EXAMPLES OF ε-EXHAUSTIVE PATHOLOGICAL SUBMEASURES

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A submeasure θ on a Boolean algebra $\mathcal{B}$ is *pathological* if it does not dominate a positive finitely additive functional. It is *ε-exhaustive* if every sequence $\{A_n\}$ of pairwise disjoint sets in $\mathcal{B}$ satisfies $\limsup_n \theta(A_n) \leq \varepsilon$. It is *exhaustive* if it is 0-exhaustive. It is *normalized* if $\theta(1_{\mathcal{B}}) = 1$.

Question 1. *Is there a normalized exhaustive pathological submeasure on the algebra of clopen subsets of the Cantor space?*

This is an equivalent reformulation of Maharam's problem ([5]) on characterization of measure algebras, also known as the Control Measure Problem. The importance of Maharam's problem largely derives from a variety of forms in which it appears (see [1], [2], [8]).

It was apparently unknown even whether there is a normalized ε-exhaustive pathological submeasure for some small $\varepsilon > 0$. The example of a normalized ε-exhaustive submeasure constructed by Roberts in [8] is not pathological, since every nonempty set has submeasure at least ε. We should note that this submeasure is *weakly pathological*: no strictly positive finitely additive functional f dominated by it satisfies $f(1_{\mathcal{B}}) = 1$. For Roberts' submeasure there is an arbitrarily long finite sequence of pairwise disjoint sets of submeasure 1, and by [3] the existence of an exhaustive submeasure with this property is equivalent to the existence of a normalized exhaustive pathological submeasure.

A game. For a submeasure ψ on a Boolean algebra $\mathcal{B}$ consider a game $E(\psi)$ for players I and II. Player I challenges by plaing reals $\varepsilon_i > 0$, and II responds by finding normalized ε_i-exhaustive submeasures $\phi_i \leq \phi_{i-1}$ (where $\phi_0 = \psi$). Player I also plays a decreasing sequence of countable ordinals.

I	$\alpha_1 < \omega_1$, $\varepsilon_1 > 0$		$\alpha_2 < \alpha_1$, $\varepsilon_2 > 0$		$\dots$	$\alpha_n < \alpha_{n-1}$, $\varepsilon_n > 0$	
II		$\phi_1 \leq \psi$		$\phi_2 \leq \phi_1$	$\dots$		$\phi_n \leq \phi_{n-1}$

It is required that each ϕ_i is a normalized ε_i-exhaustive submeasure. The first player who is unable to make a move loses.

Theorem 2. *There is a normalized submeasure ψ on the algebra of clopen subsets of the Cantor space K such that*

(1) *ψ is pathological,*
(2) *ψ is normalized,*
(3) *II wins the game $E(\psi)$.*

In particular, ψ dominates a normalized ε-exhaustive pathological submeasure for every $\varepsilon > 0$.

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The submeasure ψ is taken from a family of 'simple pathological submeasures' constructed by Talagrand [9]. Our construction has two independent ingredients. One is introduction of the notion of *potentially exhaustive submeasure* in §3. The other ingredient is a result about sequences of barriers of finite subsets of $\mathbb{N}$ and interval selectors (see the beginning of §2 and Theorem 2.3).

On the other hand, by Theorem 5.3, if $\varepsilon_1 < 1/2$ then the submeasure ϕ_1 used in the winning strategy for player II in $E(\psi)$ provided in our proof of Theorem 2 does not dominate a normalized exhaustive submeasure.

1. A submeasure on the countable product

Fix a sequence X_n, θ_n ($n \in \mathbb{N}$) so that X_n is a finite set and θ_n is a normalized submeasure on X_n (that is, on the power set of X_n). From this sequence we define a submeasure ψ on $X = \prod_{n=1}^{\infty} X_n$. Let

$$T_n = \prod_{i=1}^{n} X_i, \qquad T = \bigcup_{n=1}^{\infty} T_n.$$

For $t \in T$ let $|t|$ be the unique m such that $t \in T_m$, and let

$$[t] = \{x \in X : (\forall n < |t|) x(n) = t(n)\}.$$

Consider X as a topological space with the compact metric topology generated by $\{[t] : t \in T\}$ and let $\mathcal{A} = \mathrm{cl}(X)$ denote the algebra of clopen subsets of X.

For $A \subseteq X$ and $m \in \mathbb{N}$ let

$$[A]_m = \bigcup \{[t] : |t| = m \text{ and } [t] \cap A \neq \emptyset\}$$

and write $\mathcal{A}_n = \{B \in \mathcal{A} : [B]_n = B\}$. This is a finite subalgebra of $\mathcal{A}$ and by compactness we have $\mathcal{A} = \bigcup_n \mathcal{A}_n$.

Lemma 1.1. (1) *For every A and m we have $[A]_m \supseteq [A]_{m+1} \supseteq A$ for all m.*
(2) *If $A \subseteq X$ is clopen, then $A = [A]_m$ for all large enough m.*
(3) *If $[A]_m \not\subseteq [B]_m$, then $[t] \subseteq [A]_m \setminus [B]_m$ for some $t \in T_m$.* □

For $n \in \mathbb{N}$ and a subset F of

$$\Gamma = \bigcup_{n=1}^{\infty} \{n\} \times X_n,$$

define

$$\langle F \rangle = \{x \in X : (\exists (n,k) \in F) x(n) = k\},$$

$$F[n] = \{k : (n,k) \in F\}, \quad F[< n] = \bigcup_{i=1}^{n-1} \{i\} \times F[i], \quad F[\geq n] = F \setminus F[< n],$$

$$w(F) = \sum_{n=1}^{\infty} \theta_n(F[n]),$$

$$S(F) = \{n : (\exists k)(n,k) \in F\}.$$

Then $\langle F \rangle$ is the set covered by F, $w(F)$ is the weight of F and $S(F)$ is the support of F.

Lemma 1.2. *If $F \subseteq \Gamma$ is finite then $\langle F \rangle$ is a clopen subset of X, and $[\langle F \rangle]_m = \langle F \rangle$ for $m = \max(S(F)) + 1$.* □

For $t \in T$ define a submeasure ψ_t on $\mathcal{A}$ by

$$\psi_t^0(A) = \inf_{F \subseteq \Gamma} \{w(F) : A \cap [t] \subseteq \langle F \rangle, F \subseteq \Gamma \text{ and } |t| \leq \min S(F)\},$$

$$\psi_t(A) = \min(1, \psi_t^0(A)).$$

Lemma 1.3. (4) *For all $F^1, F^2 \subseteq \Gamma$ we have $\langle F^1 \cup F^2 \rangle = \langle F^1 \rangle \cup \langle F^2 \rangle$.*

(5) *If $A \in \mathrm{cl}(X)$ and for some n we have $\psi_t(A) < \varepsilon$ for all $t \in T_n$, then $\psi(A) < \varepsilon |T_n|$.*

Proof. (4) Obvious.

(5) For each $t \in T_n$ fix $F^t \subseteq \Gamma$ such that $\langle F^t \rangle \supseteq A \cap [t]$ and $w(F^t) < \varepsilon$. Let $F = \bigcup_{t \in T_n} F^t$. Then $w(F) < \varepsilon |T_n|$, and by (4) we have $\langle F \rangle \supseteq A$. □

The following is one of the key properties of ψ used in our construction.

Lemma 1.4. *Assume $t \in T$, $m \geq |t|$, $B \in \mathcal{A}_m$, and $C \in \mathcal{A}$. Then*

$$\psi_t(B \cap C) \geq \min(\psi_t(B), \min\{\psi_s(C) : |s| = m, [s] \cap B \neq \emptyset \text{ and } s \supseteq t\}).$$

In particular, if $\psi_s(C) \geq \psi_t(B)$ for all $s \in T_m$ such that $s \supseteq t$, then $\psi_t(B \cap C) = \psi_t(B)$.

Proof. Let $\delta = \min(\psi_t(B), \min\{\psi_s(C) : |s| = m, [s] \cap B \neq \emptyset\})$. Fix F such that $|t| < \min(S(F))$ and $w(F) < \delta$. We need to check that $B \cap C \not\subseteq \langle F \rangle$. Since $w(F[< m]) < \delta$, $\langle F[< m] \rangle \not\supseteq B$. Since both B and $\langle F[< m] \rangle$ belong to $\mathcal{A}_m$, by (3) of Lemma 1.1 there is $s \in T_m$, $s \supseteq t$ such that $[s] \subseteq B \setminus \langle F[< m] \rangle$. Since $\psi_s(C) \geq \delta$, we have

$$(C \cap [s]) \setminus \langle F[\geq m] \rangle \neq \emptyset.$$

But $C \cap [s] \subseteq B$ and therefore $\langle F \rangle \not\supseteq B \cap C$. Since F was arbitrary, we have $\psi_t(B \cap C) \geq \delta$. □

For $U \subseteq X$ and $n \in \mathbb{N}$ define

$$(U)_n = ([U^{\complement}]_n)^{\complement} = \bigcup \{[t] : |t| = n, [t] \subseteq U\}.$$

By (1) of Lemma 1.1 for $m \leq n$ we have

$$(U)_m \subseteq (U)_n \subseteq U.$$

Lemma 1.5. *Assume $m \leq n$, $\langle F[< n] \rangle \supseteq (U)_n$, and $X_i \not\subseteq F[i]$ for all $i \in [m, n)$. Then $\langle F[< m] \rangle \supseteq (U)_m$.*

Proof. Assume otherwise, that $(U)_m \setminus \langle F[< m] \rangle \neq \emptyset$. By (3) of Lemma 1.1, there is $t \in T_m$ such that $[t] \subseteq (U)_m \setminus \langle F[< m] \rangle$. For each $i \in [m, n)$ fix $s(i) \in X_i \setminus F[i]$, and let $s(i) = t(i)$ if $i < m$. Then $[s] \subseteq [t]$, therefore $[s] \subseteq (U)_n$. By the choice, $[s] \cap \langle F[< n] \rangle = \emptyset$, and therefore $\langle F[< n] \rangle \not\supseteq (U)_n$. □

Lemma 1.6. *If $U \subseteq X$ is open and $\psi((U)_n) < 1$ for all n, then there is an $F \subseteq \Gamma$ such that $\langle F \rangle \supseteq U$ and $w(F[< n]) < 1$ for all n.*

Proof. For $n \in \mathbb{N}$ let

$$\mathcal{P}_n = \{F \subseteq \Gamma : \langle F \rangle \supseteq (U)_n, S(F) \subseteq n \text{ and } w(F[< n]) < 1\}.$$

If $F \in \mathcal{P}_n$, then $w([F < k]) < 1$ for all $k \leq n$, and in particular $X_k \not\subseteq F[k]$. Therefore Lemma 1.5 implies that $\langle F[< m] \rangle \supseteq (U)_m$, and therefore $F[< m] \in \mathcal{P}_m$ whenever $m \leq n$. ence $\bigcup_{n=1}^{\infty} \mathcal{P}_n$ is a finitely branching tree with respect to the ordering $F \preceq G$ if $F = G[< n]$ for some n. By the assumption, each $\mathcal{P}_n$ is nonempty, hence by König's lemma there exists F such that $F[< n] \in \mathcal{P}_n$ for all n, and this concludes the proof. □

2. Barriers and interval selectors

Let Fin denote the family of all finite subsets of $\mathbb{N}$. For $F \subseteq \mathrm{Fin}$ we say that g is an *interval selector for* F if g is a function with domain F such that

(6) $g(s) = [m, n)$ for some $m < n$ in s,
(7) $g(s) \cap g(t) = \emptyset$ for all distinct s and t in F.

The following lemma is implicit in [8].

Lemma 2.1. *If $F \subseteq \mathrm{Fin}$ is such that $|F| \leq \min\{|s| : s \in F\}| - 1$, then F has an interval selector.*

Proof. Assume $|F| = N$ and $\min\{|s| : s \in F\} \geq N + 1$. Let $F = \{s_1, \dots, s_N\}$ be an enumeration and let $\{m^i_1, \dots, m^i_{N+1}\}$ be an increasing enumeration of the first $N + 1$ elements of s_i. Re-enumerate F so that $n^i_{i+1} \leq n^{i+1}_{i+1}$ for $i = 1, \dots, N - 1$. Then $g(s_i) = [n^i_i, n^i_{i+1})$ is an interval selector for F. □

For s and t in Fin we write

$$s \sqsubseteq t$$

to denote that $s = t \cap m$ for some $m \in \mathbb{N}$, and we write

$$s \uplus t$$

to denote $s \cup t$ while at the same time asserting that $\max(s) < \min(t)$.

Definition 2.2. *A family $\mathcal{F} \subseteq \mathrm{Fin}$ is* well-founded *if there is no infinite $\sqsubseteq$-increasing sequence included in $\mathcal{F}$. A family $\mathcal{F} \subseteq \mathrm{Fin}$ is a* barrier *if for every infinite $A \subseteq \mathbb{N}$ there is $m \in \mathbb{N}$ such that $A \cap m \in \mathcal{F}$.*

Every barrier of finite rank is included in $[\mathbb{N}]^m = \{s \subseteq \mathbb{N} \,|\, |s| = m\}$ for some $m \in \mathbb{N}$. If $\mathcal{F}_i$ $(i \in \mathbb{N})$ are barriers, then

$$\triangledown_{i=1}^{\infty}\mathcal{F}_i = \{s \in \mathrm{Fin} \,|\, s \in \mathcal{F}_{\min(s)}\}$$
$$\mathcal{F}_1 \sqcup \mathcal{F}_2 = \{s_1 \uplus s_2 | s_1 \in \mathcal{F}_1, s_2 \in \mathcal{F}_2\}$$
$$\bigsqcup_{i=1}^{k} \mathcal{F}_i = \{s_1 \uplus s_2 \uplus \dots \uplus s_k | s_i \in \mathcal{F}_i, i \leq k\}$$

are barriers as well. As a matter of fact, every barrier can be obtained from $[\mathbb{N}]^1$ by iterating the $\triangledown$ operation. See [7] for more on barriers.

For barriers $\mathcal{F}_1$ and $\mathcal{F}_2$ and $m \in \mathbb{N}$ we write

$$\mathcal{F}_2 \preceq_m \mathcal{F}_1$$

if for every $s \in \mathcal{F}_1$ such that $\min(s) \geq m$ there is $t \in \mathcal{F}_2$ such that $t \subseteq s$.

Consider the set $P \subseteq (\omega_1 \times \mathbb{N})^{<\mathbb{N}}$ consisting of all t such that $\langle t_0(i) | i < |t|\rangle$ is a decreasing sequence of ordinals. Ordered by the end-extension, $\sqsubseteq$, it is a well-founded tree of rank ω_1. Let $\mathbf{B}$ be the family of all barriers on $\mathbb{N}$.

Theorem 2.3. *There is a function* $\xi\colon P \to \mathbf{B}$ *such that for every* $t \in P$ *and* $F_i \subseteq \xi(t \restriction i)$ *satisfying* $|F_i| \le t_1(i-1)$ *for* $i \le |t|$ *the set*

$$\bigcup_{i=1}^{|t|} F_i$$

has an interval selector.

Proof. We construct ξ by recursion. Let $\xi(\langle 0, m\rangle) = [\mathbb{N}]^{m+1}$. Assume $\xi(t)$ is defined for all t such that $t_0(0) < \alpha$. Let $\{\mathcal{F}_i | i \in \mathbb{N}\}$ enumerate all $\{\xi(t) | t_0(0) < \alpha\}$, and let

$$\mathcal{F} = \triangledown_{n=1}^{\infty}\left(\bigsqcup_{i=1}^{n} \mathcal{F}_i\right).$$

Note that for every i there is m such that $\mathcal{F}_i \preceq_m \mathcal{F}$.

Define $\xi(\langle \alpha, m\rangle) = [\mathbb{N}]^{m+1} \sqcup \mathcal{F}$. If $t_0(0) = \alpha$ and $|t| \ge 2$, let $\bar{t} \in P$ be such that

$$\begin{aligned} |\bar{t}| &= |t| - 1 \\ \bar{t}_0(i) &= t_0(i+1), \text{ for } 1 \le i < |t|, \\ \bar{t}_1(i) &= t_1(i+1) + t_1(0), \text{ for } 1 \le i \le |t|. \end{aligned}$$

Let $\bar{m} \in \mathbb{N}$ be minimal such that $\xi(\bar{t} \restriction i) \preceq_{\bar{m}} \mathcal{F}$ for all $i \le |\bar{t}|$. Define

$$\xi(t) = \{s | s \setminus \bar{m} \in \xi(\bar{t})\}.$$

This defines $\xi\colon P \to \mathbf{B}$. Note that if $s \in \xi(t)$ and $|s \cap k| < t_1(0)$, then $s \setminus k$ includes a member of $\mathcal{F}$.

For $t \in P$ we need to prove that if $F_i \subseteq \xi(t \restriction i)$ are such that $|F_i| = t(i-1)$ for $i \le |t|$, then $\bigcup_i F_i$ has an interval selector. The proof is by induction on $\alpha = t_0(0)$. Assume that the statement holds for all $\beta < \alpha$ and fix t, F_i $(i \le |t|)$. If $|t| = 1$, then F_1 has an interval selector by Lemma 2.1. Assume $|t| \ge 2$, and let $\bar{m}^i$, $\bar{t}^i$ be as in the definition of $\xi(t \restriction i)$, for $i \ge 2$. Define

$$\begin{aligned} F_0' &= \{s \in F_1 | \, |s \cap \bar{m}^2| \ge t_1(0)\} \\ F_1' &= F_2 \cup \{s \setminus \bar{m}^2 | s \in F_1 \setminus F_1'\} \\ F_{i-1}' &= F_i, \text{ for } i \ge 3. \end{aligned}$$

By Lemma 2.1, F_0' has an interval selector g_1 such that $g_1(s) \subseteq \bar{m}^2$ for all s. For $s \in F_0 \setminus F_0'$ the set $s \setminus \bar{m}^2$ includes an element of $\xi(\bar{t}^2)$. Therefore $F_1' \subseteq \xi(\bar{t}^2)$, and it has size at most $t_1(0) + t_1(1) = (\bar{t}^2)_1(0)$. Hence if $\bar{t} = \bar{t}_{|t|}$, then $F_i' \subseteq \xi(\bar{t} \restriction i)$ $(i < |\bar{t}|)$ satisfy the conditions and by the inductive assumption this family has an interval selector, g_2. But $\bar{m}_i \ge \bar{m}_2$ for all i, therefore $g_2(s) \cap \bar{m}_2 = \emptyset$ for all s, hence $g_1 \cup g_2$ is an interval selector for $\bigcup_i F_i$. □

3. Potentially exhaustive submeasures

If ψ_1, ψ_2 are submeasures on a Boolean algebra $\mathcal{B}$, then $\phi = \psi_1 \wedge \psi_2$ is a submeasure defined by

$$\phi(A) = \inf\{\psi_1(B) + \psi_2(C) : A = B \cup C\}.$$

If $\mathcal{S}$ is a subset of a Boolean algebra $\mathcal{B}$ and $f\colon \mathcal{S} \to [0,1]$ then $\phi = \gamma[\mathcal{S}, f]$ is a submeasure defined by

$$\phi(C) = \min\left(1, \inf_{F\subseteq S}\left\{\sum_{B\in F} f(B) : F \subseteq \mathcal{A} \text{ and } \bigcup F \supseteq C\right\}\right).$$

If $f(B) = \varepsilon$ for all $B \in \mathcal{S}$, write $\gamma[\mathcal{S}, \varepsilon]$ instead of $\gamma[\mathcal{S}, f]$.

Definition 3.1. *A submeasure ϕ on $\mathcal{A}$ is* potentially exhaustive *if for every $B \in \mathcal{A}$, every $\delta > 0$ and every sequence A_n ($n \in \mathbb{N}$) of pairwise disjoint sets in $\mathcal{A}$ there is $B' \subseteq B$ in $\mathcal{A}$ such that $\phi(B') = \phi(B)$ and*

$$\phi(B' \cap A_i) < \delta.$$

for all but finitely many i.

Small sets. Let ψ be a submeasure on the algebra $\mathcal{A}$ of clopen subsets of $X = \prod_{i=1}^{\infty} X_i$, where X_i are finite sets and the topology on X is the product topology. Recall that

$$\mathcal{A}_m = \{B \in \mathcal{A} : [B]_m = B\}.$$

An $A \in \mathcal{A}$ is m-small (with respect to ψ) if for every $B \in \mathcal{A}_m$ such that $\psi(B) = 1$ we have $\psi(B \setminus A) = 1$. If $m < n$ then an $A \in \mathrm{cl}(X)$ is *(m,n)-small* (with respect to ψ) if $[A]_n$ is m-small (with respect to ψ). If $s = \{m_1, \dots, m_k\} \in \mathrm{Fin}$ (with $m_1 < m_2 < \dots < m_k$), then $A \in \mathrm{cl}(X)$ is *s-small* (with respect to ψ) if $A = \bigcap_{i=1}^{k-1} A_i$ and each A_i is (m_i, m_{i+1})-small for $i \leq k-1$. If $\mathcal{F} \subseteq \mathrm{Fin}$, then $A \in \mathrm{cl}(X)$ is *$\mathcal{F}$-small* (with respect to ψ) if there is $s \in \mathcal{F}$ such that A is s-small. Let $\mathcal{S}_{\mathcal{F}}$ denote the family of all $\mathcal{F}$-small sets. We will suppress writing 'with respect to ψ' whenever the choice of ψ is clear from the context.

Lemma 3.2. *If $F \subseteq \mathrm{cl}(X)$, each $A \in F$ is s_A-small for some $s_A \in \mathrm{Fin}$, and $\{s_A : A \in F\}$ has an interval selector g, then $\psi(X \setminus \bigcup F) = 1$.*

Proof. Enumerate $F = \{A_1, \dots, A_k\}$ so that if we write $s_i = s_{A_i}$, $k_i = \min(g(s_i))$, $l_i = \max(g(s_i))$, then $k_i < l_i < k_{i+1}$ for all i. Since A_i is (k_i, l_i)-small, we can recursively find $E_i \in \mathcal{A}_{l_i}$ such that $\psi(E_i) = 1$, $E_i \cap A_i = \emptyset$, and $E_{i+1} \subseteq E_i$ for all i. Then $X \setminus \bigcup F \supseteq E_k$ and the conclusion follows. □

Lemma 3.3. *Assume ψ is a potentially exhaustive submeasure on $\mathrm{cl}(X)$, where $X = \prod_{i=1}^{\infty} X_i$ and all X_i are finite. Then for every sequence A_n ($n \in \mathbb{N}$) of pairwise disjoint clopen subsets of X, every $m \in \mathbb{N}$, and every $\varepsilon > 0$ there is $C \in \mathrm{cl}(X)$ such that*

(8) *C is m-small, and*
(9) *$\psi(A_i \setminus C) < \varepsilon$ for all but finitely many i.*

Proof. Let B_i ($i \leq k$) be an enumeration of all sets in $\mathcal{A}_m$ such that $\psi(B_i) = 1$. For each i find $B_i' \subseteq B_i$ such that $\psi(B_i' \cap A_j) < \varepsilon/k$ for all but finitely many j and $\psi(B_i') = 1$. Let $B = \bigcup_{i=1}^{k} B_i'$. Then $\psi(B \cap B_i) = 1$ for all $i \leq k$, and $\psi(B \cap A_j) < \varepsilon$ for all but finitely many i. Therefore $C = X \setminus B$ is as required. □

Lemma 3.4. *Under the assumptions of Lemma 3.3, for every infinite $Z \subseteq \mathbb{N}$ and every $\delta > 0$ we can find an increasing sequence n_i of elements of Z and sets $C_i \in \mathcal{A}_{n_{i+1}}$ such that for all i*

(10) *$C_{i+1} = D \cap C_i$ for some (n_i, n_{i+1})-small D,*

(11) $\psi(A_j \setminus C_i) < \delta$ *for all but finitely many* j.

Proof. The construction is by recursion. Assume C_i, n_i $(i \leq i_0)$ are chosen so that both (10) and the following strengthening of (11).

(12) $\psi(A_j \setminus C_i) < 2^{-i-1}\delta$

hold for $i \leq i_0$ and all but finitely j. Apply Lemma 3.5 to the sequence $A'_j = A_j \cap C_{i_0}$ to obtain C satisfying (10) and (12) for all but finitely many A'_j. Then $C_{i_0+1} = C_{i_0} \cap C$ still satisfies these conditions. Pick n_{i_0+1} so that $C_{i_0+1} \in \mathcal{A}_{n_{i_0+1}}$.

If C_i and n_i are constructed in this manner, then $\psi(A_j \setminus C_i) \leq \sum_{k=0}^{i} 2^{-k-1}\delta < \delta$ for all i and all but finitely many j. □

Lemma 3.5. *Assume* ψ *is a potentially exhaustive submeasure on* $\mathrm{cl}(X)$, *where* $X = \prod_{i=1}^{\infty} X_i$. *If* $\mathcal{F} \subseteq \mathrm{Fin}$ *is a barrier, then the submeasure* $\phi = \psi \wedge \gamma[\mathcal{S}_{\mathcal{F}}, \varepsilon]$ *is* ε*-exhaustive.*

Proof. Fix a sequence A_n $(n \in \mathbb{N})$ of pairwise disjoint sets in $\mathrm{cl}(X)$ and $\delta > 0$. Let Z be an infinite subset of $\mathbb{N}$ such that every infinite subset of Z has an initial segment in $\mathcal{F}$. Find n_i, and C_i as in Lemma 3.4. There is k such that $\{n_1, \dots, n_k\} \in \mathcal{F}$, and therefore $\phi(C_k) \leq \varepsilon$. But $\psi(A_j \setminus C_k) < \delta$ for all large enough j, hence $\phi(A_j) < \varepsilon + \delta$ for all such A_j. Since $\delta > 0$ was arbitrarily small, the submeasure is ε-exhaustive. □

Let X_n $(n \in \mathbb{N})$ be finite sets, let θ_n be a submeasure on X_n, and let tree T and submeasures ψ_t $(t \in T)$ be as defined in §1.

Definition 3.6. *For a submeasure* θ *on* X *let* c_θ *be a function on* $\mathcal{P}(X)$ *defined by*

$$c_\theta(A) = \min\left\{m : (\exists B_1, \dots, B_m) A = \bigcup_{i=1}^{m} B_i \text{ and } \max_i \theta(B_i) < 1\right\}.$$

We write c_θ *for* $c_\theta(X)$.

Note that c_θ is a submeasure. We will write c_n instead of c_{θ_n}.

Lemma 3.7. *We have* $c_{\psi_t} \geq \min_{n \geq |t|} c_n$, *and* ψ_t *is normalized for every* $t \in T$.

Proof. Assume $k < \min_{n \geq |t|} c_n$ and $A_1, \dots, A_k \subseteq [t]$ are such that $\psi_t(A_i) < 1$ for all $i \leq k$. We need to check that

$$[t] \setminus \bigcup_{i=1}^{k} A_i \neq \emptyset.$$

For each $i \leq k$ fix $F_i \subseteq \Gamma$ such that $\langle F_i \rangle \supseteq A_i$ and $w(F_i) < 1$, and $\min(S(F_i)) \geq |t|$. Since for each $n \geq |t|$ we have $k < c_n$ we can pick $x(n) \in X_n \setminus \bigcup_{i=1}^{k} F_i[n]$. Let $x(n) = t(n)$ for $n < |t|$, and then $x \in [t] \setminus \bigcup_{i=1}^{k} A_i$.

Now we check that ψ_t is normalized. By definition, $\psi_t(X) \leq 1$. Since $c_n \geq 2$ for all n, we have $c_{\psi_t} \geq 2$ and therefore $\psi_t(X) \geq 1$. □

We record a related fact with a very similar proof.

Lemma 3.8. *If* $F \subseteq \Gamma$ *is such that* $c_n(\langle \{n\} \times F[n] \rangle) < c_n(X_n)$ *for all* $n \geq |t|$, *then* $[t] \setminus \langle F \rangle \neq \emptyset$ *for every* t *such that* $|t| < \min(S(F))$. □

Lemma 3.9. *If* $B \in \mathcal{A}_m$, $F \subseteq \Gamma$, $\langle F[< m] \rangle \not\supseteq B$ *and* $c_n(\langle F[n] \rangle) < c_n(X_n)$ *for all* $n \geq m$, *then* $\langle F \rangle \not\supseteq B$.

Proof. If B and F are as above, then B and $\langle F[< m]\rangle$ belong to $\mathcal{A}_m$. By (3) of Lemma 1.1 there is $s \in T_m$ such that $[s] \subseteq B \setminus \langle F[< m]\rangle$. Since $c_n(\langle F[n]\rangle) < c_n(X_n)$ for all $n \geq m$, by Lemma 3.8 we have $[s] \setminus \langle F[\geq m]\rangle \neq \emptyset$. □

Lemma 3.10. *Assume θ_n is a submeasure on X_n such that $c_n \geq 3$ for all n. For every $t \in T$, every sequence A_n ($n \in \mathbb{N}$) of pairwise disjoint clopen subsets of X and every $\varepsilon > 0$ there is a clopen $B \subseteq [t]$ such that $\psi_t(B) = 1$ and that $\psi_t(B \cap A_n) < \varepsilon$ for all but finitely many n.*

Proof. First assume that $\psi_t(\bigcup_{i=1}^m A_i) = 1$ for some m. Then $B = \bigcup_{i=1}^m A_i \cap [t]$ satisfies $\psi_t(B) = 1$ and $\psi_t(B \cap A_n) = 0$ for all but finitely many n.

So we may assume that $\psi_t(\bigcup_{i=1}^m A_i) < 1$ for all m. Since each level of T_n is finite, for every n there is a large enough m_0 such that for all $m \geq m_0$ we have $\left(\bigcup_{i=0}^m A_i\right)_n = \left(\bigcup_{i=0}^\infty A_i\right)_n$. So $U = \bigcup_{i=0}^\infty A_i$ satisfies $\psi((U)_n) < 1$ for all n. By Lemma 1.6, there is $F \subseteq \Gamma$ such that $w(F[< n]) < 1$ for all n and $\langle F\rangle \supseteq \bigcup_{i=1}^\infty A_i$. Let m be such that $w(F[\geq m]) < \varepsilon$, and let $B = [t] \setminus \langle F[< m]\rangle$. Then $B \cap A_i \subseteq \langle F[\geq m]\rangle$, hence $\psi_t(B \cap A_i) < \varepsilon$ for all i. Since $c_n \geq 3$ for all n, by Lemma 3.7 we have $c_{\psi_t} \geq 3$. Since $w(F[< m]) < 1$, we have $\psi_t(B) = 1$. □

Lemma 3.11. *Using the notation and assumptions of Lemma 3.10, ψ is potentially exhaustive.*

Proof. Fix a sequence A_n ($n \in \mathbb{N}$) of pairwise disjoint sets in $\mathrm{cl}(X)$ and $B \in \mathrm{cl}(X)$. Let n be such that $B \in \mathcal{A}_n$. By Lemma 3.10, for each $t \in T_n$ we can find $B_t \subseteq B \cap [t]$ such that $\psi_t(B_t \cap A_i) < \varepsilon/|T_n|$ for all but finitely many i and $\psi_t(B_t) = 1$. Let $B' = \bigcup_{t \in T_n} B_t$. Then for all but finitely many i we have $\psi(B' \cap A_i) < \varepsilon$ by (5) of Lemma 1.3. By Lemma 1.4 applied with $C = B'$ we have $\psi(B') = \psi(B)$, as required. □

4. Proof of Theorem 2

We will construct a sequence θ_m ($m \in \mathbb{N}$) of submeasures such that with ψ as in §1 player II wins $E(\psi)$. Although essentially any sequence of 'increasingly pathological' submeasures $\{\theta_n\}$ would do, we will provide a concrete (and well-known) example. Fix a function $\sigma\colon \mathbb{N} \to \mathrm{Fin}$ such that $|\sigma(n)| \geq 2n$ for all n and the sets $\sigma(m)$ are pairwise disjoint. Let $I^\sigma = \bigcup_m \sigma(m)$ and

$$X^\sigma = \{C \subseteq I^\sigma : |C \cap \sigma(n)| \geq \sigma(n)/2 \text{ for all } n\}.$$

Write

$$[Z]^{\geq k} = \{s \subseteq Z : |s| \geq k\}.$$

If $X_{n,m} = [m]^{\geq m/2}$, we can identify X^σ with $\prod_{n=1}^\infty X_{n,\sigma(n)}$. Let $\theta_{n,m}$ be a submeasure on $X_{n,m}$ defined by

$$\theta_{n,m}(A) = \frac{1}{n} \min\{|F| : F \subseteq \sigma(n) \text{ and } (\forall a \in A)(\exists k \in F) k \in a\}.$$

Lemma 4.1. *For all $m \geq 2n$ we have $c_{n,m} = \lceil (m+1)/2(n-1)\rceil$.*

Proof. If $l = \lceil (m+1)/2(n-1)\rceil$ and F_i ($i \leq l$) are pairwise disjoint subsets of m such that $|F_i| = n-1$, then $F = \bigcup_{i=1}^l F_i$ has size $l(n-1) > m/2$, and therefore every $a \in X_{n,m}$ intersects F. Therefore if $A_i = \{a \in X_{n,m} : a \cap F_i \neq \emptyset\}$ then $\theta_{n,m}(A_i) = (n-1)/n$, and $\bigcup_{i=1}^l A_i = X_{n,m}$, so $c_{n,m} \leq l$.

On the other hand, assume $l < \lceil (m+1)/2(n-1) \rceil$ and A_i $(i \leq l)$ are subsets of $X_{n,m}$, each of submeasure < 1. Let $F_i \subseteq m$ be of size $< n$ and such that $A_i \subseteq \{a : a \cap F_i \neq \emptyset\}$. Then $\bigcup_{i=1}^{l} F_i$ has size at most $m/2$, and therefore $\bigcup_{i=1}^{l} A_i$ does not cover $X_{n,m}$, so $c_{n,m} > l$. □

Let ψ^σ be the submeasure $\psi_{\langle\rangle}$ obtained from the sequence $\theta_{n,\sigma(n)}, X_{n,\sigma(n)}$ as in §1. This is the 'simple example of a pathological submeasure' defined in [9].

Lemma 4.2. *If $\sigma(n) > 6n$ for all n, then ψ^σ is potentially exhaustive.*

Proof. By Lemma 4.1, $c_n \geq \lceil (\sigma(n)+1)/2(n-1) \rceil \geq 3$ for all n, so the conclusion follows by Lemma 3.11. □

Proof of Theorem 2. We will take X for our copy of the Cantor set, and ψ^σ as defined above for ψ (here $\sigma(i) > 6i$ for all i). We will describe a winning strategy for player II. For a position in $E(\psi)$ in which it is II's turn, after I has played $\alpha_1, \varepsilon_1, \dots, \alpha_k, \varepsilon_k$, let $t^k \in (\omega_1 \times \mathbb{N})^k$ be such that $t^k_0(i-1) = \alpha_i$ and $t^k_1(i-1) > 1/\varepsilon_i$ for $1 \leq i \leq k$. Player II responds by playing

$$\phi_k = \psi \wedge \bigwedge_{i=1}^{k} \gamma[\mathcal{S}_{\xi(t^k \restriction i)}, \varepsilon_i/2],$$

Note that $t^1 \sqsubseteq t^2 \sqsubseteq \dots \sqsubseteq t^k$, and therefore $\phi_1 \geq \phi_2 \geq \dots \geq \phi_k$, so this is a legal strategy for II.

We prove that this is a winning strategy.

To prove that $\phi_k(X) = 1$ we need to check that if $X = A \cup \bigcup_{j=1}^{k} F_j$ and $F_j \subseteq \mathcal{S}_{\xi(t^k \restriction j)}$, then $\psi(A) + \sum_{j=1}^{k} \varepsilon_j |F_j| \geq 1$. Assume $\sum_{j=1}^{k} \varepsilon_j |F_j| < 1$. Therefore $|F_j| < 1/\varepsilon_j$ for all j. By Theorem 2.3 there is an interval selector g for $\{s_B : B \in \bigcup_{j=1}^{k} F_j\}$. By Lemma 3.2, $\psi(X \setminus \bigcup_{j=1}^{k} F) = 1$. Therefore $\psi(A) = 1$ and $\phi_k(X) = 1$.

The submeasure ψ is potentially exhaustive by Lemma 4.2. Since each $\xi(t^k \restriction i)$ is a barrier, ϕ_i is ε_i-exhaustive by Lemma 3.5. This concludes the proof. □

Variations on $E(\psi)$. Let $E^\infty(\psi)$ be a version of $E(\psi)$ in which I is not required to play ordinals. Therefore I is always able to make a move, and if II wins, $\psi_\infty = \inf_n \psi_n$ is a normalized exhaustive pathological submeasure. Let $E'(\psi)$ be another modification of $E(\psi)$, obtained by dropping only the requirement that $\alpha_1 < \omega_1$. Both games are open and therefore determined. If there is ψ such that player I does not have a winning strategy for $E'(\psi)$ and there is a measurable cardinal (or $x^\#$ exists, where x is a real coding ψ) then classical argument using indiscernibles (see [6]) shows that II wins $E^\infty(\psi)$. Therefore under these assumptions there is a normalized exhaustive pathological submeasure below ψ. As a matter of fact, it is not difficult to see that Kunen–Martin theorem (see [4]) implies a bit more. Let $E''(\psi)$ be a modification of $E(\psi)$ in which I is required to play ordinals less than ω_2. Assuming that I does not have a winning strategy in this game, there is an infinite decreasing sequence of normalized pathological submeasures $\{\phi_i\}$ such that each ϕ_i is ε_i-exhaustive, hence $\inf_i \phi_i$ is a normalized pathological exhaustive submeasure.

Talagrand ([9]) proved that if there is a normalized pathological exhaustive submeasure, then there is one on some atomless subalgebra $\mathcal{B}$ of $\mathrm{cl}(X^\sigma)$ dominated by $\psi^\sigma \restriction \mathcal{B}$, for some $\sigma \colon \mathbb{N} \to \mathrm{Fin}$ such that $|\sigma(n)| \geq 2n$ for all n. This suggests a variation of $E(\psi)$ such that the existence of a winning strategy for II for some σ is equivalent to the existence of a normalized pathological exhaustive submeasure.

However, results of the next section diminish the hope that these observations may lead to the solution of Maharam's problem.

5. Limitations

We now prove three results giving some limitations to what kind of submeasure can be constructed by using methods exploited in this paper. The assumptions are not optimal and theorems can easily be strengthened, but already the versions presented here show that the methods introduced above alone cannot lead to the solution of Maharam's problem, unless supplemented with some substantial new ideas.

Theorem 5.1. *Assume $\sigma(n) \geq 2n$ for all n. If $\varepsilon_n > 0$, $\lim_n \varepsilon_n = 0$, and each $\mathcal{F}_n$ ($n \in \mathbb{N}$) includes a barrier, then the submeasure $\phi = \psi^\sigma \bigwedge_{n=1}^\infty \gamma[\mathcal{S}_{\mathcal{F}_n}, \varepsilon_n]$ vanishes.*

Theorem 5.2. *For every $k \in \mathbb{N}$ there is a $1/k$-exhaustive normalized submeasure on $\mathcal{A}$ such that for every normalized $\theta \leq \phi$ and every $\varepsilon < 1/k$, θ is not ε-exhaustive.*

Theorem 5.3. *Assume $\mathcal{F}$ is well-founded and contains a barrier, and $\phi = \gamma[\mathcal{S}_\mathcal{F}, \varepsilon]$ is computed using ψ^σ for σ satisfying $\sigma(n) \geq 2n$ for all n. If $\phi' \leq \phi$ is exhaustive, then $\phi'(X) \leq 2\varepsilon$. In particular, if $\varepsilon < 1/2$ then ϕ does not dominate a normalized exhaustive submeasure.*

We will be using the notation and terminology of §1. For $Y \subseteq X_m$ write

$$\bar{Y} = \{x \in X | x(m) \in Y\}.$$

Note that (assuming $\theta_n(X_n) \geq 1$ for all n)

$$\psi(\bar{Y}) = \theta_m(Y).$$

Lemma 5.4. *If $m \in \mathbb{N}$ and $Y_i \subseteq X_{m+i}$ ($i \in \mathbb{N}$) and $\theta_{m+i}(Y_i) = 1$ for all i, then the sets*

$$A_k = X \setminus \bigcap_{i=1}^k \bar{Y}_i$$

are m-small for every k.

Proof. By induction on k. For $k = 1$, for every $t \in T_m$ we have $\psi_t(X \setminus A_1) = \theta_m(Y_1) = 1$. Lemma 1.4 implies $\psi(B \setminus A_1) = \psi(B)$ for every $B \in \mathcal{A}_m$. Since $X \setminus A_{k+1} = (X \setminus A_k) \cap \bar{Y}_{k+1}$ and $\psi_t(\bar{Y}_{k+1}) = 1$ for all $t \in T_{m+k+1}$, we have $\psi(B \setminus A_{k+1}) = \psi((B \setminus A_k) \cap \bar{Y}_{k+1}) = \psi(B)$ by Lemma 1.4. □

Lemma 5.5. *Assume ψ is obtained from X_n, θ_n ($n \in \mathbb{N}$) such that for infinitely many n there is a partition $X_n = X_n^0 \dot\cup X_n^1$ satisfying $\theta_n(X_n^0) = \theta_n(X_n^1) = 1$. Then for every $s \in \mathrm{Fin}$ of size at least 2 there is an s-small set A such that $X \setminus A$ is $(\min(s), \max(s))$-small.*

Proof. Let $s = \{n_1, \dots, n_k\}$, where $n_1 < n_2 < \cdots < n_k$ and $k \geq 2$. By Lemma 1.4, each $\bar{X}_{n_i}^0$ is (n_i, n_{i+1})-small, so $A = \bigcap_{i=1}^{k-1} \bar{X}_{n_i}^0$ is s-small. By Lemma 5.4, $X \setminus A = \bigcup_{i=1}^{k-1} \bar{X}_{n_i}^1$ is (n_1, n_k)-small. □

Lemma 5.6. *Under the assumptions of Lemma 5.5, for every $m \in \mathbb{N}$ there is a sequence A_n ($n \in \mathbb{N}$) of pairwise disjoint clopen subsets of X such that each $A_n^{\complement}$ is m-small.*

Proof. Let n_i ($i \in \mathbb{N}$) be the increasing enumeration of all $n_i > m$ such that $X_{n_i} = X_i^0 \dot\cup X_i^1$ so that $\theta_{n_i}(X_i^0) = \theta_{n_i}(X_i^1) = 1$ for all i. Let $A_k = \bigcap_{i=1}^{k-1} \bar{X}_i^0 \cap \bar{X}_k^1$. Then X_l^0 separates A_l from A_k for $l < k$, so the sets are pairwise disjoint. By Lemma 5.4, each $A_k^{\complement} = \bigcup_{i=1}^{k-1} \bar{X}_i^1 \cup \bar{X}_i^0$ is m-small. □

Lemma 5.7. *If $\sigma(n) \geq 2n$ then $\psi = \psi^\sigma$ satisfies the assumptions of Lemma 5.5.*

Proof. Fix m and $i = i_m \in \sigma(m)$ and let

$$X_m^0 = \{x \subseteq \sigma(m) : i \in x\}.$$

Then $\theta_m(X_m^0) = 1$ because if $u \subseteq \sigma(m)$ has size less than n, then there is $x \in X_m^0$ such that $x \cap u = \emptyset$. Similarly, $X_m^1 = X_m \setminus X_m^0$ satisfies $\theta_m(X_m^1) = 1$. □

Proof of Theorem 5.1. Fix $\varepsilon > 0$ and find an infinite $C \subseteq \mathbb{N}$ so that $\sum_{i \in C} \varepsilon_i < \varepsilon$. To simplify the notation, assume $C = \mathbb{N}$. We will find $k \in \mathbb{N}$ and $A_i \in \mathcal{S}_{\mathcal{F}_i}$ ($i \leq k$) so that $X = \bigcup_{i=1}^k A_i$.

Fix an infinite $D \subseteq \mathbb{N}$ such that $\mathcal{F}_1$ is a barrier on D. Recursively find $n_1 < n_2 < \dots$ and $s_i \in \mathcal{F}_i$ ($i \geq 2$) so that $n_i \in D$, $n_i < \min(s_i)$ and $\max(s_i) < n_{i+1}$ for all i. Then for some k we have $s_1 = \{n_i | i \leq k+1\} \in \mathcal{F}_1$. By Lemma 5.5 and Lemma 5.7, for $2 \leq i \leq k$ there is an s_i-small A_i such that $X \setminus A_i$ is (n_i, n_{i+1})-small. Therefore $A_1 = X \setminus \bigcup_{i=2}^k A_i$ is s_1-small, and $A_1, \dots, A_k$ are as required.

So $\phi(X) \leq \sum_{i=1}^k \varepsilon_i < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $\phi(X) = 0$. □

Proof of Theorem 5.3. In this proof, s-small means s-small with respect to ψ. Assume $\phi' \leq \gamma[\psi, \varepsilon]$ is exhaustive. We claim that if $s \in \mathrm{Fin} \setminus \{\emptyset\}$ and $B \in \mathcal{A}$ is s-small, then $\phi'(B) \leq \varepsilon$. Assume not. Since $\mathcal{F}$ is a barrier and every $s \in \mathcal{F}$ satisfies the claim, there is $t \in \mathrm{Fin}$ and a t-small B such that $\phi'(B) > \varepsilon$ yet for every $s \supsetneq t$ every s-small C satisfies $\phi'(C) \leq \varepsilon$. Let $m = \max(t)$. By Lemma 5.6 and Lemma 5.7, there is a sequence of pairwise disjoint sets A_n with m-small complements. Since $B \cap A_n^{\complement}$ is $t \cup \{k_n\}$ small for a large enough k_n and ϕ' is exhaustive, we have

$$\phi'(B) \leq \liminf_n (\phi'(B \cap A_n^{\complement}) + \phi'(B \cap A_n)) \leq \varepsilon.$$

If $A_1 \in \mathcal{A}_k$ then both A_1 and $A_1^{\complement}$ are (m,k)-small, and therefore we have $\phi'(X) \leq \phi'(A_1) + \phi'(A_1^{\complement}) = 2\varepsilon$. □

It remains to prove Theorem 5.2. Although it is a consequence of other results from this note that such a submeasure exists, we will provide a construction of a simple submeasure with the required properties.

Lemma 5.8. *Under the assumptions of Lemma 5.5, for every $k \geq 2$ there is $s \in [\mathbb{N}]^k$ and (s, ψ)-small sets $B_1, \dots, B_k$ such that $\bigcup_{i=1}^k B_i = X$. Moreover,* $\min(s)$ *can be chosen to be arbitrarily large.*

Proof. First we prove that for for every $m \in \mathbb{N}$ there are sets $Z_1, \dots, Z_k$ in $\mathcal{A}$ satisfying the following:

(a) $Z_i \cap Z_j = \emptyset$ if $i \neq j$,
(b) $\psi_t(Z_i) = 1$ for all $t \in T_m$ and $i \leq k$.

The proof is by induction on k. If $k = 2$, let $Z_1 = \bar{X}^0_m$ and $Z_2 = \bar{X}^1_m$. By Lemma 1.4, these sets are as required.

Assume that the assertion is true for k and prove it for $k+1$. Fix m, and find $Z_1, \dots, Z_k$ satisfying (a) and (b). Find n large enough such that $Z_i \in \mathcal{A}_n$ for all $i \leq k$, and consider a partition $X = X^0_n \dot\cup X^1_n$ such that $\theta_n(X^0_n) = \theta_n(X^1_n) = 1$. Then let $Z'_k = Z_k \cap \bar{X}^0_n$, $Z'_{k+1} = Z_k \cap \bar{X}^1_n$, and $Z'_i = Z_i$ if $i < k$. By Lemma 1.4, $\psi_t(Z'_k) = \psi_t(Z'_{k+1}) = 1$ for all $k \in T_m$, therefore the sets Z'_i $(i \leq k+1)$ are as required.

Find $n_1 < n_2 < \dots < n_k$ and sets Z^j_i $(1 \leq j \leq k-1,\ 1 \leq i \leq k)$ such that for every j we have

(c) $Z^j_1, \dots, Z^j_k$ satisfy (a) and (b) with $m = n_j$,

(d) $Z^j_i \in \mathcal{A}_{n_{j+1}}$ for all $i \leq k$.

Then for $i \leq k$ let $B_i = X \setminus \bigcup_{j=1}^{k-1} Z^j_i$.

Claim 1. $X = \bigcup_{i=1}^k B_i$.

Proof. Fix $x \in X$. For every $j \leq k-1$ there is at most one $i(x,j) \leq k$ such that $x \in X^i_{i(x,j)}$. If $i \leq k$ and $i \notin \{i(x,j) : j \leq k-1\}$ then $x \in B_i$. □

Let $s = \{n_1, \dots, n_k\}$.

Claim 2. *Each B_i $(i \leq k)$ is (s,ψ)-small.*

Proof. Recall that $\psi_t(Z^j_i) = 1$ for all i,j and $t \in T_{n_j}$. Therefore by Lemma 1.4, if $C \in \mathcal{A}_{n_j}$ then $\psi(C \cap Z^j_i) = \psi(C)$. Since $B_i \supseteq Z^j_i$, the conclusion follows. □

This concludes the proof of Lemma. □

Proof of Theorem 5.2. Fix σ such that $\sigma(n) > 6n$ for all n, and let

$$\phi = \psi \wedge \gamma[[\mathbb{N}]^k, 1/k].$$

Then ϕ is $1/k$-exhaustive by Lemma 3.5 and Lemma 3.11. If $F \subseteq [\mathbb{N}]^k$ has size $< k$, then F has an interval selector by Lemma 2.1, so by Lemma 3.2 $\psi(X \setminus \bigcup F) = 1$, hence ϕ is normalized. By Lemma 5.7 and Lemma 5.8 we can find $s \in [\mathbb{N}]^{k-1}$ and (s,ψ)-small sets $B_1, \dots, B_{k-1}$ that cover X. Let $m = \max(s)$. By Lemma 5.6 we can find pairwise disjoint sets A_n $(n \in \mathbb{N})$ in $\mathcal{A}$ such that each $A^{\complement}_n$ is m-small. If m_n is such that $A_n \in \mathcal{A}_{m_n}$, then $B_i \cap A^{\complement}_n$ is $s \cup \{m_n\}$-small for each $i \leq k$.

Therefore $A^{\complement}_n$ is covered by $k-1$ many $[\mathbb{N}]^k$-small sets and $\psi(A^{\complement}_n) \leq (k-1)/k$. If $\theta \leq \phi$ is normalized, it satisfies $1 \leq \theta(A_n) + \theta(A^{\complement}_n)$. Since $\theta(A^{\complement}_n) \leq (k-1)/k$ for all n, we must have $\limsup_n \theta(A_n) = 1/k$, and therefore θ cannot be ε-exhaustive for $\varepsilon < 1/k$. □

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