NONSEPARABLE UHF ALGEBRAS I: DIXMIER'S PROBLEM

ILIJAS FARAH AND TAKESHI KATSURA

Abstract. There are three natural ways to define UHF (uniformly hyperfinite) C*-algebras, and all three definitions are equivalent for separable algebras. In 1967 Dixmier asked whether the three definitions remain equivalent for not necessarily separable algebras. We give a complete answer to this question. More precisely, we show that in small cardinality two definitions remain equivalent, and give counterexamples in other cases. Our results do not use any additional set-theoretic axioms beyond the usual axioms, namely ZFC.

1. Introduction

Let A be a C*-algebra and let ε be a positive number. For an element x of A and a subset $\mathcal{F}$ of A, we write $x \in_\varepsilon \mathcal{F}$ if there exists $y \in \mathcal{F}$ such that $\|x - y\| < \varepsilon$. For two subsets $\mathcal{F}, \mathcal{G}$ of A, we write $\mathcal{F} \subset_\varepsilon \mathcal{G}$ if $x \in_\varepsilon \mathcal{G}$ for all $x \in \mathcal{F}$. For each $n \in \mathbb{N}$, we denote by $M_n(\mathbb{C})$ the unital C*-algebra of all $n \times n$ matrices with complex entries. A C*-algebra which is isomorphic to $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ is called a *full matrix algebra*.

Definition 1.1. A C*-algebra A is said to be

- *uniformly hyperfinite* (or *UHF*) if A is isomorphic to a tensor product of full matrix algebras.
- *approximately matricial* (or *AM*) if it has a directed family of full matrix subalgebras with dense union.
- *locally matricial* (or *LM*) if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$, there exists a full matrix subalgebra M of A with $\mathcal{F} \subset_\varepsilon M$,

For a definition of tensor products, see Definition 2.15. The property LM was called *matroid* in [6, Definition 1.1]. A UHF algebra is unital by definition, and it is easy to see that UHF implies AM and that AM implies LM. In [8, Theorem 1.13], Glimm shows that a unital separable LM algebra is UHF (see also [6, Remark 1.3 and Theorem 1.6]). Thus for separable C*-algebras, the three conditions UHF, unital AM and unital LM coincide. Dixmier asked whether these three conditions coincide for general C*-algebras in [6, Problem 8.1]. We show that this is not the case. To state our results precisely, we need the following notion.

Date: June 7, 2009.
1991 *Mathematics Subject Classification.* 46L05, 03E75.
I.F. partially supported by NSERC.

Definition 1.2. The *character density* $\chi(A)$ of a C*-algebra A is the smallest cardinality of a dense subset of A.

Hence A is separable if and only if its character density $\chi(A)$ is the first infinite cardinal $\aleph_0$. Note that $\chi(A)$ is equal to the smallest cardinality of an infinite generating subset of A.

The following are our main results which completely answer [6, Problem 8.1]. Note that $\aleph_1$ is the smallest uncountable cardinal.

Theorem 1.3.
(1) *For a C*-algebra with character density at most $\aleph_1$, AM and LM are equivalent.*
(2) *For every cardinal $\kappa > \aleph_1$, there exists a unital LM algebra with character density κ which is not AM.*
(3) *For every cardinal $\kappa \geq \aleph_1$, there exists a unital AM algebra with character density κ which is not UHF.*

Proof.
(1) Follows from Proposition 5.2 and Proposition 5.6.
(2) Follows from Proposition 6.10 and Proposition 6.12.
(3) Follows from Proposition 4.5. □

In (3), we can also control the representation density (defined in Definition 7.1) of the example (Theorem 7.17). In particular, we distinguish between AM algebras and UHF algebras faithfully represented on a separable Hilbert space.

Results similar to (1) and (2) hold for approximately finite-dimensional (AF) algebras.

Definition 1.4. A C*-algebra A is said to be

- *approximately finite-dimensional* (or *AF*) if it has a directed family of finite-dimensional subalgebras with dense union.
- *locally finite-dimensional* (or *LF*) if for any finite subset $\mathcal{F}$ of A and any $\varepsilon > 0$, there exists a finite-dimensional subalgebra D of A with $\mathcal{F} \subset_\varepsilon D$.

It is easy to see that AF implies LF. In [3, Theorem 2.2] Bratteli proved that for a separable C*-algebra, AF and LF are equivalent. We get the following.

Theorem 1.5.
(1)' *For a C*-algebra with character density at most $\aleph_1$, AF and LF are equivalent.*
(2)' *For every cardinal $\kappa > \aleph_1$, there exists an LF algebra with character density κ which is not AF.*

Proof.
(1)' Follows from Proposition 5.6.
(2)' Follows from Proposition 6.10 and Proposition 6.12. □

A C*-algebra is AM (resp. AF) if and only if it is obtained as a direct limit of full matrix algebras (resp. finite-dimensional algebras) over a general directed set (not necessarily a sequence). On the other hand, it is not hard to see that a C*-algebra is LM (resp. LF) if and only if it is obtained as a direct limit of (separable) AM (resp. AF) algebras (Lemma 2.12). Hence the two theorems above imply the following.

Corollary 1.6. *The classes of AM algebras and AF algebras are not closed under taking direct limits.*

In the sequel to this paper [7] we show that the classification problems for UHF and AM algebras are significantly different.

Organization of the paper. In §2 we set up the toolbox used in the paper. In §3 we use the Jiang–Su algebra to distinguish LM algebras from UHF algebras. σ-complete directed systems are used in §4 to distinguish between AM and UHF algebras. The relation between AM and LM algebras as well as the one between AF and LF algebras are explained in §5 and §6. In §7 we introduce the representation density, and using it distinguish between AM algebras and UHF algebras faithfully represented on a given Hilbert space.

2. Preliminary

In the present section we fix the terminology and prove some standard facts from set theory, σ-complete directed systems and tensor products (respectively).

2.1. **Set theory.** By $X \sqcup Y$ we denote the disjoint union of sets X and Y. If $f\colon X \to Y$ and $Z \subseteq X$ then we write $f[Z] = \{f(z) : z \in Z\}$ instead of the notation $f(Z)$ commonly accepted outside of set theory. Let us denote the cardinality of a set X by $|X|$. The countable infinite cardinal and the smallest uncountable cardinal are denoted by $\aleph_0$ and $\aleph_1$, respectively. The smallest uncountable ordinal is denoted by ω_1.

Lemma 2.1. *Let X be a set. For each $x \in X$, choose a countable subset $Y_x \subset X$ with $x \in Y_x$. If $|X| > \aleph_1$ then one can find two elements $x, y \in X$ such that $x \notin Y_y$ and $y \notin Y_x$.*

Proof. Take $Z \subset X$ with $|Z| = \aleph_1$. Choose $x \in X \setminus \bigcup_{z\in Z} Y_z$ and $y \in Z \setminus Y_x$. Then x and y are as required. □

Remark 2.2. The conclusion may be false if $|X| \leq \aleph_1$. To see this consider $X = \omega_1$ and $Y_x = \{y \in \omega_1 : y \leq x\}$ for $x \in \omega_1$.

Definition 2.3. A directed set Λ is said to be *σ-complete* if every countable directed $Z \subseteq \Lambda$ has the supremum $\sup Z \in \Lambda$.

The ordered set ω_1 is σ-complete. The following is another σ-complete directed set considered in this paper.

Definition 2.4. For an infinite set X, we denote by $[X]^{\aleph_0}$ the set of all countable infinite subsets of X, considered as a directed set with respect to the inclusion.

Definition 2.5. Let Λ be a σ-complete directed set. A subset Λ_0 of Λ is said to be *closed* if for every countable directed $Z \subseteq \Lambda_0$ we have $\sup Z \in \Lambda_0$, and *cofinal* if for every $\lambda \in \Lambda$ there exists $\lambda_0 \in \Lambda_0$ such that $\lambda \preceq \lambda_0$.

A closed and cofinal subset is called a *club*.

A club is an abbreviation of a *closed and unbounded* set. The condition '*unbounded*' (meaning 'not having an upper bound') is equivalent to 'cofinal' for totally ordered sets such as ω_1, but is strictly weaker than 'cofinal' for general directed sets. A widely accepted custom among set theorists is calling closed and *cofinal* subsets of $[X]^{\aleph_0}$ *closed and unbounded* sets (or *clubs*). Reluctantly, we continue this unfortunate abuse of terminology in our paper. This can be justified by that ω_1 and $[X]^{\aleph_0}$ are the only σ-complete directed sets that we will consider from the next section on.

Lemma 2.6. *Let Λ be a σ-complete directed set. Let Λ_0 and Λ_0' be clubs of Λ and $\phi\colon \Lambda_0 \to \Lambda_0'$ be an order isomorphism. Then there exists a club Λ_{00} of Λ such that $\Lambda_{00} \subset \Lambda_0 \cap \Lambda_0'$ and $\phi\restriction_{\Lambda_{00}} = \mathrm{id}$.*

Proof. Set $\Lambda_{00} := \{\lambda \in \Lambda_0 \cap \Lambda_0' : \phi(\lambda) = \lambda\}$. It is easy to see that Λ_{00} is closed. We will see that it is cofinal. Take $\lambda \in \Lambda$. Since Λ_0 is cofinal, there exists $\lambda_1 \in \Lambda_0$ with $\lambda \preceq \lambda_1$. Since Λ_0' is cofinal, there exists $\lambda_1' \in \Lambda_0'$ with $\lambda_1 \preceq \lambda_1'$ and $\phi(\lambda_1) \preceq \lambda_1'$. Recursively, we can find $\lambda_n \in \Lambda_0$ and $\lambda_n' \in \Lambda_0'$ for $n = 1, 2, \dots$ such that

$$\lambda_n \preceq \lambda_n', \quad \phi(\lambda_n) \preceq \lambda_n', \quad \lambda_n' \preceq \lambda_{n+1}, \quad \phi^{-1}(\lambda_n') \preceq \lambda_{n+1}.$$

Then

$$\lambda_{00} := \sup\{\lambda_n\}_{n=1}^{\infty} = \sup\{\lambda_n'\}_{n=1}^{\infty} \in \Lambda_0 \cap \Lambda_0'$$

satisfies $\phi(\lambda_{00}) = \lambda_{00}$. Thus we have found $\lambda_{00} \in \Lambda_{00}$ with $\lambda \preceq \lambda_{00}$. □

2.2. **σ-complete directed families of subalgebras.** By a subalgebra of a C*-algebra we mean a C*-subalgebra, and by a unital subalgebra of a unital C*-algebra we mean a C*-subalgebra containing the unit of the original C*-algebra. By a directed family $\{A_\lambda\}_{\lambda\in\Lambda}$ of subalgebras of a C*-algebra A, we mean that Λ is a directed set, and $\lambda \preceq \mu$ if and only if $A_\lambda \subset A_\mu$.

Definition 2.7. A directed family $\{A_\lambda\}_{\lambda\in\Lambda}$ of subalgebras of a C*-algebra A is said to be *σ-complete* if Λ is σ-complete and for every countable directed $Z \subseteq \Lambda$, $A_{\sup Z}$ is the closure of the union of $\{A_\lambda\}_{\lambda\in Z}$.

In other words, a directed family $\{A_\lambda\}_{\lambda\in\Lambda}$ is σ-complete if $\overline{\bigcup_{\lambda\in Z} A_\lambda}$ is in the family for every countable directed $Z \subseteq \Lambda$.

Lemma 2.8. *Let A be a C*-algebra, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a σ-complete directed family of subalgebras of A with dense union. Then for a club $\Lambda_0 \subset \Lambda$, the restriction $\{A_\lambda\}_{\lambda\in\Lambda_0}$ is also a σ-complete directed family with dense union.*

Proof. The restriction $\{A_\lambda\}_{\lambda\in\Lambda_0}$ is σ-complete because Λ_0 is closed, and its union is dense because Λ_0 is cofinal. □

Lemma 2.9. *Every C*-algebra A has a σ-complete directed family of separable subalgebras with dense union.*

Proof. We can take the family of all separable subalgebras of A ordered by the inclusion. □

Lemma 2.10. *Let A be a C*-algebra, and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a σ-complete directed family of separable subalgebras of A with dense union. For every separable subalgebra A_0 of A there exists $\lambda \in \Lambda$ such that $A_0 \subset A_\lambda$.*

Proof. Let $\{a_1, a_2, \ldots\}$ be a dense sequence of A_0. For each $n \in \mathbb{N}$, one can inductively find $\lambda_n \in \Lambda$ such that $a_i \in_{1/n} A_{\lambda_n}$ for $i = 1, 2, \ldots, n$ and $\lambda_{n-1} \preceq \lambda_n$ because the family $\{A_\lambda\}_{\lambda\in\Lambda}$ is directed and its union is dense in A. Then $\lambda := \sup\{\lambda_n : n \in \mathbb{N}\} \in \Lambda$ satisfies $A_0 \subset A_\lambda$. □

By the lemma above, we can see that the union of a σ-complete directed family is automatically closed.

Proposition 2.11. *Let A and B be C*-algebras, and $\{A_\lambda\}_{\lambda\in\Lambda}$ and $\{B_{\lambda'}\}_{\lambda'\in\Lambda'}$ be σ-complete directed families of separable subalgebras of A and B with dense union. Let $\Phi\colon A \to B$ be an isomorphism. Then there exist clubs $\Lambda_0 \subset \Lambda$ and $\Lambda'_0 \subset \Lambda'$ and an order isomorphism $\phi\colon \Lambda_0 \to \Lambda'_0$ such that $\Phi[A_\lambda] = B_{\phi(\lambda)}$ for all $\lambda \in \Lambda_0$. If $\Lambda = \Lambda'$, then one can take $\Lambda_0 = \Lambda'_0$ and $\phi = \mathrm{id}$.*

Proof. Let Λ_0 be the set of all $\lambda \in \Lambda$ such that there exists $\lambda' \in \Lambda'$ with $\Phi[A_\lambda] = B_{\lambda'}$. Similarly we define $\Lambda'_0 \subset \Lambda'$ as the set of all $\lambda' \in \Lambda'$ such that there is $\lambda \in \Lambda$ with $\Phi^{-1}[B_{\lambda'}] = A_\lambda$. Then there exists an order isomorphism $\phi\colon \Lambda_0 \to \Lambda'_0$ such that $\Phi[A_\lambda] = B_{\phi(\lambda)}$ for all $\lambda \in \Lambda_0$. We are going to show that $\Lambda_0 \subset \Lambda$ is a club. It is clear that Λ_0 is closed. Take $\lambda \in \Lambda$. Since A_λ is separable, there exists $\lambda'_1 \in \Lambda'$ such that $\Phi[A_\lambda] \subseteq B_{\lambda'_1}$ by Lemma 2.10. By the same reason, there exists $\lambda_1 \in \Lambda$ such that $\Phi^{-1}[B_{\lambda'_1}] \subseteq A_{\lambda_1}$. Then we have $A_\lambda \subseteq A_{\lambda_1}$. In this way, we can find sequences

$$A_\lambda \subseteq A_{\lambda_1} \subseteq A_{\lambda_2} \subseteq A_{\lambda_3} \subseteq \cdots$$
$$B_{\lambda'_1} \subseteq B_{\lambda'_2} \subseteq B_{\lambda'_3} \subseteq \cdots$$

such that $B_{\lambda'_n} \subseteq \Phi[A_{\lambda_n}]$ and $\Phi[A_{\lambda_n}] \subseteq B_{\lambda'_{n+1}}$ for $n = 1, 2, \ldots$. Let $\lambda_0 \in \Lambda$ and $\lambda'_0 \in \Lambda'$ be the supremums of $\{\lambda_n\}_{n=1}^\infty$ and $\{\lambda'_n\}_{n=1}^\infty$. Then we have $A_{\lambda_0} = \overline{\bigcup_{n=1}^\infty A_{\lambda_n}}$ and $B_{\lambda'_0} = \overline{\bigcup_{n=1}^\infty B_{\lambda'_n}}$. Since $\Phi[A_{\lambda_0}] = B_{\lambda'_0}$, we get $\lambda_0 \in \Lambda_0$. This shows that Λ_0 is cofinal, and hence it is a club. Similarly $\Lambda'_0 \subset \Lambda'$ is a club. This shows the former assertion. The latter assertion follows from Lemma 2.6. □

Lemma 2.12. *A C*-algebra A is LF if and only if it has a σ-complete directed family of separable AF algebras with dense union.*

Proof. We only need to prove the direct implication. We see that A has a σ-complete directed family of separable subalgebras $\{A_\lambda\}_{\lambda\in\Lambda}$ with dense union by Lemma 2.9. Since by [3, Theorem 2.2] every separable LF algebra is AF, it suffices to show that the set Λ_0 of all $\lambda \in \Lambda$ such that A_λ is LF is a club. Clearly Λ_0 is closed. To show that Λ_0 is cofinal, it suffices to see that for any separable subalgebra A_0 of A, there exists a separable subalgebra A_0' containing A such that for any finite subset $\mathcal{F}$ of A_0 and any $\varepsilon > 0$, there exists a finite-dimensional subalgebra M of A_0' with $\mathcal{F} \subset_\varepsilon M$. This is easy to see. □

In the same way, one can show that a C*-algebra A is LM if and only if it has a σ-complete directed family of separable AM subalgebras with dense union.

Remark 2.13. Lemma 2.12 is just a special case of the downward Löwenheim–Skolem theorem for logic of metric structures ([1]). Similar arguments have been used by C*-algebraists to reflect properties of nonseparable algebras to separable subalgebras (see [2, II.8.5]) such as for example simplicity or the existence of the unique trace.

2.3. **Tensor products.** In this subsection, we give a definition and some properties of tensor products of C*-algebras. We only deal with nuclear C*-algebras for which there is no distinction between the minimal tensor products and the maximal ones. We are interested in tensor products of possibly uncountably many unital C*-algebras, and for this purpose the maximal tensor products are easier to treat than the minimal ones.

Definition 2.14. A family $\{A_x\}_{x\in X}$ of subalgebras of a C*-algebra A is said to *mutually commute* if for distinct $x, y \in X$, every element of A_x commutes with every element of A_y.

Definition 2.15. For a family $\{A_x\}_{x\in X}$ of unital C*-algebras, its (maximal) *tensor product* $\bigotimes_{x\in X} A_x$ is the C*-algebra having (an isomorphic copy of) A_x as unital subalgebras for $x \in X$ satisfying the following two properties:

(1) the family $\{A_x\}_{x\in X}$ of subalgebras of $\bigotimes_{x\in X} A_x$ mutually commutes, and its union $\bigcup_{x\in X} A_x$ generates $\bigotimes_{x\in X} A_x$.
(2) for a unital C*-algebra B and a family $\{\varphi_x\}_{x\in X}$ of unital $*$-homomorphisms $\varphi_x\colon A_x \to B$ such that $\{\varphi_x[A_x]\}_{x\in X}$ is a mutually commuting family of unital subalgebras of B, there exists a unital $*$-homomorphism $\varphi\colon \bigotimes_{x\in X} A_x \to B$ such that $\varphi\restriction_{A_x} = \varphi_x$ for all $x \in X$.

When $A_x = A$ for all $x \in X$, we simply write $\bigotimes_X A$ for $\bigotimes_{x\in X} A_x$.

It is not difficult to see that the tensor product exists and is unique. A nice exposition of tensor products of C*-algebras can be found e.g., in [4], and universal C*-algebras are defined e.g., in [2, II.8.3].

For two families $\{A_x\}_{x\in X_1}$ and $\{A_x\}_{x\in X_2}$ of unital C*-algebras, the tensor product $\bigotimes_{x\in X_1 \amalg X_2} A_x$ is naturally isomorphic to

$$\Big(\bigotimes_{x\in X_1} A_x\Big) \otimes \Big(\bigotimes_{x\in X_2} A_x\Big).$$

We identify these two tensor products. In particular, we can and will consider $\bigotimes_{x\in Y} A_x$ as a unital subalgebra of $\bigotimes_{x\in X} A_x$ for a subset Y of X. We use the convention that $\bigotimes_{x\in Y} A_x = \mathbb{C}$ for $Y = \emptyset$.

The following is easy to see.

Lemma 2.16. *Let $\{A_x\}_{x\in X}$ be an infinite family of unital C*-algebras, and set $A = \bigotimes_{x\in X} A_x$. Then $\{\bigotimes_{x\in\lambda} A_x\}_{\lambda\in[X]^{\aleph_0}}$ is a σ-complete directed system of subalgebras of A with dense union.* □

Lemma 2.17. *If $A = \bigotimes_{x\in X} A_x$, X is infinite, and each A_x is separable and not $\mathbb{C}$, then the character density $\chi(A)$ of A is equal to $|X|$.*

Proof. Fix a countable dense $C_x \subseteq A_x$ for each x. Their union has cardinality $|X|$ and generates A. This shows $\chi(A) \le |X|$. Take a subset $Z \subseteq A$ with cardinality less than $|X|$. For each $z \in Z$, there exists $\lambda_z \in [X]^{\aleph_0}$ with $z \in \bigotimes_{x\in\lambda_z} A_x$ by Lemma 2.16. Since the set $\bigcup_{z\in Z} \lambda_z \subset X$ has cardinality less than $|X|$, we can find $x \in X$ outside of this set. Since A_x is not $\mathbb{C}$, Y is not dense in A. Hence $\chi(A) = |X|$. □

Lemma 2.18. *Assume $A = \bigotimes_{x\in X} A_x$, $B = \bigotimes_{y\in Y} B_y$ and all A_x and all B_y are unital, separable, and not $\mathbb{C}$. Let $\Phi\colon A \to B$ be an isomorphism. Then there exist partitions $X = \sqcup_{z\in Z} X_z$ and $Y = \sqcup_{z\in Z} Y_z$ of X and Y into disjoint nonempty countable subsets indexed by a same set Z such that*

$$\Phi[\textstyle\bigotimes_{x\in X_z} A_x] = \bigotimes_{y\in Y_z} B_y$$

for all $z \in Z$.

Proof. Consider families $\{X_z\}_{z\in Z}$ and $\{Y_z\}_{z\in Z}$ of disjoint nonempty countable subsets of X and Y, respectively, indexed by a same set Z such that we have $\Phi[\bigotimes_{x\in X_z} A_x] = \bigotimes_{y\in Y_z} Y_z$ for every $z \in Z$. By Zorn's lemma, there exists maximal one $\{X_z\}_{z\in Z}$ and $\{Y_z\}_{z\in Z}$ among such families. If we set $X' := X \setminus \bigcup_{z\in Z} X_z$ and $Y' := Y \setminus \bigcup_{z\in Z} Y_z$ then we have $\Phi[\bigotimes_{x\in X'} A_x] = \bigotimes_{y\in Y'} Y_z$. Thus X' is nonempty if and only if Y' is nonempty. Suppose, to derive a contradiction, both X' and Y' are nonempty. By applying the argument in the proof of Lemma 2.6 (see also Lemma 2.16), we find non-empty countable $X_0 \subseteq X'$ and $Y_0 \subseteq Y'$ such that $\Phi[\bigotimes_{x\in X_0} A_x] = \bigotimes_{y\in Y_0} B_y$. This contradicts the assumed maximality of $\{X_z\}_{z\in Z}$ and $\{Y_z\}_{z\in Z}$. Hence both X' and Y' are empty, and the maximal families $\{X_z\}_{z\in Z}$ and $\{Y_z\}_{z\in Z}$ are what we want. □

Let A and B be unital C*-algebras. Since we consider A and B as unital subalgebras of $A\otimes B$, each $a \in A$ and each $b \in B$ are considered as elements of $A \otimes B$. Thus the product $ab \in A \otimes B$ makes sense whereas this element is usually denoted by $a \otimes b \in A \otimes B$. Similarly, for a family $\{A_x\}_{x\in X}$ of unital C*-algebras and a finite family $\{a_x\}_{x\in Y}$ of elements with $a_x \in A_x$ for $x \in Y \subset X$, we denote by $\prod_{x\in Y} a_x \in \bigotimes_{x\in X} A_x$ the product of $\{a_x\}_{x\in Y}$. Note that this product does not depend on the order of multiplications because the family $\{a_x\}_{x\in Y}$ in $\bigotimes_{x\in X} A_x$ mutually commutes.

By the universality, a family $\{\alpha_x\}_{x\in X}$ of automorphisms α_x on A_x determines the automorphism α on $\bigotimes_{x\in X} A_x$ with $\alpha\restriction_{A_x}=\alpha_x$ which we denote by $\bigotimes_{x\in X}\alpha_x$. For a subset $Y\subset X$ and a family $\{\alpha_x\}_{x\in Y}$ of automorphisms α_x on A_x, we denote by $\bigotimes_{x\in Y}\alpha_x$ the automorphism $\bigotimes_{x\in X}\alpha_x$ of $\bigotimes_{x\in X} A_x$ where $\alpha_x=\mathrm{id}_{A_x}$ for $x\in X\setminus Y$.

For a unitary u of a unital C*-algebra A, an automorphism $\mathrm{Ad}\,u$ on A is defined by $\mathrm{Ad}\,u(a)=uau^*$ for $a\in A$. Let $\{A_x\}_{x\in X}$ be a family of unital C*-algebras. Take a subset $Y\subset X$ and unitaries $u_x\in A_x$ for $x\in Y$. Then we get an automorphism $\bigotimes_{x\in Y}\mathrm{Ad}\,u_x$ on $A=\bigotimes_{x\in X}A_x$. When Y is finite, we get $\bigotimes_{x\in Y}\mathrm{Ad}\,u_x=\mathrm{Ad}\,u$ where $u=\prod_{x\in Y}u_x\in A$, but in general, $\bigotimes_{x\in Y}\mathrm{Ad}\,u_x$ is not in the form $\mathrm{Ad}\,u$ for a unitary u of A.

We use the following well-known fact without mentioning. We give its proof for the reader's convenience.

Lemma 2.19. *Let A and B be unital C*-algebras, and $A_0\subset A$ and $B_0\subset B$ be unital subalgebras. Then we have $(A_0\otimes B_0)\cap B=B_0$ in $A\otimes B$.*

Proof. Take a state φ of A. Define a linear map $E\colon A\otimes B\to B$ by $E(ab)=\varphi(a)b$ for $a\in A$ and $b\in B$. Since $E(b)=b$ for $b\in B$ and $E(A_0\otimes B_0)\subset B_0$, we get $(A_0\otimes B_0)\cap B\subset B_0$. The inverse inclusion is easy to see. □

The next lemma is also standard, but we give the proof for the reader's convenience.

Lemma 2.20. *Let A and B be unital subalgebras of a unital C*-algebra D commuting with each other. If A is LM, then the natural map from $A\otimes B$ to the C*-subalgebra $C^*(A\cup B)$ of D generated by $A\cup B\subset D$ is an isomorphism.*

Proof. We first show the statement in the case that A is a full matrix algebra $M_n(\mathbb{C})$. Let $\{e_{i,j}\}_{i,j=1}^n$ be a matrix unit of $A\cong M_n(\mathbb{C})$. Then every element of $A\otimes B$ can be written as $\sum_{i,j=1}^n e_{i,j}b_{i,j}$ for $b_{i,j}\in B$. In $C^*(A\cup B)\subset D$, we have

$$b_{i',j'}=\sum_{k=1}^n e_{k,i'}\Big(\sum_{i,j=1}^n e_{i,j}b_{i,j}\Big)e_{j',k}$$

for $i',j'=1,2,\dots,n$. Hence if an element $\sum_{i,j=1}^n e_{i,j}b_{i,j}\in A\otimes B$ is sent to $0\in D$ by the natural map $A\otimes B\to D$, then $b_{i,j}=0$ for all i,j which implies $\sum_{i,j=1}^n e_{i,j}b_{i,j}=0$ in $A\otimes B$. Thus when A is a full matrix algebra, the natural map $A\otimes B\to C^*(A\cup B)$ is injective, and hence an isomorphism.

Now suppose that A is LM. Let $\pi\colon A\otimes B\to C^*(A\cup B)$ be the natural map. Take $x\in A\otimes B$. Take $\varepsilon>0$ arbitrarily. Then there exist $a_1,a_2,\dots,a_n\in A$ and $b_1,b_2,\dots,b_n\in B$ such that

$$\Big\|x-\sum_{i=1}^n a_ib_i\Big\|<\varepsilon.$$

Since A is LM, we may assume (by perturbing a_i's slightly if necessarily) that $a_1,a_2,\dots,a_n\in M$ for some unital full matrix subalgebra M of A. Then by the first part of the proof, we have

$$\Big\| \sum_{i=1}^{n} a_i b_i \Big\| = \Big\| \pi\Big(\sum_{i=1}^{n} a_i b_i \Big) \Big\|.$$

Hence we get

$$\big| \|x\| - \|\pi(x)\| \big| < 2\varepsilon.$$

Since ε was arbitrary, we have $\|x\| = \|\pi(x)\|$. This shows that the natural map $\pi\colon A \otimes B \to C^*(A \cup B)$ is injective, and hence an isomorphism. □

Because of this lemma, we may be careless to use the notation $A \otimes B$ of tensor products when A or B is LM. Note that this lemma is false if we replace LM by LF. To see this, just consider $A = B = D = \mathbb{C} \oplus \mathbb{C}$.

2.4. **Relative commutants.** For a subset A of a C*-algebra B, we denote by $Z_B(A)$ the *relative commutant* (or *centralizer*) of A inside B;

$$Z_B(A) := \{b \in B : ab = ba \text{ for all } a \in A\}$$

which is a subalgebra of B if A is closed under the $*$-operation (for example if A is a subalgebra). We avoid the common notation $A' \cap B$ for $Z_B(A)$ in order to increase the readability of certain formulas. For a subset A of a C*-algebra B, we denote by $C^*(A)$ the subalgebra generated by A. Note that $Z_B(C^*(A)) = Z_B(A)$ for a subset A closed under the $*$-operation. We also note that $Z_B(A_1 \cup A_2) = Z_B(A_1) \cap Z_B(A_2)$.

Lemma 2.21. *Let A and D be unital C*-algebras. If A is LM, then $Z_{A\otimes D}(A) = D$.*

Proof. It is clear from the definition of tensor products that $Z_{A\otimes D}(A) \supset D$. Take $x_0 \in Z_{A\otimes D}(A)$. For any $\varepsilon > 0$, there exist elements $a_1, \ldots, a_n \in A$ and $d_1, \ldots, d_n \in D$ such that $\|x_0 - \sum_{i=1}^n a_i d_i\| < \varepsilon$. Since A is LM, we may assume that $a_1, \ldots, a_n$ are in a full matrix unital subalgebra M of A. Let $E\colon A\otimes D \to A\otimes D$ be a contractive linear map defined by $E(x) = \int_U uxu^* du$ for $x \in A\otimes D$ where du is the normalized Haar measure on the unitary group U of M. Since $x_0 \in Z_{A\otimes D}(A)$, we have $E(x_0) = x_0$. For $a \in M$ and $d \in D$, we have $E(ad) = \mathrm{tr}(a)d$ where $\mathrm{tr}\colon M \to \mathbb{C}$ is the normalized trace. Hence we have $\|x_0 - \sum_{i=1}^n \mathrm{tr}(a_i)b_i\| < \varepsilon$. This means that $x_0 \in_\varepsilon D$. Since ε was arbitrary, $x_0 \in D$. Thus we get $Z_{A\otimes D}(A) \subset D$, and therefore $Z_{A\otimes D}(A) = D$. We are done. □

By letting $D = \mathbb{C}$ in the lemma above, we see that the center $Z_A(A)$ of an LM algebra A is $\mathbb{C}$. Thus one can write the conclusion of Lemma 2.21 as $Z_{A\otimes D}(A) = Z_A(A)\otimes D$. It is natural to expect that $Z_{A\otimes D}(A_0) = Z_A(A_0)\otimes D$ holds for general unital C*-algebras A and D and a unital subalgebra A_0 of A. However this turns out to be a difficult question, and the authors could not get this equality without assuming the nuclearity of D. A nice exposition of nuclearity of C*-algebras can be found e.g., in [4].

Lemma 2.22. *Let A and D be unital C*-algebras, and A_0 a unital subalgebra of A. Suppose that D is nuclear. Then $Z_{A\otimes D}(A_0) = Z_A(A_0)\otimes D$.*

Proof. Clearly we have $Z_A(A_0)\otimes D \subseteq Z_{A\otimes D}(A_0)$. Let

$$F := \{c \in A\otimes D : (\mathrm{id}\otimes\omega)(c) \in Z_A(A_0) \text{ for all } \omega \in D^*\}.$$

For $a \in A \subset A\otimes D$ and $c \in A\otimes D$, we have

$$(\mathrm{id}\otimes\omega)(ac) = a(\mathrm{id}\otimes\omega)(c), \quad (\mathrm{id}\otimes\omega)(ca) = (\mathrm{id}\otimes\omega)(c)a$$

for all $\omega \in D^*$. Hence we get $Z_{A\otimes D}(A_0) \subseteq F$. We claim that $F = Z_A(A_0)\otimes D$. This equality is usually referred to as the *slice map property* of the triple $(D, A, Z_A(A_0))$ (see [4, Definition 12.4.3]). By [4, Theorem 12.4.4 (2)] (see [4, Definition 12.4.1] and note nuclear⇔CPAP⇒SOAP), the triple $(D, A, Z_A(A_0))$ has the slice map property because D is nuclear. Thus we have $Z_A(A_0)\otimes D = Z_{A\otimes D}(A_0)$. □

Definition 2.23. Let A be a unital C*-algebra, and A_0 a unital subalgebra of A. We say that A_0 is *complemented* in A if $C^*(A_0 \cup Z_A(A_0)) = A$.

In a tensor product $A = \bigotimes_{x\in X} A_x$ of unital C*-algebras A_x, a subalgebra $A_Y = \bigotimes_{x\in Y} A_x$ is complemented for every subset Y of X.

Proposition 2.24. *Let A be a unital C*-algebra. Suppose that there exists a unital C*-algebra D such that $A\otimes D$ is a tensor product of separable nuclear C*-algebras. Then for a σ-complete directed system $\{A_\lambda\}_{\lambda\in\Lambda}$ of separable subalgebras of A with dense union, there exists a club $\Lambda_0 \subset \Lambda$ such that for each $\lambda \in \Lambda_0$, A_λ is complemented in A.*

Proof. By Lemma 2.9 take a σ-complete directed system $\{D_{\lambda'}\}_{\lambda'\in\Lambda'}$ of separable subalgebras of D with dense union. Then $\{A_\lambda \otimes D_{\lambda'}\}_{(\lambda,\lambda')\in\Lambda\times\Lambda'}$ is a σ-complete directed system of separable subalgebras of $A\otimes D$ with dense union. Since $A\otimes D$ is a tensor product of separable C*-algebras, $A\otimes D$ has a σ-complete directed system of separable complemented subalgebras with dense union by Lemma 2.16. Hence by Proposition 2.11, there exists a club $C \subset \Lambda\times\Lambda'$ such that $A_\lambda\otimes D_{\lambda'}$ is complemented in $A\otimes D$ for all $(\lambda,\lambda') \in C$. Let Λ_0 be the image of C under the map $\Lambda\times\Lambda' \ni (\lambda,\lambda') \mapsto \lambda \in \Lambda$. Then $\Lambda_0 \subset \Lambda$ is a club. Thus it remains to prove that A_λ is complemented in A if $A_\lambda \otimes D_{\lambda'}$ is complemented in $A\otimes D$ for some $D_{\lambda'} \subset D$.

Since $A\otimes D$ is a tensor product of nuclear C*-algebras, D is nuclear by [4, Proposition 10.1.7]. Hence for each $\lambda \in \Lambda_0$ we get $Z_A(A_\lambda)\otimes D = Z_{A\otimes D}(A_\lambda)$ by Lemma 2.22. This implies

$$\begin{aligned} A\otimes D &= C^*\big((A_\lambda\otimes D_{\lambda'}) \cup Z_{A\otimes D}(A_\lambda\otimes D_{\lambda'})\big) \\ &\subset C^*\big((A_\lambda\otimes D) \cup Z_{A\otimes D}(A_\lambda)\big) \\ &= C^*\big((A_\lambda\otimes D) \cup (Z_A(A_\lambda)\otimes D)\big) \\ &= C^*(A_\lambda \cup Z_A(A_\lambda))\otimes D. \end{aligned}$$

This shows that A_λ is complemented in A and finishes the proof. □

When we apply this proposition in Proposition 4.5, we use the fact that a UHF algebra is a tensor product of separable nuclear C*-algebras because full matrix algebras are nuclear.

3. LM but not UHF

Proposition 3.2 gives examples of unital LM algebras that are not UHF, answering part of [6, Problem 8.1]. Recall that $\mathcal{Z}$ is the *Jiang–Su algebra*. We shall need the following properties of $\mathcal{Z}$ proved in [9]:

- $\mathcal{Z}$ is a unital, separable C*-algebra which is not UHF.
- $\bigotimes_{\aleph_0} \mathcal{Z} \cong \mathcal{Z}$.
- $A \otimes \mathcal{Z} \cong A$ for any infinite-dimensional separable UHF algebra A.

Definition 3.1. The UHF algebra $\bigotimes_{\aleph_0} M_2(\mathbb{C})$ is called the *CAR algebra*.

Proposition 3.2. *For two sets X and Y, define $A_{X,Y} := \bigotimes_X M_2(\mathbb{C}) \otimes \bigotimes_Y \mathcal{Z}$. Suppose that X is infinite. Then we have the following.*

(1) *$A_{X,Y}$ is an LM algebra with $\chi(A_{X,Y}) = |X| + |Y|$.*
(2) *$A_{X,Y}$ is UHF if and only if $|X| \geq |Y|$.*
(3) *$A_{X,Y} \otimes D$ is UHF for any UHF algebra D with $\chi(D) \geq \chi(A_{X,Y})$.*

Proof. Since X is infinite, we can identify $A_{X,Y}$ with $\bigotimes_X A \otimes \bigotimes_Y \mathcal{Z}$ where A is the CAR algebra. For each $\lambda \in [X]^{\aleph_0}$ and $\lambda' \in [Y]^{\aleph_0}$ we set

$$D_{\lambda,\lambda'} := \bigotimes_{\lambda} A \otimes \bigotimes_{\lambda'} \mathcal{Z} \subset A_{X,Y}$$

Then $D_{\lambda,\lambda'}$ is the CAR algebra for all λ and λ'. Since $\{D_{\lambda,\lambda'}\}$ is a σ-complete directed system with dense union, we see that $A_{X,Y}$ is LM. By Lemma 2.17, we have $\chi(A_{X,Y}) = |X| + |Y|$. This shows (1).

By rearranging the factors, we see that $A_{X,Y}$ is UHF if $|X| \geq |Y|$ and that $A_{X,Y} \otimes D$ is UHF for a UHF algebra D with $\chi(D) \geq \chi(A_{X,Y})$. It remains to show that $A_{X,Y}$ is UHF only if $|X| \geq |Y|$. For the sake of obtaining a contradiction, assume that $|X| < |Y|$ and $A_{X,Y}$ is UHF. Then by Lemma 2.18, there exists a partition $X \sqcup Y = \sqcup_{z \in Z} C_z$ of $X \sqcup Y$ into disjoint countable subsets such that $D_z := \bigotimes_{X \cap C_z} A \otimes \bigotimes_{Y \cap C_z} \mathcal{Z}$ is UHF for all $z \in Z$. Since $|X| < |Y|$, there exists $z \in Z$ with $C_z \subset Y$. For such z, $D_z \cong \mathcal{Z}$ is not UHF. This is a contradiction. □

By Proposition 3.2, the unital C*-algebra $A_{X,Y}$ is LM but not UHF if $|X| < |Y|$. When $|X| = \aleph_0$ and $|Y| = \aleph_1$, we see that $A_{X,Y}$ is AM by Theorem 1.3 (1). In the other case, we do not know whether $A_{X,Y}$ is AM or not.

Problem 3.3. Let X, Y be sets such that $\aleph_0 \leq |X| < |Y|$ and $\aleph_1 < |Y|$. Is $A_{X,Y} = \bigotimes_X M_2(\mathbb{C}) \otimes \bigotimes_Y \mathcal{Z}$ AM?

4. AM but not UHF

In this section, for each infinite set X we define a unital AM-algebra B_X with $\chi(B_X) = |X|$, and show that B_X, or even $B_X \otimes D$ for a unital C*-algebra D, is not UHF when $|X| \geq \aleph_1$.

Lemma 4.1. *A C*-algebra generated by two self-adjoint unitaries v, w with $vw = -wv$ is always isomorphic to $M_2(\mathbb{C})$.*

Proof. A C*-algebra A generated by two self-adjoint unitaries v, w with $vw = -wv$ is spanned (as a vector space) by 4 elements $\{1, v, w, vw\}$, and it is noncommutative. Hence it is isomorphic to $M_2(\mathbb{C})$ which is the unique noncommutative C*-algebra with dimension ≤ 4. The concrete isomorphism from A to $M_2(\mathbb{C})$ is given by sending v and w to the unitaries

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in $M_2(\mathbb{C})$. □

Let us take a set X. For each $x \in X$, let A_x be a C*-algebra generated by two self-adjoint unitaries v_x, w_x with $v_x w_x = -w_x v_x$. By Lemma 4.1, A_x is isomorphic to $M_2(\mathbb{C})$. We define a UHF algebra A_X by $A_X := \bigotimes_{x \in X} A_x \cong \bigotimes_X M_2(\mathbb{C})$. We define an automorphism α on A_X by $\alpha := \bigotimes_{x \in X} \operatorname{Ad} v_x$. Note that $\alpha^2 = \mathrm{id}$. Let $\{e_{i,j}\}_{i,j=1}^2$ be a system of matrix units of $M_2(\mathbb{C})$, and define an embedding

$$\iota\colon A_X \ni a \mapsto a e_{1,1} + \alpha(a) e_{2,2} \in A_X \otimes M_2(\mathbb{C}).$$

Let $u \in A_X \otimes M_2(\mathbb{C})$ be a self-adjoint unitary defined by $u := e_{1,2} + e_{2,1}$. Set $B_X := C^*(\iota(A_X) \cup \{u\})$. We consider A_X as a unital subalgebra of B_X and omit ι. Then we have $ua = \alpha(a)u$ for $a \in A_X$ and $B_X = \{au + a' : a, a' \in A_X\}$.

Remark 4.2. The C*-algebra B_X is nothing but the crossed product $A_X \rtimes_\alpha (\mathbb{Z}/2\mathbb{Z})$.

For $Y \subseteq X$, we denote by A_Y the subalgebra $\bigotimes_{x \in Y} A_x \subset A_X \subset B_X$, and define $B_Y := C^*(A_Y \cup \{u\}) \subset B_X$. It is easy to see that $A_Y \subset A_X$ is globally invariant under α, and hence $B_Y = \{au + a' : a, a' \in A_Y\}$.

Lemma 4.3. *If Y is infinite then $Z_{B_X}(A_Y) = A_{X \setminus Y}$.*

Proof. Since $Z_{A_X}(A_Y) = A_{X \setminus Y}$ by Lemma 2.21, it suffices to show that $Z_{B_X}(A_Y) \subset A_X$. Take $au + a' \in Z_{B_X}(A_Y)$ with $a, a' \in A_X$, and we will show $a = 0$.

For any $\varepsilon > 0$ there is a finite $F \subseteq X$ such that $a \in_\varepsilon A_F$. Since Y is infinite, pick $y \in Y \setminus F$. The unitary $w_y \in A_Y$ satisfies $uw_y = -w_y u$. Hence $w_y(au + a') = (au + a')w_y$ yields $(w_y a + a w_y)u + (w_y a' - a' w_y) = 0$. Since $bu + b' = 0$ for b, b' in A_X implies $b = b' = 0$, we have $w_y a = -a w_y$. Thus we get

$$\|a\| = \|w_y a\| = \|w_y a + w_y a\|/2 = \|w_y a - a w_y\|/2.$$

Since $a \in_\varepsilon A_F$ and w_y commutes with A_F, we have $\|a\| = \|w_y a - a w_y\|/2 < \varepsilon$. Since ε was arbitrary, $a = 0$. We are done. □

Lemma 4.4. *If $Y \subsetneq X$ and Y is infinite, then B_Y is not complemented in B_X.*

Proof. Since $B_Y = C^*(A_Y \cup \{u\})$, we have

$$Z_{B_X}(B_Y) = Z_{B_X}(A_Y) \cap Z_{B_X}(\{u\}) = A_{X\setminus Y} \cap Z_{B_X}(\{u\})$$

by Lemma 4.3. Hence

$$C^*(B_Y \cup Z_{B_X}(B_Y)) = \{au + a' : a, a' \in A_Y \otimes (A_{X\setminus Y} \cap Z_{B_X}(\{u\}))\}$$

which does not contain $w_y \in A_{X\setminus Y}$ for $y \in X \setminus Y$. □

Proposition 4.5. (1) *If X is infinite, then B_X is a unital AM algebra with $\chi(B_X) = |X|$.*
(2) *If X is uncountable, then B_X is not UHF and $B_X \otimes D$ is not UHF for any unital C^*-algebra D.*

Proof. (1) Suppose X is infinite.

Let us set

$$\Lambda = \{(F, y) : F \subset X \text{ finite, and } y \in X \setminus F\}$$

and define

$$D_{(F,y)} = C^*\big(B_F \cup \{w_y\}\big) \subset B_X.$$

for $(F, y) \in \Lambda$. Then it is clear that $\{D_{(F,y)}\}_{(F,y)\in\Lambda}$ is a directed family with dense union. It remains to show that $D_{(F,y)}$ is a full matrix algebra for $(F, y) \in \Lambda$. Take $(F, y) \in \Lambda$ with $|F| = n \in \mathbb{N}$. Then we have $A_F \cong M_{2^n}(\mathbb{C})$, and the restriction of α to A_F coincides with $\mathrm{Ad}\, v$ where

$$v = \prod_{x\in F} v_x \in A_F.$$

Then the two self-adjoint unitaries uv and w_y in $D_{(F,y)}$ satisfy $w_y(uv) = -(uv)w_y$ and commute with A_F. By Lemma 4.1, the subalgebra of $D_{(F,y)}$ generated by uv and w_y is isomorphic to $M_2(\mathbb{C})$, and commute with A_F. Since this subalgebra and A_F generates $D_{(F,y)}$, $D_{(F,y)}$ is isomorphic to $M_{2^{n+1}}(\mathbb{C})$. We are done.

(2) Suppose X is uncountable. Then $\{B_Y\}_{Y\in[X]^{\aleph_0}}$ is a σ-complete directed family of separable subalgebra of B_X with dense union. By Lemma 4.4, neither one of these subalgebras is complemented. By Proposition 2.24, $B_X \otimes D$ cannot be UHF for any unital C*-algebra D. □

Note that an example of a unital LM algebra A that is not UHF given in Proposition 3.2 has the property that $A \otimes D$ is UHF for some UHF algebra D, but the one given in Proposition 4.5 does not have this property.

The following answers [6, Problem 8.3] negatively although it was certainly known.

Corollary 4.6. *There is a proper subalgebra A of the CAR algebra B such that A is also CAR algebra and $Z_B(A) = \mathbb{C}1$. In particular, $B \neq A \otimes Z_B(A)$.*

Proof. Use Proposition 4.5 with $X = \mathbb{N}$. Then A_X is the CAR algebra. The C*-algebra B_X is also the CAR algebra because it is a separable unital LM algebra obtained as a direct limit of algebras of the form $M_{2^n}(\mathbb{C})$ by the proof of Proposition 4.5. By Lemma 4.3, we have $Z_{B_X}(A_X) = \mathbb{C}1$. □

5. AM = LM and AF = LF for character density $\leq \aleph_1$

We first show AM = LM + AF. We use the following well-known result repeatedly. Recall that a finite-dimensional C*-algebra D is isomorphic to a direct sum of finitely many full matrix algebras (e.g., [5, Theorem III.1.1]), and the cardinality $|F|$ of a system F of matrix units of D as defined after [5, Theorem III.1.1] coincides with the dimension of D.

Lemma 5.1 ([5, Corollary III.3.3])**.** *Given $d \in \mathbb{N}$, there exists $\delta > 0$ so that if D is a finite-dimensional subalgebra of a C*-algebra A with a system F of matrix units such that $|F| = d$ and B is a subalgebra of A such that $F \subseteq_\delta B$, there exists a unitary u in the unitization of A satisfying $uDu^* \subseteq B$ and commuting with $D \cap B$.*

Moreover, for a previously given $\varepsilon > 0$ in addition to d, there exists $\delta > 0$ such that one can choose u as above so that $\|u - 1\| < \varepsilon$. □

Proposition 5.2. *A C*-algebra is AM if and only if it is LM and AF.*

Proof. We only need to prove that if a C*-algebra A is LM and AF, then it is AM. Take a directed family $\{D_\lambda\}_{\lambda\in\Lambda}$ of finite-dimensional subalgebras of A with dense union. To show that A is AM, it suffices to show that for any $\lambda \in \Lambda$ there exists a full matrix subalgebra M containing D_λ and contained in $D_{\lambda'}$ for some $\lambda' \succeq \lambda$. Then the set of such full matrix subalgebras is directed and has dense union.

Take $\lambda \in \Lambda$. Let F be a system of matrix units of D_λ. Let $\delta > 0$ be as in Lemma 5.1 for $d = |F| \in \mathbb{N}$. Since A is LM, it has a full matrix subalgebra M_0 such that $F \subseteq_\delta M_0$. By Lemma 5.1, there exists a unitary u in the unitization of A satisfying $uD_\lambda u^* \subseteq M_0$. Let F' be a system of matrix units of u^*M_0u. Let $\delta' > 0$ be as in Lemma 5.1 for $d = |F'|$. Since $\{D_\lambda\}_{\lambda\in\Lambda}$ has dense union, there exists $\lambda' \in \Lambda$ such that $\lambda' \succeq \lambda$ and $F' \subseteq_{\delta'} D_{\lambda'}$. By Lemma 5.1, there exists a unitary u' in the unitization of A satisfying $u'(u^*M_0u)u'^* \subseteq D_{\lambda'}$ and commuting with $(u^*M_0u) \cap D_{\lambda'}$. Since $D_\lambda \subset (u^*M_0u) \cap D_{\lambda'}$, the full matrix subalgebra $M := u'(u^*M_0u)u'^*$ of A satisfies $D_\lambda \subseteq M \subseteq D_{\lambda'}$. This completes the proof. □

By Proposition 5.2, the statement AM = LM is reduced to AF = LF because LM implies LF. Thus we only show that AF = LF for character density at most $\aleph_1$ although the same argument as below works for showing AM = LM directly by just changing "F" to "M" and "finite-dimensional" to "full matrix" in all statements and proofs.

Lemma 5.3. *Let A be a separable AF algebra. For an increasing sequence $\{D_n\}_{n\in\mathbb{N}}$ of finite-dimensional subalgebras of A there exists an increasing sequence $\{D'_n\}_{n\in\mathbb{N}}$ of finite-dimensional subalgebras with dense union such that $\bigcup_{n\in\mathbb{N}} D_n \subseteq \bigcup_{n\in\mathbb{N}} D'_n$.*

Proof. This is well-known to specialists, and can be shown in a similar way to [5, Theorem III.3.5]. For the reader's convenience, we give a proof.

Let $\{B_k\}_{k\in\mathbb{N}}$ be an increasing sequence of finite-dimensional subalgebras of A with dense union. We construct inductively an increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ in $\mathbb{N}$ and a sequence $\{u_n\}_{n\in\mathbb{N}}$ of unitaries in the unitization of A with $\|u_n - 1\| < 2^{-n}$ such that for each $n \in \mathbb{N}$ the finite-dimensional algebra

$$u_n \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_n^*$$

is contained in B_{k_n} and commutes with u_{n+1}. We first construct k_1 and u_1. Choose $k_1 \in \mathbb{N}$ such that $F \subset_\delta B_{k_1}$ where F is a system of matrix units of D_1 and $\delta > 0$ be as in the latter statement of Lemma 5.1 for $d = |F|$ and $\varepsilon = 2^{-1}$. By Lemma 5.1, there exists a unitary u_1 in the unitization of A satisfying $u_1 D_1 u_1^* \subseteq B_{k_1}$ and $\|u_1 - 1\| < 1/2$. Suppose that $k_1, \ldots, k_{n-1} \in \mathbb{N}$ and unitaries $u_1, \ldots, u_{n-1}$ were chosen. Choose $k_n \in \mathbb{N}$ such that $k_n > k_{n-1}$ and $F' \subset_{\delta'} B_{k_n}$ where F' is a system of matrix units of

$$u_{n-1} \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_{n-1}^*$$

and $\delta' > 0$ be as in the latter statement of Lemma 5.1 for $d = |F'|$ and $\varepsilon = 2^{-n}$. Lemma 5.1 gives us a unitary u_n in the unitization of A satisfying

$$u_n u_{n-1} \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_{n-1}^* u_n^* \subseteq B_{k_n},$$

commuting with

$$u_{n-1} \cdots u_2 u_1 D_{n-1} u_1^* u_2^* \cdots u_{n-1}^* \subseteq u_{n-1} \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_{n-1}^* \cap B_{k_n}$$

and satisfying $\|u_n - 1\| < 2^{-n}$. Thus we get the desired sequences $\{k_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$.

Since $\|u_n - 1\| < 2^{-n}$ for all $n \in \mathbb{N}$ and $\sum_{n\in\mathbb{N}} 2^{-n} = 1 < \infty$, the sequence $\{u_n \cdots u_2 u_1\}_{n\in\mathbb{N}}$ converges to a unitary u in the unitization of A. Since

$$\begin{aligned} u_n \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_n^* &= u_{n+1} u_n \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_n^* u_{n+1}^* \\ &\subseteq u_{n+1} u_n \cdots u_2 u_1 D_{n+1} u_1^* u_2^* \cdots u_n^* u_{n+1}^*, \end{aligned}$$

$u_n \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_n^*$ commutes with u_{n+2}. By repeating this argument, one can see that $u_n \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_n^*$ commutes with u_m for all $m > n$. Hence we get $u D_n u^* = u_n u_{n-1} \cdots u_2 u_1 D_n u_1^* u_2^* \cdots u_{n-1}^* u_n^* \subseteq B_{k_n}$. We set $D'_n := u^* B_{k_n} u$ for $n \in \mathbb{N}$. Then $\{D'_n\}_{n\in\mathbb{N}}$ is an increasing sequence of finite-dimensional subalgebras with dense union such that $\bigcup_{n\in\mathbb{N}} D_n \subseteq \bigcup_{n\in\mathbb{N}} D'_n$. ☐

In the next lemma, for two families $\Upsilon = \{D_\lambda\}_{\lambda\in\Lambda}$ and $\Upsilon' = \{D'_\lambda\}_{\lambda\in\Lambda'}$ of subalgebras, $\Upsilon \subseteq \Upsilon'$ means that $\Lambda \subseteq \Lambda'$ and $D_\lambda = D'_\lambda$ for each $\lambda \in \Lambda$.

Lemma 5.4. *Let A be a separable AF algebra contained in a separable AF algebra A'. For a countable directed family Υ of finite-dimensional subalgebras of A with dense union, there exists a countable directed family Υ' of finite-dimensional subalgebras of A' with dense union such that $\Upsilon \subseteq \Upsilon'$.*

Proof. Let us write $\Upsilon = \{D_\lambda\}_{\lambda\in\Lambda}$. Since Λ is countable, we can choose a subsequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of Λ such that $\bigcup_{\lambda\in\Lambda} D_\lambda = \bigcup_{n\in\mathbb{N}} D_{\lambda_n}$. By Lemma 5.3, there exists an increasing sequence $\{D'_n\}_{n\in\mathbb{N}}$ of finite-dimensional subalgebras of A' with dense union such that $\bigcup_{n\in\mathbb{N}} D_{\lambda_n} \subset \bigcup_{n\in\mathbb{N}} D'_n$. For each $\lambda \in \Lambda$, there exists $n \in \mathbb{N}$ such that $D_\lambda \subseteq D'_n$ because D_λ is finite-dimensional. Let $\Lambda' := \Lambda \amalg \mathbb{N}$, ordered by requiring that Λ and $\mathbb{N}$ have their natural orderings and $\lambda \preceq n$ if $D_\lambda \subseteq D'_n$. Then the family $\Upsilon' := \{D'_\lambda\}_{\lambda\in\Lambda'}$ defined by $D'_\lambda := D_\lambda$ for $\lambda \in \Lambda$ satisfies the desired properties. □

Lemma 5.5. *Each LF algebra of character density at most $\aleph_1$ has a σ-complete directed family of separable AF subalgebras with dense union indexed by the ordinal ω_1.*

Proof. Let A be an LF algebra with $\chi(A) \leq \aleph_1$. Fix a dense subset $\{x_\gamma : \gamma \in \omega_1\}$ of A, and define $A_\lambda := C^*(\{x_\gamma : \gamma < \lambda\})$ for each $\lambda \in \omega_1$. Then $\{A_\lambda\}_{\lambda\in\omega_1}$ is a σ-complete directed family of separable subalgebras of A. By Lemma 2.12, A also has a σ-complete direct family of separable AF subalgebras with dense union. By Proposition 2.11 applied with $\mathrm{id}\colon A \to A$, there is a club $\Lambda \subseteq \omega_1$ such that A_λ is AF for $\lambda \in \Lambda$. As ordered sets, Λ is isomorphic to ω_1, and $\{A_\lambda\}_{\lambda\in\Lambda}$ is the desired family. □

Proposition 5.6. *Each LF algebra of character density at most $\aleph_1$ is an AF algebra.*

Proof. Let A be an LF algebra with $\chi(A) \leq \aleph_1$. Let $\{A_\xi\}_{\xi\in\omega_1}$ be a σ-complete directed family of separable AF subalgebras of A with dense union as in Lemma 5.5. Using transfinite recursion, we are going to construct an increasing family of countable directed families Υ_ξ of finite-dimensional subalgebras whose union is dense in A_ξ for each $\xi \in \omega_1$. For $\xi = 0$, choose an increasing sequence of finite-dimensional subalgebras of A_0 with dense union, and set it Υ_0. If Υ_ξ has been defined, then $\Upsilon_{\xi+1}$ is defined using Lemma 5.4. If η is a limit ordinal and Υ_ξ has been defined for all $\xi < \eta$, let $\Upsilon_\eta = \bigcup_{\xi<\eta} \Upsilon_\xi$. Since A_η is the closure of the union of $\{A_\xi\}_{\xi<\eta}$, Υ_η is as required.

Finally let $\Upsilon = \bigcup_{\xi\in\omega_1} \Upsilon_\xi$. Then this is a directed family of finite-dimensional subalgebras of A with dense union. Thus A is an AF algebra. □

The example of the following section easily shows that the version of Lemma 5.4 for nonseparable algebras is false.

6. AM $\neq$ LM and AF $\neq$ LF for character density $> \aleph_1$

In this section, we construct an LM algebra which is not AF. This C*-algebra shows the difference between the classes of AM and LM algebras as well as between the classes of AF and LF algebras. To show that a given C*-algebra is not AF, we use the following criterion.

The converse direction in the following lemma was proved by George Elliott, following a remark by Tamas Matrai, during the first author's talk at a set theory seminar in Toronto in April 2009.

Lemma 6.1. *A C*-algebra A is AF if and only if there exists a map $\rho\colon A \to A$ such that $\|a - \rho(a)\| < 1$ for every $a \in A$ and $C^*(\{\rho(a)\}_{a\in F})$ is finite-dimensional for every finite subset F of A.*

Proof. Assume A is AF and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a directed family of finite-dimensional subalgebras of A with dense union. For each $a \in A$ there exists $\lambda_a \in \Lambda$ such that there exists $\rho(a) \in A_{\lambda_a}$ with $\|a - \rho(a)\| < 1$. For every finite subset F of A there exists $\lambda \in \Lambda$ such that $\lambda \succeq \lambda_a$ for all $a \in F$. Then $C^*(\{\rho(a)\}_{a\in F}) \subset A_\lambda$ is finite-dimensional.

Now assume that $\rho\colon A \to A$ is as in the statement of the lemma. If Λ is the family of all finite subsets of A then $A_\lambda = C^*(\{\rho(a)\}_{a\in\lambda})$ form a directed family of finite-dimensional subalgebras of A. Fix $a \in A$ and $\varepsilon > 0$. Let $\lambda = \{a/\varepsilon\}$. Then $\varepsilon\rho(a/\varepsilon) \in A_\lambda$ and $\|a - \varepsilon\rho(a/\varepsilon)\| < \varepsilon$. Since a and ε were arbitrary, we conclude A is AF. □

We also use the following lemma (for the case when A is the CAR algebra) in the proof of Proposition 6.12

Lemma 6.2. *Let A be a unital LM subalgebra of a unital C*-algebra B. Take $a_1, a_2, \dots, a_n \in A$ and $b_1, b_2, \dots, b_n \in Z_B(A)$. If $(a_i)_{i=1}^n$ is linearly independent in A and $\sum_{i=1}^n a_i b_i = 0$ in B, then we have $b_i = 0$ for all i.*

Proof. Since A is LM, the natural map from $A \otimes Z_B(A)$ to B is injective by Lemma 2.20. It is well known that the inclusion map from the algebraic tensor product of A and $Z_B(A)$ to the (maximal) tensor product $A \otimes Z_B(A)$ is injective (see [2, II.9.1.3]). The conclusion follows from these lemmas. □

Definition 6.3. We say that a pair (v_1, v_2) of self-adjoint unitaries v_1, v_2 in a unital C*-algebra is *generic* if the family

$$\begin{aligned}&\big((v_1v_2)^n, (v_1v_2)^n v_1\big)_{n\in\mathbb{Z}}\\ &= (1, v_1, v_2, v_1v_2, v_2v_1, v_1v_2v_1, v_2v_1v_2, v_1v_2v_1v_2, v_2v_1v_2v_1, v_1v_2v_1v_2v_1, \dots)\end{aligned}$$

is linearly independent.

In other words, (v_1, v_2) is generic if and only if the map sending the natural generators of the group algebra $\mathbb{C}((\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}))$ to v_1, v_2 is injective.

Lemma 6.4. *Let v_1, v_2, w_1, w_2 be the four self-adjoint unitaries in the C^*-algebra $C([0,1], M_2(\mathbb{C}))$ defined by*

$$v_1(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad v_2(t) = \begin{pmatrix} \cos(\pi t) & \sin(\pi t) \\ \sin(\pi t) & -\cos(\pi t) \end{pmatrix},$$
$$w_1(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad w_2(t) = \begin{pmatrix} -\sin(\pi t) & \cos(\pi t) \\ \cos(\pi t) & \sin(\pi t) \end{pmatrix}$$

for $t \in [0,1]$. Then v_1, v_2, w_1, w_2 satisfy $v_1 w_1 = -w_1 v_1$, $v_2 w_2 = -w_2 v_2$ and the pair (v_1, v_2) is generic.

Proof. It is routine to check the two equalities $v_1 w_1 = -w_1 v_1$ and $v_2 w_2 = -w_2 v_2$. That the pair (v_1, v_2) is generic comes from the fact that $\{\cos(n\pi t) + \sqrt{-1}\sin(n\pi t)\}_{n\in\mathbb{Z}}$ is linearly independent in $C([0,1])$. We leave the details to the readers. □

Let X be an infinite set, and $[X]^2$ be the set of all subsets of X with cardinality 2. For $\xi = \{x, y\} \in [X]^2$ let A_ξ be the CAR algebra. We fix four self-adjoint unitaries $v_{x,y}, v_{y,x}, w_{x,y}, w_{y,x}$ in A_ξ such that $v_{x,y} w_{x,y} = -w_{x,y} v_{x,y}$, $v_{y,x} w_{y,x} = -w_{y,x} v_{y,x}$ and the pair $(v_{x,y}, v_{y,x})$ is generic. Such unitaries exist by Lemma 6.4 because there exists a unital embedding from $C([0,1], M_2(\mathbb{C}))$ to the CAR algebra.

We define a UHF algebra $A_{[X]^2}$ by $A_{[X]^2} = \bigotimes_{\xi\in[X]^2} A_\xi \cong \bigotimes_{[X]^2\times\aleph_0} M_2(\mathbb{C})$. For a subset Y of X, we set $A_{[Y]^2} = \bigotimes_{\xi\in[Y]^2} A_\xi \subset A_{[X]^2}$.

Definition 6.5. For a set X, we denote by G_X the abelian group consisting of all finite subsets of X where the operation is the symmetric difference Δ.

We often identify an element x of X with a subset $\{x\}$ of X. Thus the group G_X is generated by the family $\{x\}_{x\in X}$ of mutually commuting involutions. Hence G_X is isomorphic to the group $\bigoplus_X(\mathbb{Z}/2\mathbb{Z})$ of the direct sum of $|X|$ copies of $\mathbb{Z}/2\mathbb{Z}$.

For $g \in G_X$ we define an automorphism α_g on $A_{[X]^2}$ by

$$\alpha_g = \bigotimes_{x\in g \text{ and } y\notin g} \operatorname{Ad} v_{x,y}.$$

If $x \notin g$ define unitaries $V_{g;x}$ and $V_{x;g}$ in $A_{[X]^2}$ via

$$V_{g;x} = \prod_{y\in g} v_{y,x} v_{x,y} \qquad \text{and} \qquad V_{x;g} = \prod_{y\in g} v_{x,y} v_{y,x}.$$

Lemma 6.6. *If $x \notin g$ then $\alpha_g \circ \alpha_x = \operatorname{Ad}(V_{g;x}) \circ \alpha_{g\cup\{x\}}$ and $\alpha_{g\cup\{x\}} \circ \alpha_x = \operatorname{Ad}(V_{x;g}) \circ \alpha_g$.*

Proof. Note that $v_{x,y}$ and $v_{z,t}$ commute unless $z = y$ and $x = t$. Using $x \notin g$ we have

$$\begin{aligned}\alpha_g \circ \alpha_x &= \Big(\bigotimes_{y\in g \text{ and } z\notin g} \operatorname{Ad} v_{y,z}\Big) \circ \Big(\bigotimes_{z\neq x} \operatorname{Ad} v_{x,z}\Big) \\ &= \Big(\bigotimes_{y\in g} \operatorname{Ad} v_{y,x} \circ \bigotimes_{y\in g \text{ and } z\notin g\cup\{x\}} \operatorname{Ad} v_{y,z}\Big) \circ \Big(\bigotimes_{z\in g} \operatorname{Ad} v_{x,z} \circ \bigotimes_{z\notin g\cup\{x\}} \operatorname{Ad} v_{x,z}\Big) \\ &= \operatorname{Ad}(V_{g;x}) \circ \alpha_{g\cup\{x\}}\end{aligned}$$

This proves the first equality. Since α_x is an involution and $V_{x;g} = V^*_{g;x}$, the first equality implies the second equality. □

Let us choose a faithful representation $A_{[X]^2} \subset B(H)$ on some Hilbert space H (see Section 7 for one construction of such a representation). Let $\ell^2(G_X, H)$ be the Hilbert space consisting of functions $\xi : G_X \to H$ with $\sum_{g\in G_X} \|\xi(g)\|^2 < \infty$. We embed $A_{[X]^2}$ into $B(\ell^2(G_X, H))$ by

$$(a\xi)(g) = \alpha_g(a)\xi(g) \in H$$

for $a \in A_{[X]^2}$, $\xi \in \ell^2(G_X, H)$ and $g \in G_X$. For each $x \in X$, we define $u_x \in B(\ell^2(G_X, H))$ by

$$\begin{aligned}(u_x\xi)(g) &= V_{g;x}\,\xi(g\cup\{x\}) \in H \\ (u_x\xi)(g\cup\{x\}) &= V_{x;g}\,\xi(g) \in H\end{aligned}$$

for $\xi \in \ell^2(G_X, H)$ and $g \in G_X$ with $x \notin g$.

Lemma 6.7. *For each $x \in X$, u_x is a self-adjoint unitary such that $\operatorname{Ad} u_x$ and α_x agree on $A_{[X]^2} \subset B(\ell^2(G_X, H))$.*

Proof. For $g \in G_X$ such that $x \notin g$ the subspace $\ell^2(\{g, g\cup\{x\}\}, H) \subset \ell^2(G_X, H)$ is invariant for u_x, and u_x is represented on it as

$$u_x = \begin{pmatrix} 0 & V_{g;x} \\ V_{x;g} & 0 \end{pmatrix}.$$

This shows that u_x is a self-adjoint unitary. To show that $\operatorname{Ad} u_x$ and α_x agree on $A_{[X]^2} \subset B(\ell^2(G_X, H))$, it suffices to see

$$\begin{aligned}\operatorname{Ad}(V_{x;g}) \circ \alpha_g &= \alpha_{g\cup\{x\}} \circ \alpha_x \\ \operatorname{Ad}(V_{g;x}) \circ \alpha_{g\cup\{x\}} &= \alpha_g \circ \alpha_x\end{aligned}$$

which is Lemma 6.6. □

By Lemma 6.7 we see that for $\{x, y\} \in [X]^2$ and $z \in X$ we have $\operatorname{Ad} u_x \restriction_{A_{\{x,y\}}} = \operatorname{Ad} v_{x,y}$, and $\operatorname{Ad} u_z \restriction_{A_{\{x,y\}}} = \mathrm{id}$ if $z \notin \{x, y\}$. In particular, u_z commutes with $v_{x,y}$ unless $y = z$.

Lemma 6.8. *For $\{x, y\} \in [X]^2$ the two self-adjoint unitaries $u_xv_{x,y}$ and $u_yv_{y,x}$ commute.*

Proof. Take $\{x, y\} \in [X]^2$. First note that for $h \in G_X$ we have $\alpha_h(v_{x,y}) = v_{y,x}v_{x,y}v_{y,x}$ if $y \in h$ and $x \notin h$, and $\alpha_h(v_{x,y}) = v_{x,y}$ otherwise.

Fix $g \in G_X$ such that $x \notin g$ and $y \notin g$. The subspace

$$H_g = \ell^2\big(\{g, g \cup \{x\}, g \cup \{y\}, g \cup \{x, y\}\}, H\big) \subset \ell^2(G_X, H)$$

is invariant for each of u_x, u_y, $v_{x,y}$ and $v_{y,x}$. Using the observation in the beginning of this proof, we see that u_x, u_y, $v_{x,y}$ and $v_{y,x}$ are represented on H_g by

$$u_x = \begin{pmatrix} 0 & V_{g;x} & 0 & 0 \\ V_{x;g} & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{g\cup\{y\};x} \\ 0 & 0 & V_{x;g\cup\{y\}} & 0 \end{pmatrix}, \quad v_{x,y} = \begin{pmatrix} v_{x,y} & 0 & 0 & 0 \\ 0 & v_{x,y} & 0 & 0 \\ 0 & 0 & v_{y,x}v_{x,y}v_{y,x} & 0 \\ 0 & 0 & 0 & v_{x,y} \end{pmatrix},$$

$$u_y = \begin{pmatrix} 0 & 0 & V_{g;y} & 0 \\ 0 & 0 & 0 & V_{g\cup\{x\};y} \\ V_{y;g} & 0 & 0 & 0 \\ 0 & V_{y;g\cup\{x\}} & 0 & 0 \end{pmatrix}, \quad v_{y,x} = \begin{pmatrix} v_{y,x} & 0 & 0 & 0 \\ 0 & v_{x,y}v_{y,x}v_{x,y} & 0 & 0 \\ 0 & 0 & v_{y,x} & 0 \\ 0 & 0 & 0 & v_{y,x} \end{pmatrix}.$$

Using the computations such as $V_{x;g\cup\{y\}} = V_{x;g}v_{x,y}v_{y,x}$, we see that $u_xv_{x,y}$ and $u_yv_{y,x}$ are represented on H_g by

$$u_xv_{x,y} = \begin{pmatrix} 0 & V_{g;x}v_{x,y} & 0 & 0 \\ V_{x;g}v_{x,y} & 0 & 0 & 0 \\ 0 & 0 & 0 & V_{g;x}v_{y,x} \\ 0 & 0 & V_{x;g}v_{y,x} & 0 \end{pmatrix},$$

$$u_yv_{y,x} = \begin{pmatrix} 0 & 0 & V_{g;y}v_{y,x} & 0 \\ 0 & 0 & 0 & V_{g;y}v_{x,y} \\ V_{y;g}v_{y,x} & 0 & 0 & 0 \\ 0 & V_{y;g}v_{x,y} & 0 & 0 \end{pmatrix}.$$

The unitaries $V_{g;x}, V_{x;g}, V_{g;y}, V_{y;g}, v_{x,y}$ and $v_{y,x}$ occurring in entries of these two matrices commute with each others except that $v_{x,y}$ does not commute with $v_{y,x}$. Using this fact, one can show that both $(u_xv_{x,y})(u_yv_{y,x})$ and $(u_yv_{y,x})(u_xv_{x,y})$ are equal to

$$\begin{pmatrix} 0 & 0 & 0 & V_{g;x}V_{g;y} \\ 0 & 0 & V_{x;g}V_{g;y}v_{x,y}v_{y,x} & 0 \\ 0 & V_{g;x}V_{y;g}v_{y,x}v_{x,y} & 0 & 0 \\ V_{x;g}V_{y;g} & 0 & 0 & 0 \end{pmatrix}.$$

Therefore $u_xv_{x,y}$ and $u_yv_{y,x}$ commute. □

Let

$$B_{[X]^2} := C^*(A_{[X]^2} \cup \{u_x\}_{x\in X}) \subset B\big(\ell^2(G_X, H)\big).$$

For a subset $Y \subset X$, we define

$$B_{[Y]^2} := C^*(A_{[Y]^2} \cup \{u_x\}_{x\in Y}) \subset B_{[X]^2}.$$

Remark 6.9. The C*-algebra $B_{[X]^2}$ does not depend on the choices of embeddings $A_{[X]^2} \subset B(H)$, and is isomorphic to a cocycle crossed product $A_{[X]^2} \rtimes_{(\alpha,c)} G_X$ for an appropriate cocycle action (α, c) (see [10] for definitions of cocycle actions and cocycle crossed products). In fact, the proof of Proposition 6.10 shows that any C*-algebra generated by $A_{[X]^2} \cup \{u_x\}_{x\in X}$ with the relations in Lemma 6.7 and Lemma 6.8 is isomorphic to $B_{[X]^2}$.

Proposition 6.10. *The C*-algebra $B_{[X]^2}$ is a unital LM algebra with $\chi(B_{[X]^2}) = |X|$.*

Proof. By Lemma 2.17, we have $\chi(A_{[X]^2}) = |X|$. This implies $\chi(B_{[X]^2}) = |X|$.

We are going to show that $B_{[X]^2}$ is a direct limit of CAR algebras. This implies that $B_{[X]^2}$ is LM. For a finite subset $F \subset X$ and an injective map $\iota\colon F \to X \setminus F$, define a subalgebra $D_{(F,\iota)} \subset B_{[X]^2}$ by

$$D_{(F,\iota)} := C^*\big(B_{[F]^2} \cup \{w_{x,\iota(x)}\}_{x\in F}\big) \subset B_{[X]^2}.$$

The family $\{D_{(F,\iota)}\}_{(F,\iota)}$ of subalgebras is directed because X is infinite, and its union is dense in $B_{[X]^2}$. Thus it suffices to show that $D_{(F,\iota)}$ is the CAR algebra for every finite subset $F \subset X$ and every injective map $\iota\colon F \to X \setminus F$.

Take a finite subset $F \subset X$ and an injective map $\iota\colon F \to X \setminus F$. For $x \in F$, we define

$$u'_x := u_x \prod_{y\in F\setminus\{x\}} v_{x,y} \in D_{(F,\iota)}.$$

which is a self-adjoint unitary. Since Lemma 6.7 shows

$$\operatorname{Ad} u_x \restriction_{A_{[F]^2}} = \alpha_x \restriction_{A_{[F]^2}} = \operatorname{Ad}\Big(\prod_{y\in F\setminus\{x\}} v_{x,y}\Big) \restriction_{A_{[F]^2}},$$

u'_x commutes with the subalgebra $A_{[F]^2}$. The family $\{u'_x\}_{x\in F}$ mutually commutes by Lemma 6.8. For each $x \in F$, the self-adjoint unitary $w_{x,\iota(x)} \in D_{(F,\iota)}$ commutes with $A_{[F]^2}$ and $\{w_{y,\iota(y)}, u'_y\}_{y\in F\setminus\{x\}}$, and satisfies $u'_x w_{x,\iota(x)} = -w_{x,\iota(x)} u'_x$. Therefore $C^*(u'_x, w_{x,\iota(x)})$ is isomorphic to $M_2(\mathbb{C})$ for $x \in F$ by Lemma 4.1, and the family

$$\{C^*(u'_x, w_{x,\iota(x)})\}_{x\in F} \cup \{A_{[F]^2}\}$$

mutually commutes. Since $D_{(F,\iota)}$ is generated by these mutually commuting subalgebras, we get

$$D_{(F,\iota)} = A_{[F]^2} \otimes \bigotimes_{x\in F} C^*(u'_x, w_{x,\iota(x)}) \cong \bigotimes_{|[F]^2|\times\aleph_0+|F|} M_2(\mathbb{C}).$$

We are done. ∎

Lemma 6.11. *Let Y be a nonempty proper subset of X. Take $x \in Y$ and $y \in X \setminus Y$. Then every element in $B_{[Y]^2} \subset B_{[X]^2}$ can be written as $av_{x,y} + a'$ for $a, a' \in Z_{B_{[X]^2}}(A_{\{x,y\}})$.*

Proof. Since $v_{x,y}$ is a self-adjoint unitary in $A_{\{x,y\}}$, the set of all elements in the form $av_{x,y}+a'$ for $a,a' \in Z_{B_{[X]^2}}(A_{\{x,y\}})$ is a subalgebra of $B_{[X]^2}$. Hence it suffices to show that the generators $A_{[Y]^2} \cup \{u_z\}_{z\in Y}$ of $B_{[Y]^2}$ are in this form. We have $A_{[Y]^2} \subset Z_{B_{[X]^2}}(A_{\{x,y\}})$ since $y \notin Y$. We have $u_z \in Z_{B_{[X]^2}}(A_{\{x,y\}})$ for $z \in Y\setminus\{x\}$. Finally, we get $u_x = (u_x v_{x,y})v_{x,y}$ and $u_x v_{x,y} \in Z_{B_{[X]^2}}(A_{\{x,y\}})$. We are done. □

Proposition 6.12. *If $|X| > \aleph_1$ then $B_{[X]^2}$ is not AF.*

Proof. For the sake of obtaining a contradiction, assume that $B_{[X]^2}$ is AF. Then by Lemma 6.1 there exists a family $\{b_x\}_{x\in X}$ in $B_{[X]^2}$ with $\|u_x - b_x\| < 1$ for all $x \in X$ such that $C^*(\{b_x\}_{x\in F}) \subset B_{[X]^2}$ is finite-dimensional for all finite subsets F of X.

For each $x \in X$, there exists a countable subset Y_x of X with $x \in Y_x$ such that $b_x \in B_{[Y_x]^2}$. Since $|X| > \aleph_1$, we can apply Lemma 2.1 to get $\{x,y\} \in [X]^2$ such that $x \notin Y_y$ and $y \notin Y_x$. By Lemma 6.11, there exists $a_x, a'_x, a_y, a'_y \in Z_{B_{[X]^2}}(A_{\{x,y\}})$ such that $b_x = a_x v_{x,y} + a'_x$ and $b_y = a_y v_{y,x} + a'_y$. Since $\|u_x - b_x\| < 1$, we have

$$\big\|\big((u_x - b_x) - w_{x,y}(u_x - b_x)w_{x,y}\big)/2\big\| < 1.$$

We have

$$\big(b_x - w_{x,y} b_x w_{x,y}\big)/2 = \big((a_x v_{x,y} + a'_x) - (-a_x v_{x,y} + a'_x)\big)/2 = a_x v_{x,y},$$

and similarly $(u_x - w_{x,y} u_x w_{x,y})/2 = u_x$. Hence we get $\|u_x - a_x v_{x,y}\| < 1$. Thus $\|u_x v_{x,y} - a_x\| < 1$. Since $u_x v_{x,y}$ is a unitary, a_x is an invertible element. Similarly, one can show that a_y is also invertible.

By the assumption, $C^*(\{b_x, b_y\})$ is finite-dimensional. Therefore $\{(b_x b_y)^n\}_{n=0}^{\infty}$ is linearly dependent. Hence there exist $N \in \mathbb{N}$ and $\lambda_0, \lambda_1, \ldots, \lambda_N \in \mathbb{C}$ with $\lambda_N \neq 0$ such that $\sum_{n=0}^{N} \lambda_n (b_x b_y)^n = 0$. We can write

$$\sum_{n=0}^{N} \lambda_n (b_x b_y)^n = \sum_{n=0}^{N} \lambda_n \big((a_x v_{x,y} + a'_x)(a_y v_{y,x} + a'_y)\big)^n = \sum_{v\in V} f_v v$$

where

$$V := \{1, v_{x,y}, v_{y,x}, v_{x,y}v_{y,x}, v_{y,x}v_{x,y}, v_{x,y}v_{y,x}v_{x,y}, v_{y,x}v_{x,y}v_{y,x}, \ldots, (v_{x,y}v_{y,x})^N\}$$

and for each $v \in V$, $f_v \in Z_{B_{[X]^2}}(A_{\{x,y\}})$ is a sum of products of $\lambda_0, \lambda_1, \ldots, \lambda_N \in \mathbb{C}$ and $a_x, a'_x, a_y, a'_y \in Z_{B_{[X]^2}}(A_{\{x,y\}})$. Since $V \subset A_{\{x,y\}}$ is linearly independent, we get $f_v = 0$ for all $v \in V$ by Lemma 6.2. In particular, $f_{(v_{x,y}v_{y,x})^N} = \lambda_N (a_x a_y)^N \in Z_{B_{[X]^2}}(A_{\{x,y\}})$ is 0. This cannot happen because $\lambda_N \neq 0$ and both a_x and a_y are invertible. Thus we get a contradiction. We are done. □

Remark 6.13. When $|X| = \aleph_0$, $B_{[X]^2}$ is a UHF algebra (in fact CAR algebra) by Glimm's theorem [8, Theorem 1.13]. When $|X| = \aleph_1$, $B_{[X]^2}$ is a unital AM algebra by Proposition 6.10 and Theorem 1.3 (1). In this case one can show that $B_{[X]^2}$ is not UHF in a similar (but much more complicated) way to the proof of Proposition 4.5 (2) (see [7]).

Remark 6.14. As we pointed out in Remark 4.2, the examples in Section 4 of unital AM algebras which are not UHF are obtained as crossed products of UHF algebras by the group $\mathbb{Z}/2\mathbb{Z}$. The examples in this section of unital LM algebras which are not AM are obtained as *cocycle* crossed products (see Remark 6.9). However we do not know the following.

Problem 6.15. Find an example of a unital LM algebra which is not AM such that it is obtained as a crossed product of a unital AM (or UHF) algebra by a discrete group.

Remark 6.16. We can solve the non-unital version of this problem using the examples in this section. In fact, by [10, Corollary 3.7] the tensor product $B_{[X]^2} \otimes K$ is obtained as an (ordinary) crossed product of $A_{[X]^2} \otimes K$ by the group G_X where $K := K\big(\ell^2(G_X)\big)$ is the non-unital AM algebra of all compact operators on the Hilbert space $\ell^2(G_X)$. Thus for every cardinal $\kappa > \aleph_1$, there exists an example of a *non-unital* LM algebra with character density κ which is not AM such that it is obtained as a crossed product of a non-unital AM algebra by a discrete group. Note that $B_{[X]^2} \otimes K$ is not AM if $B_{[X]^2}$ is not AM because every corner of an AM algebra is AM.

The same comments can be applied to LF and AF instead of LM and AM.

7. Representation density and character density

The purpose of this section is to give an answer to the half of the question raised by Masamichi Takesaki when the second author gave a talk on this paper. We could not answer the other half (Problem 7.19). The proof uses the construction (Proposition 7.12) that was given by Bruce Blackadar when the first author gave a talk. Both authors would like to thank Masamichi Takesaki and Bruce Blackadar.

For a Hilbert space H, we also denote by $\chi(H)$ the smallest cardinality of a dense subset of A. Note that for an infinite set X, we get $\chi(H) = |X|$ if and only if H is isomorphic to $\ell^2(X)$.

Definition 7.1. The *representation density* $\chi_{\mathrm{r}}(A)$ of a C*-algebra A is the smallest cardinal $\chi(H)$ where H is a Hilbert space on which A can be faithfully represented.

Note that both the representation density χ_{r} and the character density χ (Definition 1.2) are monotonic in the sense that if A is a subalgebra of B then the density of B is not smaller than the density of A.

Since these cardinal invariants of C*-algebras were apparently not considered previously, the reader will hopefully excuse us for starting this section by listing a few trivial statements.

Lemma 7.2. *For every C*-algebra A we have that*

$$\chi(A) \geq \sup\big\{|X| : X \textit{ is a family of commuting projections in } A\big\}$$
$$\chi_{\mathrm{r}}(A) \geq \sup\big\{|X| : X \textit{ is a family of nonzero orthogonal projections in } A\big\}.$$

Proof. For the first part note that if p and q are distinct commuting projections then $\|p-q\| = 1$. The second part is obvious. □

Lemma 7.3. *For every infinite-dimensional Hilbert space H we have*

$$\chi\big(\mathcal{B}(H)\big) = |\mathcal{B}(H)| = 2^{\chi(H)}.$$

Proof. Let us choose an infinite set X with $|X| = \chi(H)$, and identify H with $\ell^2(X)$. For a subset $Y \subset X$, let $p_Y \in \mathcal{B}(H)$ be the projection onto the subspace $\ell^2(Y) \subset H$. Then $\{p_Y\}_{Y\subset X}$ is a family of commuting projections of size $2^{|X|}$. Thus we have $\chi(\mathcal{B}(H)) \geq 2^{|X|}$ by Lemma 7.2. For $x, y \in X$, $p_{\{x\}}\mathcal{B}(H)p_{\{y\}}$ is one dimensional, and the map

$$\mathcal{B}(H) \ni T \mapsto \big(p_{\{x\}}Tp_{\{y\}}\big)_{x,y\in X} \in \prod_{x,y\in X}\big(p_{\{x\}}\mathcal{B}(H)p_{\{y\}}\big) \cong \prod_{x,y\in X}\mathbb{C}$$

is injective. Hence we get $\chi(\mathcal{B}(H)) \leq |\mathcal{B}(H)| \leq |\mathbb{C}|^{|X\times X|} = 2^{|X|}$. We are done. □

If $K = 2^{2^{\aleph_0}}$ with the product topology then $C(K)$ is an abelian C*-algebra with character density $2^{\aleph_0}$ and representation density $\aleph_0$. The first claim follows by Lemma 7.2 and by the fact that K has $2^{\aleph_0}$ distinct clopen sets. The second claim follows from the fact that K is, being a product of $2^{\aleph_0}$ separable spaces, separable. See also Corollary 7.7, Theorem 7.17 and Problem 7.19.

Lemma 7.4. *For every C*-algebra A we have $\chi_{\mathrm{r}}(A) \leq \chi(A) \leq 2^{\chi_{\mathrm{r}}(A)}$.*

Proof. Choose a subset $X \subset A$ with $|X| = \chi(A)$. For each $x \in X$, there exists a cyclic representation $\pi_x \colon A \to \mathcal{B}(H_x)$ with $\|\pi_x(x)\| = \|x\|$ (see [2, Corollary II.6.4.9]). Since H_x has a cyclic vector for π_x, we have $\chi(H_x) \leq \chi(A)$. Then the representation

$$\pi := \bigoplus_{x\in X}\pi_x \colon A \to \mathcal{B}\Big(\bigoplus_{x\in X} H_x\Big)$$

is faithful, and

$$\chi\Big(\bigoplus_{x\in X} H_x\Big) = \sum_{x\in X}\chi(H_x) \leq |X| \times \chi(A) = \chi(A)$$

Hence $\chi_{\mathrm{r}}(A) \leq \chi(A)$. The second inequality $\chi(A) \leq 2^{\chi_{\mathrm{r}}(A)}$ follows from Lemma 7.3. □

Lemma 7.5. *Let $X_0 \ni x \mapsto \xi_x \in H$ be a map from a set X_0 to a Hilbert space H such that $|X_0| > \chi(H)$. Then for every $\varepsilon > 0$, there exists $X_1 \subset X_0$ with $|X_1| > \chi(H)$ such that $\|\xi_x - \xi_y\| < \varepsilon$ for every $x, y \in X_1$.*

Proof. Choose a dense subset $Y \subset H$ with $|Y| = \chi(H)$. For each $x \in X_0$ there exists $\eta(x) \in Y$ such that $\|\xi_x - \eta(x)\| < \varepsilon/2$. Since $|X_0| > \chi(H) = |Y|$, there exists $\eta \in Y$ such that the set $X_1 := \{x \in X_0 : \eta(x) = \eta\} \subset X_0$ satisfies $|X_1| > \chi(H)$. Then for every $x, y \in X_1$, we get

$$\|\xi_x - \xi_y\| \leq \|\xi_x - \eta\| + \|\xi_y - \eta\| < \varepsilon. \qquad \square$$

Proposition 7.6. *For a family $\{A_x\}_{x\in X}$ of nonabelian unital C*-algebras, the representation density of the tensor product $A = \bigotimes_{x\in X} A_x$ is at least $|X|$.*

Proof. Assume the contrary and fix a faithful representation $\pi \colon A \to \mathcal{B}(H)$ for a Hilbert space H with $|X| > \chi(H)$. Note that this assumption implies that X is uncountable. For each $x \in X$, fix a_x and b_x in the unit ball of A_x such that $a_x b_x \neq b_x a_x$. Since π is faithful, we can choose a vector $\xi_x \in H$ such that

$$\pi(a_x b_x - b_x a_x)\xi_x \neq 0.$$

Since X is uncountable, there exist $\delta > 0$ and a subset $X_0 \subset X$ with $|X_0| > \chi(H)$ such that for all $x \in X_0$ we have

$$\|\pi(a_x b_x - b_x a_x)\xi_x\| \geq \delta.$$

Set $\varepsilon = \delta/4 > 0$. In this proof, we write $a \approx_\varepsilon b$ if $\|a - b\| < \varepsilon$. Since $|X_0| > \chi(H)$, we can apply Lemma 7.5 to $\{\xi_x\}_{x\in X_0}$ and $\varepsilon > 0$ to get $X_1 \subset X_0$ with $|X_1| > \chi(H)$ such that $\xi_x \approx_\varepsilon \xi_y$ for every $x, y \in X_1$. By applying Lemma 7.5 three more times to $\{\pi(a_x)\xi_x\}_{x\in X_1}$ and so on, we get $X_4 \subset X_1$ with $|X_4| > \chi(H)$ such that

$$\pi(a_x)\xi_x \approx_\varepsilon \pi(a_y)\xi_y, \quad \pi(b_x)\xi_x \approx_\varepsilon \pi(b_y)\xi_y,$$
$$\pi(b_x a_x)\xi_x \approx_\varepsilon \pi(b_y a_y)\xi_y$$

for every $x, y \in X_4$. Since $|X_4| > \chi(H) \geq \aleph_0$, we can take two distinct $x, y \in X_4$. Then we have

$$\begin{array}{c} \pi(a_x b_x)\xi_x = \pi(a_x)\pi(b_x)\xi_x \approx_\varepsilon \pi(a_x)\pi(b_y)\xi_y = \pi(a_x b_y)\xi_y \approx_\varepsilon \pi(a_x b_y)\xi_x \\ \hfill \| \\ \pi(b_x a_x)\xi_x \approx_\varepsilon \pi(b_y a_y)\xi_y = \pi(b_y)\pi(a_y)\xi_y \approx_\varepsilon \pi(b_y)\pi(a_x)\xi_x = \pi(b_y a_x)\xi_x \end{array}$$

because $a_x \in A_x \subset A$ and $b_y \in A_y \subset A$ commute. Thus we get

$$\big\|\pi(a_x b_x - b_x a_x)\xi_x\big\| < 4\varepsilon = \delta,$$

which is a contradiction. This completes the proof. $\square$

Corollary 7.7. *If A is a UHF algebra then $\chi(A) = \chi_{\mathrm{r}}(A)$.* $\square$

With the possible exception of the algebras $A_{X,Y}$ as defined in §3, each example of an AM, or even LM, algebra given so far has a UHF subalgebra with the same character density. Since the algebras $A_{X,Y}$ are tensor products of separable algebras, Proposition 7.6 implies that for each AM or LM algebra A so far defined in this paper we have $\chi(A) = \chi_{\mathrm{r}}(A)$. We are going to show that $\chi(A)$ can be any cardinality between $\chi_{\mathrm{r}}(A)$ and $2^{\chi_{\mathrm{r}}(A)}$ for unital AM algebras A.

Let X be an infinite set. As in Section 4, let A_x be a C*-algebra generated by two self-adjoint unitaries v_x, w_x with $v_x w_x = -w_x v_x$ for each $x \in X$, and let $A_X := \bigotimes_{x\in X} A_x$. By Lemma 4.1, $A_x \cong M_2(\mathbb{C})$ for each $x \in X$ and hence $A_X \cong \bigotimes_X M_2(\mathbb{C})$ is a UHF algebra. For each $Y \subseteq X$, we set

$$A_Y := \bigotimes_{x\in Y} A_x \subset A_X.$$

We are going to use the GNS representation of A_X associated with the unique tracial state of A_X. For the reader's convenience we explain what it is. For each finite subset $F \subset X$, there exists a unique linear functional $\tau_F \colon A_F \to \mathbb{C}$ satisfying the trace condition $\tau_F(ab) = \tau_F(ba)$ for $a, b \in A_F$ and the normalized condition $\tau_F(1) = 1$. If $|F| = n$, then we have $\tau_F = 2^{-n}\mathrm{Tr}$ where Tr is the usual trace of $A_F \cong M_{2^n}(\mathbb{C})$. It is easy to see that τ_F is positive and faithful, that is, $\tau_F(a^*a) > 0$ for all $a \in A_F \setminus \{0\}$. Let $A_X^{\mathrm{fin}} := \bigcup_{F\subset X} A_F \subset A_X$ where F runs all finite subsets of X. By the uniqueness of the tracial state τ_F, we get $\tau_{F'} \restriction_{A_F} = \tau_F$ for two finite subsets $F \subset F' \subset X$. Thus we get a linear map $\tau \colon A_X^{\mathrm{fin}} \to \mathbb{C}$ such that $\tau \restriction_{A_F} = \tau_F$ for every finite subset $F \subset X$. Although we do not need it, we would like to remark that τ can be extended to the unique tracial state of A_X (cf. [5, Lemma I.9.5]). We define an inner product on A_X^{fin} by $A_X^{\mathrm{fin}} \times A_X^{\mathrm{fin}} \ni (a, b) \mapsto \tau(ab^*) \in \mathbb{C}$. Then the completion H_X of A_X^{fin} with respect to the norm coming from the inner product defined as above becomes a Hilbert space. The embedding from A_X^{fin} to H_X is denoted by $A_X^{\mathrm{fin}} \ni a \mapsto \hat{a} \in H_X$. The image of this embedding is dense in H_X. For each finite subset $F \subset X$ and each $a \in A_F$, it is easy to see that the map $\hat{b} \mapsto \widehat{ab}$ extends to a bounded operator on H_X. Thus we get a $*$-homomorphism $\pi_F \colon A_F \to B(H_X)$ such that $\pi_F(a)(\hat{b}) = \widehat{ab}$ for $a \in A_F$ and $b \in A_X^{\mathrm{fin}}$. We have $\pi_{F'} \restriction_{A_F} = \pi_F$ for two finite subsets $F \subset F' \subset X$. Since the family $\{\pi_{\{x\}}[A_{\{x\}}]\}_{x\in X}$ mutually commutes, we get a representation $\pi \colon A_X \to B(H_X)$ such that $\pi \restriction_{A_F} = \pi_F$ for every finite subset $F \subset X$. This representation is called the GNS representation associated with τ. Since $\pi(a)(\widehat{a^*}) = \widehat{aa^*} \neq 0$ for all $F \subset X$ and all $a \in A_F \setminus \{0\}$, π is injective. In order to simplify the notation we identify A_X with the subalgebra $\pi[A_X]$ of $B(H_X)$.

Lemma 7.8. *We have $\chi(H_X) = |X|$.*

Proof. Since the union of finite-dimensional subspaces $\{\hat{a} \in H_X \mid a \in A_F\}$ for finite subsets $F \subset X$ is dense in H_X, we have $\chi(H_X) \leq |X|$. For distinct $x, y \in X$, we have $\tau(u_x u_y) = 0$ because

$$\begin{aligned} \tau(u_x u_y) &= \tau(w_x(w_x u_x u_y)) = \tau((w_x u_x u_y) w_x) \\ &= \tau(w_x u_x (w_x u_y)) = \tau(w_x(-w_x u_x) u_y) = -\tau(u_x u_y). \end{aligned}$$

Hence we get

$$\|\widehat{u_x} - \widehat{u_y}\|^2 = \tau((u_x - u_y)(u_x - u_y)) = \tau(2 - 2u_x u_y) = 2$$

for all $x, y \in X$ with $x \neq y$. This shows that $\chi(H_X) \geq |X|$. Thus we get $\chi(H_X) = |X|$. □

We can consider the power-set $\mathcal{P}(X)$ of a set X as an abelian group with respect to the symmetric difference. This group is naturally isomorphic to the direct product of X copies of $\mathbb{Z}/2\mathbb{Z}$. For $g \in \mathcal{P}(X)$ consider an automorphism of A_X defined by

$$\alpha_g = \bigotimes_{x \in g} \mathrm{Ad}\, v_x.$$

Then α defines an action of $\mathcal{P}(X)$ on A_X. For each $g \in \mathcal{P}(X)$, the automorphism α_g preserves the subalgebra $A_F \subset A_X$ and satisfies $\tau_F \circ \alpha_g = \tau_F$ for every finite subset $F \subset X$. Hence we get an element $u_g \in B(H_X)$ such that $u_g(\widehat{b}) = \widehat{\alpha_g(b)}$ for $b \in A_X^{\mathrm{fin}}$.

Lemma 7.9. *The elements $\{u_g\}_{g \in \mathcal{P}(X)} \subset B(H_X)$ are self-adjoint unitaries satisfying $u_g a u_g = \alpha_g(a)$ and $u_g u_h = u_{gh}$ for $a \in A_X \subset B(H_X)$ and $g, h \in \mathcal{P}(X)$.*

Proof. Take $g \in \mathcal{P}(X)$. Since α_g preserves τ, the element $u_g^* \in B(H_X)$ satisfies $u_g^*(\widehat{b}) = \widehat{\alpha_g^{-1}(b)}$ for $b \in A_X^{\mathrm{fin}}$. Hence u_g is a unitary. This is self-adjoint because $\alpha_g^{-1} = \alpha_g$. The latter two equalities follow from the equations $\alpha_g(a\alpha_g(b)) = \alpha_g(a)b$ and $\alpha_g(\alpha_h(b)) = \alpha_{gh}(b)$ for $b \in A_X$. □

Definition 7.10. For an infinite set X and a subgroup $\Gamma \subset \mathcal{P}(X)$ we define

$$B_{X,\Gamma} := C^*(A_X \cup \{u_g\}_{g \in \Gamma}) \subset B(H_X).$$

Remark 7.11. One can show that $B_{X,\Gamma}$ is isomorphic to the crossed product $A_X \rtimes_\alpha \Gamma$. In particular B_X in Section 4 is isomorphic to $B_{X,\Gamma}$ for $\Gamma = \{\emptyset, X\} \cong \mathbb{Z}/2\mathbb{Z}$.

Proposition 7.12. *The C^*-algebra $B_{X,\Gamma}$ satisfies $\chi(B_{X,\Gamma}) = |X| + |\Gamma|$ and $\chi_{\mathrm{r}}(B_{X,\Gamma}) = |X|$.*

Proof. We have $\chi(A_X) = |X|$ by Lemma 2.17. On the other hand, we have $\chi(C^*(\{u_g\}_{g\in\Gamma})) \geq |\Gamma|$ by Lemma 7.2 because $\{(u_g + 1)/2\}_{g \in \Gamma}$ is a family of commuting projections. Since $B_{X,\Gamma}$ is generated by A_X and $\{u_g\}_{g\in\Gamma}$, we get

$$|X| + |\Gamma| = \max\{|X|, |\Gamma|\} \leq \chi(B_{X,\Gamma}) \leq |X| + |\Gamma|$$

This shows $\chi(B_{X,\Gamma}) = |X| + |\Gamma|$. Since $B_{X,\Gamma} \subset B(H_X)$, we have $\chi_{\mathrm{r}}(B_{X,\Gamma}) \leq \chi(H_X) = |X|$ by Lemma 7.8. We also have $\chi_{\mathrm{r}}(B_{X,\Gamma}) \geq \chi_{\mathrm{r}}(A_X) = |X|$ by Corollary 7.7. Hence we get $\chi_{\mathrm{r}}(B_{X,\Gamma}) = |X|$. □

Proposition 7.13. *The unital C^*-algebra $B_{X,\Gamma}$ is AM if every finite subset of Γ is included in a subgroup generated by $g_1, g_2, \ldots, g_n \in \Gamma$ which are infinite and mutually disjoint.*

Proof. Take mutually disjoint infinite elements $g_1, g_2, \dots, g_n \in \Gamma$. Take a finite subset F of X and choose $x_i \in g_i \setminus F$ for $i = 1, 2, \dots, n$. Let Λ be the set of all such data $\lambda = (\{g_i\}_{i=1}^n, F, \{x_i\}_{i=1}^n)$, and define

$$D_\lambda := C^*\big(\{u_{g_i}\}_{i=1}^n \cup A_F \cup \{w_{x_i}\}_{i=1}^n\big) \subset B_{X,\Gamma}.$$

By the assumption of Γ, the family $\{D_\lambda\}_{\lambda\in\Lambda}$ of subalgebras is directed and its union is dense in $B_{[X]^2}$. We are going to show $D_\lambda \cong M_{2^{m+n}}(\mathbb{C})$ for $\lambda = (\{g_i\}_{i=1}^n, F, \{x_i\}_{i=1}^n)$ as above where $m = |F|$. This implies that $B_{[X]^2}$ is AM, and hence completes the proof. For $i \in \{1, 2, \dots, n\}$ define

$$u_i' = u_{g_i} \prod_{x\in F\cap g_i} v_x \in D_\lambda.$$

Since

$$\operatorname{Ad} u_{g_i} \restriction_{A_F} = \operatorname{Ad}\Big(\prod_{x\in F\cap g_i} v_x\Big) \restriction_{A_F},$$

u_i' is a self-adjoint unitary and commutes with the subalgebra A_F. It is easy to see that the family $\{u_i'\}_{i=1}^n$ mutually commutes. Since $x_i \in g_i \setminus F$ and g_i is disjoint from g_j for $j \neq i$, we have that w_{x_i} commutes with A_F and $\{u_j', w_{x_j}\}_{j\neq i}$. Finally w_{x_i} and u_i' anti-commute because so do w_{x_i} and u_{g_i}. Therefore $C^*(u_i', w_{x_i})$ is isomorphic to $M_2(\mathbb{C})$ for $i \in \{1, 2, \dots, n\}$ by Lemma 4.1, and the family

$$\{C^*(u_i', w_{x_i})\}_{i=1}^n \cup \{A_F\}$$

mutually commutes. Since D_λ is generated by these mutually commuting subalgebras, we get

$$D_\lambda = \Big(\bigotimes_{i=1}^n C^*(u_i', w_{x_i})\Big) \otimes A_F \cong \bigotimes_{n+|F|} M_2(\mathbb{C}) \cong M_{2^{n+m}}(\mathbb{C}),$$

as required. □

Remark 7.14. For finite $g \in \mathcal{P}(X)$, we have $\alpha_g = \operatorname{Ad}\big(\prod_{x\in g} v_x\big)$. From this fact, one can show that $B_{X,\Gamma}$ is not AM if Γ contains a finite nonempty element g (one can also show that $B_{X,\Gamma}$ is always AF). Thus in order for $B_{X,\Gamma}$ to be AM it is necessary that every $g \in \Gamma\setminus\{\emptyset\}$ is infinite. One can show that this is also sufficient although its proof becomes significantly complicated compared with Proposition 7.13. We shall not need such generality for proving Theorem 7.17.

Remark 7.15. One can show that $B_{X,\Gamma}$ is not UHF when $|X| \geq \aleph_1$ and $\Gamma \neq \{\emptyset\}$ in a similar way to the proof of Proposition 4.5 (2). We omit the proof because we do not need this (see the proof of Theorem 7.17 for some special cases). One can also show that $Z_{B_{X,\Gamma}}(A_X) = \mathbb{C}1$ holds when every $g \in \Gamma\setminus\{\emptyset\}$ is infinite (even in the case $\chi(A_X) < \chi(B_{X,\Gamma})$). This shows that a generalization of question [6, Problem 8.3] for nonseparable AM algebras has a very strong negative answer (see Corollary 4.6). The authors would like to thank Bruce Blackadar for pointing out the phenomenon $Z_{B_{X,\Gamma}}(A_X) =$

$\mathbb{C}1$. This strong phenomenon does not occur for UHF algebras because we can show $\chi(Z_B(A)) = \chi(B)$ for a subalgebra A of a UHF algebra B with $\chi(A) < \chi(B)$, and hence in this case $Z_B(A)$ is huge.

Lemma 7.16. *For every cardinal κ with $|X| \leq \kappa \leq 2^{|X|}$, there exists a subgroup $\Gamma \subset \mathcal{P}(X)$ with $|\Gamma| = \kappa$ such that every finite subset of Γ is included in a subgroup generated by $g_1, g_2, \ldots, g_n \in \Gamma$ which are infinite and mutually disjoint.*

Proof. Take a subset $Y \subset \mathcal{P}(X)$ with $|Y| = \kappa$. Let Γ_0 be the Boolean subalgebra of $\mathcal{P}(X)$ generated by Y, that is the smallest subset of $\mathcal{P}(X)$ containing Y and closed under taking unions, intersections and complements. Then Γ_0 is a subgroup of $\mathcal{P}(X)$ with $|\Gamma_0| = \kappa$. Choose a bijection $\iota\colon X \times \mathbb{N} \to X$ and define an injective homomorphism

$$\varphi\colon \mathcal{P}(X) \ni g \mapsto \iota[g \times \mathbb{N}] \in \mathcal{P}(X).$$

Let $\Gamma := \varphi[\Gamma_0] \subset \mathcal{P}(X)$. Then every finite subset of Γ is included in a finite Boolean subalgebra of Γ. If $g_1, g_2, \ldots, g_n \in \Gamma$ are the atoms of this subalgebra then they clearly satisfy the requirements. □

Theorem 7.17. *For every pair of infinite cardinals κ and ν with $\kappa \geq \aleph_1$ and $\nu \leq \kappa \leq 2^\nu$, there exists a unital AM algebra of character density κ and representation density ν which is not UHF.*

Proof. For $\kappa = \nu \geq \aleph_1$, the example B_X in Proposition 4.5 for $|X| = \kappa$ is a unital AM algebra of character density κ and representation density ν which is not UHF. Suppose $\nu < \kappa \leq 2^\nu$. Take a set X with $|X| = \nu$. By Lemma 7.16, there exists a subgroup $\Gamma \subset \mathcal{P}(X)$ with $|\Gamma| = \kappa$ satisfying the assumption of Proposition 7.13. Then $B_{X,\Gamma}$ is a unital AM algebra of character density κ and representation density ν by Proposition 7.12 and Proposition 7.13. This is not UHF by Corollary 7.7. □

From Theorem 7.17 we have the following.

Corollary 7.18. *There is a unital AM algebra faithfully represented on a separable Hilbert space that is not a UHF algebra.* □

This corollary answers a half of the question raised by Masamichi Takesaki. The following is the other half which we could not answer.

Problem 7.19. Is there an LM algebra faithfully represented on a separable Hilbert space which is not AM?

Since $\chi\big(B(\ell^2(\mathbb{N}))\big) = 2^{\aleph_0}$, by Theorem 1.3 (1) there is no such a C*-algebra if we assume the continuum hypothesis $2^{\aleph_0} = \aleph_1$. We do not know what happens if we do not assume the continuum hypothesis.

Acknowledgment. Many of the results presented in this paper were proved while both authors were visiting the Fields Institute in Fall 2007 and in early 2008. I.F. would like to thank colleagues from his department, in particular Juris Steprāns and Man Wah Wong, for making his stay at the Fields Institute possible. Both authors would like to thank George Elliott, Toshihiko Masuda, Narutaka Ozawa, N. Christopher Phillips, Juris Steprāns, Reiji Tomatsu and Andrew Toms for illuminating conversations.

References

[1] I. Ben Yaacov, A. Berenstein, C.W. Henson, and A. Usvyatsov, *Model theory for metric structures*, Model Theory with Applications to Algebra and Analysis, Vol. II (Z. Chatzidakis et al., eds.), Lecture Notes series of the London Math. Society., no. 350, Cambridge University Press, 2008, pp. 315–427.

[2] B. Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.

[3] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[4] N. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, Amer. Math. Soc., Providence, RI, 2008.

[5] K.R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, vol. 6, Amer. Math. Soc., Providence, RI, 1996.

[6] J. Dixmier, *On some C^*-algebras considered by Glimm*, J. Functional Analysis **1** (1967), 182–203.

[7] I. Farah and T. Katsura, *Nonseparable UHF algebras II: Classification*, in preparation, 2009.

[8] J.G. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.

[9] X. Jiang and H. Su, *On a simple unital projectionless C^*-algebra*, Amer. J. Math. **121** (1999), no. 2, 359–413.

[10] J.A. Packer and I. Raeburn, *Twisted crossed products of C^*-algebras*, Math. Proc. Cambridge Philos. Soc. **106** (1989), no. 2, 293–311.

Department of Mathematics and Statistics, York University, 4700 Keele Street, North York, Ontario, Canada, M3J 1P3, and Matematicki Institut, Kneza Mihaila 34, Belgrade, Serbia

URL: http://www.math.yorku.ca/∼ifarah

E-mail address: ifarah@mathstat.yorku.ca

Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kouhoku-ku, Yokohama, JAPAN, 223-8522

E-mail address: katsura@math.keio.ac.jp