A UNIVERSAL PATHOLOGICAL SUBMEASURE

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We construct a universal pathological submeasure on the algebra of closed and open subsets of the Cantor space.

A function $\theta$ on a Boolean algebra $\mathcal{B}$ is a submeasure if $A \subseteq B$ implies $\theta(A) \leq \theta(B)$, $\theta(A \cup B) \leq \theta(A) + \theta(B)$ and $\theta(0_B) = 0$. It is pathological if it does not dominate a positive finitely additive functional. It is normalized if $\theta(1_B) = 1$. Assume $\phi$ and $\psi$ are submeasures on Boolean algebras $\mathcal{A}$ and $\mathcal{B}$, respectively. We write $\phi \preceq \psi$ if there is a homomorphism $\Phi: \mathcal{A} \to \mathcal{B}$ such that for every $A \in \mathcal{A}$ we have $\phi(A) \leq \psi(\Phi(A))$. Talagrand constructed ([2]) a family of submeasures $\{\psi^\sigma\}$ such that for every pathological submeasure $\phi$ we have $\phi \preceq \psi^\sigma$ for some $\psi^\sigma$.

By $|a|$ we denote the cardinality of $a$.

If $X_n$ is a finite set and $\theta_n$ is a submeasure on $\mathcal{P}(X_n)$ for $n \in \mathbb{N}$, define a submeasure $\psi = \psi(X_n, \theta_n)$ as follows. We may assume $\mathbb{N} = \bigcup_n X_n$ is a disjoint partition and identify $\mathcal{P}(\mathbb{N})$ with $X = \prod_n X_n$. The submeasure $\psi$ is defined on the clopen algebra of $\mathcal{P}(\mathbb{N})$, considered with its Cantor-set topology. For $F \subseteq \mathbb{N}$ define

$$\langle F \rangle = \{x \in \mathbb{N} | x \cap F \neq \emptyset\}$$

and $w(F) = \sum_n \theta_n(F \cap X_n)$.

For $A \subseteq \mathcal{P}(\mathbb{N})$ let $\psi(A) = \inf\{w(F) | \langle F \rangle \supseteq A\}$; this is a submeasure. If $2n \leq m$ define a submeasure $\eta_{n,m}$ on $\mathcal{P}(m)$ via

$$\eta_{n,m}(A) = \min(1, \inf\{b : b \leq m \text{ and } \langle \forall a \in A \rangle a \cap b \neq \emptyset\}).$$

If $X_{n,m} = \{a \leq m : |a| \geq m/2\}$, let $\theta_{n,m}$ be the restriction of $\eta_{n,m}$ to $\mathcal{P}(X_{n,m})$.

For a function $\sigma: \mathbb{N} \to \mathbb{N}$ such that $\sigma(n) \geq 2n$ for all $n$ let $\psi^\sigma = \psi(X_{n,\sigma(n)}, \theta_{n,\sigma(n)})$. In [2] it was proved that each $\psi^\sigma$ is normalized, pathological, and that for every pathological submeasure $\phi$ there is a $\sigma$ such that $\phi \preceq \psi^\sigma$.

Throughout we assume $X_n$ is finite and $\theta_n$ is a normalized (not necessarily pathological) submeasure on $\mathcal{P}(X_n)$. A pathological submeasure $\psi$ is universal if $\phi \preceq \psi$ for every pathological $\phi$.

**Theorem 1.** There exists a universal pathological submeasure $\psi$ on the algebra of clopen subsets of the Cantor space.

The 1947 unsolved problem of D. Maharam can be reformulated as a question whether there is a normalized pathological submeasure $\phi$ which is exhaustive: for every pairwise disjoint sequence $\{A_i\}$ we have $\limsup_n \phi(A_n) = 0$ (see [1]). By Theorem 1, this is equivalent to requiring $\phi \preceq \psi$ in addition.

**Lemma 2.** Assume $a_i \subseteq X_i$, $i < m$, are such that $1/n \leq \Theta_i(a_i)$ and $\Theta_i(X_i \setminus a_i) = 1$ for all $i$. Then $\eta_{n,m} \preceq \psi$ and $\theta_{n,m} \preceq \psi$. 

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Proof. Let $H : \times_{i=0}^{m-1} X_i \to \mathcal{P}(m)$ be $H(x) = \{i : x \in a_i\}$ (using $m = \{0, \ldots, m - 1\}$).

Assume $\psi(H^{-1}(A)) \leq p < 1$, and fix $F$ such that $H^{-1}(A) \subseteq \langle F \rangle$ and $w(F) = p < 1$. Let $b = \{i : F \cap X_i \subseteq a_i\}$. Assume $c \cap b = \emptyset$. We will prove $c \notin A$. For each $i \in c$ we can pick $x(i) \in a_i \setminus F \cap X_i$ since $i \notin b$. For $i \notin c$ pick $x(i) \in X_i \setminus F \cap X_i$ since $w(F) < 1$. Then $x \notin \langle F \rangle$ and $H(x) = c$, therefore $c \notin A$. We conclude that $A \subseteq \{c : c \cap b = \emptyset\}$ and therefore $\eta_{n,m}(A) \leq |b|/n \leq w(F) = \psi(H^{-1}(A))$.

Let $Z = H(X_{n,m})$ and pick $i \in X_{n,m}$. If $H' : \times_i X_i \to X_{n,m}$ is $H'(x) = H(x)$ if $x \in Z$ and $H'(x) = i$ otherwise, then $(H')^{-1}$ witnesses $\theta_{n,m} \leq \psi$.

Lemma 3. If $a_i \subseteq X_i$ so that $\Theta_i(a_i) \leq 2/3$ for $i < 3$ then $a = \times_{i=0}^{2} a_i$ satisfies

(i) $\min \Theta_i(a_i) = \psi(a)$ and

(ii) $\psi(\times_{i=0}^{2} X_i \setminus a) = 1$.

Proof. (i) Taking $F = \{i\} \times a_i$ we get $\Theta_i(a_i) \geq \psi(a)$. If $\langle F \rangle \supseteq a$ and $F \cap X_i \supseteq a_i$ for all $i$, then $\langle F \rangle \setminus a$ is nonempty. So we must have $w(F) \geq \Theta_i(a_i)$ for some $i$.

(ii) If $w(F) < 1$, then $X_i \setminus F$ is nonempty for all $i$. Moreover, for some $i$ we have $w(F \cap X_i) < 1/3$, and therefore we can find $x_i \in X_i \setminus (F \cap X_i \cup a_i)$, so $\langle F \rangle$ does not cover $\times_{i=0}^{2} X_i \setminus a$.

Proof of Theorem 1. We shall prove that every diffuse normalized pathological submeasure of the form $\psi = \psi(X_n, \theta_n)$ is universal. (A submeasure on $\mathcal{B}$ is diffuse if for every $\varepsilon > 0$ there is a finite partition of $1_\mathcal{B}$ into sets of submeasure $\leq \varepsilon$.) This implies that every $\psi^\sigma$ is universal.

Fix $\varepsilon$ and a pathological submeasure $\phi$. By [2] there is $a$ such that $\phi \leq \psi^\sigma$. If for some $\varepsilon > 0$ there is $x \in \times_n X_n$ such that $\theta_n(\{x(n)\}) \geq \varepsilon$ for all $n$, then $\psi(A) \geq \varepsilon$ for any $A$ containing $x$. Since $\psi$ is diffuse, for every $\varepsilon > 0$, for infinitely many $n$ we have $\sup_{\mathcal{B} X_n} \theta_n(\{i\}) \leq \varepsilon$. Thus the set $\{\theta_n(a) : a \subseteq X_n\}$ is $\varepsilon$-dense in $[0, 1]$. By Lemma 3 we can find pairwise disjoint finite sets $I_j$ such that for $Y_j = \times_{i \in I_j} X_i$ equipped with $\psi_j = \psi(X_i, \Theta_i : i \in I_j)$ there is $a_j \subseteq Y_j$ satisfying $1/j \leq \psi_j(a_j)$ and $\psi_j(Y_j \setminus a_j) = 1$. Note that $\psi_Y = \psi(Y_j, \psi_j)$ satisfies $\psi_Y \succeq \psi(X_n, \theta_n)$.

Lemma 2 implies $\theta_{j, \sigma(j)} \succeq \psi_j$, and thus $\psi^\sigma \succeq \psi(Y_j, \psi_j : j \in \mathbb{N})$, consequently $\phi \succeq \psi(X_n, \theta_n)$.

Not every diffuse $\psi(X_n, \theta_n)$ is pathological. Let $X_n = 2^n$, let $\theta_n$ be the normalized counting measure on $X_n$. Then $A \subseteq \times_n X_n$ be $\{x(\forall n)x(n+1) \mid n = x(n)\}$ is closed and the restriction of $\psi$ to the clopen algebra of $A$ is finitely additive and normalized.

References
