CONVOLUTIONS OF PATHOLOGICAL SUBMEASURES

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This note presents the proof of the main result of [2], that there is a compact group G and a pathological submeasure on the algebra of clopen subsets of G whose convolution with the Haar measure dominates the Haar measure; this is Theorem 5 below. Throughout ϕ, ψ, θ are normalized submeasures. Haar measure on a compact group G is denoted by ν_G. If I is a finite set then ν_I is the normalized counting measure on I. A submeasure is *pathological* if it does not dominate a (finitely additive) nonvanishing measure.

By 'ϕ is a submeasure on I' we mean that ϕ is a submeasure whose domain is $\mathcal{P}(I)$. If ϕ is a submeasure on I then for any n define $\tilde{\phi}$ on I^n as follows. If $i < n$ then $p_i \colon I^n \to I$ is the projection, $p_i(x) = x(i)$. Let $\phi_i(A) = \phi(p_i[A])$; it is a submeasure and it is pathological if ϕ is pathological. Then

$$\tilde{\phi}(A) = \int \phi_i(A)\, d\nu_n(i)$$

is a normalized submeasure on I^n. For a submeasure ϕ on I let $\varepsilon_\phi = \inf\{\phi(A) : A \subseteq I \text{ is nonempty}\}$. Throughout we assume ϕ is a measure on a finite set and $\phi(A) > 0$ for every nonempty A, hence $\varepsilon_\phi > 0$ always.

Lemma 1. *Assume ϕ is a submeasure on a finite set I and $\delta < \min(\varepsilon_\phi/2, 1/|I|)$. Then for n large enough we have*

$$\nu_{I^n}(A) + \delta \leq \tilde{\phi}(A)$$

for all $A \subseteq I^n$ such that $0 < \tilde{\phi}(A) < 1$.

Proof. Fix n satisfying $((|I|-1)/|I|)^n < \varepsilon_\phi/2$ and $A \subseteq I^n$ such that $0 < \tilde{\phi}(A) < 1$. Let $S = \{i < n : p_i[A] = I\}$. Since $\tilde{\phi}(A) < 1$, we have $|S| \leq n-1$. With $D_j = p_j^{-1}(p_j[A])$ for $j \notin S$ we have $\nu_{I^n}(D_j) \leq 1 - 1/|I|$, so

$$\nu_{I^n}(A) \leq \left(\frac{|I|-1}{|I|}\right)^{n-|S|}.$$

With $\varepsilon = \varepsilon_\phi$ we have $\tilde{\phi}(A) \geq \frac{1}{n}(|S| + \varepsilon(n - |S|))$. Let $\alpha = |S|/n$. Then $0 \leq \alpha \leq 1 - 1/n$ and we need to prove

$$\left(\frac{|I|}{|I|-1}\right)^{n\alpha - n} + \delta \leq \alpha + \varepsilon(1-\alpha).$$

When $\alpha = 0$ it reduces to $((|I|-1)/|I|)^n \leq \varepsilon - \delta$, which follows by the condition on n and $\delta < \varepsilon/2$. When $\alpha = 1 - 1/n$ it reduces to $(|I|-1)/|I| + \delta \leq 1$, and this follows by $\delta < 1/|I|$. Since the left-hand side of () is convex in α, the inequality follows. $\square$

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For submeasures ϕ on I and ψ on J define $\theta = \phi \times \psi$ on $I \times J$ by (P_I and P_J are the projections from $I \times J$ to I and J, respectively)

$$\theta(A) = \inf\{\phi(X) + \psi(Y) : A \subseteq p_I^{-1}(X) \cup p_J^{-1}(Y)\}.$$

Then θ is normalized if ϕ and ψ are. It is pathological if at least one of ϕ and ψ is pathological. Also—still assuming ϕ and ψ are normalized—we have $\theta(P_I^{-1}(A)) = \phi(A)$ and $\theta(P_J^{-1}(B)) = \psi(B)$ for all $A \subseteq I$ and $B \subseteq J$). On $I^n \times J^m$ for $i \in I$ and $j \in J$ define θ_{ij} via $\theta_{ij}(A) = \theta(p_{ij}[A])$ and on $\tilde{\theta}$ via

$$\tilde{\theta}(A) = \iint \theta_{ij}(A)\, d\nu_n(i)\, d\nu_m(j).$$

Our next goal is Lemma 3 below, and the following approximation lemma is the key to its proof.

Lemma 2. *Assume ϕ is a submeasure on I, $n \in \mathbb{N}$, and*

(1) *$\nu_{I^n}(B) + \delta_0 \leq \tilde{\phi}(B)$ for all $B \subseteq I^n$ such that $0 < \tilde{\phi}(B) < 1$.*

Also assume ψ is a submeasure on J. If $\delta_1 = \min(\delta_0/3, \varepsilon_\phi/2, \varepsilon_\psi/2, 1/(|I \times J|))$ then for a large enough m the following holds. For every $A \subseteq I^n \times J^m$ such that $0 < \tilde{\theta}(A) < 1$ and every $i < n$ there is $Z_i \subseteq I^n \times J^m$ such that $Z_i = p_I^{-1}(Z_i^0)$ for some $Z_i^0 \subseteq I^n$ satisfying

(1) *$\nu_{I^n \times J^m}(A \setminus Z_i) \leq \delta_1$, and*

(2) *for all $j < m$ we have $\theta_{ij}(Z_i) \leq \delta_1 + \int \theta_{ik}(A)\, d\nu_m(k)$.*

We shall postpone the proof of Lemma 2 until we show the following is its consequence.

Lemma 3. *Assume ϕ is a submeasure on I and $n \in \mathbb{N}$ are such that (1) holds. Moreover assume ψ is a submeasure on J and $m \in \mathbb{N}$ are such that for every $A \subseteq I^n \times J^m$ and $i < n$ there is Z_i as in Lemma 3. Then we have*

$$\nu_{I^n \times J^m}(A) + \delta_1 \leq \tilde{\theta}(A)$$

for all $A \subseteq I^n \times J^m$ satisfying $0 < \tilde{\theta}(A) < 1$.

Proof. Fix $A \subseteq I^n \times J^m$ such that $0 < \tilde{\theta}(A) < 1$. If Z_i are as obtained in Lemma 2, let $Z = \bigcap_{i<n} Z_i$. Then we have

(3) $\nu(A \setminus Z) \leq \sum_{i<n} \nu(A \setminus Z_i) \leq \delta_1$

and

$$\begin{aligned}\tilde{\theta}(Z) = \iint \theta_{ij}(Z)\, d\nu_n(i)\, d\nu_m(j) &\leq \iint \theta_{ij}(Z_i)\, d\nu_n(i)\, d\nu_m(j) \\ &\leq \delta_1 + \iiint \theta_{ik}(A)\, d\nu_m(k)\, d\nu_n(i)\, d\nu_m(j) = \delta_1 + \tilde{\theta}(A),\end{aligned}$$

(using (2) in the nontrivial inequality) therefore

(4) $\tilde{\theta}(Z) \leq \delta_1 + \tilde{\theta}(A)$.

We check A is as required. If $0 < \tilde{\theta}(Z) < 1$ then by the assumption

(5) $\nu_{I^n \times J^m}(Z) + \delta_0 = \nu_{I^n}(p_{I^n}[Z]) + \delta_0 \leq \tilde{\phi}(p_{I^n}(Z)) = \tilde{\theta}(Z)$.

Using (3), the property of δ_1, (5) and (4) we obtain $\nu_{I^n \times J^m}(A) + 2\delta_1 \leq \nu_{I^n \times J^m}(Z) + 3\delta_1 \leq \nu_{I^n \times J^m}(Z) + \delta_0 \leq \tilde{\theta}(Z) \leq \delta_1 + \tilde{\theta}(A)$, hence $\nu_{I^n \times J^m}(A) + \delta_1 \leq \tilde{\theta}(A)$.

Consider the case when $\tilde{\theta}(Z) = 0$. Then by our convention we have $Z = \emptyset$ and $\nu_{I^n \times J^m}(A) \leq \delta_1$ by (3). Since $A \neq \emptyset$ we have $\theta_{ij}(A) \geq \min(\varepsilon_\phi, \varepsilon_\psi)$ for all i, j and therefore $\tilde{\theta}(A) \geq \min(\varepsilon_\phi, \varepsilon_\psi)$ and $\nu_{I^n \times J^m}(A) + \delta_1 \leq 2\delta_1 \leq \tilde{\theta}(A)$ by the requirement on δ_1.

Finally assume $\tilde{\theta}(Z) = 1$. Then $\tilde{\theta}(A) \leq 1 - \delta_1$ by (4). Since $\tilde{\theta}(A) < 1$ we have $p_{ij}[A] \neq I \times J$ for some i, j. Hence $\nu_{I^n \times J^m}(A) \leq 1 - \frac{1}{|I| \times |J|} \leq 1 - 2\delta_1$ and $\nu_{I^n \times J^m}(A) + \delta_1 \leq 1 - \delta_1 \leq \tilde{\theta}(A)$. □

Proof of Lemma 2. Pick m large enough to satisfy $\left(\frac{|J|-1}{|J|}\right)^{2^{-|I|-1} m\delta_1} \leq \delta_1$. Fix $i < n$ and let $a = \int \theta_{ik}(A)\, d\nu_m(k)$. If $a + \delta_1 \geq 1$ then let $Z_i = I \times J$. Condition (1) is immediate and (2) follows since $\tilde{\theta}(Z) = 1 \leq \delta_1 + a$.

We may therefore assume $a + \delta_1 < 1$. Let

$$T = \{j \in m : \theta_{ij}(A) < a + \delta_1\}$$

Now $a \geq (1 - \nu_m(T))(a + \delta_1 a) \geq a(1 - \nu_m(T))(1 + \delta_1)$ (since $a \leq 1$). Consequently $1 \geq (1 - \nu_m(T))(1 + \delta_1) = 1 + \delta_1 - \nu_m(T)(1 + \delta_1)$ and

(6) $\nu_m(T) \geq \delta_1/(1 + \delta_1) \geq \delta_1/2$.

Every $j \in T$ satisfies $\theta_{ij}(A) < a + \delta_1$, so we can fix $X_j \subseteq I$ and $Y_j \subseteq J$ such that (with p_i and p_j considered as projections from $I^n \times J^m$ to I and J, respectively) $A \subseteq p_i^{-1}(X_j) \cup p_j^{-1}(Y_j)$ and $\phi(X_j) + \psi(Y_j) < a + \delta_1$. (Note that $X_j \neq I$ and $Y_j \neq J$.) There is $X \subseteq I$ such that $V = \{j < m : X_j = X\}$ satisfies $\nu_m(V) \geq 2^{-|I|}\nu_m(T) \geq 2^{-|I|-1}\delta_1$, by (6), hence

(7) $|V| \geq 2^{-|I|-1} m\delta_1$.

Let $Z_i = p_i^{-1}(X)$. Then $\theta_{ij}(Z) = \phi(X) < a + \delta_1$ hence (2) holds. Now check (1). For $j \in V$ we have $A \setminus Z_i \subseteq p_j^{-1}(Y_j)$. Then $Y_j \neq J$ and the independence of $p_j^{-1}(Y_j)$ imply

$$\nu(A \setminus Z_i) \leq \left(\frac{|J|-1}{|J|}\right)^{|V|} \leq \left(\frac{|J|-1}{|J|}\right)^{2^{-|I|-1} m\delta_1} \leq \delta_1,$$

by the choice of m. □

In the sequel we consider pathological submeasures on the algebra of clopen subsets of an infinite compact metric group. Accordingly we modify our convention and say ϕ is a submeasure on G if its domain is equal to the algebra $\mathrm{cl}(G)$ of clopen subsets of G. It is well-known (e.g., [3]) that there is a sequence of finite groups H_i and submeasures ϕ_i on H_i ($i \in \mathbb{N}$) such that $\phi(A) = \inf \sum_i \{\phi_i(X_i) : X_i \subseteq H_i, \bigcup_i p_i^{-1}(X_i) \supseteq A\}$ is a normalized pathological submeasure on $H = \prod_i H_i$. We extend the notation introduced earlier as follows. On a group $K = \prod_i H_i^{L_i}$ for $y \in L = \prod_i L_i$ define the projection $p_y \colon K \to H$ by $p_y(x)(i) = x(i)(y)$. If ϕ is a submeasure on H, then $\phi_y(A) = \phi(p_y[A])$ is a submeasure on K, and so is $\tilde{\theta}$ defined by

$$\tilde{\theta}(A) = \int \phi_y(A)\, d\nu_L(y).$$

All groups are considered with the product topology.

Lemma 4. *Given H_i, ϕ_i as above, there is a sequence L_i ($i \in \mathbb{N}$) of finite groups such that $\tilde{\theta}$ satisfies $\tilde{\theta}(A) \geq \nu_K(A)$.*

Proof. Let $\varepsilon_n = \inf\{\phi(A) : A \subseteq K$ is a clopen set supported in $\prod_{i<n} H_i^{L_i}\}$ and let $\tilde{\theta}_n$ be the restriction of $\tilde{\theta}$ to to the algebra of clopen sets supported in $K_n = \prod_{i\leq n} H_i^{L_i}$. We identify $\tilde{\theta}_n$ with a submeasure on K_n in a natural way.

Recursively pick L_i, δ_i ($i \in \mathbb{N}$) as follows. Let $\delta_1 = \min(\varepsilon_1/2, 1/|H_1|)$. By Lemma 1 for every large enough L_1 every $A \subseteq H_1^{L_1}$ such that $0 < \tilde{\theta}_1(A) < 1$ satisfies $\tilde{\theta}_1(A) \geq \nu_{H_1^{L_1}}(A) + \delta_1$. Using Lemma 3 recursively find δ_n, L_n so that for all $A \subseteq K_n$ satisfying $0 < \tilde{\theta}_n(A) < 1$ we have $\nu_{K_n}(A) + \delta_n \leq \tilde{\theta}_n(A)$. Since every $A \in \mathrm{cl}(G)$ is supported in some K_n, we conclude that $\tilde{\theta} \geq \nu_{K_n}$ on $\mathrm{cl}(G)$. ☐

Theorem 5. *There is a compact metric group G and a pathological submeasure ψ on G whose convolution with the Haar measure is nonpathological; more precisely, $\nu_G \leq \nu_G * \psi$.*

Proof. Let $G = L \times K$ with L, K as in Lemma 4. For $A \subseteq G$ and $z \in L$ consider the vertical section, $A_z = \{x \in K : (z, x) \in A\}$. Define $\psi(A)$ for $A \in \mathrm{cl}(G)$ via (ϕ and ϕ_z as defined in Lemma 4)

$$\psi(A) = \int \phi_z(A_z)\, d\nu_L(z).$$

We claim ψ is a pathological submeasure. Assume $\mu \leq \psi$ is a finitely additive measure on $\mathrm{cl}(G)$. Extend μ to a Borel measure on G, also denoted by μ. By the measure desintegration theorem ([1, §452]) there is a measurable map $z \mapsto \mu_z$ such that $\mu(A) = \int_B \mu_z(A_z)\, d\nu_L(z)$ for all Borel A. If $A = B \times C \subseteq G$ this implies $\mu(A) = \int_B \mu_z(C)\, d\nu_L(z) \leq \int \phi_z(C)\, d\nu_L(z)$. For each $C \in \mathrm{cl}(K)$ there is a ν_L-null $Z_C \subseteq L$ such that $\mu_z(C) \leq \phi_z(C)$ for all $z \notin Z_C$. Since $\mathrm{cl}(K)$ is countable, $\nu_L(\{z \in L : \mu_z(C) \leq \phi_z(C)$ for all $C \in \mathrm{cl}(K)\}) = 1$ and therefore μ_z vanishes for ν_L-almost all z. Hence μ vanishes as well, concluding the proof that ψ is pathological.

It remains to check $\nu_G \leq \nu_G * \psi$. Fix $A \in \mathrm{cl}(G)$. Then $\nu_G * \psi(A) = \int \psi(t + A)\, d\nu_G(t) = \iint \phi_z((t + A)_z)\, d\nu_G(t)\, d\nu_L(z)$. If $t = (t_0, t_1)$ for $t_0 \in L$ and $t_1 \in K$, then $(t + A)_z = t_1 + A_{z-t_0}$, hence

$$\begin{aligned}\int \psi(t + Y)\, d\nu_G(t) &= \int \phi_z(t_1 + A_{z-t_0})\, d\nu_L(z)\, d\nu_L(t_0)\, d\nu_K(t_1)\\ &= \int \phi_z(t_1 + A_{t_0})\, d\nu_L(z)\, d\nu_L(t_0)\, d\nu_K(t_1).\end{aligned}$$

Lemma 4 implies $\int \phi_z(t_1 + A_{t_0})\, d\nu_L(z) \geq \nu_K(t_1 + A_{t_0}) = \nu_K(A_{t_0})$ and finally $\int \psi(t + A)\, d\nu_G(t) \geq \int \nu_K(A_{t_0})\, d\nu_L(t_0) = \nu_G(A)$. ☐

References

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