of compact operators.

(3) If A is abelian then $p \sim q$ if and only if $p = q$.

The following example requires some minimal knowledge of vector bundles; see e.g., [41].

Example 3.10. Projections of $C(X, M_n(\mathbb{C}))$ are maps $f\colon X \to M_n(\mathbb{C})$ such that $f(x)$ is a projection for all $x \in X$. By identifying a projection in $M_n(\mathbb{C})$ with a subspace of $\mathbb{C}^n$ one sees that projections of $C(X, \mathcal{K})$ are vector bundles over X. Murray-von Neuman equivalence of these projections is the usual equivalence of vector bundles.

Lemma 3.11. *Assume p and q are projections in A such that $\|p-q\| < 1/2$. Then $p \sim q$. If A is unital then there is moreover a unitary u such that $u^*pu = q$.*

Proof. We first prove the case when A is unital. Let $a = pq + (1-p)(1-q)$. Since $1-p$ and $1-q$ are at a distance $\|p-q\| < 1/2$, the distance from a to $1 = p^2 + (1-p)^2$ is < 1. By Lemma 1.4 a is invertible. By Lemma 3.1 we have $a = |a|u$ for a unitary u.

One easily checks that $p_1 = aqa^{-1}$ satisfies $p_1^2 = p_1 = p_1^*$, and is therefore a projection. Similarly, $p_2 = a(1-p)a^{-1}$ is a projection and $p_1 + p_2 = 1$. By inspecting the definition of a one sees that $pp_1 = p_1$ and $(1-p)p_2 = p_2$, By Exercise 1.6.4 we conclude that $p = aqa^{-1}$. By Lemma 3.8 we conclude $p = uqu^*$. Then $v = uq$ is a partial isometry such that $vv^* = p$ and $v^*v = q$.

Now assume A is not unital. By the above, in the unitization of A there exists a unitary u such that $uqu^* = p$. The partial isometry $v = uq$ as above belongs to A and witnesses $p \sim q$. □

Lemma 3.11 can be improved; see Exercise 3.5.7.

3.5. Exercises

3.5.1. Prove that $p \sim q$ is equivalent to the existence of v such that $v^*pv = q$ and $vqv^* = p$. Also prove that such v is necessarily a partial isometry.

3.5.2. If $F\colon A \to B$ is a *-homomorphism and p and q are projections in A, show that $p \sim q$ implies $F(p) \sim F(q)$. Give an example showing that the converse may fail.

3.5.3. Let $\Phi\colon A \to B$ be a unital *-homomorphism between C*-algebras and let a and b be such that $b = \Phi(a)$.

(1) Prove that if a is normal (self-adjoint, positive, unitary, projection, partial isometry) then b is normal (self-adjoint, positive, unitary, projection, partial isometry).
(2) Assume b is self-adjoint (positive) Prove that we can choose a' such that $\Phi(a') = b$ and a' is self-adjoint (positive).
(3) Provide examples showing that b can be normal (unitary, projection, partial isometry, respectively) while no a' satisfying $\Phi(a') = b$ is normal (unitary, projection, partial isometry, respectively).
(Hint: For projections and partial isometries consider abelian algebras and use Exercise 2.2.6.)

3.5.4. Find a C*-algebra A and two Murray-von Neumann equivalent projections that are not conjugate.
(Hint: Try $\mathcal{B}(H)$.)

Let $\mathcal{P}(A)$ denote the set of all projections of a C*-algebra A.

3.5.5. Two projections are *homotopic* (in a C*-algebra A) if they belong to the same path-connected component of $\mathcal{P}(A)$. Prove that being homotopic implies being conjugate by a unitary.

(Hint: Lemma 3.11.)

3.5.6. Prove that $\|p - q\| < 1$ implies p and q are homotopic.

(Hint: The path $tp + (1-t)q$, for $0 \leq t \leq 1$, consists of nonzero positive elements. Use the continuous functional calculus to morph it into a path consisting of projections.)

3.5.7. Prove that $\|p-q\| < 1$ implies p and q are conjugate and in particular $p \sim q$.

(Hint: Combine Exercise 3.5.5 and Exercise 3.5.6.)

3.5.8. Prove that there exists $\varepsilon > 0$ such that for all A and projections p and q in A we have $p \sim q$ if and only if there exists $a \in A$ such that $\|aa^* - p\| < \varepsilon$ and $\|a^*a - q\| < \varepsilon$.

(Hint: Lemma 3.5 and Lemma 3.11.)

3.5.9. Prove that the following two properties of a C*-algebra A are equivalent.

(1) the set of invertible self-adjoint elements in the unitization of A is dense in the set of all self-adjoint elements in the unitization of A.
(2) Linear combinations of projections are dense in A.

(Hint: Every element of A is a linear combination of two self-adjoint operators. Use continuous functional calculus.)

C*-algebras A satisfying either of the statements from Exercise 3.5.9 have *real rank zero*.

Recall that on projections we define a relation $p \leq q$ if and only if $pq = p$ (Exercise 1.6.3). A nonzero projection p in a C*-algebra is *minimal* if the only projections $q \leq p$ are 0 and p.

3.5.10. Prove that a projection p in a real rank zero algebra A is minimal if and only if pAp is isomorphic to $\mathbb{C}$. Then prove that all minimal projections in a simple real rank zero algebra are Murray-von Neumann equivalent.

(Hint: If p and q are minimal projections in a real rank zero algebra prove that the vector space $pAq = \{paq : a \in A\}$ is one-dimensional.)

3.6. Traces

A *trace* of a C*-algebra A is a state τ such that $\tau(ab) = \tau(ba)$ for all a and b. We record an immediate consequence of the definition of $\sim$.

Lemma 3.12. *If τ is a trace on A and $p \sim q$ then $\tau(p) = \tau(q)$.* □

Let

$$T(A) = \{\tau \in A^* : \tau \text{ is a trace}\}.$$

If τ and σ are traces and $0 < t < 1$ then $t\tau + (1-t)\sigma$ is a trace. Therefore $T(A)$ is a convex subset of the unit sphere of A^*. Also, since being a trace is a closed condition, by Birkhoff–Alaoglu theorem $T(A)$ is compact in the weak*-topology. Being a compact and convex set, by the Krein–MIlman theorem $T(A)$ is the closure of the convex hull of its extreme points.

On $M_n(\mathbb{C})$ define the normalized trace via (below a stands for the matrix $(a_{ij})_{i\leq n, j\leq n}$)

$$\mathrm{tr}(a) = \frac{1}{n}\sum_{j=1}^{n} a_{jj}$$

Lemma 3.13. *If p and q are projections in $M_n(\mathbb{C})$ then $p \sim q$ if and only if $\mathrm{tr}(p) = \mathrm{tr}(q)$, and $\mathrm{tr}(p) = k/n$ where $0 \leq k \leq n$ is the dimension of the range of p.*

Proof. The direct implication is Lemma 3.12. The converse implication is an exercise in linear algebra. □

Lemma 3.14. *Functional* tr *is a unique trace on $M_n(\mathbb{C})$.*

Proof. This is of course a standard linear algebra fact. Fix a and b and note that the i-th diagonal entry of ab is equal to $\sum_{j<n} a_{ij}b_{ji}$, and therefore $\mathrm{tr}(ab) = \frac{1}{n}\sum_{i<n}\sum_{j<n} a_{ij}b_{ji}$. Analogous argument shows that $\mathrm{tr}(ba)$ has the same value. A similar computation shows that $\mathrm{tr}(aa^*) = \frac{1}{n}\sum_{i,j} a_{ij}\overline{a}_{ij} \geq 0$, and tr is therefore positive. Finally, $\mathrm{tr}(1) = 1$ is clear.

In order to check the uniqueness assume σ is a trace of $M_n(\mathbb{C})$. By Lemma 3.12 and Example 3.9 (2) all rank one projections have the same trace. Therefore $\sigma(p) = 1/n$ for all rank one projections p. This implies that for diagonal matrices a $\sigma(a) = \mathrm{tr}(a)$.

It only remains to note that every off-diagonal matrix unit necessarily has trace 0. □

Lemma 3.15. *There is a unital *-homomorphism from $M_n(\mathbb{C})$ into $M_k(\mathbb{C})$ if and only if n divides k. All unital *-homomorphisms from $M_n(\mathbb{C})$ into $M_k(\mathbb{C})$ are conjugate.*

Proof. This can be proved in many ways and our proof is not the shortest. If $\Phi\colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is a unital *-homomorphism then $\tau(a) = \mathrm{tr}(\Phi(a))$ defines a trace on $M_n(\mathbb{C})$. By Lemma 3.14 trace τ coincides with tr. Since a *-homomorphsm sends projections to projections, by Lemma 3.13 we conclude that $1/n = m/k$ for some m, concluding the proof.

In order to prove the second part, assume Φ and Ψ are unital *-homomorphisms of $M_n(\mathbb{C})$ into $M_k(\mathbb{C})$. Fix a minimal projection $p \in M_n(\mathbb{C})$, for example $p = \mathrm{diag}(1, 0, 0, \dots, 0)$. Then $\Phi(p)$ and $\Phi(q)$ are projections in $M_k(\mathbb{C})$ each with trace $1/n$ Therefore $A = \Phi(p)M_k(\mathbb{C})\Phi(p)$ and $B = \Psi(p)M_k(\mathbb{C})\Psi(p)$ are both isomorphic to $M_{k/n}(\mathbb{C})$. By Exercise 2.6.12 we have that $M_k(\mathbb{C}) \cong M_n(A) \cong M_n(B)$. Therefore an isomorphism $\alpha\colon A \to B$ extends to an automorphism α' of $M_k(\mathbb{C})$ such that $\Psi = \alpha' \circ \Phi$. By the easy finite-dimensional case of Exercise 1.2.7 we have that α is inner and therefore for some unitary $u \in M_k(\mathbb{C})$ we have $\Psi = \mathrm{Ad}\, u \circ \Phi$, as required. □

Note that if $F\colon B \to C$ is a unital *-homomorphism and τ is a trace of C then $\tau \circ F$ is a trace of B. The map $T(C) \ni \tau \mapsto \tau \circ F \in T(B)$ is continuous and affine.

Lemma 3.16. *Assume* $A = \lim_n A_n$ *is unital. If each* A_n *has a unique trace then* A *has a unique trace. More generally,* $T(A) = \varprojlim T(A_n)$.

Proof. We prove only the first assertion. Since A is unital, all but finitely many *-homomorphisms from A_n to A are unital. Let τ_n be the unique trace of A_n. Then $\tau_{n+1} \restriction A_n = \tau_n$. Therefore $\tau' = \lim_n \tau_n$ is a well-defined trace on a dense subset of A. Since trace is norm-continuous, τ' has a unique extension to a trace of A.

Assume σ is a trace of A. Then $\sigma \restriction A_n = \tau_n$ for all n and therefore σ and τ agree on a dense subset of A. Since σ is a continuous functional, $\sigma = \tau$.

In order to prove the second assertion use Exercise 2.2.10 in addition to the above and observe that the functor is contravariant. □

Example 3.17. (1) Assume A is unital and abelian. By the Gelfand–Naimark theorem $A = C(X)$ for a compact Hausdorff space X. By the Riesz Representation theorem, every continuous functional ϕ of A is of the form $\phi(f) = \int f\, d\mu$ for a finite Radon measure μ on X. If ϕ is a state then μ_ϕ is a probability measure. Since A is abelian the condition $\phi(ab) = \phi(ba)$ is automatic and therefore all states are traces.

Therefore $T(A)$ is affinely homeomorphic to $P(X)$, the space of Radon probability measures on X.

(2) By Lemma 3.14 $T(M_n(\mathbb{C}))$ is a singleton for every n.

(3) Furthermore, Lemma 3.16 implies that every UHF algebra carries a unique trace.

(4) Let $A = B \oplus C$. Then clearly $T(A) = \{\lambda\tau + (1-\lambda)\sigma : \tau \in T(B), \sigma \in T(C), 0 < \lambda < 1\}$. Therefore if A is a direct sum of n matrix algebras by (2) we have that $T(A)$ is affinely homeomorphic to the n-simplex, Δ_n.

(5) By (4) every trace τ of $\bigoplus_{i<n} M_{n(i)}(\mathbb{C})$ is determined by vector $(\lambda_i^\tau : i < n)$ in $[0,1]^n$ such that $\sum_{i<n} \lambda_i^\tau = 1$. Therefore if $A = \bigoplus_{i<n} M_{n(i)}(\mathbb{C})$, $B = \bigoplus_{i<k} M_{k(i)}(\mathbb{C})$, $F\colon A \to B$ is a unital *-homomorphism and τ is a trace on B. then $\sigma(a) = \tau(F(a))$ is a trace on A. If p_i is the identity of $M_{n(i)}(\mathbb{C})$ and q_j is the identity of $M_{k(j)}(\mathbb{C})$ then $\sigma(p_i)$ is uniquely determined by $\tau(q_j)$ and the Bratteli diagram of F.

3.7. Exercises

3.7.1. Assume $n \leq k$. Classify all (not necessarily unital) *-homomorphisms of $M_n(\mathbb{C})$ into $M_k(\mathbb{C})$, up to conjugacy.

(Hint: Consider the image of the identity and apply Lemma 3.15, which gives the unital *-homomorphism case.)

3.7.2. Prove that $T(A)$ is a weak*-compact convex subset of A^* (the Banach space dual of A).

3.7.3. Let X be a compact metric space and assume A has a unique trace. Prove that $T(C(X,A))$ is affinely homeomorphic to $\mathcal{P}(X)$, the space of Borel probability measures on X.

(Hint: See Example 3.17 (1).)

4. UHF algebras and AF algebras

4.1. UHF algebras

C*-algebras that are infinite tensor products of full matrix algebras $M_n(\mathbb{C})$ are said to be *uniformly hyperfinite* (shortly UHF). For separable algebras this is equivalent to being a unital direct limit of full matrix algebras. In separable case we have therefore have algebras of the form $\lim_j M_{n(j)}(\mathbb{C})$ for a sequence $n(j)$ such that $n(j)$ divides $n(j+1)$ for all j (Lemma 3.15). Also by the same lemma, the choice of *-homomorphisms in the unital case is inconsequential for the isomorphism type of the direct limit.

A *generalized integer* (regrettably also known as *supernatural number*) is a formal product of the form $\prod_j p_j^{k(j)}$ where p_j, for $j \in \mathbb{N}$, is an increasing enumeration of the primes and $k(j) \in \mathbb{N} \cup \{\infty\}$ (with $0 \in \mathbb{N}$).

To a separable UHF algebra $A = \lim_j M_{n(j)}(\mathbb{C})$ we associate a generalized integer $\mathbf{k}$ such that $k(j)$ is the largest k such that p_j^k divides $n(l)$

for some l, or ∞ if the set of such k is unbounded. One can show that the generalized integer uniquely determines a separable UHF algebra up to the isomorphism.

Let me again emphasize that UHF algebras are by definition unital. Non-unital algebras that are direct limits of full matrix algebras are called matroid algebras, approximately matricial (AM) algebras, or stabilized UHF algebras (note that the latter terminology is somewhat misleading, since they are not necessarily stable; see Exercise 4.3.6).

Example 4.1. (1) The CAR (Canonical Anticommutation Relation) algebra is the UHF algebra which is a direct limit of $M_{2^n}(\mathbb{C})$ for $n \in \mathbb{N}$. It is often denoted by M_{2^∞}.

(2) One can similarly define M_{3^∞} as the direct limit of $M_{3^n}(\mathbb{C})$ for $n \in \mathbb{N}$.

(3) The *universal UHF algebra* is the UHF algebra corresponding to the generalized integer $\prod_j p_j^\infty$.

Let D_n denote the subalgebra consisting of all diagonal matrices in $M_n(\mathbb{C})$. Then D_n is a maximal abelian subalgebra of $M_n(\mathbb{C})$ isomorphic to $\mathbb{C}^n$. If $A = \lim_j M_{n(j)}(\mathbb{C})$ is a UHF algebra then algebras $D_{n(j)}$ form a directed system and their limit D is the *diagonal masa* in A (cf. Exercise 4.3.2). If A is unital and infinite-dimensional then by Exercise 4.3.3 its diagonal masa is isomorphic to $C(2^{\mathbb{N}})$. Therefore the CAR algebra can be considered as a noncommutative version of the Cantor space. (It is customary to identify compact Hausdorff space X and the C*-algebra $C(X)$, since compact Hausdorff spaces and unital abelian C*-algebras form equivalent categories.)

Lemma 4.2. *Every UHF algebra A has a unique trace τ. The values of τ on projections of A are all numbers of the form k/n, where $k \in \mathbb{N}$ and n is a natural number that divides n_A.*

Proof. Each $M_n(\mathbb{C})$ has a unique trace (Lemma 3.14) and the conclusion follows by Lemma 3.16. □

We note that Glimm's result applies to an apparently larger class of algebras (see Theorem 4.4).

Theorem 4.3 (Glimm)**.** *Separable UHF algebras A and B are isomorphic if and only if they have the same generalized integer.*

Proof. If $\mathbf{k}_A \neq \mathbf{k}_B$ then by Lemma 4.2 A and B are not isomorphic.

Assume $\mathbf{k}_A = \mathbf{k}_B$. Write $A = \lim_j M_{n(j)}(\mathbb{C})$ and $B = \lim_j M_{m(j)}(\mathbb{C})$. By going to subsequences of $n(j)$ and of $m(j)$ we may assume that for all j we have that $n(j)$ divides $m(j)$ and $m(j)$ divides $n(j+1)$. By Lemma 3.15 we can fix a *-homomorphism $\phi_j \colon M_{n(j)}(\mathbb{C}) \to M_{m(j)}(\mathbb{C})$ and a *-homomorphism $\psi_j \colon M_{m(j)}(\mathbb{C}) \to M_{n(j+1)}(\mathbb{C})$ for every j. By the second part of the same lemma we may choose these maps so that all triangles in Figure 1 commute.

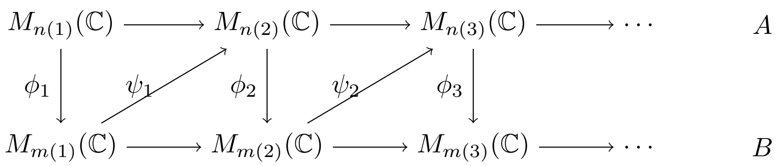

Fig. 1. Proof of Glimm's theorem

Therefore $\Phi_0 = \bigcup_j \phi_j$ is a well-defined *-homomorphism from a dense subalgebra of A into B. It is an isometry by Lemma 2.7 and it therefore extends to a *-homoorphism $\Phi \colon A \to B$. By the same argument we have a *-homomorphism $\Psi \colon B \to A$ extending $\bigcup_j \psi_j$. We claim that $\Psi \circ \Phi$ is the identity on A. It suffices to check this for the dense subalgebra $\bigcup_j M_{n(j)}(\mathbb{C})$. Indeed, for any j it is the identity on $M_{n(j)}(\mathbb{C})$ by the commutativity of the above diagram. Similarly $\Phi \circ \Psi$ is the identity on B, and therefore $\Phi \colon A \to B$ is a *-isomorphism. $\square$

4.2. An another look at the UHF algebras

A C*-algebra A is *locally matricial* (or LM) if for every $\varepsilon > 0$ and every finite $F \subseteq A$ there exist n and a *-homomorphism $\Phi \colon M_n(\mathbb{C}) \to A$ such that $F \subseteq_\varepsilon \Phi(M_n(\mathbb{C}))$. (Recall that $K \subseteq_\varepsilon L$ means that $\inf_{y \in L} \|x - y\| \leq \varepsilon$ for all $x \in K$.) In other words, for every finite subset of A there exists a full matrix subalgebra B of A such that each element of F is within ε of B.

Infinite tensor products of unital C*-algebras were defined in §2.5.7.

Theorem 4.4 (Glimm)**.** *For a separable unital C*-algebra the following are equivalent.*

(1) A is a tensor product of full matrix algebras.
(2) A is a direct limit of full matrix algebras (i.e., it is UHF), and
(3) A is LM.

An algebra as in (1) is necessarily unital, and (2) and (3) remain equivalent in the case when A is not necessarily unital.

Proof. We prove only the equivalence of (2) and (3) (but see the hint to Exercise 4.3.1). By the definition of direct limit (2) implies (3), even without the separability requirement. The implication from (3) to (2) is a consequence of Lemma 3.6 □

As a corollary to Theorem 4.4 and Theorem 4.3, to each unital separable LM algebra one can associate a generalized integer and that this generalized integer is a complete isomorphism invariant for unital separable LM algebras.

The following theorem taken from [30] shows that the situation in non-separable case is quite different. Recall that a *density character* of a C*-algebra is the minimal cardinality of a dense subset.

Theorem 4.5. *(1) There exist a unital C*-algebra of density character $\aleph_1$ that is a direct limit of full matrix algebras but not a tensor product of full matrix algebras.*
(2) Every LM algebra of density character $\leq \aleph_1$ is a direct limit of full matrix algebras.
(3) There exists a unital LM algebra of density character $\aleph_2$ that is not a direct limit of full matrix algebras. □

All algebras constructed in [30] are indistinguishable from the CAR algebra by their Elliott invariant, Cuntz semigroup, or any other known C*-algebraic invariant (see [31]).

4.3. Exercises

4.3.1. Prove the equivalence of (1) and (2) in Theorem 4.4: a unital separable C*-algebra is a tensor product of full matrix algebras if and only if it is a unital direct limit of full matrix algebras.

(Hint: Exercise 2.6.12.)

4.3.2. Prove that the diagonal masa (see the paragraph before Lemma 4.2) is a masa (i.e., a maximal abelian C*-subalgebra).

4.3.3. Prove that the diagonal masa of a separable infinite-dimensional UHF algebra is isomorphic to $C(2^{\mathbb{N}})$, where $2^{\mathbb{N}}$ denotes the Cantor space.

(Hint: It is a direct limit of finite-dimensional abelian C*-algebras. Prove that it is isomorphic to $C(X)$ for X a compact metrizable zero-dimensional space without isolated points.)

4.3.4. Let A be a UHF algebra. Show that its generalized integer $\mathbf{k}_A$ is uniquely defined as the number whose finite divisors are those n such that $M_n(\mathbb{C})$ has a unital *-homomorphism into A.

4.3.5. Let A and B be unital separable UHF algebras. Prove that A is elementarily equivalent to B (in the logic of metric structures, [29]) if and only A is isomorphic to B.

(Hint: [7].)

4.3.6. A C*-algebra is *stable* if it is isomorphic to its stabilization (see §2.5.6). Construct a direct limit of full matrix algebras that is neither unital nor stable.

(Hint: First prove that a stable algebra cannot have a finite trace and then construct a nonunital direct limit of full matrix algebras with a finite trace.)

4.3.7 (Dixmier)**.** Classify separable non-unital direct limits of full matrix algebras.

(Hint: First classify pairs (A, p) where A is a non-unital direct limit of full matrix algebras and $p \in A$ is a projection. Being familiar with classification of rank one torsion-free abelian groups may help, but beware of Exercise 4.3.6.)

4.3.8. Characterize when two separable UHF algebras have isomorphic corners (see §2.5.9).

4.3.9. Let A and B be separable UHF algebras. Prove that A is isomorphic to a unital subalgebra of B if and only if $\mathbf{k}_A$ divides $\mathbf{k}_B$.

4.4. Bratteli diagrams

The following lemma is a consequence of the Artin–Wedderburn theorem but we sketch a direct proof in Exercise 4.7.6 below.

Lemma 4.6. *Every finite-dimensional C*-algebra is *-isomorphic to a direct sum of finitely many full matrix algebras that each of these full matrix algebras is a minimal (nontrivial) ideal of the algebra.* □

We shall introduce a tool for describing *-homomorphisms $\Phi\colon A \to B$ between finite-dimensional C*-algebras. By Lemma 4.6 every such algebra is a direct sum of its minimal ideals each of which is isomorphic to a full matrix algebras. Recall that there exists a unital *-homomorphism from $M_n(\mathbb{C})$ into $M_k(\mathbb{C})$ if and only if n divides k (Lemma 3.15). If $n \leq k$ and $\Phi\colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is a *-homomorphism (not necessarily unital) then its *multiplicity* is the rank of Φ-image of any minimal projection in $M_n(\mathbb{C})$ (since Φ preserves Murray–von Neumann equivalence, the rank is well-defined). In other words, if Φ sends a to $\operatorname{diag}(a, a, \dots, a, 0, \dots, 0)$ then its multiplicity is the number of the occurrences of a on the right-hand side. (If you haven't tackled Exercise 3.7.1 yet, now is a good time.)

Bratteli diagram of $\Phi \colon A \to B$ is a bipartite graph whose vertices on the left correspond to the minimal ideals of A and vertices on the right correspond to the minimal ideals of B. These vertices may be labelled by numbers indicating the dimension of the corresponding algebra. Two vertices are connected by k edges if and only if the multiplicity of the map between them is k.

An example is in order. Consider a unital *-homomorphism between $M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ and $M_6(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_6(\mathbb{C})$ defined by

$$(a, b) \mapsto (\operatorname{diag}(a, a, a), \operatorname{diag}(a, b), \operatorname{diag}(b, b)).$$

The Bratteli diagram describing this map is given in Figure 2.

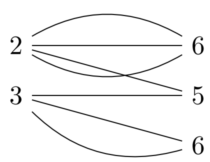

Fig. 2. A Bratteli diagram

By Lemma 4.7, *-homomorphism described by a Bratteli diagram is unique up to unitary conjugacy.

When describing a unital AF algebra $A = \lim_n A_n$ by a Bratteli diagram we put together diagrams of each $\Phi_n \colon A_n \to A_{n+1}$. For convenience we also let $A_1 = \mathbb{C}$ and assume that all Φ_n are unital. Under these conventions the labels of vertices can be omitted since the dimension of any of the full matrix algebras can be determined by adding the multiplicities of vertices from the earlier levels.

Some examples of C*-algebras defined via Bratteli diagrams are given in figures 3–7.

Lemma 4.7. *Every unital *-homomorphism between finite direct sums of matrix algebras corresponds to a Bratteli diagram. Moreover, any two unital *-homomorphisms with the same Bratteli diagram are conjugate.*

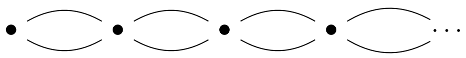

Fig. 3. This diagram describes the unital directed system $M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \dots$ and its limit, the CAR algebra M_{2^∞}.

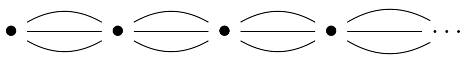

Fig. 4. This diagram represents M_{3^∞}.

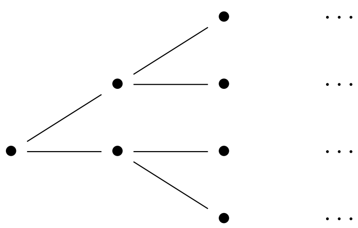

Fig. 5. In this diagram every node splits into two nodes. Note that all nodes in the diagram correspond to the abelian algebra $\mathbb{C}$. Therefore the n-th level corresponds to the algebra $\mathbb{C}^{2^n}$ and it is not difficult to prove that the direct limit is the algebra $C(2^{\mathbb{N}})$.

Proof. This is an almost immediate consequence of Lemma 3.15. First note that if $F\colon M_n(\mathbb{C}) \to M_k(\mathbb{C})$ is a non-unital *-homomorphism then $p = F(1)$ is a projection. Also, $pM_k(\mathbb{C})p$ is isomorphic to $M_m(\mathbb{C})$ where m is the rank of p. By Lemma 3.15 m is a multiple of n.

In order to prove the second part of the lemma, fix a unital *-homomorphism $F\colon \bigoplus_{i<N} M_{n(i)}(\mathbb{C}) \to \bigoplus_{i<K} M_{k(i)}(\mathbb{C})$. Let p_i be the identity of $M_{n(i)}(\mathbb{C})$ and let q_j be the identity of $M_{k(j)}(\mathbb{C})$. Then $F(\sum_i p_i) = \sum_i F(p_i) = 1$ and $r_{ij} = q_j F(p_i)$ is a projection since q_j is a central projection. Then $q_j = \sum_i r_{ij}$ and the rank of r_{ij} is by the above a multiple of the rank of p_i for all i and j. This data gives a Bratteli diagram. □

4.5. Exercises

4.5.1. If A and B are separable AF algebras given by the same Bratteli diagram then they are isomorphic. The converse may fail.

(Hint: Apply Lemma 4.7 along the diagram.)

4.5.2. Characterize Bratteli diagrams of simple AF algebras.

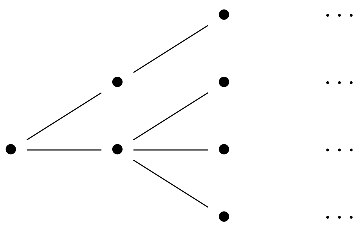

Fig. 6. In this diagram the n-th level has 2^n nodes, each one corresponding to $\mathbb{C}$. Hence the algebra is a direct limit of $\mathbb{C}^{2^n}$, just like the algebra $C(2^{\mathbb{N}})$ from the previous example. However, in the present case only one of the nodes on the n-th level splits, and it splits into 2^n other nodes. One can prove that the direct limit is the algebra $C(\omega + 1)$, where $\omega + 1$ is a converging sequence together with its limit.

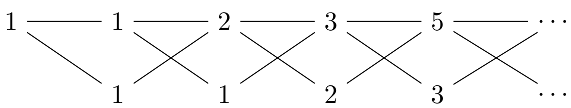

Fig. 7. In this example nodes are marked by numbers for readability. The n-th level of this diagram corresponds to algebra $M_{F(n)}(\mathbb{C}) \oplus M_{F(n+1)}(\mathbb{C})$, where $F(n)$ is the n-th Fibonacci number. It can be shown that this algebra, called *Fibonacci algebra*, is a simple, unital AF algebra with a unique trace that is not a UHF algebra.

(Hint: Given a Bratteli diagram of A, identify its subsets whose direct limits are ideals of A.)

4.5.3. Which algebra corresponds to the Bratteli diagram given in Figure 8?

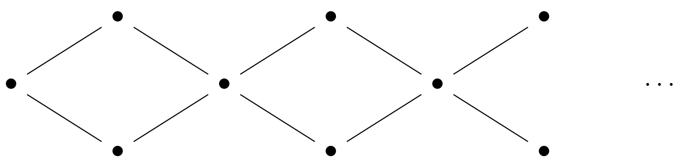

Fig. 8. Diagram for Exercise 4.5.3

4.5.4 (Bratteli)**.** By Exercise 4.5.1, to each Bratteli diagram $\mathbb{D}$ one can associate the unique AF algebra $A(\mathbb{D})$. Describe the equivalence relation on Bratteli diagrams defined by $\mathbb{D}_1 \mathrel{\mathrm{E}} \mathbb{D}_2$ iff $A(\mathbb{D}_1) \cong A(\mathbb{D}_2)$.

4.5.5. Construct a Bratteli diagram such that the corresponding AF algebra A is simple and $T(A)$ is affinely homeomorphic to $[0, 1]$.

Hint: see Figure 9.

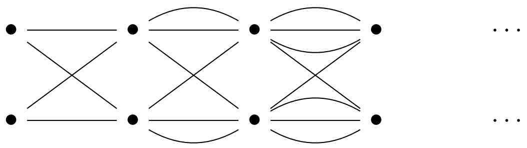

Fig. 9. Hint for Exercise 4.5.5

4.6. AF algebras

A C*-algebra is *AF* (approximately finite) if it is a direct limit of finite-dimensional C*-algebras.

If A is a subalgebra of B, X is a subset of B, and $\varepsilon > 0$, we write

$$X \subseteq_\varepsilon A$$

if every element of X is within ε of an element of A.

Definition 4.8. A C*-algebra B is an *LF-algebra* (locally finite) if for every finite $F \subseteq B$ there are a finite-dimensional subalgebra C of A such that $X \subseteq_\varepsilon C$.

Clearly every AF algebra is an LF algebra. Analogously to the case of UHF algebras (Theorem 4.4), the converse is true for separable AF algebras (Exercise 4.7.10). However, this fails for separable AH algebras: direct limits of AH algebras need not be AH ([11]; see §6 for the definition of AH algebras). Analogous statements also fail for nonseparable AF, and even AM, algebras ([30]).

4.7. Exercises

4.7.1. Prove that for $n \in \mathbb{N}$ every n-dimensional abelian C*-algebra is isomorphic to $\mathbb{C}^n$ (with the max norm).

A projection p in A is *minimal* if it is nonzero and the only projections $\leq p$ are p and 0.

4.7.2. Prove that UHF algebras have no minimal projections.

4.7.3. Assume D is a masa (maximal abelian C*-subalgebra) in A. Prove that every minimal projection of D is a minimal projection of A.

4.7.4. Prove that if A is infinite dimensional then every masa in A is infinite-dimensional.

(Hint: Use Exercise 4.7.3. It works even if A has no nontrivial projections.)

In the following two exercises we sketch a direct proof of the instance of Artin–Wederburn theorem for C*-algebras (Lemma 4.6).

4.7.5. Assume A is a C*-algebra that is both simple and finite-dimensional. Prove that $A \cong M_n(\mathbb{C})$ for some n.

(Hint: First let D be a maximal abelian subalgebra of A and apply Exercise 4.7.1. Then apply Exercise 3.5.10.)

4.7.6. Assume A is a finite-dimensional C*-algebra. Prove that it is a direct sum of full matrix algebras.

(Hint: Apply Exercise 4.7.5 to each minimal ideal of A.)

Recall that $p \sim q$ implies $\tau(p) = \tau(q)$ for all traces τ. While the converse in general fails, it holds in some well-behaved classes of C*-algebras. The case of UHF algebras was already used earlier.

4.7.7. Assume p and q are projections in an AF algebra A. Prove that if $\tau(p) = \tau(q)$ for all traces τ then $p \sim q$.

4.7.8. Prove that a Bratteli diagram corresponds to an abelian AF algebra if and only if it is a tree. In this case, the corresponding AF algebra is isomorphic to $C(X)$ where X is the set of all branches through this tree.

4.7.9. Prove that a unital abelian algebra is AF if and only if it is of the form $C(X)$ for a zero-dimensional space X.

4.7.10. Prove that every separable LF algebra is AF.

(Hint: This is similar to the proof in the case of UHF algebras, i.e., that LM implies AM. See Theorem 4.4. All computations are given in detail in [12].)

4.7.11. Check that if A is a direct sum of n full matrix algebras then $T(A)$ is affinely homeomorphic to the $n-1$-dimensional simplex.

Given a unital *-homomorphism $\Phi\colon A \to B$ between finite-dimensional C*-algebras, use the above to describe which traces in $T(A)$ are Φ^*-images of traces in B (see Exercise 2.2.10).

4.7.12. Prove that the Fibonacci algebra has a unique trace.
(Hint: Exercise 4.7.11.)

5. The functor K_0

We are ready to define the classifying invariant for AF-algebras, $K_0(A)$. We shall define it only in the unital case since the general case involves some additional technicalities (see e.g., [68]). We first define the *Murray-von Neumann semigroup* of a C*-algebra A, denoted by $V(A)$.

The underlying set of $V(A)$ is $\mathcal{P}(M_\infty(A))/\sim$. (See §2.5.6 for its definition.) One could consider $\mathcal{K} \otimes A$ instead (see §2.5.6). Since $\mathcal{K} \otimes A$ is the completion of $M_\infty(A)$, Lemma 3.3 implies that every projection in $\mathcal{K} \otimes A$ is equivalent to a projection in $M_\infty(A)$.

Note that a projection $p \in M_n(A)$ is Murray-von Neumann equivalent to $\mathrm{diag}(0, p) \in M_{2n}(A)$ (here 0 is the zero matrix in $M_n(A)$). This device of 'moving projections away down the diagonal' is used to define addition in $V(A)$. For projections p and q in $M_\infty(A)$ we can find n such that both p and q belong to $M_n(A)$ and define $[p] \oplus [q]$ in $V(A)$ to be the equivalence class of the projection $\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$ in $M_{2n}(A)$. By conjugating with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we see that this projection is equivalent to $\begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$. The associativity of $\oplus$ is also easy to check and therefore $\oplus$ defines an operation on $V(A)$ that turns it into an abelian semigroup.

Note that $p + q = r$ implies $[p] \oplus [q] = [r]$, but certainly not vice versa; in particular $p + q$ need not be a projection.

Recall that a *Grothendieck group* of a semigroup $(V, +)$ is defined as follows. On V^2 define equivalence relation $\approx$ via $(f, g) \approx (f', g')$ if $f + g' = f' + g$. The addition of equivalence classes is defined coordinatewise, by $[(f_1, g_1)] + [(f_2, g_2)] = [(f_1 + f_2, g_1 + g_2)]$. This construction results in an abelian group.

We define $K_0(A)$ to be the Grothendieck group of $(V(A), \oplus)$. Also, by letting $K_0^+(A)$ be the image of $V(A)$ we provide an ordered group structure. Finally, in the unital case $K_0(A)$ is the ordered abelian group with order unit,

$$(K_0(A), K_0^+(A), [1_A]),$$

where $[1_A]$ denotes the equivalence class of the identity.

A word of caution is due at this point. The ordering on $K_0(A)$ can behave in a very unusual way. In some cases it contains elements p and q

such that $np > (n+1)q$ for some n, while $p \not\succ q$. This behaviour can be exhibited by analyzing nontrivial vector bundles (cf. Example 3.10) and it is a key ingredient in the construction of many pathological C*-algebras (see §6.0.1).

We shall use an abbreviation and an abuse of notation and write $K_0(A)$ instead of $(K_0(A), K_0(A)^+, [1_A])$ whenever there is no danger of confusion.

5.1. Computation of K_0 in some simple cases

5.1.1. *K_0 of $M_n(\mathbb{C})$* Since $M_n(\mathbb{C}) \otimes \mathcal{K}$ is isomorphic to $\mathcal{K}$, we have that $V(M_n(\mathbb{C}))$ is isomorphic to $(\mathbb{N}, +)$ (with $0 \in \mathbb{N}$). Note that $V(M_n(\mathbb{C}))$ and $V(M_k(\mathbb{C}))$ can be distinguished if one keeps track of the $\sim$-equivalence class of the identity of the algebra, $[1_A]$.

After rescaling, $K_0(M_n(\mathbb{C}))$ becomes $(\mathbb{Z}[1/n], \mathbb{Z}^+[1/n], 1)$.

5.1.2. *K_0 of $\mathcal{B}(H)$.* We are assuming H is infinite-dimensional. Then we have $[p] + [1] = [p]$ for all p, and therefore $K_0(\mathcal{B}(H)) = \{0\}$.

5.1.3. *K_0 of the Calkin algebra Calkin algebra*, denoted $\mathcal{C}(H)$, is the quotient $\mathcal{B}(H)/\mathcal{K}(H)$. Gelfand–Naimark–Segal theorem implies that it is a C*-algebra. All nonzero projections in the Calkin algebra are Murray–von Neumann equivalent. This extends to $\mathcal{K} \otimes \mathcal{C}(H)$ and therefore $K_0(\mathcal{C}(H)) = \{0\}$.

5.1.4. *K_0 of the CAR algebra* By §5.1.1, for every UHF algebra A we have that $K_0(A)$ is a direct limit of copies of $\mathbb{Z}$, with the positive part being exactly the positive integers. For the CAR algebra, the unit of the n-th copy of $\mathbb{Z}$ is 2^n. Therefore $K_0(M_{2^\infty})$ is isomorphic to the group of dyadic rationals, $\{k/2^n : k \in \mathbb{Z}, n \in \mathbb{N}\}$, with $[1_A] = 1$ and the positive part being exactly the positive dyadic rationals.

5.1.5. *K_0 of other UHF algebras* Let A be a UHF algebra corresponding to the generalized integer $\mathbf{k}$. The above argument shows that

$$K_0(A) = \{m/k : m \in \mathbb{Z} \text{ and } k \text{ divides } \mathbf{k}\}$$

with $[1_A] = 1$ and the natural positive part.

Note that for projections $p \in A$ the equivalence class $[p]$ exactly corresponds to the normalized trace $\mathrm{tr}(p)$.

5.1.6. *K_0 of a *-homomorphism* If $\Phi\colon A \to B$ is a *-homomorphism then by Exercise 3.5.2 it extends to a semigroup homomorphism from $V(A)$ to $V(B)$. If A, B and Φ are unital then Φ can be canonically extended to a *-homomorphism from $M_\infty(A)$ to $M_\infty(B)$ and we therefore have a homomorphism

$$K_0(\Phi)\colon (K_0(A), K_0(A)^+, [1_A]) \to (K_0(B), K_0(B)^+, [1_B]).$$

If Φ is an isomorphism then so is $K_0(\Phi)$, but the converse may fail. The following lemma shows that K_0 is continuous with respect to inductive limits.

Lemma 5.1. *If A_i, for $i \in I$ and F_{ij}, for $i < j$ is a unital directed system and $A = \lim_i A_i$ then $K_0(A_i)$, for $i \in I$ and $K_0(F_{ij})$ for $i < j$ in I is a directed system and $K_0(A) = \lim_i K_0(A_i)$.*

Proof. The first claim is an immediate consequence of the above discussion. By Lemma 3.3 every projection in $A \otimes \mathcal{K}$ is Murray-von Neumann equivalent to an image of a projection in some $A_i \otimes \mathcal{K}$ and therefore $V(A) = \lim_i V(A_i)$ and $K_0(A) = \lim_i K_0(A_i)$. □

5.2. Exercises

5.2.1. Describe $K_0(A \oplus B)$ in terms of A and B.

5.2.2. Use Exercise 5.2.1 and Lemma 5.1 to show that K_0 of an AF algebra is a direct limit of groups of the form $\mathbb{Z}^{n(i)}$, for $n(i) \in \mathbb{N}$, with their natural ordering.

5.3. Cancellation property

An abelian semigroup $(S, +)$ has the *cancellation property* if $x+y = z+y$ implies $x = z$. This is equivalent to stating that in the Grothendieck group of $(S, +)$ no two distinct elements of S belong to the same equivalence class. A C*-algebra A has the cancellation property if its Murray-von Neumann semigroup has cancellation property.

Lemma 5.2. *A direct limit of algebras with cancellation property has cancellation property.*

Proof. Assume $A = \lim_n A_n$ does not have cancellation property. Not having cancellation property is witnessed by the following objects in $\mathcal{K} \otimes A$: three projections, p, q and r, one partial isometry, v, such that $vv^* = p + r$ and $v^*v = q + r$ and the absence of a partial isometry w such that $ww^* = p$ and $w^*w = q$. By Lemma 3.3, we may assume p, q and r all belong to

$M_n(A_n)$ for a large enough n. By Lemma 3.5, we may assume that v also belongs to $M_n(A_n)$. Therefore A_n does not have cancellation property. □

Lemma 5.3. *Every AF algebra has cancellation property.*

Proof. By Lemma 5.2 it suffices to show that every finite-dimensional C*-algebra A has cancellation property. This can be proved in a variety of ways. For example, $V(A)$ is isomorphic to the free abelian semigroup with k generators where k is the number of full matrix direct summands of A, and this semigroup does not have cancellation property. □

In the category of K_0-groups homomorphisms are group homomorphisms that preserve both positivity and the unit.

Note that if p, q and r are projections in $M_\infty(A)$ then $p+q=r$ implies $[p]+[q]=[r]$. The converse is false since $p+q$ need not be a projection. The following weak converse indicates why K_0 is defined in $\mathcal{P}(M_\infty(A))$ instead of $\mathcal{P}(A)$.

Lemma 5.4. *If p and q are projections in $M_\infty(A)$ then there exists $q' \in M_\infty(A)$ such that $q \sim q'$ and $p+q$ is a projection.*

Proof. Let n be such that p and q both belong to $M_n(A)$ and identify them with $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$, respectively, in $M_{2n}(A)$. Let v be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (a block-matrix in $M_{2n}(A)$). Then $q' = (\operatorname{Ad} v)q = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$ is as required. □

Lemma 5.5. *Assume A is finite-dimensional and B has cancellation property. Then every homomorphism of preordered abelian groups with order unit $\phi\colon (K_0(A), K_0(A)^+, [1_A]) \to (K_0(B), K_0(B)^+, [1_B])$ is of the form $K_0(\Phi)$ for a unital *-homomorphism $\Phi\colon A \to B$.*

Proof. Choose n such that there are minimal projections p_i, for $i < n$, in A satisfying $\sum_{i<n} p_i = 1$. Since ϕ is positive, Lemma 5.4 implies that for each i we can choose a projection $q_i \in M_\infty(B)$ such that $\phi([p_i]) = [q_i]$. Since ϕ is a homomorphism we have that $\sum_{i<n}[q_i] = [1_B]$. (Here 1_B denotes the unit of B.) This means that for some projection r we have $\operatorname{diag}(q_0, q_1, \dots, q_{n-1}, r) \sim \operatorname{diag}(1_B, r)$, but by the cancellation property of B we have $\operatorname{diag}(q_0, q_1, \dots, q_{n-1}) \sim 1_B$. Denote the right hand side by q and let v be a partial isometry in $M_\infty(B)$ such that $vqv^* = 1_B$. Then $r_i = vq_iv^*$ is a projection in $M_\infty(B)$ such that $\sum_{i<n} r_i = 1_B$. Therefore r_i, for $i < n$, are orthogonal projections in B.

Now we can shed the scaffolding provided by $M_\infty(B)$ and work in B. Also, $r_i \sim r_j$ if and only if $p_i \sim p_j$. Let $\approx$ be an equivalence relation on n defined by $i \approx j$ if and only if $p_i \sim p_j$ in A. For each such pair choose a partial isometry $w_{ij} \in A$ such that

(1) $w_{ij}w_{ij}^* = p_i$ and $w_{ij}^* w_{ij} = p_j$, and
(2) $w_{ij}w_{kl} = w_{il}\delta_{jk}$

for all i, jk, l.

We can recursively choose partial isometries v_{ij} for all $i \approx j$ that, together with r_i for $i < n$, generate a finite-dimensional unital subalgebra of B isomorphic to A and satisfy equalities corresponding to the above. Now define $\Phi\colon A \to B$ by $\Phi(p_i) = r_i$, $\Phi(w_{ij}) = v_{ij}$ for $i \approx j$ and extend it linearly. Then $\Phi\colon A \to B$ is a unital *-homomorphism and $K_0(\Phi) = \phi$. □

5.4. Classification of AF algebras

A *dimension group* is a direct limit of ordered groups with order unit of the form $(\mathbb{Z}^n, (\mathbb{Z}^+)^n, [e])$ where $e \in (\mathbb{Z}^=)^n$. It is an easy consequence of Lemma 5.1 that K_0 of a separable unital AF algebra is a dimension group. The converse is also not difficult to prove: a countable ordered group is a dimension group if and only if it is equal to K_0 of a separable unital C*-algebra (actually one still has the equivalence if both separability and countability are dropped). A first-order characterization of dimension groups was given by Effros–Handelman–Shen (see [16] or [12]).

Lemma 5.6. *Assume $A = \lim_n A_n$ and $B = \lim_n B_n$. Also assume $\phi_n\colon A_n \to B_n$ and $\psi_n\colon B_n \to A_{n+1}$ are *-homomorphisms such that the diagram in Figure 10 commutes.*

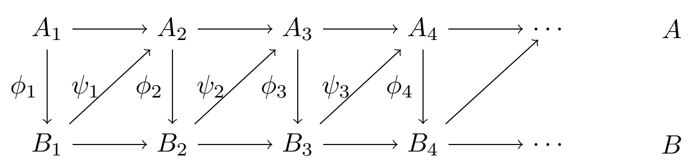

Fig. 10.

Then A is isomorphic to B.

Proof. Let $F_n\colon A_n \to A$ denote the canonical *-homomorphism for each $n \in$

$\mathbb{N}$. Since the diagram commutes,

$$\phi(a) = \phi_n(F_n a) \text{ if } a \in A_n$$

is a well-defined *-homomorphism from a dense subset of A into B. Since each ϕ_n is a contraction, so is ϕ and therefore ϕ extends to a *-homomorphism from A into B. One analogously defines $\psi\colon B \to A$ as a direct limit of ψ_n for $n \in \mathbb{N}$. Since $\psi \circ \phi$ is id_A and $\phi \circ \psi$ is id_B, we conclude that ϕ and ψ are *-isomorphisms. □

Two remarks regarding Lemma 5.6 are in order. First, maps ϕ_n and ψ_n are not required to be isomorphisms or even injections. All we need is commutation of the diagram. Second, this lemma is about direct limits of arbitrary structures.

Proof of the following lemma is very similar to the proof of Lemma 5.6.

Lemma 5.7. *Assume $A = \lim_n A_n$ and $B = \lim_n B_n$. Also assume $\phi_n\colon A_n \to B_n$ are *-homomorphisms such that the diagram in Figure 11 commutes.*

$$\begin{array}{ccccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & \cdots & & A \\
\phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow & & \phi_4 \downarrow & & & & \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & \cdots & & B
\end{array}$$

Fig. 11.

*Then there is a *-homomorphism from A to B that extends $\bigcup_n \phi_n$.* □

The assumption that the algebra be unital is not needed in Theorem 5.8 or in Theorem 5.9. However, the definition of K_0 of a nonunital algebra is not given in these notes.

Theorem 5.8 (Elliott, [18])**.** *Two separable unital AF algebras are isomorphic if and only if their ordered K_0 groups are isomorphic.*

Proof. Write $A = \lim_n A_n$ and $B = \lim_n B_n$, where all A_n and all B_n are finite dimensional and all connecting maps are unital. Then $K_0(A) = \lim_n K_0(A_n)$ and $K_0(B) = \lim_n K_0(B_n)$. Let $\phi\colon K_0(A) \to K_0(B)$ be an ordered unit group isomorphism. Each $K_0(A_n)$ is finitely generated, and therefore we can go to a subsequence of A_n and B_n so that, after re-enumerating, we have that ϕ sends $K_0(A_n)$ into $K_0(B_n)$ and ϕ^{-1} sends $K_0(B_n)$ into $K_0(B_{n+1})$ for all n.

The plan is to apply Lemma 5.6 with appropriately chosen *-homomorphisms Φ_n and Ψ_n, for $n \in \mathbb{N}$. By applying Lemma 5.5 for each n in both directions we can recursively choose a *-homomorphism $\Phi_n\colon A_n \to B_n$ and a *-homomorphism $\Psi'_n\colon B_n \to A_n$ which lift the corresponding maps between K_0 groups. Assume Φ_n and Ψ'_n were chosen. We shall modify Ψ'_n to make the triangle between A_n, B_n and A_{n+1} in the diagram from Lemma 5.6 commute. We have two *-homomorphisms from A_n to A_{n+1}, namely $F_{n,n+1}\colon A_n \to A_{n+1}$ given by the directed system and $\Psi'_n \circ \Phi_n$. Lemma 4.7 applies to this pair and gives a unitary u_n in A_{n+1} such that $\operatorname{Ad} u_n \circ \Psi'_n \circ \Phi_n = F_{n,n+1}$. Then $\Psi_n = \operatorname{Ad} u \circ \Psi'_n$ is as required. Once all Φ_n and Ψ_n are chosen to make the whole diagram commute, Lemma 5.6 implies $A \cong B$. □

The above proof shows a bit more. Let us say that two *-homomorphisms $\Phi_j\colon A \to B$, for $j = 0, 1$ are *approximately unitarily equivalent* if there is a sequence of unitaries u_n, for $n \in \mathbb{N}$, in B such that $\Phi_0 \circ \operatorname{Ad} u_n$ converges to Φ_1 pointwise.

Theorem 5.9 (Elliott, [18]). *If A and B are separable unital C*-algebras then for every positive unital group homomorphism*

$$\Phi\colon (K_0(A), K_0(A)^+, [1_A]) \to (K_0(B), K_0(B)^+, [1_B])$$

*there exists a unital *-homomorphism $\Phi\colon A \to B$ such that $\phi = K_0(\Phi)$. Moreover, $K_0(\Phi) = K_0(\Psi)$ if and only if Φ and Ψ are approximately unitarily equivalent. Moreover, if ϕ is an isomorphism then so is Φ.*

Proof. The proof of the first statement is similar to the above and uses Lemma 5.7. The approximate unitary equivalence of *-homomorphisms whose K_0 coincide is a consequence of Lemma 4.7 applied along the finite stages of the diagram. The last sentence is Theorem 5.8. □

5.5. Exercises

5.5.1. Prove that $K_0(A)$ is countable if A is separable.
(Hint: Lemma 3.11.)

5.5.2. Prove that if A has cancellation property and p, q are projections in A such that $p \sim q$, then $1 - p \sim 1 - q$. Find a C*-algebra in which this is not true, and which therefore does not have the cancellation property.

(Hint: For the second part use Example 3.9 (1).)

5.5.3. Prove that we could have defined $V(A)$ and $K_0(A)$ in $\mathcal{P}(A \otimes \mathcal{K})$ (see §2.5.6) instead of $\mathcal{P}(M_\infty(A))$.

(Hint: Lemma 3.11)

5.5.4. Compute K_0 of the UHF algebra corresponding to the generalized integer $\prod_j p_j^{k(j)}$.

5.5.5. Prove that K_0 of the Fibonacci algebra is (with $s = (1 + \sqrt{5})/2$) $(\mathbb{Z}^2, \{(m, n) : sm + n \geq 0\}, (1, 0))$.

5.5.6. Prove the assertions made in §5.1.6.

5.5.7. A *state* on a preordered abelian group with order unit $(G, G^+, [1])$ is a homomorphism $\chi \colon G \to \mathbb{R}$ such that $\chi[G^+] \subseteq \mathbb{R}^+$ and $\chi(1) = 1$. Prove that a trace $\tau \in T(A)$ defines a state of $(K_0(A), K_0(A)^+, [1_A])$.

5.5.8 (Approximate intertwining, aka Elliott intertwining)**.** Assume $A = \lim_n A_n$ and $B = \lim_n B_n$. Furthermore assume $F_n \subseteq A_n$, for $n \in \mathbb{N}$, is an increasing sequence of finite subsets of A with dense union, and $G_n \subseteq B_n$, for $n \in \mathbb{N}$ is an increasing sequence of finite subsets of B with dense union.

Finally, assume $\phi_n \colon A_n \to B_n$ and $\psi_n \colon B_n \to A_{n+1}$ are *-homomorphisms such that the n-th triangle of the diagram in Figure 12 commutes up to 2^{-n} on both F_n and G_n.

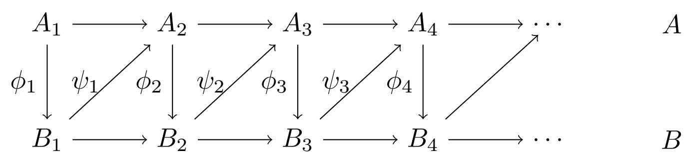

Fig. 12. An approximate intertwining argument

Prove that $A \cong B$.

5.5.9. Prove that $K_0(C([0, 1], A)$ is isomorphic to $K_0(A)$ for all A.

6. Elliott's program

Around 1990, Geoge Elliott conjectured that separable, unital, nuclear, simple C*-algebras can be classified by K-theoretic invariants. Nuclear C*-algebras can be defined in several equivalent but apparently rather different ways (e.g., Definition 2.9), but for our purposes the actual definition is irrelevant. See e.g., [4] or [70].

The Elliott invariant of an algebra A is the sixtuple

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)^+, [1_A]_0, K_1(A), T(A), r_A \colon T(A) \to S(K_0(A)).$$

We have already defined K_0 group and the tracial space $T(A)$. Since $p \sim q$ implies $\tau(p) = \tau(vv^*) = \tau(v^*v) = \tau(q)$, every trace τ on A defines a state (unital, positive, additive map into $\mathbb{R}$) on $K_0(A)$, and r_A is the coupling map that associates states to traces (see Exercise 5.5.7). $K_1(A)$ is an another countable (in case when A is separable) abelian group (see [70]).

Fix a family of C*-algebras $\mathbb{B}$. We would like to consider it as the family of *building blocks* of a larger family of C*-algebras. This can be formalized in (at least) two ways.

Definition 6.1. A C*-algebra is an *A*$\mathbb{B}$*-algebra* if it is a direct limit of a directed system of algebras in $\mathbb{B}$.

Example 6.2. (1) If $\mathbb{B}$ consists of all full matrix algebras $M_n(\mathbb{C})$ for $n \in \mathbb{N}$, then A$\mathbb{B}$-algebras are UHF algebras.

(2) if $\mathbb{B}$ consists of all finite-dimensional C*-algebras, then A$\mathbb{B}$-algebras are the *approximately finite*, or *AF*, algebras.

(3) Recall that $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathbb{B}$ be the class of all direct sums of algebras of the form $C(\mathbb{T}, M_n(\mathbb{C}))$. Then we arrive at the class of *AT algebras* (see [70]; T stands for $\mathbb{T}$).

(4) If $\mathbb{B}$ consists of all algebras of the form $C([0,1], M_n(\mathbb{C}))$ and their direct sums, then we have the class of *AI algebras* (see [70]).

(5) Take the class of all algebras of the form $C(X, M_n(\mathbb{C}))$ where X is a compact metric space and close it under direct sums and corners (§2.5.9) to obtain $\mathbb{B}$. Then A$\mathbb{B}$ is the class of *AH algebras* (H stands for 'homogeneous') (see [70]).

After several spectacular successes ([65], [54], [22], [55], [21], [70]), counterexamples to Elliott's conjecture were found by Rørdam and Toms ([71] and [80]). While these examples rule out functorial classification as conjectured by Elliott, they don't give information about the descriptive complexity of the isomorphism relation of separable nuclear simple unital C*-algebras.

In recent years Elliott's program has been revitalized by influx of new ideas, including invariants such as the Cuntz semigroup (see [6], [9]), regularity properties such as $\mathcal{Z}$-stability ([87]), and a technical tour de force

([23]), among other developments. See [20] for a survey and the upcoming [88] for the current state of the art.

6.0.1. *Failure of cancellation* The following example was first used by Villadsen ([83]) and it appears in one form or another in all known counterexamples to Elliott's program. Recall that S^2 is the unit sphere in three-dimensional (real) Euclidean space and consider the C*-algebra $A = C(S^2, M_2(\mathbb{C}))$ (see §2.5.8). By Exercise 2.6.9, projections in A are continuous maps from S^2 into $\mathcal{P}(M_2(\mathbb{C}))$, the space of projections in $M_2(\mathbb{C})$. Apart from the 'trivial' projections corresponding to constant maps, this algebra has nontrivial projections. Recall that $\mathbb{C}P^1$ is the *complex projective space*, the space of all lines in $\mathbb{C}^2$. It is homeomorphic to the space $\mathcal{P}(M_2(\mathbb{N}))$ of all 1-dimensional projections in $M_2(\mathbb{C})$. It is also homeomorphic to S^2. The *Bott projection* p in A corresponds to a natural homeomorphism of S^2 onto $\mathcal{P}(M_2(\mathbb{C}))$ (i.e., to the *Hopff vector bundle*). It is not Murray-von Neumann equivalent to the trivial rank one projection, but there is a partial isometry $v \in A$ such that $p + vpv^* = 1$.

Villadsen used Bott projection and a clever direct limit construction to construct an AH algebra A such that $K_0(A)$ contains a non-positive element x such that nx is positive for some n ([83]).

6.1. Exercises Exercises given below illustrate basic constructions of AH algebras. See [82] for analysis of the classification problem for such algebras. In the following exercises X and Y are compact metric spaces, and the property of being metric is mostly unnecessary.

6.1.1. Prove that $T(C(X, M_n(\mathbb{C}))$ is affinely homeomorphic to $\mathcal{P}(X)$, the space of Borel probability measures on X (cf. Exercise 3.7.3), where to a measure ν one associates trace

$$\tau_\nu(f) = \int \mathrm{tr}(f(x))d\nu(x).$$

6.1.2. Let $A_n = C(X, M_{2^n}(\mathbb{C}))$. If $x_{n,j}$, for $j < n$ are points in X define a unital *-homomorphism $\Phi_n \colon A_n \to A_{n+1}$ by the block matrix

$$\Phi_n(f) = \mathrm{diag}(f, f(x_{n,0}), f(x_{n,1}), \dots, f(x_{n,n-1})).$$

(i) Prove that the direct limit of this system A is simple if for every m the set $\bigcup_{n\geq m}\{x_{n,j} j < n\}$ is dense in X.

(ii) With Φ^* as in Exercise 2.2.10 prove that for a trace $\tau \in T(A_{n+1})$ $\tau' = \Phi^*(\tau)$ satisfies

$$\tau'(f) = \frac{1}{2}\tau(f) + \frac{1}{2n}\sum_{j<n} \operatorname{tr}(f(x_{n,j}).$$

Assume $m = nk$ and $\phi_j : Y \to X$, for $j < k$, are continuous maps. Then a unital *-homomorphism $\Phi : C(X, M_n(\mathbb{C})) \to C(Y, M_m(\mathbb{C}))$ is defined by

$$\Phi(f) = \operatorname{diag}(f \circ \phi_0, f \circ \phi_1, \ldots, f \circ \phi_{k-1}).$$

A *-homomorphism of this form is a *standard *-homomorphism with characteristic functions ϕ_j, for $j < n$.*

6.1.3. Given a standard *-homomorphism $\Phi : A \to B$ as above, describe the map $\Phi^* : T(B) \to T(A)$.

6.1.4. Assume $A_n = C([0,1], M_{k(n)}(\mathbb{C}))$ and $\Phi_n : A_n \to A_{n+1}$ is a standard unital *-homomorphism and let $A = \lim_n A_n$. Prove that $K_0(A)$ is isomorphic (as a preordered abelian group with order unit) with K_0 of the UHF algebra $\lim_n M_{k(n)}(\mathbb{C})$.

(Hint: Recall that the projections in $C([0,1], M_n(\mathbb{C}))$ correspond to continuous functions from $[0,1]$ into $\mathcal{P}(M_n(\mathbb{C}))$, and therefore have well-defined rank. In order to prove that $p \sim q$ if and only if their ranks coincide, one uses Lemma 3.5 together with the compactness and homotopic triviality of $[0,1]$.)

6.1.5. Let $A_n = C([0,1], M_{2^n}(\mathbb{C}))$ and let $\Phi_n : A_n \to A_{n+1}$ be the standard *-homomorphism with characteristic functions $\phi_0(x) = x/2$ and $\phi_1(x) = (1+x)/2$. Let $A = \lim_n A_n$.

(i) Prove that A is simple.

(ii) Compute $K_0(A)$.

(iii) Prove that A has a unique trace.

By the last exercise, Exercise 5.5.7, and the fact that K_1 of both A and the CAR algebra is trivial, one has that $\operatorname{Ell}(A) \cong \operatorname{Ell}(M_{2^\infty})$. It is then a consequence of Elliott's classification of simple AI algebras ([21]) that A is isomorphic to the CAR algebra.

6.1.6 (Dimension drop algebras)**.** Fix m and n in $\mathbb{N}$ and identify $M_{mn}(\mathbb{C})$ with $M_m(\mathbb{C}) \otimes M_n(\mathbb{C})$ (Exercise 2.6.11). Consider the following subalgebra of $C([0,1], M_{mn}(\mathbb{C}))$.

$$\mathcal{Z}_{m,n} = \{f : f(0) \in M_m(\mathbb{C}) \otimes 1_n \text{ and } f(1) \in 1_m \otimes M_n(\mathbb{C})\}$$

Prove that the only projections in $\mathcal{Z}_{m,n}$ are 0 and 1 (i.e., "$\mathcal{Z}_{m,n}$ is projectionless") if and only if m and n are relatively prime.

6.1.7. Prove that $T(\mathcal{Z}_{m,n})$ is affinely homeomorphic to $\mathcal{P}([0,1])$, the space of probability measures on $[0,1]$.

(Hint: See Exercise 6.1.1.)

6.1.8. Construct a direct limit of projectionless dimension drop algebras that is simple and has a unique trace.

(Hint: For the trace part keep in mind Exercise 2.2.10(2).)

The solution to Exercise 6.1.8 is uniquely defined up to the isomorphism. It is the notorious *Jiang–Su algebra* $\mathcal{Z}$ ([45]). Its Elliott invariant is equal to the Elliott invariant of $\mathbb{C}$, and moreover $\mathrm{Ell}(\mathcal{Z} \otimes A) = \mathrm{Ell}(A)$ for all well-behaved (as well as many misbehaved) C*-algebras A. A C*-algebra A is *$\mathcal{Z}$-stable* if $A \otimes \mathcal{Z} \cong A$, and by the above one can only hope to classify $\mathcal{Z}$-stable algebras by their Elliott invariants. By remarkable results of Wilhelm Winter, this is true in a number of cases ([82], [86], [87]).

7. Abstract classification

Main references for the remaining sections are [33] and [32]. Recall that a topological space X is *Polish* if it is separable and completely metrizable. A subset of X is *analytic* if it is a continuous image of a Borel set in some Polish space. An equivalence relation E on X is analytic if it is an analytic subset of X^2.

The theory of abstract classification can be traced back to the work of Mackey on classification of representations of locally compact metrizable groups, and in particular to the following.

Definition 7.1 (Mackey)**.** An equivalence relation E on X is *smooth* if there is a Borel-measurable $f\colon X \to \mathbb{R}$ such that

$$x\,E\,y \text{ iff } f(x) = f(y).$$

Example 7.2. Similarity of $n \times n$ complex Hermitian matrices is smooth. This is because one can associate to M the list of its eigenvalues (in the increasing order, with multiplicities).

Smooth equivalence relations are effectively classifiable.

Example 7.3. Classification of rank 1 torsion-free abelian groups.

These are exactly the subgroups of $\mathbb{Q}$. To every such group Γ we can associate a generalized integer $\prod_j p_j^{k(j)}$ (see §4.1) as follows. Choose $a \in \Gamma$ and consider $\{n : (\exists b \in \Gamma) nb = a\}$. This set is the set of all divisors of a generalized integer $\mathbf{k}(\Gamma, a)$.

Note that $\mathbf{k}$ depends on the choice of a, but only up to a finite factor. A straightforward recursive construction shows that Γ_1 and Γ_2 are isomorphic if and only if for some (equivalently, all) $a_1 \in \Gamma_1$ and $a_2 \in \Gamma_2$ the corresponding generalized integers coincide up to a finite factor, i.e., there exist nonzero m_1 and m_1 in $\mathbb{N}$ such that $m_1\mathbf{k}(\Gamma_1, a_1) = m_2\mathbf{k}(\Gamma_2, a_2)$.[d]

One could prove that the equivalence relation in Example 7.3 is not smooth. However, it is generally accepted as a simple and natural classification result. In order to compare the complexity of equivalence relations on Polish spaces (and therefore of different classification problems), [36] and [40] have independently introduced the following definition

Definition 7.4. Assume E, F are equivalence relations on Polish spaces X, Y, respectively. Then E is *Borel reducible* to F, or $E \leq_B F$, if there is a Borel-measurable map $f \colon X \to Y$ such that

$$x \, E \, y \Leftrightarrow f(x) \, F \, f(y).$$

Note that $E \leq_B = \mathbb{R}$ if and only if E is smooth. The relation $E \leq_B F$ can be interpreted in the following ways.

(1) Borel cardinality of X/E is $\leq$ than the Borel cardinality of Y/F.
(2) Classification problem for E is simpler than the classification problem for F.
(3) F-equivalence classes are complete invariants for E-equivalence classes.

Thesis 7.5. *Almost all classical classification problems deal with analytic equivalence relations on Polish spaces.*

Thesis 7.6. *In almost all cases, the space of invariants has a Polish topology and the computation of invariants is given by a Borel-measurable function.*

In order to verify a particular instance of these two theses, one needs to

(i) Define a Polish space X whose elements correspond to objects that need to be classified,

[d]We consider generalized integers as formal infinite products.

(ii) Define a Polish space Y whose elements correspond to the intended invariants,

(iii) Check that the map that associates invariant $I(x)$ to each object $x \in X$ is Borel.

Once all three steps are verified, the particular classification problem is subject to the well-developed abstract theory of classification (e.g., [44] and [15]), and one can proceed to estimate the complexity of this classification problem.

7.1. Effros Borel space

A *standard Borel space* is a pair (X, Σ) where X is a set and Σ is a σ-algebra of subsets of X with the property that X carries a Polish topology τ such that Σ is the corresponding σ-algebra of Borel sets. The actual topology on X is irrelevant for our purposes and by a classical result of Kuratowski all uncountable Polish spaces are Borel-isomorphic. Therefore, considering a standard Borel space instead of a Polish space as the ambient space for our classification problems makes our setting more canonical. However, if the topology on X is particularly natural we shall use it.

7.1.1. *Spaces of countable structures* Fix any countable signature σ. The set $\mathcal{M}_\sigma$ of all countable models of σ can naturally be identified with a subset of $\mathcal{P}(\mathbb{N})$, and the latter carries the compact metric topology. For details see [3]. A case of particular interest for us is when σ is the signature of ordered groups with order unit. This is worked out in [32].

The tsomorphism relation on this space is analytic. To see this, note that the space of all triples (A, B, f) where A and B are in $\mathcal{M}_\sigma$ and $f\colon \mathbb{N} \to \mathbb{N}$ is an isomorphism between A and B is closed. The set $\{(A, B) \in \mathcal{M}_\sigma : A \cong B\}$ is the projection of this set of triples and therefore analytic.

For example, the space $\mathbf{G}$ of preordered countable groups with order unit looks as follows. The space $\mathbf{G}$ consists of all triples (F, G, H) which code $(K, K^+, [1])$ as follows. We may assume that the underlying set of K is $\mathbb{N}$, and $F \subseteq \mathbb{N}^3$ is the set $\{(m, n, k) : m +_K n = k\}$. Also $G = K$ and $H = 1_K$.

7.1.2. *Compact metric spaces* Every compact metric space is homeomorphic with a subspace of the Hilbert cube $[0, 1]^{\mathbb{N}}$. Thus the space $K([0, 1]^{\mathbb{N}})$ of all closed subsets of $[0, 1[^{\mathbb{N}}$ is the space of all compact metric spaces. The Hausdorff metric on this space,

$$d_H(K, L) = \inf_{\varepsilon \geq 0}(K \subseteq_\varepsilon L \text{ and } L \subseteq_\varepsilon L)$$

turns it into a compact metric space.

The homeomorphism relation on this space is analytic. To see this, identify a homeomorphism f between K and L with its graph. This graph is a closed subset of the square and the set of all triples (K, L, f) such that K and L are compact subsets of $[0,1]^{\mathbb{N}}$ and $f\colon K \to L$ is (with the above identification) a closed subset of the compact metric space $K([0,1]^{\mathbb{N}})^2 \times K(([0,1]^{\mathbb{N}})^2)$. Like in the previous example, its continuous image is analytic.

7.1.3. *Separable Banach spaces* In order to define a Borel space of all separable Banach spaces we shall need some lemmas. All Banach spaces are real, although the complex case neither involves any additional complexity nor brings additional simplicity (as in the case of C*-algebras).

Lemma 7.7. *Every separable Banach space is isometric with a subspace of $C(\{0,1\}^{\mathbb{N}})$ of all continuous functions from the Cantor space into $\mathbb{R}$.*

Proof. This is related to Exercise 2.2.10. A Banach space X is isometric to a subspace of a Banach space Y if and only if there is an affine continuous surjection f from the unit ball of Y^* onto the dual ball of X^*. By the Birkhoff–Alaoglu theorem the unit ball of a separable Banach space is compact in the weak*-topology. We can therefore fix a continuous surjection of $\{0,1\}^{\mathbb{N}}$ onto the unit ball of X^* and extend it affinely to the unit ball of $C(\{0,1\}^{\mathbb{N}})$. □

Let $Z = C(\{0,1\}^{\mathbb{N}})$ and let $F(Z)$ denote the space of all closed subsets of Z. If Y is not locally compact, the Hausdorff metric used in §7.1.2 will not be separable, In order to define a Borel structure on $F(Z)$ we use the following result.

Theorem 7.8 (Effros). *If Y is a Polish space then the space $F(Y)$ of closed subsets of Y with respect to the σ-algebra generated by the sets $\{K \in F(Y) : K \cap U \neq \emptyset\}$ for $U \subseteq Y$ open is a standard Borel space.*

Proof. The idea is to fix a metric compactification Y of X and take advantage of the fact that a subspace of a Polish space is Polish if and only if it is G_δ, in both directions. For defails see [48, §12.C]. □

Since it is not difficult to check that for a Banach space Z the set $\{Y \in F(Z) : Y \text{ is a linear subspace}\}$ is Effros-Borel, we have a standard Borel space of all separable Banach spaces.

There are (at least) three natural equivalence relations on this space:

(i) (linear) isometry,
(ii) isomorphism (i.e., the existence of a linear homeomorphism) and
(iii) bi-embeddability (the existence of linear isometric embedding of X into Y and a linear isometric embedding of Y into X).

It is again not difficult to check that all three relations are analytic. Complexities of these equivalence relations were computed in [57] and [35].

7.1.4. *Von Neumann algebras with a separable predual* (A reader not familiar with these may want to skip this paragraph.) Every such von Neumann algebra is isomorphic to a weakly closed subalgebra of $\mathcal{B}(H)$ for a separable complex Hilbert space X. Since $\mathcal{B}(H)$ is weakly separable, one can consider this space with respect to the Effros Borel structure. However, unlike in the case of Banach spaces, this space carries a natural Polish topology called *Effros–Maréchal* topology. See [74] and [85] for more.

7.1.5. *Separable C*-algebras* In both cases of Banach spaces and von Neumann algebras there exists a universal separable object ($C(\{0,1\}^{\mathbb{N}})$ and $\mathcal{B}(H)$, respectively hence the Effros Borel structure provides the setting for analysis of classification problems. This is not the case with C*-algebras. By a result of Junge and Pisier ([47]) there is no universal separable C*-algebra. However, there are at least two different Borel spaces of separable C*-algebras.

The following space was defined by Kechris ([49]). It takes advantage of a slight refinement of the GNS theorem (Theorem 2.3), to the effect that every separable C*-algebra is isomorphic to a subalgebra of $\mathcal{B}(H)$ for the separable Hilbert space H. The space $\mathcal{B}(H)$ becomes a standard Borel space when equipped with the Borel structure generated by the weakly open subsets. Let

$$\Gamma = \mathcal{B}(H)^{\mathbb{N}},$$

and equip this with the product Borel structure. For each $\gamma \in \Gamma(H)$ we let $C^*(\gamma)$ be the C*-algebra generated by γ. If we identify each $\gamma \in \Gamma(H)$ with $C^*(\gamma)$, then $\Gamma(H)$ parameterizes all separable C*-algebras acting on H. This gives us a standard Borel parameterization of the category of all separable C*-algebras. The relation $\gamma_1 \sim \gamma_2$ is an analytic equivalence relation (see [32] or Exercise 7.4.2).

The above can be considered as the space of concrete separable C*-algebras. One can also define the space of abstract separable C*-algebras and prove that there is a Borel isomorphism between these spaces that respects the corresponding isomorphism relations (see [32]).

7.2. Computation of the Elliott invariant is Borel

By §7.1.5 and §7.1.1 we have standard Borel spaces Γ of separable C*-algebras and $\mathbf{G}$ countable ordered groups with order unit, respectively. The following lemma (and corresponding facts for other C*-algebraic invariants) taken from [32] largely justifies taking the descriptive set theoretic view of Elliott's program.

Let $\Gamma_1 = \{\gamma \in \Gamma : \gamma(0) = 1\}$—the (clearly Borel) space of unital separable C*-algebras.

Lemma 7.9. *There is a Borel map* $\mathbb{K}\colon \Gamma_1 \to \mathbf{G}$ *such that* $\mathbb{K}(\gamma)$ *is isomorphic to* $(K_0(C^*(\gamma)), K_0(C^*(\gamma)^+, [1_A])$.

Sketch of the proof. The details can be found in [32].

First we need a Borel map $\mathbb{P}$ that sends Γ to Γ so that $\mathbb{P}(\gamma)$ enumerates a countable dense set of projections in $\mathcal{P}(M_\infty(C^*(\gamma)))$. For simplicity, we shall instead only construct $\mathbb{P}_1$ such that $\mathbb{P}_1(\gamma)$ enumerates a countable dense set of $\mathcal{P}(C^*(\gamma))$. Let $\mathfrak{p}_n$, for $n \in \mathbb{N}$, enumerate all *-polynomials over $\mathbb{Q} + i\mathbb{Q}$ in variables x_j, for $j \in \mathbb{N}$, with the property that $\mathfrak{p}_n(x) = \mathfrak{p}_n(x)^*$. Then $\{\mathfrak{p}_j(\gamma) : j \in \mathbb{N}\}$ enumerates a countable dense subset of the set of self-adjoint operators in $C^*(\gamma)$.

Let $f\colon \mathbb{R} \to \mathbb{R}$ be any function whose iterates, f^n for $n \in \mathbb{N}$, uniformly converge to some function g such that $g(x) = 0$ if $x \leq 1/4$ and $g(x) = 1$ if $x \geq 3/4$. If $\mathrm{sp}(\mathfrak{p}_j(\gamma)) \cap [1/4, 3/4] = \emptyset$ then $f^n(\mathfrak{p}_j(\gamma))$ converges to a projection in the norm topology.

A verification that $\mathbb{P}_1(\gamma) = (q_j : j \in \mathbb{N})$ with $q_j = \lim_n f^n(\mathfrak{p}_j(\gamma))$ if this sequence is norm-convergent and $q_j = 0$ otherwise is a Borel map is straightforward. Clearly, $\mathbb{P}(\gamma)$ is an enumeration of a dense set of projections in $C^*(\gamma)$.

By Lemma 3.11, the range of $\mathbb{P}(\gamma)$ intersects all Murray–von Neumann equivalence classes in $\mathcal{P}(M_\infty(C^*(\gamma))$. We need to check that the map $\gamma \mapsto \sim_\gamma$ that associates a binary relation on $\mathbb{N}$ to γ such that $m \sim_\gamma n$ if and only if projections $\mathbb{P}(\gamma)(m)$ and $\mathbb{P}(\gamma)(n)$ are Murray-von Neumann equivalent in $M_\infty(C^*(\gamma))$. But this is a consequence of the fact that we have an effective enumeration of a dense subset of $C^*(\gamma)$ akin to $\mathfrak{p}_j(\gamma)$, for $j \in \mathbb{N}$, and Exercise 3.5.8.

Similar arguments show that the operation $\oplus$ and the Groethendieck construction can be effectively defined on $\mathbb{P}(\gamma)$. This gives a Borel map that sends γ to an element of the Borel space $\mathbf{G}$ of preordered countable groups with order unit that codes $(K_0(C^*(\gamma)), K_0(C^*(\gamma))^+, [1_{C^*(\gamma)}])$. □

The following was proved in [32].

Theorem 7.10. *There is a standard Borel space of Elliott invariants* Ell *and there is a Borel map* $\Phi\colon \Gamma \to \mathrm{Ell}$ *such that* $\Phi(\gamma)$ *represents* $\mathrm{Ell}(C^*(\gamma))$. □

7.3. Comparing complexities of analytic equivalence relations

In this section we introduce some important classes of analytic equivalence relations.

7.3.1. *Relation* E_0 On $\{0,1\}$ define $x \,\mathrm{E}_0\, y$ if $(\forall^\infty n)x(n) = y(n)$. This equivalence relation is F_σ and a simple Baire category argument shows that it is not smooth. By [40], E_0 is the minimal non-smooth Borel equivalence relation. (It is, however, not the minimal non-smooth analytic equivalence relation; see Exercise 7.4.3.) It should be noted that the combinatorial essence of this result, called *Glimm–Effros dichotomy*, first appeared in Glimm's [38] as a device for embedding M_{2^∞} into C*-algebras with non-smooth dual (non-type I C*-algebras) as a subquotient.

7.3.2. *Essentially countable equivalence relations* A Borel equivalence relation all of whose classes are countable is said to be *countable.* A Borel equivalence relation which is Borel-reducible to a countable equivalence relation is *essentially countable.* By the above, E_0 is the $\leq_B$-minimal essentially countable equivalence relation. Orbit equivalence relation of the shift action of the free group on infinitely many generators on its power-set the $\leq_B$-maximal essentially countable equivalence relation and by a result of Adams and Kechris the $\leq_B$ ordering on essentially countable equivalence relations has a very rich structure ([1]).

7.3.3. *Countable structures* An equivalence relation E is *classifiable by countable structures* if E is Borel-reducible to the isomorphism relation of countable structures in some countable signature (see §7.1.1). By a result of Hjorth–Kechris–Louveau, this is equivalent to E being Borel-reducible to an orbit equivalence relation of a continuous action of a closed subgroup of S_∞ on $\mathcal{P}(\mathbb{N})$.

The $\leq_B$-maximal analytic equivalence relation in this class is the graph isomorphism relation.

7.3.4. *Orbit equivalence relations* If G is a Polish group that acts continuously on a Polish space X, then E^X_G is the orbit equivalence relation,

$$x \,\mathrm{E}^X_G\, y \text{ iff } (\exists g \in G) g.x = y.$$

Being a projection of the closed set $\{(x,y,g) : g.x = y\}$, it is clearly analytic. By a result of Becker and Kechris ([3]), considering Borel actions instead of continuous ones does not result in a larger class of equivalence relations.

This class also has the $\leq_B$-maximal equivalence relation. It is the shift action of the isometry group G of the Urysohn space on the Effros Borel space of its closed subsets, denoted $E^{X_\infty}_{G_\infty}$.

7.3.5. *Turbulence* Being classifiable by countable structures is strictly weaker than being below a Polish group action. This is a result of Hjorth ([43]) and it utilizes the notion of *turbulence*, a dynamical property of group action. This is the main tool used in §8.1.

7.3.6. *The dark side* On $[0,1]^{\mathbb{N}}$ define $x \mathrel{\mathrm{E}_1} y$ if $(\forall^\infty n) x(n) = y(n)$. This equivalence relation is F_σ. By [50], E_1 is an immediate $\leq_B$ successor of E_0 and it is not Borel-reducible to any orbit equivalence relation of a Polish group action. Therefore being 'above E_1' is a measure of an equivalence relation not being simply classifiable.

In [72] Rosendal isolated the $\leq_B$-maximal K_σ equivalence relation, E_{K_σ}. Its underlying space is the space of nondecreasing functions in $\mathbb{N}^{\mathbb{N}}$ and we let $f \mathrel{\mathrm{E}_{K_\sigma}} g$ if and only if there is n such that $f(j) \leq g(j)+n$ and $g(j) \leq f(j)+n$ for all n.

There exists a maximal analytic equivalence relation $E_{\Sigma^1_1}$, isolated in [57]. Both bi-embeddability of separable Banach spaces ([57]) and the isomorphism of separable Banach spaces ([35]) are bireducible with the maximal analytic equivalence relation.

7.4. Exercises

7.4.1. Compare classification of rank 1 torison-free abelian groups (Example 7.3) with the classification of UHF algebras (§4.1) and nonunital separable direct limits of full matrix algebras algebras (Exercise 4.3.7).

7.4.2. Prove that both the isomorphism and bi-embeddability of C*-algebras are analytic subsets of the square of Kechris's space Γ (§7.1.5).

7.4.3. On the space of compact subspaces of the Hilbert cube defined in §7.1.2 consider the equivalence relation $x \mathrel{\mathrm{E}} y$ iff either both x and y are uncountable or both x and y are countable and homeomorphic. Then show that this relation is neither smooth nor $\geq_B E_0$.

(Hint: Count the equivalence classes.)

The following exercise gives a Borel space of UHF algebras.

7.4.4. Prove that there exists a universal separable UHF algebra Q. Also prove that not all unital infinite-dimensional subalgebras of this algebra are UHF, but its UHF subalgebras form a Borel subset of $F(Q)$.

7.4.5. Do the nonunital case of Exercise 7.4.4.

8. Estimating the complexity of the isomorphism of C*-algebras

Upper and lower bounds for the complexity of the isomorphism relation for C*-algebras were proved in [33].

8.1. Turbulence: A lower bound for complexity

Dynamical properties of Polish group actions affect the complexity of the orbit equivalence relation. We shall use the following classical result as a warmup.

Proposition 8.1. *Assume $G \curvearrowright X$ is a continuous action of a Polish group on a Polish space such that*

(1) every orbit is dense, and
(2) every orbit is meager.

Then the orbit equivalence relation is not smooth.

Proof. Assume $f\colon X \to \mathbb{R}$ is a Borel map such that $x \mathrel{\mathrm{E}_G^X} y$ implies $f(x) = f(y)$. Let Y be a dense G_δ subset of X such that the restriction of f to Y is continuous. Since $x \mapsto g.x$ is a homeomorphism for all $g \in G$, the set $\{(g, x) : g.x \in Y\}$ is by Kuratowski–Ulam theorem comeager in $G \times X$. Therefore there exists $x \in X$ such that $g.x \in Y$ for comeager many $g \in G$. But this implies that the intersection of the orbit of x with Y is dense in Y. Since f is constant on this set, it is by the continuity constant on Y. Since all orbits are meager, f cannot be a reduction of E_G^X to $=_{\mathbb{R}}$. □

Let $G \curvearrowright X$ be a continous action of a Polish group on a Polish space. Fix $x \in X$, a symmetric open neighborhood U of e_G and an open neighbourhood V of x. On V define a graph by letting $\{x, y\}$ be an edge if $x \neq y$ and there exists $g \in U$ such that $g.x = y$. Since $U = U^{-1}$, this relation is symmetric. Let $\mathcal{O}(x, U, V)$ be the connected component of x in this graph. Then $\mathcal{O}(x, U, V)$ is the set of points that can be reached from x by taking small steps (smallness being measured by U) while staying inside V.

Definition 8.2 (Hjorth, [43])**.** An action $G \curvearrowright X$ as above is *turbulent* if

(1) every orbit is dense,
(2) every orbit is meager, and
(3) the closure of every local orbit of every $x \in X$ has a nonempty interior.

In the presence of (1) and (2), (3) above is equivalent to the assertion that the closure of every local orbit intersects every G-orbit (exercise!). There are some other equivalent reformulations of the definition, and for all purposes it suffices to assume that the set of points x satisfying (3) is comeager (such actions are *generically turbulent*).

Theorem 8.3 (Hjorth, [43]). *If the action $G \curvearrowright X$ is turbulent then E_G^X is not classifiable by countable structures.* □

Example 8.4. (1) Consider c_0 as an additive group. It is a Polish group with respect to its norm topology. Then the shift action of ℓ_2 on $\mathbb{R}^{\mathbb{N}}$ is turbulent.

In order to prove this, fix $x = (x_n), U, V$ as above and fix $y \in \mathbb{R}^{\mathbb{N}}$. We shall prove that the orbit of y intersects the closure of $\mathcal{O}(x, U, V)$. Then there exist $\varepsilon > 0$ and k such that

$$U \supseteq \{(g_n) \in c_0 : |g_n| < \varepsilon \text{ for all } n\}$$

and

$$V \supseteq \{(z_n) \in \mathbb{R}^{\mathbb{N}} : |x_n - z_n| < \varepsilon \text{ for all } n < k\}.$$

Let $z = (z_n) \in \mathbb{R}^{\mathbb{N}}$ be such that $z_n = x_n$ for $n < k$ and $z_n = y_n$ for $n \geq k$. Then $z \in V \cap [y]$. We shall prove that $z \in \overline{\mathcal{O}(x, U, V)}$. Fix $m \in \mathbb{N}$ and let $K = \max_{n \leq m} |x_n - z_n|$. If $j > K/\varepsilon$, then we can find a sequence $x = z^0$, z^1,$\dots z^j$ such that the first m coordinates of z^j coincide with the first m coordinates of z and $z^i - z^{i+1} \in U$ for all $i < j$. Therefore all z^i belong to $\mathcal{O}(x, U, V)$ and since m was arbitrary z is an accumulation point of $\mathcal{O}(x, U, V)$.

(2) The above proof shows that the action of any classical Banach space ℓ_p, for $p \geq 1$ on $\mathbb{R}^{\mathbb{N}}$ is turbulent,

(3) The following example will be used later. Consider the Polish group $G = \mathbb{Z}^{\mathbb{N}}$ and let

$$G_0 = \{g \in G : \lim_{n \to \infty} \frac{g(n)}{n} = 0\}.$$

Then G_0 is a Polish subgroup of G, and the coset action is turbulent.

Here is our main application of turbulence.

Theorem 8.5 (Farah–Toms–Törnquist, [33])**.** *The isomorphism relation $\cong_{AI}$ of simple, separable, unital AI algebras has the following two properties:*

(1) it is not classifiable by countable structures.
(2) for any countable signature $\mathcal{L}$ the isomorphism relation of countable $\mathcal{L}$-models is Borel-reducible to it.

The proof of this theorem proceeds in two steps, via using $\mathcal{H}^2$, the homeomorphism relation of compact subsets of $[0,1]^2$: (i) showing that $\mathcal{H}^2$ has these properties (Lemma 8.6) and (ii) $\mathcal{H}^2 \leq_B \cong_{AI}$ (Lemma 8.9). A curious feature of (ii) is that we shall use classification result to prove a non-classification result. The following is a slight improvement of a result of Hjorth used in [43, 4.21].

Lemma 8.6. *(1) The homeomorphism relation of compact subspaces of $[0,1]^2$ is not classifiable by countable structures.*
(2) If $\mathcal{L}$ is any countable signature then the isomorphism relation of countable $\mathcal{L}$-models is Borel-reducible to the homeomorphism of compact subsets of $[0,1]^2$.

Proof. (1) It is notationally convenient to work with $[-1,1]^2$ instead of $[0,1]^2$.

Let $G = \mathbb{Z}^{\mathbb{N}}$. Note that G is a Polish group when given the product group structure and product topology. We let

$$G_0 = \{g \in G : \lim_{n\to\infty} \frac{g(n)}{n} = 0\}.$$

By [43, 4.16] G_0 acts turbulently on G by translation. Let

$$T_1 = \{(x,y) \in [-1,1] \times [0,1] : |x| \leq y\}$$

and for $n \in \mathbb{N}$, let in general

$$T_n = \bigcup_{k=1}^{n} [(2k-2,0) + T_1].$$

Then the T_n are compact and connected and for $m \neq n$ we have $T_m \not\cong_{\mathrm{hom}} T_n$. Fix an order-preserving homeomorphism $f : \mathbb{R} \to (-1,1)$ such that $f(0) = 0$. For each $m \in \mathbb{Z}$ and $n,k \in \mathbb{N}$ and let

$$J_{m,n,k} \subseteq [\frac{1}{2n-1}, \frac{1}{2n} - \frac{|m|}{2n(2n-1)(|m|+1)}] \times [f\left(\frac{2m-1}{n}\right), f\left(\frac{2m}{n}\right)]$$

be a closed set homeomorphic to T_k. Note that $J_{m,n,k} \subseteq [\frac{1}{2n-1}, \frac{1}{2n}] \times [-1,1]$ and that for all n,

$$\lim_{m\to\pm\infty} \sup\{\operatorname{diam}(J_{m,n,k}) : k \in \mathbb{N}\} \to 0.$$

Fix a bijection $\varphi : \mathbb{Z}\times\mathbb{N} \to \mathbb{N}$ and define for each $g \in G$ a set $K(g) \subseteq [-1,1]^2$ by

$$X(g) = \left(\bigsqcup_{m,n} J_{m,n,\varphi(m+g(n),n)}\right) \sqcup (\partial([0,1]\times[-1,1]) \cup [-1,0]\times\{0\}) . \quad (*)$$

It is easy to see that $X(g)$ is compact for all $g \in G$, and that the map

$$G \to K([-1,1]^2) : g \mapsto X(g)$$

is Borel, where $K([-1,1]^2)$ denotes the compact hyperspace of $[-1,1]^2$. Note that $(*)$ provides a decomposition of each $X(g)$ into mutually non-homeomorphic connected components.

By the (m,n) component of $X(g)$ we mean the set

$$J_{m,n,\varphi(m+g(n),n)},$$

and by the n'th column of $X(g)$ we mean the set

$$X(g)_n = \bigsqcup_m J_{m,n,\varphi(m+g(n),n)}.$$

It is clear that the set of components of the n'th column

$$\{J_{m,n,\varphi(m+g(n),n)} : m \in \mathbb{N}\}$$

does not depend on $g \in G$. On the other hand, if $n_0 \neq n_1$, then the components of $X(g)_{n_0}$ are not homeomorphic to any of the components of $X(g)_{n_1}$. Therefore any homeomorphism from $X(g)$ to $X(h)$ must map $X(g)_n$ to $X(h)_n$, for all $n \in \mathbb{N}$. Moreover, any homeomorphism between $X(g)$ and $X(h)$ must fix the point $(0,0)$.

Claim 8.7. *For all $g \in G$, if $(x_n, y_n) \in \bigsqcup_{m,n} J_{m,n,\varphi(m+g(n),n)}$ is a sequence such that $(x_n, y_n) \to (0,0)$ and $x_n \in [\frac{1}{2n-1}, \frac{1}{2n}]$ for all n, then the function $\theta : \mathbb{N} \to \mathbb{Z}$ defined by*

$\theta(n) = k \iff$ *the connected component of (x_n, y_n) isomorphic to $T_{\varphi(k,n)}$*

satisfies $\theta - g \in G_0$.

Proof. For each $n \in \mathbb{N}$, let $m_n \in \mathbb{Z}$ be such that

$$y_n \in [f\left(\frac{2m_n - 1}{n}\right), f\left(\frac{2m_n}{n}\right)].$$

Since $y_n \to 0$ it follows that $f(\frac{2m_n}{n}) \to 0$, and so $\frac{m_n}{n} \to 0$. By definition of $X(g)$ the connected component in which (x_n, y_n) lies in is isomorphic to $T_{\varphi(m_n+g(n),n)}$, and so

$$\theta(n) = m_n + g(n).$$

Thus $\theta - g \in G_0$. □

Claim 8.8. *For all $g, h \in G$, $X(g)$ is homeomorphic $X(h)$ if and only if $g - h \in G_0$.*

Proof. Suppose first that $X(g)$ and $X(h)$ are homeomorphic, and let $\hat{\pi}: X(g) \to X(h)$ witness this. Fix a sequence $(x'_n, y'_n) \in X(g)$ such that $(x'_n, y'_n) \in J_{0,n,\varphi(g(n),n)}$ for all $n \in \mathbb{N}$. Then $(x'_n, y'_n) \to (0,0)$, and so since $\hat{\pi}(0,0) = (0,0)$ we have $(x_n, y_n) = \hat{\pi}(x'_n, y'_n) \to 0$. Since $X(g)_n$ is mapped to $X(h)_n$ if holds for all n that $x_n \in [\frac{1}{2n-1}, \frac{1}{2n}]$. Moreover,

$$g(n) = k \iff \text{the connected component of } (x_n, y_n) \text{ isomorphic to } T_{\varphi(k,n)}$$

and so $g - h \in G_0$ by Claim 1.

Suppose conversely that $z = g - h \in G_0$. Then define $\pi : X(g) \to X(h)$ by letting

$$\pi \restriction \partial([0,1] \times [-1,1]) \cup [-1,0] \times \{0\} = \mathrm{id},$$

and for each $n \in \mathbb{N}$ and $m \in \mathbb{Z}$ letting $\pi \restriction J_{m,n,\varphi(m+g(n),n)}$ be a homeomorphism

$$J_{m,n,\varphi(m+g(n),n)} \to J_{m+z(n),n,\varphi(m+z(n)+h(n),n)}.$$

To see that π is a homeomorphism it suffices to see that π is continuous, since it is clearly 1-1 and onto. To see this, for each $x \in \bigsqcup_{m,n} J_{m,n,\varphi(m+g(n),n)}$ let $m_x \in \mathbb{Z}$ and $n_x \in \mathbb{N}$ be such that

$$x \in [\frac{1}{2n_x - 1}, \frac{1}{2n_x}] \times [f\left(\frac{2m_x - 1}{n_x}\right), f\left(\frac{2m_x}{n_x}\right)]$$

Then by the definition of π we have

$$\begin{aligned} d(x,\pi(x)) \leq\ & d\left(f\left(\frac{2m_x}{n_x}\right), f\left(\frac{2(m_x+z(n_x))}{n_x}\right)\right) + \\ & d\left(f\left(\frac{2m_x-1}{n_x}\right), f\left(\frac{2m_x}{n_x}\right)\right) + \\ & d\left(\frac{2m_x-1}{n_x}, \frac{2m_x}{n_x} - \frac{|m_x|}{2n_x(2n_x-1)(|m_x|+1)}\right) + \\ & d\left(\frac{2(m_x+z(n_x))-1}{n_x}, \frac{2(m_x+z(n_x))}{n_x} - \frac{|m_x+z(n_x)|}{2n_x(2n_x-1)(|m_x+z(n_x)|+1)}\right) \end{aligned}$$

which shows that $d(x,\pi(x)) \to 0$ as $x \to \partial([0,1]\times[-1,1])$. Thus π is continuous. □

(2) follows from the argument suggested in [43, 4.22] but I shall provide a slightly different proof. By [36], it suffices to reduce the isomorphism of countable graphs. With $[\mathbb{Z}]^2 = \{(m,n) : m < n\}$ fix a bijection $\phi\colon [\mathbb{Z}]^2 \to \mathbb{N}$. Let $E_{m,n}$ be the union of straight lines in $\mathbb{R}^3$ connecting $(m,0,0)$ with $(0,\phi(m,n),\phi(m,n))$ and $(0,\phi(m,n),\phi(m,n))$ with $(n,0,0)$. These lines are pairwise disjoint and $\bigcup_{m,n} E_{m,n}$ is closed.

Let L_n be union of three straight lines connecting $(m,0,0)$ and $(m,0,-1)$, $(m,0,0)$ and $(m,-1,-1)$ and $(m,0,0)$ and $(m,-1,0)$ for all n.

To a countable graph $G = (\mathbb{N},\mathcal{E})$ associate the set

$$X(G) = \bigcup_{n\in\mathbb{Z}} L_n \cup \bigcup_{\{m,n\}\in\mathcal{E}} L_{m,n}.$$

This map is clearly continuous. If graphs G and H are isomorphic, then $X(G)$ and $X(H)$ are clearly isomorphic. Now assume $X(G)$ and $X(H)$ are homeomorphic. Homeomorphism has to send each endpoint of the form $(m,0,-1)$, $(m,-1,-1)$, or $(m,-1,0)$ to a point of the same form. Therefore each $(m,0,0)$ goes to some $(f(m),0,0)$ and It is then easy to show that f is an isomorphism between G and H. □

The following lemma completes the proof of Theorem 8.5.

Lemma 8.9. *There is a Borel map Φ from $\mathcal{K}([0,1]^{\mathbb{N}})$, the space of compact subspaces of $[0,1]^{\mathbb{N}}$, to Γ such that each $A(K) = C^*(\Phi(K))$ is a simple, unital AI algebra and $A(K) \cong A(L)$ if and only if K and L are homeomorphic.*

Proof. In [78] Thomsen constructed (among other things) for every compact metric K algebra $A(K)$ such that

(1) $A(K)$ is a simple, unital AI algebra,

(2) $K_0(A(K)) = (\mathbb{Q}, \mathbb{Q}^+, [1])$,
(3) $K_1(A(K)) = \{0\}$,
(4) $T(A(K))$ is affinely homeomorphic to $\mathcal{P}(K)$, the space of Borel probability measures on K.

In [33] it was demonstrated that Thomsen's construction can be performed effectively. Clearly, if K is not homeomorphic to L then $A(K)$ and $A(L)$ are not isomorphic. For the converse we use a result of Elliott who proved that simple, separable AI algebras are classified by their Elliott invariant (see [70]). (SInce $(\mathbb{Q}, \mathbb{Q}^+, [1])$ has the unique state, the pairing function ρ is uniquely determined in each $A(K)$.) $\square$

8.2. Below a group action: An upper bound for complexity

While the isomorphism of separable Banach spaces is the $\leq_B$ maximal analytic equivalence relation ([35]), the isomorphism of von Neumann factors is reducible to an orbit equivalence relation of a Polish group action (classical, see e.g., [74]). This is not surprising since Banach spaces are much wilder objects than von Neumann algebras (note, however, that the isometry relation of separable Banach spaces is reducible to an orbit equivalence relation of a Polish group action, by [57]).

C*-algebras are not as wild as arbitrary Banach spaces and not as well-behaved as von Neumann algebras, and one can ask what is the complexity of the isomorphism of separable C*-algebras. While [33] left the general problem open, it did show that the isomorphism of algebras relevant to Elliott's program is below a group action. The proof, however, took a detour and used some of the deepest results on the structure of separable nuclear C*-algebras. This detour gives us an excuse to introduce a fascinating object.

8.3. Cuntz algebra $\mathcal{O}_2$.

Let H be a separable complex Hilbert space with orthonormal basis e_n, for $n \in \mathbb{N}$. The equations

$$s(e_n) = e_{2n} \qquad t(e_n) = e_{2n+1} \text{ for all } n$$

uniquely define linear operators s and t. We have that

$$s^*s = 1, \qquad t^*t = 1, \text{ and } ss^* + tt^* = 1 \tag{8.1}$$

since ss^* and tt^* are mutually orthogonal projections.

Consider the C*-algebra generated by s and t. The unit in this algebra satisfies the following definition (note that the property of p is computed relative to the ambient C*-algebra A, and recall that for projections p and q we write $p \leq q$ if $pq = p$).

Definition 8.10. A projection p in A is *properly infinite* if there are projections $q \sim p$ and $r \sim p$ such that $q + r \leq p$.

The first separable simple C*-algebra in which the unit is properly infinite was constructed by Dixmier. He considered $C^*(s,t)/J$ with the above s and t, where J is the maximal ideal of $C^*(s,t)$. However, moding out by J was not necessary.

Theorem 8.11 (Cuntz, [10])**.** *If s and t are isometries satisfying* (8.1) *then the algebra $C^*(s,t)$ is simple. Moreover, any two algebras generated in this way are isomorphic.* □

This algebra generated by s and t is denoted by $\mathcal{O}_2$ and we shall spend some time investigating its properties.

Lemma 8.12. *If s and t satisfy* (8.1) *then $\mathcal{O}_2$ is the closed linear span of all monomials of the form $\prod_{i\leq m} x_i \prod_{j\leq n} y_j^*$, where $\{x_j, y_i\} \subseteq \{s,t\}$.*

Proof. We need to show that any monomial in s, t, s^* and t^* is either 0 or equal to a monomial of the form $\prod_{i\leq m} x_i \prod_{j\leq n} y_j^*$, where $\{x_j, y_i\} \subseteq \{s,t\}$. Since $s^*s = t^*t = 1$, it will suffice to prove that $s^*t = t^*s = 0$. But by (8.1) we have $ss^*t + tt^*t = t$, and therefore $ss^*t = 0$. By multiplying by s^* on the left we obtain $s^*t = 0$. A proof of $t^*s = 0$ is similar. □

The structure of $\mathcal{O}_2$ is discussed below in exercises 8.4.1 and 8.4.2.

Every separable simple C*-algebra, including $\mathcal{O}_2$, has outer automorphisms. The following lemma shows that automorphisms of $\mathcal{O}_2$ are coded by its unitaries. However, not only that α_u is distinct from $\mathrm{Ad}\, u$, but moreover $u \mapsto \alpha_u$ is not an action of the unitary group $\mathcal{O}_2$ on $\mathcal{O}_2$.

Lemma 8.13. *Every automorphism α of $\mathcal{O}_2$ is determined by*

$$\alpha(s) = us \text{ and } \alpha(t) = ut$$

for some unitary u. Conversely, every unitary $u \in \mathcal{O}_2$ uniquely determines an automorphism α_u of $\mathcal{O}_2$ such that $\alpha_u(s) = us$ and $\alpha_u(t) = ut$.

Proof. Let $u = \alpha(s)s^* + \alpha(t)t^*$. We claim that u is a unitary. By applying Lemma 8.12 we have $uu^* = \alpha(s)\alpha(s)^* + \alpha(t)\alpha(t)^* = \alpha(ss^* + tt^*) = 1$. Similarly, $u^*u = 1$ and therefore u is a unitary. Using Lemma 8.12 again we get $us = \alpha(s)$ and $ut = \alpha(t)$. □

More information on this fascinating object, as well as its relatives $\mathcal{O}_n$ for $n \in \{2, 3, \dots, \infty\}$, including the proofs of the results below, can be found in [10] and in [70]. The definition of approximate unitary equivalence was given in the paragraph before Theorem 5.9.

Theorem 8.14. *Every endomorphism $\beta\colon \mathcal{O}_2 \to \mathcal{O}_2$ is approximately unitarily equivalent to the identity map.* □

By the following remarkable result $\mathcal{O}_2$ tensorially absorbs exactly the algebras that are subject of the Elliott's program. Its precursor, the proof that $\mathcal{O}_2 \otimes \mathcal{O}_2$ is isomorphic to $\mathcal{O}_2$, was proved by Elliott using Theorem 8.14 and a clever use of ultrapowers and an approximate intertwining argument (see [70]). The difficult proof of Theorem 8.15 proof can be found in [53] or (with fewer details) in [70].

Theorem 8.15 (Kirchberg)**.** *Every separable nuclear unital C*-algebra is isomorphic to a unital subalgebra of $\mathcal{O}_2$.*

Also, A is separable, nuclear, simple and unital C-algebra if and only if $A \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.* □

Unlike the case of Thomsen's theorem (see the proof of Theorem 8.5), we were unable to find a direct Borel proof of Kirchberg's theorem. Its known proofs are not taking place in the 'Borel world' but are instead considering embeddings into the corona of the stabilization of $\mathcal{O}_2$. Nevertheless, Kirchberg's theorem can be used to prove its Borel version (cf. Exercise 8.4.6).

I can now give a rough sketch of the proof of an upper bound for the complexity of the isomorphism relation. By $\mathrm{SA}(\mathcal{O}_2)$ we denote the space of subalgebras of $\mathcal{O}_2$, with respect to the Effros Borel structure.

Theorem 8.16 (Farah–Toms–Törnquist, [33])**.** *The isomorphism relation of separable, simple, unital and nuclear C*-algebras is Borel-reducible to an orbit equivalence relation of a Polish group action.*

Sketch of a proof. More precisely, the set $\Gamma_N = \{\gamma \in \Gamma : C^*(\gamma)$ is simple, unital and nuclear$\}$ is Borel. There is a Borel reduction $\Phi\colon N \to \mathrm{SA}(\mathcal{O}_2)$ such that $C^*(\gamma_1) \cong C^*(\gamma_2)$ if and only if $\Phi(\gamma_1)\, \mathrm{E}^{\mathrm{SA}(\mathcal{O}_2)}_{\mathrm{Aut}(\mathcal{O}_2)}\, \Phi(\gamma_2)$.

By a Borel version of Kirchberg's Theorem 8.15 proved in [33] there is a Borel map Φ from N into $\mathrm{SA}(\mathcal{O}_2)$ such that $A = \Phi(\gamma)$ is a unital subalgebra of $\mathcal{O}_2$ isomorphic to $C^*(\gamma)$ with the property that the *relative commutant* of A in $\mathcal{O}_2$,

$$B = \{b \in \mathcal{O}_2 : ab = ba \text{ for all } a \in A\}$$

is isomorphic to $\mathcal{O}_2$ and $C^*(A, B)$ is both equal to $\mathcal{O}_2$ and is naturally isomorphic to $A \otimes \mathcal{O}_2$.

Clearly we only need to check that if $C^*(\gamma_1) \cong C^*(\gamma_2)$ then there is $\alpha \in \mathrm{Aut}(\mathcal{O}_2)$ that sends $\Phi(\gamma_1)$ onto $\Phi(\gamma_2)$. Fix an isomorphism α_0 between $\Phi(\gamma_1)$ and $\Phi(\gamma_2)$. Let B_1 and B_2 be the relative commutants of these algebras. Since they are both isomorphic to $\mathcal{O}_2$, we can fix an isomorphism $\alpha_2 \colon B_1 \to B_2$. Then $\alpha = \alpha_1 \otimes \alpha_2$ is as required. □

8.4. Exercises

8.4.1. Let s and t be the generators of $\mathcal{O}_2$. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the linear span of $\prod_{j \le n} x_j \prod_{j \le n} y_j^*$, where $\{x_j, y_j\} \subseteq \{s, t\}$. Prove that $\mathcal{F}_n$ is isomorphic to $M_{2^n}(\mathbb{C})$.

(Hint: First consider $n = 1$ and check that ss^*, tt^*, st^* and ts^* are the matrix units in $M_2(\mathbb{C})$ (cf. Exercise 2.6.7). Then go up.)

8.4.2. Using notation of the previous exercise, show that the 'balanced' products $\prod_{j \le n} x_j \prod_{j \le n} y_j^*$, for $n \in \mathbb{N}$ and $\{x_j, y_j\} \subseteq \{s, t\}$ generate a subalgebra isomorphic to the CAR algebra.

Then prove that $\mathcal{O}_2$ is generated by the CAR algebra A and a partial isometry s such that $a \mapsto sas^*$ is an endomorphism sending A onto a corner pAp where $p \in A$ is a projection of trace $1/2$.

8.4.3. Show that all nonzero projections in $\mathcal{O}_2$ are Murray-von Neumann equivalent and that $K_0(\mathcal{O}_2)$ is trivial.

8.4.4. Prove that $\mathcal{O}_2$ has no normalized traces.

(Hint: $1_{\mathcal{O}_2}$ is properly infinite.)

One can prove that $K_1(\mathcal{O}_2)$ is also trivial, and therefore by the previous two exercises the Elliott invariant of $\mathcal{O}_2$ is equal to the Elliott invariant of the C*-algebra $\{0\}$ (assuming that we accept the latter as a C*-algebra). Compare this with Theorem 8.15 and think $0 \cdot A = 0$.

8.4.5. Let A_n, F_{mn}, for $m \le n$, $m, n \in \mathbb{N}$, be a directed unital system of C*-algebras all of them isomorphic to $\mathcal{O}_2$. Show that the direct limit is isomorphic to $\mathcal{O}_2$.

(Hint: Theorem 8.14 and approximate intertwining.)

I don't know what happens if one considers a transfinite direct limit of an $\aleph_1$-directed sequence of copies of $\mathcal{O}_2$, or whether such algebra of density character $\aleph_1$ is uniquely determined.[e] I also don't know whether the analogue of Kirchberg's embedding theorem (see Theorem 8.15) holds for nuclear, or exact, C*-algebras of density character $\aleph_1$.

Note that the conclusion of the following exercise is strictly weaker than what is required in the proof of Theorem 8.16.

8.4.6. Using notation from the proof of Theorem 8.16, prove that there exists function $\Phi\colon N \to \mathrm{SA}(\mathcal{O}_2)$ that is C-measurable.

(Hint: Use Jankov, von Neumann selection theorem.)

9. Concluding remarks

9.1. The Borel-reducibility diagram

Figure 13 summarizes some of the known Borel reductions, concentrating on results relevant to operator algebras.[f]

All classes of C*-algebras occurring in the diagram are separable and unital (unless otherwise specified). Unless otherwise specified, the equivalence relation on a given class is the isomorphism relation. The bi-reducibility between the isomorphism for UHF algebras and bi-embeddability of UHF algebras is an immediate consequence of Exercise 4.3.9, or rather of its (straightforward) Borel version. Classification of compact metric spaces up to isometry is due to Gromov. Borel bi-reducibility between abelian C*-algebras and compact metric spaces was proved in [32]. Borel reductions from compact metric spaces to Choquet simplexes to simple AI algebras (as well as the definition of the latter) are given in [33]. A Borel-reduction of Choquet simplexes to the isometry of Banach spaces is given by sending a Choquet simplex K to the Banach space of affine functions on K.

A Borel version of Elliott's reduction of simple AI algebras to Elliott invariant follows from Elliott's classification result and the fact that the computation of the Elliott invariant is Borel was proved in [32] (see Lemma 8.9). Equireducibility of the maximal orbit equivalence relation of a Polish group action with the isometry of Polish spaces and the isometry of Banach spaces was proved in [8] and [61], respectively.

[e] Added on April 20, 2013: Now I do know. It isn't. The tensor product of the algebra constructed in [26] with $\mathcal{O}_2$ is not isomorphic to $\bigotimes_{\aleph_1} \mathcal{O}_2$.

[f] Added in July 2013. Recently Marcin Sabok announced that the affine homeomorphism of metrizable Choquet simplexes is the maximal orbit equivalence relation ([73]).

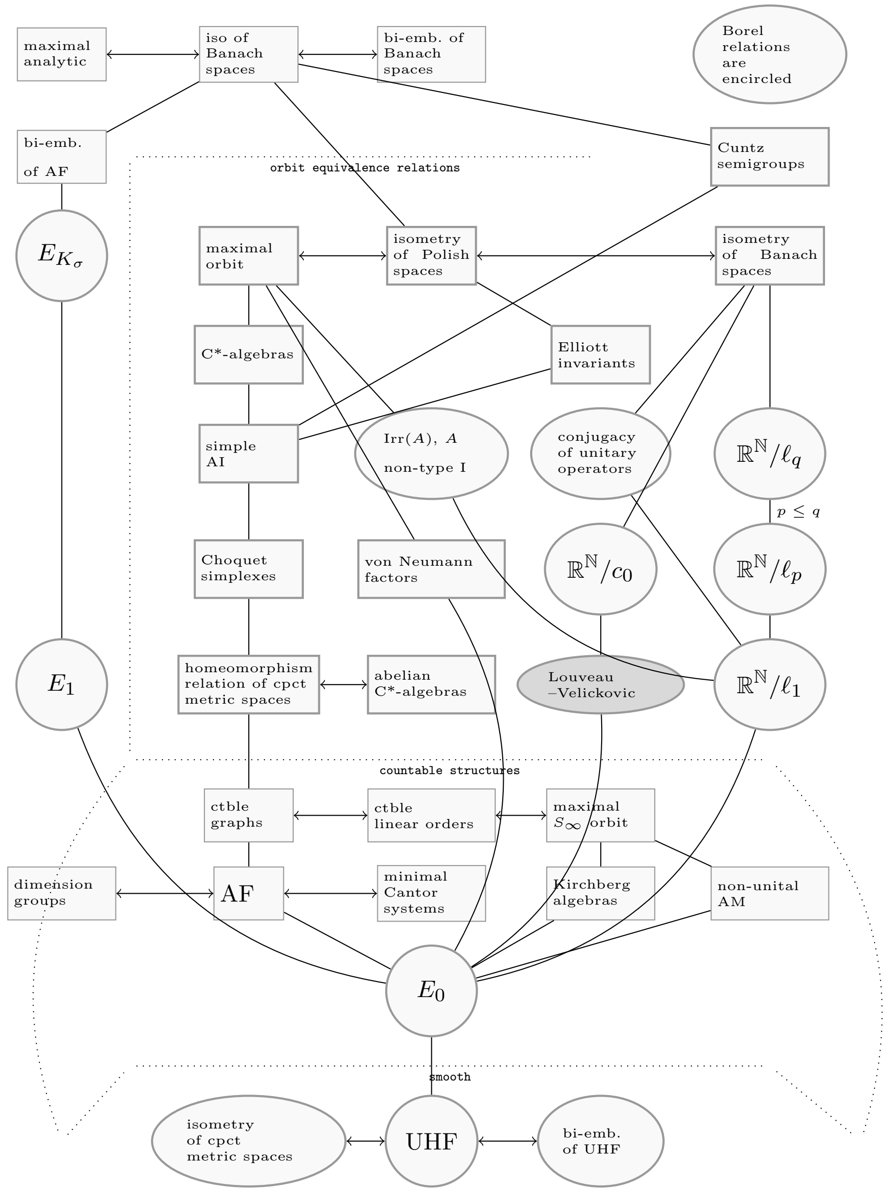

Fig. 13. Borel reducibility diagram

The reduction of the Elliott invariant to the maximal orbit equivalence relation, as well as the facts about the Cuntz semigroup, are proved in [32].

Dougherty–Hjorth ([14]) proved that $\mathbb{R}^{\mathbb{N}}/\ell_p \leq_B \mathbb{R}^{\mathbb{N}}/\ell_q$ if and only if $p \leq q$, and Hjorth proved ([42]) that the translation action of ℓ_1 on $\mathbb{R}^{\mathbb{N}}$ is a minimal turbulent action. On the other hand, I proved ([25]) that there is no minimal turbulent action Borel-reducible to $\mathbb{R}^{\mathbb{N}}/c_0$ and that the complicated structure of turbulent equivalence relations constructed in [58] occurs below orbit equivalence relation of any turbulent action Borel-reducible to $\mathbb{R}^{\mathbb{N}}/c_0$. Also, $\mathbb{R}^{\mathbb{N}}/c_0$ and $\mathbb{R}^{\mathbb{N}}/\ell_1$ are $\leq_B$-incomparable (see [25]). A class of $2^{\aleph_0}$ many F_σ equivalence relations not classifiable by countable strucutres that are *incompatible*, in the sense that any relation Borel-reducible to any two of them is classifiable by countable structures, was constructed in [24], but I could not fit this family into the picture.

The fact that $\mathbb{R}^{\mathbb{N}}/\ell_1$ is Borel-reducible to the conjugacy of unitary operators on $\mathcal{B}(H)$ was proved in [51]. As Todor Tsankov and Alain Louveau pointed out, Dellacherie proved that the conjugacy relation of unitary operators in $\mathcal{B}(H)$ is Borel. Characterization of $\leq_B$-maximal isomorphism relation of countable structures as the graph isomorphism or isomorphism of countable linear orderings is in [36] (and so is the original definition of $\leq_B$!). AF algebras are classifiable by countable structures by Elliott's Theorem 5.8 and the fact that the computation of K_0 is Borel (Lemma 7.9). The fact that the topological orbit equivalence of minimal Cantor systems is equireducible with AF algebras was proved in [37], and that these are equivalent with the isomorphism of dimension groups is in [16].

Classification of Kirchberg algebras by countable structures is a consequence of Kirchberg–Phillips theorem ([65], [54]) and Borel-computability of the Elliott invariant. Classification of separable, but not necessarily unital direct limits of full matrix algebras (i.e., non-unital AM algebras) by countable structures is given in [13]; see also Exercise 4.3.7 .

Non-classifiability of the isomorphism relation in every major class of von Neumann factors that was not already classified is given in [74] and [75]; see also [60].

Since the maximal isomorphism relation for countable structures is not Borel ([43]), it is not Borel-reducible to any Borel equivalence relation. This in particular implies that the conjugacy of unitary operators is strictly $\leq_B$-below the isometry of Polish space.

By $\operatorname{Irr}(A)$ we denote the space of irreducible representations of a C*-algebra A, or equivalently, the space of pure states of A up to the unitary conjugacy. This relation satisfies a dichotomy result: it is either smooth (when A is a type I C*-algebra, by the definition of type I algebra) or turbulent was proved independently in [52] and [27], and the latter proof gives that this relation Borel-reduces $\mathbb{R}^{\mathbb{N}}/\ell_1$ (and not $\mathbb{R}^{\mathbb{N}}/\ell_2$, as stated in this paper). The fact that this relation is Borel (actually F_σ) follows from a result of Glimm and Kadison (see [27]).

Bi-embeddability of AF algebras is proved to be above E_{K_σ} in [33, §8]. The isomorphism of separable Banach spaces is the complete analytic equivalence relation by [35]. Several results on non-classifiability of automorphisms of C*-algebras with respect to unitary equivalence were obtained in [59].

Last, but not least, the isomorphism relation of separable C*-algebras is reducible to an orbit equivalence relation of a Polish group action [19]. This result (stated as an open problem in my last lecture) was obtained in email correspondence between the authors of [19] in the wake of the June 2012 BIRS meeting on applications of descriptive set theory to functional analysis.

Some of the results presented in Figure 13 may not be optimal. For example, it is not known whether the homeomorphism of compact metric spaces is equireducible with the maximal orbit equivalence relation. See also [73].

9.2. Selected open problems

The following problem was suggested by N. Christopher Phillips.

Question 9.1. *Not necessarily self-adjoint, norm-closed algebras are well-studied (see e.g., [66]). Finite-dimensional algebras in this class are not necessarily direct sums of matrix algebras, and there is no simple classification of such algebras. Is there a descriptive set-theoretic explanation for this? For example, are these algebras classifiable by countable structures?*

Let H be the class of homogeneous algebras of the form $C(X, M_n(\mathbb{C}))$ for X compact metric and form the class AH using H as the building blocks (Definition 6.1). By [11], not every direct limit of AH algebras is AH.

Question 9.2. *Let α be the minimal ordinal such that the class of algebras obtained by closing H as above under direct limits α times is closed under direct limits. By the above, $\alpha \geq 2$. What is the value of α?*

(Warning: I don't have an impression that the C*-algebraists are too keen to find an answer to Question 9.2.)

Question 9.3. *For $2 \leq m \leq \infty$ let E_m be the homeomorphism relation of compact subspaces of $[0,1]^m$. Clearly $\mathrm{E}_n \leq_B \mathrm{E}_{n+1} \leq_B \mathrm{E}_\infty$ for all n and $\mathrm{E}_\infty \leq E^{X_\infty}_{G_\infty}$. Does any of the converses hold? In particular, is E_2 the $\leq_B$-maximal orbit equivalence relation of a Polish group action?*

The following important question is vague since it is really a large number of questions in one.

Question 9.4. *How does the complexity of the isomorphism relation of a class of C*-algebras change as the class increases? In particular, is the isomorphism relation of all C*-algebras Borel-reducible to the isomorphism relation of nuclear C*-algebras, or to the isomorphism relation of simple C*-algebras, or to the isomorphism relation of simple nuclear C*-algebras?*

Question 9.5. *What is the complexity of the isomorphism relation of (nuclear, simple) C*-algebras with the same fixed value of the Elliott invariant?*

In [79] and [81] Toms constructed infinitely, and then continuum many, nonisomorphic nuclear simple C*-algebras with the same Elliott invariant.

The present paragraph contains a vague suggestion instead of a question or a problem. By the basic theory of vector bundles ([41]), if $2\dim(X) < n$ then $C(X, M_n(\mathbb{C}))$ contains only trivial projections (cf. §6.0.1). Therefore counterexamples to Elliott's conjecture (e.g., [80]) are AH algebras that are direct limits of direct sums of algebras $C(X, M_n(\mathbb{C}))$ where $\dim(X) > n$. On the other hand, in a technical tour de force the AH algebras with 'slow dimension growth' were classified by Elliott's invariant ([23]). One could therefore speculate that the complexity of the isomorphism relation for AH algebras can be tied with very fast growing functions, especially because the latter were a source of interesting logical result for decades. George Elliott suggested that the substantial paper of Rieffel ([69]) could be a useful source for this project.

Many classification results have matching range of invariant results and it would be good to have their Borel versions. We have already mentioned [78] whose partial Borel version was used in Lemma 8.9 and the Effross–Handelman–Shen result that every dimension group corresponds to an AF algebra ([16]). A Borel version of this result is not difficult to prove and it was used implicitly above. In [77] it was proved that a pair of countable abelian groups appears as the Elliott invariant of a Kirchberg algebra if and only if the second group is free. It would also be good to have a Borel version of the result of [6], that the Cuntz semigroup is recovered functorially from the Elliott invariant for a large class of C*-algebras.

Question 9.6. *Do the above range of invariant results have Borel versions?*

It would be nice to have a general selection theorem that provides such results automatically. However, the obvious strategy falls short of obtaining Borel maps (see Exercise 8.4.6).

As pointed out before, the proof of Kirchberg's Theorem makes a detour into the nonseparable world. The following question may have an interesting answer.

Problem 9.7. What is the strength of Kirchberg's Theorem 8.15 in the sense of reverse mathematics?

It is not known whether nuclear simple separable C*-algebras of real rank zero (see Exercise 3.5.9) are classifiable by K_0 and K_1. An obvious set-theoretic take on this problem is the following.

Problem 9.8. Can separable nuclear C*-algebras of real rank zero be classified by countable structures? What about separable nuclear simple C*-algebras or real rank zero?

It is known that K_0 and K_1 do not classify non-nuclear separable C*-algebras of real rank zero (a Löwenheim–Skolem type argument is given in [65]).

A distinguishing feature of Elliott's view of the classification program of C*-algebras is its functoriality. This does not seem to be captured by the present theory of Borel equivalence relations and the following problem is an attempt to remedy this situation. Polish groupoids were defined in [67] and one can similarly define a 'Polish category' whose objects are metric structures based on Polish spaces. Some preliminary results on the following problem were obtained by the author in a joint work with S. Coskey, G. Elliott, and M. Lupini.

Problem 9.9. Develop Borel reduction theory for Polish groupoids, and more generaly for Polish categories.

References

1. Scot Adams and Alexander S. Kechris, *Linear algebraic groups and countable Borel equivalence relations*, J. Amer. Math. Soc. **13** (2000), no. 4, 909–943.
2. William Arveson, *A short course on spectral theory*, Graduate Texts in Mathematics, vol. 209, Springer-Verlag, New York, 2002.
3. H. Becker and A.S. Kechris, *The descriptive set theory of Polish group actions*, Cambridge University Press, 1996.

4. B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.
5. N. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.
6. N.P. Brown, F. Perera, and A.S. Toms, *The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras*, J. Reine Angew. Math. **621** (2008), 191–211.
7. K. Carlson, E. Cheung, I. Farah, A. Gerhardt-Bourke, B. Hart, L. Mezuman, N. Sequeira, and A. Sherman, *Omitting types and AF algebras*, Arch. Math. Logic (to appear), arXiv:1212.3576.
8. John D. Clemens, Su Gao, and Alexander S. Kechris, *Polish metric spaces: their classification and isometry groups*, Bull. Symbolic Logic **7** (2001), no. 3, 361–375.
9. K.T. Coward, G.A. Elliott, and C. Ivanescu, *The Cuntz semigroup as an invariant for C^*-algebras*, J. Reine Angew. Math. **623** (2008), 161–193.
10. Joachim Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. **57** (1977), no. 2, 173–185.
11. Marius Dădărlat and Søren Eilers, *Approximate homogeneity is not a local property*, J. Reine Angew. Math. **507** (1999), 1–13.
12. K.R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, American Mathematical Society, Providence, RI, 1996.
13. J. Dixmier, *On some C^*-algebras considered by Glimm*, J. Functional Analysis **1** (1967), 182–203.
14. Randall Dougherty and Greg Hjorth, *Reducibility and nonreducibility between l^p equivalence relations*, Trans. Amer. Math. Soc. **351** (1999), no. 5, 1835–1844.
15. Edward G. Effros, *Classifying the unclassifiables*, Group representations, ergodic theory, and mathematical physics: a tribute to George W. Mackey, Contemp. Math., vol. 449, Amer. Math. Soc., Providence, RI, 2008, pp. 137–147.
16. Edward G. Effros, David E. Handelman, and Chao Liang Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102** (1980), no. 2, 385–407.
17. Edward G. Effros and Zhong-Jin Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, vol. 23, The Clarendon Press Oxford University Press, New York, 2000.
18. G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras*, J. Algebra **38** (1976), no. 1, 29–44.
19. G.A. Elliott, I. Farah, V. Paulsen, C. Rosendal, A.S. Toms, and A. Törnquist, *The isomorphism relation of separable C*-algebras*, Math. Res. Letters (to appear), preprint, arXiv 1301.7108.
20. G.A. Elliott and A.S. Toms, *Regularity properties in the classification program for separable amenable C^*-algebras*, Bull. Amer. Math. Soc. **45** (2008), no. 2, 229–245.

21. George A. Elliott, *A classification of certain simple C^*-algebras*, Quantum and non-commutative analysis (Kyoto, 1992), Math. Phys. Stud., vol. 16, Kluwer Acad. Publ., Dordrecht, 1993, pp. 373–385.
22. ———, *The classification problem for amenable C^*-algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 922–932.
23. George A. Elliott, Guihua Gong, and Liangqing Li, *On the classification of simple inductive limit C^*-algebras. II. The isomorphism theorem*, Invent. Math. **168** (2007), no. 2, 249–320.
24. I. Farah, *Basis problem for turbulent actions I: Tsirelson submeasures*, Proceedings of XI Latin American Symposium in Mathematical Logic, Merida, July 1998, Annals of Pure and Applied Logic, vol. 108, 2001, pp. 189–203.
25. ———, *Basis problem for turbulent actions II: c_0-equalities*, Proceedings of the London Mathematical Society **82** (2001), 1–30.
26. ———, *Graphs and CCR algebras*, Indiana Univ. Math. Journal **59** (2010), 1041–1056.
27. ———, *A dichotomy for the Mackey Borel structure*, Proceedings of the 11th Asian Logic Conference In Honor of Professor Chong Chitat on His 60th Birthday (Yang Yue et al., eds.), 2011, pp. 86–93.
28. I. Farah, B. Hart, and D. Sherman, *Model theory of operator algebras I: Stability*, Bull. London Math. Soc. **45** (2013), 825–838.
29. ———, *Model theory of operator algebras II: Model theory*, Israel J. Math. (to appear), arXiv:1004.0741.
30. I. Farah and T. Katsura, *Nonseparable UHF algebras I: Dixmier's problem*, Adv. Math. **225** (2010), no. 3, 1399–1430.
31. ———, *Nonseparable UHF algebras II: Classification*, Math. Scand. (to appear), preprint arXiv:1301.6152.
32. I. Farah, A.S. Toms, and A. Törnquist, *The descriptive set theory of C*-algebra invariants*, Int. Math. Res. Notices (2013), 5196–5226, Appendix with Caleb Eckhardt.
33. ———, *Turbulence, orbit equivalence, and the classification of nuclear C*-algebras*, J. Reine Angew. Math. (to appear).
34. I. Farah and E. Wofsey, *Set theory and operator algebras*, Appalachian set theory 2006-2010 (J. Cummings and E. Schimmerling, eds.), Cambridge University Press, 2013, pp. 63–120.
35. V. Ferenczi, A. Louveau, and C. Rosendal, *The complexity of classifying separable Banach spaces up to isomorphism*, J. Lond. Math. Soc. (2) **79** (2009), no. 2, 323–345.
36. H. Friedman and L. Stanley, *A Borel reducibility theory for classes of countable structures*, The Journal of Symbolic Logic **54** (1989), 894–914.
37. Thierry Giordano, Ian F. Putnam, and Christian F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. **469** (1995), 51–111.
38. J. Glimm, *Type I C^*-algebras*, Ann. of Math. (2) **73** (1961), 572–612.
39. J. G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.

40. L.A. Harrington, A.S. Kechris, and A. Louveau, *A Glimm–Effros dichotomy for Borel equivalence relations*, Journal of the American Mathematical Society **4** (1990), 903–927.
41. A. Hatcher, *Vector bundles and K-theory*, 2003.
42. G. Hjorth, *Actions by classical Banach spaces*, The Journal of Symbolic Logic **65** (2000), 392–420.
43. ———, *Classification and orbit equivalence relations*, Mathematical Surveys and Monographs, vol. 75, American Mathematical Society, 2000.
44. ———, *Borel equivalence relations*, Handbook of set theory, 2010.
45. X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, Amer. J. Math **121** (1999), 359–413.
46. V.F.R. Jones, *Von Neumann algebras*, 2010, lecture notes, http://math.berkeley.edu/∼vfr/.
47. M. Junge and G. Pisier, *Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$*, Geom. Funct. Anal. **5** (1995), no. 2, 329–363.
48. A.S. Kechris, *Classical descriptive set theory*, Graduate texts in mathematics, vol. 156, Springer, 1995.
49. ———, *The descriptive classification of some classes of C^*-algebras*, Proceedings of the Sixth Asian Logic Conference (Beijing, 1996), World Sci. Publ., River Edge, NJ, 1998, pp. 121–149.
50. A.S. Kechris and A. Louveau, *The structure of hypersmooth Borel equivalence relations*, Journal of the American Mathematical Society **10** (1997), 215–242.
51. A.S. Kechris and N. E. Sofronidis, *A strong generic ergodicity property of unitary and self-adjoint operators*, Ergodic Theory Dynam. Systems **21** (2001), no. 5, 1459–1479.
52. D. Kerr, H. Li, and M. Pichot, *Turbulence, representations, and trace-preserving actions*, Proc. Lond. Math. Soc. (3) **100** (2010), no. 2, 459–484.
53. E. Kirchberg and N.C. Phillips, *Embedding of exact C*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. reine angew. Math. **525** (2000), 17–53.
54. Eberhard Kirchberg, *Exact* C**-algebras, tensor products, and the classification of purely infinite algebras*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 943–954.
55. Huaxin Lin, *Classification of simple C^*-algebras of tracial topological rank zero*, Duke Math. J. **125** (2004), no. 1, 91–119.
56. T.A. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs, vol. 8, American Mathematical Society, Providence, RI, 1997.
57. A. Louveau and C. Rosendal, *Complete analytic equivalence relations*, Trans. Amer. Math. Soc. **357** (2005), no. 12, 4839–4866.
58. A. Louveau and B. Velickovic, *A note on Borel equivalence relations*, Proceedings of the American Mathematical Society **120** (1994), 255–259.
59. M. Lupini, *Unitary equivalence of automorphisms of separable C*-algebras*, arXiv preprint arXiv:1304.3502 (2013).
60. M. Lupini and A. Törnquist, *Set theory and von Neumann algebras*, Appalachian set theory 2006-2010 (J. Cummings and E. Schimmerling, eds.),

Cambridge University Press, to appear.

61. Julien Melleray, *Computing the complexity of the relation of isometry between separable Banach spaces*, MLQ Math. Log. Q. **53** (2007), no. 2, 128–131.
62. Gerard J. Murphy, *C^*-algebras and operator theory*, Academic Press Inc., Boston, MA, 1990.
63. Vern Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.
64. G.K. Pedersen, *Analysis now*, Graduate Texts in Mathematics, vol. 118, Springer-Verlag, New York, 1989.
65. N.C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114 (electronic).
66. S.C. Power, *Limit algebras: an introduction to subalgebras of C^*-algebras*, Pitman Research Notes in Mathematics Series, vol. 278, Longman Scientific & Technical, Harlow, 1992.
67. Arlan Ramsay, *The mackey-glimm dichotomy for foliations and other polish groupoids*, Journal of Functional Analysis **94** (1990), no. 2, 358–374.
68. M. Rørdam, F. Larsen, and N.J. Laustsen, *An introduction to K-theory for C^* algebras*, London Mathematical Society Student Texts, no. 49, Cambridge University Press, 2000.
69. Marc A. Rieffel, *Projective modules over higher-dimensional noncommutative tori*, Canad. J. Math. **40** (1988), no. 2, 257–338.
70. M. Rørdam, *Classification of nuclear C^*-algebras*, Encyclopaedia of Math. Sciences, vol. 126, Springer-Verlag, Berlin, 2002.
71. ———, *A simple C^*-algebra with a finite and an infinite projection*, Acta Math. **191** (2003), 109–142.
72. C. Rosendal, *Cofinal families of Borel equivalence relations and quasiorders*, Journal of Symbolic Logic **70** (2005), 1325–1340.
73. Marcin Sabok, *Completeness of the isomorphism problem for separable c*-algebras*, arXiv preprint arXiv:1306.1049 (2013).
74. R. Sasyk and A. Törnquist, *Borel reducibility and classification of von Neumann algebras*, Bulletin of Symbolic Logic **15** (2009), no. 2, 169–183.
75. ———, *Turbulence and Araki-Woods factors*, J. Funct. Anal. **259** (2010), no. 9, 2238–2252.
76. P. Skoufranis, *Operator theory notes*, notes available at http://www.math.ucla.edu/%7Epskoufra/OperatorAlgebras.html, 2012.
77. Wojciech Szymański, *The range of K-invariants for C^*-algebras of infinite graphs*, Indiana Univ. Math. J. **51** (2002), no. 1, 239–249.
78. K. Thomsen, *Inductive limits of interval algebras: the tracial state space*, Amer. J. Math. **116** (1994), no. 3, 605–620.
79. A.S. Toms, *An infinite family of non-isomorphic C^*-algebras with identical K-theory*, Trans. Amer. Math. Soc. **360** (2008), no. 10, 5343–5354.
80. ———, *On the classification problem for nuclear C^*-algebras*, Ann. of Math. (2) **167** (2008), no. 3, 1029–1044.
81. ———, *Comparison theory and smooth minimal C^*-dynamics*, Comm. Math. Phys. **289** (2009), no. 2, 401–433.

82. A.S. Toms and W. Winter, *The Elliott conjecture for Villadsen algebras of the first type*, J. Funct. Anal. **256** (2009), no. 5, 1311–1340.
83. Jesper Villadsen, *Simple C^*-algebras with perforation*, J. Funct. Anal. **154** (1998), no. 1, 110–116.
84. N. Weaver, *Set theory and C^*-algebras*, Bull. Symb. Logic **13** (2007), 1–20.
85. Carl Winslow and Uffe Haagerup, *The Effros–Maréchal topology in the space of von Neumann algebras*, American Journal of Mathematics **120** (1998), 567–617.
86. W. Winter, *Decomposition rank and $\mathcal{Z}$-stability*, Invent. Math. **179** (2010), 229–301.
87. ———, *Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras*, Invent. Math. **187** (2012), 259–342.
88. ———, *Ten lectures on topological and algebraic regularity properties of nuclear C*-algebras*, CBMS conference notes, to appear.