

NONSEPARABLE UHF ALGEBRAS II: CLASSIFICATION

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ABSTRACT. For every uncountable cardinal κ there are 2^κ nonisomorphic simple AF algebras of density character κ and 2^κ nonisomorphic hyperfinite II_1 factors of density character κ . These estimates are maximal possible. All C^* -algebras that we construct have the same Elliott invariant and Cuntz semigroup as the CAR algebra.

1. INTRODUCTION

The classification program of nuclear separable C^* -algebras can be traced back to classification of UHF algebras of Glimm and Dixmier. However, it was Elliott's classification of AF algebras and real rank zero AT algebras that started the classification program in earnest (see e.g., [24] and [8]).

While it was generally agreed that the classification of nonseparable C^* -algebras is a nontractable problem, there were no concrete results to this effect. Methods from logic were recently successfully applied to analyze the classification problem for separable C^* -algebras ([15]) and II_1 factors with separable predual ([25]) and it comes as no surprise that they are also instrumental in analyzing classification of nonseparable operator algebras. We construct large families of nonseparable AF algebras with identical K-theory and Cuntz semigroup as the CAR algebra. Since the CAR algebra is a prototypical example of a classifiable algebra, this gives a strong endorsement to the above viewpoint. We also construct a large family of hyperfinite II_1 factors with predual of density character κ for every uncountable cardinal κ . Recall that a *density character* of a metric space is the least cardinality of a dense subset. While the CAR algebra is unique and there is a unique hyperfinite II_1 factor with separable predual, our results show that uniqueness badly fails in every uncountable density character κ .

For each $n \in \mathbb{N}$, we denote by $M_n(\mathbb{C})$ the unital C^* -algebra of all $n \times n$ matrices with complex entries. A C^* -algebra which is isomorphic to $M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ is called a *full matrix algebra*.

Definition 1.1. A C^* -algebra A is said to be

- *uniformly hyperfinite* (or *UHF*) if A is isomorphic to a tensor product of full matrix algebras.
- *approximately matricial* (or *AM*) if it has a directed family of full matrix subalgebras with dense union.

- locally matricial (or LM) if for any finite subset F of A and any $\varepsilon > 0$, there exists a full matrix subalgebra M of A such that $\text{dist}(a, M) < \varepsilon$ for all $a \in F$.

In [7] Dixmier remarked that in the unital case these three classes coincide under the additional assumption that A is separable and asked whether this result extends to nonseparable algebras. In [19] a pair of nonseparable AF algebras not isomorphic to each other but with the same Bratteli diagram was constructed. Dixmier's question was answered in the negative in [13]. Soon after, AM algebras with counterintuitive properties were constructed. A simple nuclear algebra that has irreducible representations on both separable and nonseparable Hilbert space was constructed in [9] and an algebra with nuclear dimension zero which does not absorb the Jiang–Su algebra tensorially was constructed in [12]. Curiously, all of these results (with the possible exception of [12]) were proved in ZFC.

Results of the present paper widen the gap between unital UHF and AM algebras even further by showing that there are many more AM algebras than UHF algebras of every uncountable density character. In §5 and §6 we prove the following.

Theorem 1.2. *For every uncountable cardinal κ there are 2^κ pairwise non-isomorphic AM algebras with character density κ . All these algebras have the same K_0 , K_1 , and Cuntz semigroup as the CAR algebra.*

Every AM algebra is LM and by Theorem 1.2 there are already as many AM algebras as there are C*-algebras in every uncountable density character. Therefore no quantitative information along these lines can be obtained about LM algebras.

Theorem 1.3. *For every uncountable cardinal κ there are 2^κ nonisomorphic hyperfinite II_1 factors with predual of density character κ .*

While there is a unique hyperfinite II_1 factor with separable predual, it was proved by Widom ([29]) that there are at least as many nonisomorphic hyperfinite II_1 factors with predual of density character κ as there are infinite cardinals $\leq \kappa$.

Note that there are at most 2^κ C*-algebras of density character κ and at most 2^κ von Neumann algebras with predual of density character κ . This is because each such algebra has a dense subalgebra of cardinality κ , and an easy counting argument shows that there are at most 2^κ ways to define $+$, \cdot , $*$ and $\|\cdot\|$ on a fixed set of size κ .

On the positive side, in Proposition 4.2 we show that Glimm's classification of UHF algebras by their generalized integers extends to nonseparable algebras. This shows that the number of isomorphism classes of UHF algebras of density character $\leq \kappa$ is equal to 2^{\aleph_0} , as long as there are only countably many cardinals $\leq \kappa$ (Proposition 4.3 and the table in §7). Hence UHF algebras of arbitrary density character are 'classifiable' in the sense of

Shelah (e.g., [26]). Note, however, that they don't form an elementary class (cf. [5]).

Two C^* -algebras are isomorphic if and only if they are isometric, and the same fact is true for II_1 factors with ℓ_2 -metric. However, in some situations there exist topologically isomorphic but not isometric structures—notably, in the case of Banach spaces. The more general problem of constructing many nonisomorphic models in a given density character was considered in [28].

Organization of the paper. In §2 we set up the toolbox used in the paper. In §3 we study K-theory and Cuntz semigroup of nonseparable LM algebras. UHF algebras are classified in §4. In §5 we prove a non-classification result for AM algebras and hyperfinite II_1 factors in regular density characters. Shelah's methods from [27], as adapted to the context of metric structures in [14] are used to extend this to arbitrary uncountable density characters in §6. In §7 we state some open problems and provide some limiting examples.

The paper requires only basic background in operator algebras (e.g., [2]) and in naive set theory. On several occasions we include remarks aimed at model theorists. Although they provide an additional insight, these remarks can be safely ignored by readers not interested in model theory.

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2. PRELIMINARIES

A cardinal κ is a *successor cardinal* if it is the least cardinal greater than some other cardinal. A cardinal that is not a successor is called a *limit cardinal*. Note that every infinite cardinal is a limit ordinal. Cardinal κ is *regular* if for $X \subseteq \kappa$ we have $\sup X = \kappa$ if and only if $|X| = \kappa$. For example, every successor cardinal is regular. A cardinal that is not regular is *singular*. The least singular cardinal is \aleph_ω and singular cardinal combinatorics is a notoriously difficult subject. A subset C of an ordinal γ is *closed and unbounded* (or *club*) if its supremum is γ and whenever $\delta < \gamma$ is such that $\sup(C \cap \delta) = \delta$ we have $\delta \in C$. A subset of an ordinal γ is called stationary if it intersects every club in γ non-trivially.

Some of the lemmas in the present paper, (e.g., Lemma 2.1) are well-known but we provide proofs for the convenience of the readers.

Lemma 2.1. *If κ is a regular cardinal then there exists a function $S: \mathcal{P}(\kappa) \rightarrow \mathcal{P}(\kappa)$ such that $S(X)\Delta S(Y)$ is stationary whenever $X \neq Y$.*

Proof. We first prove that κ can be partitioned into κ many stationary sets, Z_γ , $\gamma < \kappa$. If κ is a successor cardinal then this is a result of Ulam ([20, Corollary 6.12]). If κ is a limit cardinal, then there are κ regular cardinals below κ . For each such cardinal the set $Z_\gamma = \{\delta < \kappa : \min\{|X| : X \subseteq \delta \text{ and } \sup X = \delta\} = \gamma\}$ is stationary.

For $X \subseteq \kappa$ let $S(X) = \bigcup_{\gamma \in X} Z_\gamma$. Then clearly $S(X)\Delta S(Y)$ is stationary whenever $X \neq Y$. \square

Let $|X|$ denote the cardinality of a set X . We shall now recall some basic set-theoretic notions worked out explicitly in the case of C*-algebras in [13].

Definition 2.2. A directed set Λ is said to be σ -complete if every countable directed $Z \subseteq \Lambda$ has the supremum $\sup Z \in \Lambda$. A directed family $\{A_\lambda\}_{\lambda \in \Lambda}$ of subalgebras of a C*-algebra A is said to be σ -complete if Λ is σ -complete and for every countable directed $Z \subseteq \Lambda$, $A_{\sup Z}$ is the closure of the union of $\{A_\lambda\}_{\lambda \in Z}$.

Assume A is a nonseparable C*-algebra. Then A is a direct limit of a σ -complete directed system of its separable subalgebras ([13, Lemma 2.10]). Also, if A is represented as a direct limit of a σ -complete directed system of separable subalgebras in two different ways, then the intersection of these two systems is a σ -complete directed system of separable subalgebras and A is its direct limit ([13, Lemma 2.6]).

The following was proved in [13, remark following Lemma 2.13].

Lemma 2.3. *A C*-algebra A is LM if and only if it is equal to a union of a σ -complete directed family of separable AM subalgebras.* \square

In §6 we shall use the following well-known fact without mentioning. We give its proof for the reader's convenience.

Lemma 2.4. *Let α be an action of a group G on a unital C*-algebra A . Let $\{u_g\}_{g \in G} \subset A \rtimes_\alpha^r G$ be the implementing unitaries in the reduced crossed product. Suppose that a unital subalgebra $A_0 \subset A$ and a subgroup $G_0 \subset G$ satisfy that $\alpha_g[A_0] = A_0$ for all $g \in G_0$, and set $B_0 := C^*(A_0 \cup \{u_g\}_{g \in G_0})$. Then we have*

$$B_0 \cap A = A_0 \quad \text{and} \quad B_0 \cap \{u_g\}_{g \in G} = \{u_g\}_{g \in G_0}$$

in $A \rtimes_\alpha G$.

Proof. First note that there exists a conditional expectation E onto $A \subset A \rtimes_\alpha G$ such that $E(a) = a$ and $E(au_g) = 0$ for all $a \in A$ and $g \in G \setminus \{e\}$ (see [3, Proposition 4.1.9]). Since the linear span of $\{au_g : a \in A_0, g \in G_0\}$ is dense in B_0 , we have $E[B_0] = A_0$. This shows $B_0 \cap A = E[B_0 \cap A] = A_0$. For the same reason we have $E[B_0 u_g^*] = 0$ for all $g \in G \setminus G_0$. This shows that $u_g \notin B_0$ for $g \in G \setminus G_0$. Thus $B_0 \cap \{u_g\}_{g \in G} = \{u_g\}_{g \in G_0}$. \square

3. K -THEORY OF LM ALGEBRAS

For definition of groups $K_0(A)$ and $K_1(A)$ see e.g., [2] or [23] and for the Cuntz semigroup $\text{Cu}(A)$ see e.g., [6]. If A is a unital subalgebra of B then $K_1(A)$ is a subgroup of $K_1(B)$ and if $B = \varinjlim B_\lambda$ then $K_1(B) = \varinjlim K_1(B_\lambda)$. Both of these two properties fail for K_0 in general.

A reader familiar with the logic of metric structures ([1], [11]) will notice that in Lemma 3.1 we are only using two standard facts: (1) the family of separable elementary submodels of algebra A is σ -complete and has A as its direct limit and (2) if A_λ is an elementary submodel of A then $K_0(A_\lambda)$ is a subgroup of $K_0(A)$ and $\text{Cu}(A_\lambda)$ is a subsemigroup of $\text{Cu}(A)$.

Lemma 3.1. *If A is a nonseparable C^* -algebra then A is a union of a σ -complete directed family of separable subalgebras A_λ , $\lambda \in \Lambda$, such that for each $\lambda \in \Lambda$ we have*

- (1) $K_0(A_\lambda)$ is a subgroup of $K_0(A)$ and $K_0(A) = \varinjlim K_0(A_\lambda)$,
- (2) $\text{Cu}(A_\lambda)$ is a sub-semigroup of $\text{Cu}(A)$ and $\text{Cu}(A) = \varinjlim \text{Cu}(A_\lambda)$.

Proof. (1) As usual $p \sim q$ denotes the Murray–von Neumann equivalence of projections in algebra A , namely $p \sim q$ if and only if $p = vv^*$ and $q = v^*v$ for some v in A .

For a subalgebra B of A we have that $K_0(B) < K_0(A)$ if and only if for any two projections p and q in $B \otimes \mathcal{K}$ we have $p \sim q$ in B if and only if $p \sim q$ in A .

We need to show that the family of separable subalgebras B of A such that $K_0(B) < K_0(A)$ is closed and unbounded. Since $\|p - q\| < 1$ implies $p \sim q$, this set is closed. The following condition for all p, q in B implies $K_0(B)$ is a subgroup of $K_0(A)$:

$$\inf_{v \in B} \|vv^* - p\| + \|v^*v - q\| = \inf_{v \in A} \|vv^* - p\| + \|v^*v - q\|.$$

We can now apply a standard Löwenheim–Skolem closing-up argument similar to that in the proof of [13, Lemma 2.13] (its version also appears in [2, II.8.5.1]). Let us write $\phi(v, p, q) = \|vv^* - p\| + \|v^*v - q\|$. Starting from a separable subalgebra B_0 of A , build an increasing chain of separable subalgebras B_n of A such that for every n and all p, q in B_n we have $\inf_{v \in B_{n+1}} \phi(v, p, q) = \inf_{v \in A} \phi(v, p, q)$. Since $\|p - p'\| < 1$ implies $p \sim p'$ the subalgebra B of A generated by $\bigcup_n B_n$ satisfies the above condition for each pair of projections in it.

The assertion that $K_0(A) = \varinjlim K_0(A_\lambda)$ is automatic since $A = \bigcup_\lambda A_\lambda$.

(2) Recall that the Cuntz ordering on positive elements in algebra A is defined by $a \preceq b$ if for every $\varepsilon > 0$ there exists $x \in A$ such that $\|a - bx^*\| < \varepsilon$.

We need to show that the family of separable subalgebras B of A such that for all a and b in B we have $a \preceq b$ in B if and only if $a \preceq b$ in A is closed and unbounded. It is clearly closed. Again it suffices to assure that for a dense set of pairs a, b of positive operators in B we have $\inf_{x \in B} \|a - bx^*\| =$

$\inf_{x \in A} \|a - xb x^*\|$, and this is achieved by a Löwenheim–Skolem argument resembling one sketched in the proof of (1) above.

The assertion that $\text{Cu}(A) = \bigcup_{\lambda} \text{Cu}(A_{\lambda})$ is again automatic. \square

It is also true that if A is a nonseparable C^* -algebra with the unique trace then its separable subalgebras with the unique trace form a σ -complete directed system whose direct limit is equal to A . This follows from an argument due to N.C. Phillips (see [22]) and it can be proved by the argument of Lemma 3.1 (see also [13, Remark 2.14]).

Let us denote the set of all prime numbers by \mathcal{P} . Recall that n is a *generalized integer* (or a supernatural number) if $n = \prod_{p \in \mathcal{P}} p^{n_p}$ where $n_p \in \mathbb{N} \cup \{\infty\}$ for all p . For a unital UHF algebra A define the generalized integer $n = \prod_{p \in \mathcal{P}} p^{n_p}$ of A by

$n_p := \sup \{k \in \mathbb{N} : \text{there exists a unital homomorphism from } M_{p^k}(\mathbb{C}) \text{ to } A\}$
for each $p \in \mathcal{P}$.

Glimm ([16]) has shown that the generalized integer provides a complete invariant for isomorphism of separable unital UHF algebras. For a generalized integer n define the group

$$\mathbb{Z}[1/n] = \{k/m : k \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\}, m|n\}$$

where $m|n$ is defined in the natural way. Then for a separable UHF algebra A and its generalized integer n we have $K_0(A) = \mathbb{Z}[1/n]$.

Proposition 3.2. *An LM algebra A has a unique tracial state τ . If A is unital, then τ induces an isomorphism from $K_0(A)$ onto $\mathbb{Z}[1/n] \subset \mathbb{R}$, with n defined as above, as ordered groups. We have $K_1(A) = 0$.*

Proof. Uniqueness of the tracial state immediately follows from the fact that a nonseparable LM algebra is a σ -complete direct limit of separable UHF algebras, since they have a unique tracial state. If A is unital we fix τ so that $\tau(1) = 1$.

For projections p and q of A we have $\tau(p) = \tau(q)$ if and only if $p \sim q$. This is true for separable LM algebras and the nonseparable case follows immediately by Lemma 2.3. Therefore τ is an isomorphic embedding of $K_0(A)$ into $\mathbb{Z}[1/n]$. Since $K_1(B) = 0$ for each separable LM algebra $A = \varinjlim A_{\lambda}$ implies $K_1(A) = \varinjlim K_1(A_{\lambda})$, we have $K_1(A) = 0$ by Lemma 2.3. \square

The following is an immediate consequence of the main result of [4].

Proposition 3.3. *If A is an infinite-dimensional LM algebra then its Cuntz semigroup is isomorphic to $K_0(A)_+ \sqcup (0, \infty)$.* \square

4. CLASSIFICATION OF UHF ALGEBRAS

Lemma 4.1. *Assume $A = \bigotimes_{x \in X} A_x$, $B = \bigotimes_{y \in Y} B_y$ and all A_x and all B_y are unital, separable, simple, and not equal to \mathbb{C} . Let $\Phi: A \rightarrow B$ be an isomorphism. Then there exist partitions $X = \bigsqcup_{z \in Z} X_z$ and $Y = \bigsqcup_{z \in Z} Y_z$*

of X and Y into disjoint nonempty countable subsets indexed by the same set Z such that

$$\Phi[\bigotimes_{x \in X_z} A_x] = \bigotimes_{y \in Y_z} B_y$$

for all $z \in Z$.

Proof. Consider the set \mathbb{P} of pairs of families $(\{X_z\}_{z \in Z}, \{Y_z\}_{z \in Z})$ of disjoint nonempty countable subsets of X and Y , respectively, with some common index set Z such that we have $\Phi[\bigotimes_{x \in X_z} A_x] = \bigotimes_{y \in Y_z} B_y$ for every $z \in Z$. Order \mathbb{P} by letting

$$(\{X_z\}_{z \in Z}, \{Y_z\}_{z \in Z}) \leq (\{X'_z\}_{z \in Z'}, \{Y'_z\}_{z \in Z'})$$

if $Z \subseteq Z'$ and $X'_z = X_z$ and $Y'_z = Y_z$ for all $z \in Z$.

By Zorn's lemma, there exists a maximal one $\{X_z\}_{z \in Z}$ and $\{Y_z\}_{z \in Z}$ among such families. If we set $X' := X \setminus \bigcup_{z \in Z} X_z$ and $Y' := Y \setminus \bigcup_{z \in Z} Y_z$ then $\bigotimes_{x \in X'} A_x = \bigcap_{z \notin X'} Z_A(A_z)$ and $\bigotimes_{y \in Y'} B_y = \bigcap_{z \notin Y'} Z_B(B_z)$ by [17, Theorem 1]. Therefore $\Phi[\bigotimes_{x \in X'} A_x] = \bigotimes_{y \in Y'} B_y$. Thus X' is nonempty if and only if Y' is nonempty. Suppose, to derive a contradiction, both X' and Y' are nonempty. By applying the argument in the proof of [13, Lemma 2.6] (see also [13, Lemma 2.19]), we find non-empty countable $X_0 \subseteq X'$ and $Y_0 \subseteq Y'$ such that $\Phi[\bigotimes_{x \in X_0} A_x] = \bigotimes_{y \in Y_0} B_y$. This contradicts the assumed maximality of $\{X_z\}_{z \in Z}$ and $\{Y_z\}_{z \in Z}$. Hence both X' and Y' are empty, and the maximal families $\{X_z\}_{z \in Z}$ and $\{Y_z\}_{z \in Z}$ are what we want. \square

Proposition 4.2. *If $\kappa_p, \lambda_p, p \in \mathcal{P}$ are sequences of cardinals indexed by the prime numbers then $\bigotimes_{p \in \mathcal{P}} \bigotimes_{\kappa_p} M_p(\mathbb{C})$ and $\bigotimes_{p \in \mathcal{P}} \bigotimes_{\lambda_p} M_p(\mathbb{C})$ are isomorphic if and only if $\kappa_p = \lambda_p$ for all p .*

Proof. Only the direct implication requires a proof. The separable case is a theorem of Glimm ([16]). Assume the algebras are nonseparable, and let $X = \bigcup_{p \in \mathcal{P}} \{p\} \times \kappa_p$, $A_{(p,\gamma)} = M_p(\mathbb{C})$, $Y = \bigcup_{p \in \mathcal{P}} \{p\} \times \lambda_p$, and $B_{(p,\gamma)} = M_p(\mathbb{C})$. By Lemma 4.1 applied to the isomorphism between $\bigotimes_{x \in X} A_x$ and $\bigotimes_{y \in Y} B_y$ we can find partitions $X = \bigcup_{z \in Z} X_z$ and $Y = \bigcup_{z \in Z} Y_z$ into countable sets such that $\bigotimes_{x \in X_z} A_x$ and $\bigotimes_{y \in Y_z} B_y$ are isomorphic for each $z \in Z$. By Glimm's theorem and simple cardinal arithmetic this implies $\kappa_p = \lambda_p$ for all p . \square

By Proposition 4.2, for each UHF algebra $A = \bigotimes_{p \in \mathcal{P}} \bigotimes_{\kappa_p} M_p(\mathbb{C})$ we can define the generalized integer $\kappa(A) = \prod_{p \in \mathcal{P}} p^{\kappa_p}$ and UHF algebras are completely classified up to isomorphism by the generalized integers $\kappa(A)$ associated with them. Note that $\kappa(A)$ being well-defined hinges on Proposition 4.2. It is unclear whether $\kappa(A)$ coincides with the generalized integer obtained by a straightforward generalization of definition given for separable UHF algebras before Proposition 3.2; see Problem 7.1 and Problem 7.2. We shall avoid using this notation for generalized integers in order to avoid the confusion with powers of cardinal numbers.

Proposition 4.3. *For every ordinal γ there are $(|\gamma| + \aleph_0)^{\aleph_0}$ isomorphism classes of unital UHF algebras of density character $\leq \aleph_\gamma$.*

Proof. Let K be the set of cardinals less than or equal to \aleph_γ . Then $|K| = |\gamma| + \aleph_0$. By Proposition 4.2, the number of isomorphism classes of UHF algebras of density character $\leq \aleph_\gamma$ is equal to $|\{f : f : \mathcal{P} \rightarrow K\}| = |K|^{\aleph_0}$. \square

Note that for any ordinal γ with $0 \leq |\gamma| \leq 2^{\aleph_0}$, we have $(|\gamma| + \aleph_0)^{\aleph_0} = 2^{\aleph_0}$. Thus for such γ , there are only as many UHF algebras of density character $\leq \aleph_\gamma$ as there are separable UHF algebras (see the table in §7).

5. NON-CLASSIFICATION OF AM ALGEBRAS IN REGULAR UNCOUNTABLE DENSITY CHARACTERS

The main result of this section shows that for a regular uncountable cardinal κ there are as many AM algebras of density character κ as there are C^* -algebras of density character κ and as many hyperfinite II_1 factors of density character κ as there are II_1 factors of density character κ . The latter fact is in stark contrast with the separable case, when the hyperfinite II_1 factor is unique. While there are continuum many separable UHF algebras, one should note that all AM algebras constructed here have the same K-theory as the (unique) CAR algebra.

We first concentrate on case when $\kappa = \aleph_1$. Let Λ be the set of all limit ordinals in \aleph_1 . As an ordered set, Λ is isomorphic to \aleph_1 . For each $\xi \in \aleph_1$, let A_ξ be the C^* -algebra generated by two self-adjoint unitaries v_ξ, w_ξ with $v_\xi w_\xi = -w_\xi v_\xi$. By [13, Lemma 4.1], A_ξ is isomorphic to $M_2(\mathbb{C})$. We define a UHF algebra A by $A := \bigotimes_{\xi \in \aleph_1} A_\xi \cong \bigotimes_{\aleph_1} M_2(\mathbb{C})$. For a subset Y of \aleph_1 , we set $A_Y = \bigotimes_{\xi \in Y} A_\xi \subset A$. For $\xi \in \aleph_1$, we use the notations $[0, \xi)$ and $[0, \xi]$ to denote the subsets $\{\delta \in \aleph_1 : \delta < \xi\}$ and $\{\delta \in \aleph_1 : \delta \leq \xi\}$ of \aleph_1 . For each $\delta \in \Lambda$, we define $\alpha_\delta \in \text{Aut}(A)$ by

$$\alpha_\delta = \bigotimes_{\xi \in [0, \delta)} \text{Ad } v_\xi.$$

Then we have $\alpha_\delta^2 = \text{id}$ and $\{\alpha_\delta\}_{\delta \in \Lambda}$ commute with each other. Let G_Λ be the discrete abelian group of all finite subsets of Λ as in [13, Definition 6.5]. Define an action α of G_Λ on A by $\alpha_F := \prod_{\delta \in F} \alpha_\delta$ for $F \in G_\Lambda$ and let $B := A \rtimes_\alpha G$. For each $\delta \in \Lambda$, the unitary implementing α_δ will be denoted by $u_\delta \in B$. For a subset S of Λ , we define $B_S := C^*(A \cup \{u_\delta\}_{\delta \in S}) \subset B$. We note that B_S is naturally isomorphic to $A \rtimes_\alpha G_S$ where G_S is considered as a subgroup of G_Λ .

Definition 5.1. Let S be a subset of Λ , and λ be an element of Λ . We define a subalgebra $D_{S, \lambda}$ of B_S by

$$D_{S, \lambda} := C^*(A_{[0, \lambda)} \cup \{u_\delta\}_{\delta \in S \cap [0, \lambda)}) \subset B_S.$$

Lemma 5.2. *For each $S \subset \Lambda$ the algebra B_S is AM. Also, $\{D_{S,\lambda}\}_{\lambda \in \Lambda}$ is a σ -complete directed family subalgebras of B_S isomorphic to the CAR algebra with dense union.*

Proof. Consider a triple (F, G, H) such that $F \subset \lambda$, $G = \{\delta_1, \delta_2, \dots, \delta_m\} \subset S$ and $H = \{\xi_1, \xi_2, \dots, \xi_m\} \subset \lambda$ are finite sets, $F \cap H = \emptyset$, and

$$\xi_1 < \delta_1 < \xi_2 < \delta_2 < \xi_3 < \dots < \delta_{m-1} < \xi_m < \delta_m.$$

For such (F, G, H) define $D_{(F,G,H)} \subset B_S$ by

$$D_{(F,G,H)} := C^*(A_F \cup \{u_\delta\}_{\delta \in G} \cup \{w_\xi\}_{\xi \in H}) \subset B_S.$$

We have $A_F \cong M_{2^n}(\mathbb{C})$ where n is the cardinality of F . For each $k = 1, 2, \dots, m$, there exists a unitary $v_k \in A_F$ with $v_k a = \alpha_{\delta_k}(a) v_k$ for all $a \in A_F$. For $k = 1, 2, \dots, m$, we set $v'_k := v_k u_{\delta_k}$ which is a self-adjoint unitary in $D_{(F,G,H)}$ commuting with A_F . We define self-adjoint unitaries $\{w'_k\}_{k=1}^m$ in $D_{(F,G,H)}$ by $w'_k := w_{\xi_k} w_{\xi_{k+1}}$ for $k = 1, 2, \dots, m-1$ and $w'_m := w_{\xi_m}$. Since $F \cap H = \emptyset$, the unitaries $\{w'_k\}_{k=1}^m$ commute with A_F . It is routine to check $v'_k w'_l = w'_l v'_k$ for $k, l \in \{1, 2, \dots, m\}$ with $k \neq l$, and $v'_k w'_k = -w'_k v'_k$ for $k = 1, 2, \dots, m$. Thus by [13, Lemma 4.1] the subalgebra A'_k of $D_{(F,G,H)}$ generated by v'_k and w'_k is isomorphic to $M_2(\mathbb{C})$ for every k . The family $\{A_F\} \cup \{A'_k\}_{k=1}^m$ of unital subalgebras of $D_{(F,G,H)}$ mutually commutes, and generate $D_{(F,G,H)}$. Hence $D_{(F,G,H)}$ is isomorphic to $M_{2^{n+m}}(\mathbb{C})$.

For two such triples $(F, G, H), (F', G', H')$, we have $D_{(F,G,H)} \subsetneq D_{(F',G',H')}$ if $F \cup H \subset F'$ and $G \subset G'$. Since there exist infinitely many elements of λ between two elements of S , for arbitrary finite subsets $F \subset \lambda$ and $G \subset S$ there exists a finite subset $H \subset \lambda$ such that the triple (F, G, H) satisfies the conditions above. Therefore the family $\{D_{(F,G,H)}\}_{(F,G,H)}$ of full matrix subalgebras of $D_{S,\lambda}$ is directed. It is clear that the union of this family is dense in $D_{S,\lambda}$. Since $D_{S,\lambda}$ is separable and a unital direct limit of algebras $M_{2^k}(\mathbb{C})$, $k \in \mathbb{N}$, it is isomorphic to the CAR algebra.

Since the family $\{D_{S,\lambda}\}_{\lambda \in \Lambda}$ is clearly σ -complete and covers B_S , this completes the proof. \square

Proposition 5.3. *For every $S \subset \Lambda$, B_S is a unital AM algebra of density character \aleph_1 with the same K_0, K_1 , and the Cuntz semigroup as the CAR algebra.*

Proof. Since $\chi(A) = \aleph_1$ and $|G_\Lambda| = \aleph_1$, $\chi(B_S) = \aleph_1$. By Lemma 5.2 the algebra B_S is the direct limit of the σ -complete system $D_{S,\lambda}$, $\lambda \in \Lambda$, of its separable subalgebras each of which is isomorphic to the CAR algebra. By Lemma 3.1 and [13, Lemma 2.6], B_S has the same K_0, K_1 , and the Cuntz semigroup as the CAR algebra. \square

Lemma 5.4. *For $S \subset \Lambda$ and $\lambda \in \Lambda$, we have*

$$\begin{aligned} Z_{B_S}(D_{S,\lambda}) &= C^*(A_{\aleph_1 \setminus [0,\lambda]} \cup \{u_\delta u_{\delta'}\}_{\delta, \delta' \in S \setminus [0,\lambda]}), \\ Z_{B_S}(Z_{B_S}(D_{S,\lambda})) &= C^*(A_{[0,\lambda]} \cup \{u_\delta\}_{\delta \in S \cap [0,\lambda]}). \end{aligned}$$

In particular, $D_{S,\lambda} = Z_{B_S}(Z_{B_S}(D_{S,\lambda}))$ if and only if $\lambda \notin S$.

Proof. Let us set $D' := C^*(A_{\aleph_1 \setminus [0,\lambda]} \cup \{u_\delta u_{\delta'}\}_{\delta, \delta' \in S \setminus [0,\lambda]})$. It is clear that $A_{\aleph_1 \setminus [0,\lambda]} \subset Z_{B_S}(D_{S,\lambda})$ and $u_g \in Z_{B_S}(D_{S,\lambda})$ for $g \in G_S$ such that $|g|$ is even and $g \subset [\lambda, \aleph_1)$. Hence we get $D' \subset Z_{B_S}(D_{S,\lambda})$. Take $a \in Z_{B_S}(D_{S,\lambda})$. For any $\varepsilon > 0$, there exist a finite set $F \subseteq \aleph_1$, finite families $b_1, b_2, \dots, b_n \in A_F$ and $g_1, g_2, \dots, g_n \in G_S$ such that $b = \sum_{k=1}^n b_k u_{g_k}$ satisfies $\|a - b\| < \varepsilon$. Let $\delta_1, \delta_2, \dots, \delta_m$ be the list of $[0, \lambda) \cap (\bigcup_{k=1}^n g_k)$ ordered increasingly. Choose $H = \{\xi_1, \xi_2, \dots, \xi_m\} \subset \aleph_1 \setminus F$ such that

$$\xi_1 < \delta_1 < \xi_2 < \delta_2 < \xi_3 < \dots < \delta_{m-1} < \xi_m < \delta_m < \lambda.$$

For each $H' \subset H$, we define a self-adjoint unitary $w_{H'}$ by $w_{H'} = \prod_{\xi \in H'} w_\xi$. Let us define a linear map $E: B_S \rightarrow B_S$ by $E(x) = 2^{-m} \sum_{H' \subset H} w_{H'} x w_{H'}$. Then E is a contraction. Since $a \in Z_{B_S}(D_{S,\lambda})$, we have $E(a) = a$. Hence $\|a - E(b)\| < \varepsilon$. For $g \in G_S$ with $\delta_k \in g$ for some k , we have $E(u_g) = 0$. For $g \in G_S$ such that $g \subset [\lambda, \aleph_1)$ and $|g|$ is odd, we also have $E(u_g) = 0$. For $g \in G_S$ with $g \subset [\lambda, \aleph_1)$ and $|g|$ is even, we get $E(u_g) = u_g$. Therefore $E(b) = \sum_k b_k u_{g_k}$ where k runs over elements such that $g_k \in G_S$ satisfies that $|g_k|$ is even and $g_k \subset [\lambda, \aleph_1)$. Next let $F' = F \cap [0, \lambda)$. We define a contractive linear map $E': B_S \rightarrow B_S$ by $E'(x) = \int_U x u x u^* du$ where U is the unitary group of the finite dimensional subalgebra $A_{F'}$ of $D_{S,\lambda}$, and du is its normalized Haar measure. Since $a \in Z_{B_S}(D_{S,\lambda})$, we have $E'(a) = a$. Hence $\|a - E'(E(b))\| < \varepsilon$. For $g_k \in G_S$ such that $|g_k|$ is even and $g_k \subset [\lambda, \aleph_1)$, we have $u_{g_k} u^* = u^* u_{g_k}$ for all $u \in U$. Hence for such k , we have $E'(b_k u_{g_k}) = E'(b_k) u_{g_k}$. Since $E'(b_k) \in A_{[\lambda, \aleph_1)}$, we get $E'(E(b)) \in D'$. Since $\varepsilon > 0$ was arbitrary, $a \in D'$. Thus we have shown $Z_{B_S}(D_{S,\lambda}) = D'$.

The equality $Z_{B_S}(D') = C^*(A_{[0,\lambda]} \cup \{u_g\}_{g \in G_S, g \subset [0,\lambda]})$ can also be proved in a similar way as above. The only difference is that $\delta_1, \delta_2, \dots, \delta_m$ is now the list of $(\lambda, \aleph_1) \cap (\bigcup_{k=1}^n g_k)$ ordered increasingly, and choose $H = \{\xi_1, \xi_2, \dots, \xi_m\} \subset \aleph_1 \setminus F$ such that

$$\lambda < \xi_1 < \delta_1 < \xi_2 < \delta_2 < \xi_3 < \dots < \delta_{m-1} < \xi_m < \delta_m.$$

We leave the details to the readers. \square

Lemma 5.5. *For $S \subset \Lambda$ and $\lambda \in \Lambda$, B_S is generated by $D_{S,\lambda}$ and $Z_{B_S}(D_{S,\lambda})$ if and only if $S \subset [0, \lambda)$.*

Proof. Lemma 5.4 implies that B_S is generated by $D_{S,\lambda}$ and $Z_{B_S}(D_{S,\lambda})$ if $S \subset [0, \lambda)$. If there exists $\delta \in S \setminus [0, \lambda)$, then u_δ is not in the C^* -algebra generated by $D_{S,\lambda}$ and $Z_{B_S}(D_{S,\lambda})$. \square

Compare the following proposition to Proposition 6.6.

Proposition 5.6. *For $S \subset \Lambda$, the C^* -algebra B_S is UHF if and only if S is bounded. In this case, B_S is isomorphic to $A \cong \bigotimes_{\aleph_1} M_2(\mathbb{C})$.*

Proof. When S is unbounded, the σ -complete system $\{D_{S,\lambda}\}_{\lambda \in \Lambda}$ in Lemma 5.2 satisfies that B_S is not generated by $D_{S,\lambda}$ and $Z_{B_S}(D_{S,\lambda})$ for all λ by

Lemma 5.5. Hence B_S is not a UHF algebra. When $S \subset [0, \lambda)$ for some $\lambda \in \aleph_1$, then we have $B_S = D_{S,\lambda} \otimes A_{[\lambda, \aleph_1)}$ by Lemma 5.5. By Lemma 5.2, $D_{S,\lambda}$ is the CAR algebra. Hence $B_S \cong \bigotimes_{\aleph_1} M_2(\mathbb{C})$. \square

Proposition 5.7. *Let S and S' be two subsets of Λ . If B_S and $B_{S'}$ are isomorphic, then there exists a club Λ_0 in Λ such that $\Lambda_0 \cap (S \Delta S') = \emptyset$.*

Proof. Assume $\Phi: B_S \rightarrow B_{S'}$ is an isomorphism. By [13, Proposition 2.12], there exists a club $\Lambda_0 \subset \Lambda$ such that $\Phi[D_{S,\lambda}] = D_{S',\lambda}$ for all $\lambda \in \Lambda_0$. For $\lambda \in \Lambda_0$, $\lambda \in S$ if and only if $\lambda \in S'$ by Lemma 5.4. Thus we have $\Lambda_0 \cap (S \Delta S') = \emptyset$. \square

Proof of Theorem 1.2. By Lemma 2.1 we can fix a family $S_0(X)$, $X \subseteq \aleph_1$, of subsets of \aleph_1 such that $S_0(X) \Delta S_0(Y)$ is stationary whenever $X \neq Y$. Since Λ is a club in \aleph_1 , the sets $S(X) = \Lambda \cap S_0(X)$ retain this property.

Therefore the algebras $B_{S(X)}$, $X \subseteq \aleph_1$, are nonisomorphic by Proposition 5.7. By Proposition 5.3 these algebras have the same K -theory and Cuntz semigroup as the CAR algebra. \square

For any uncountable regular cardinal κ one can define $< \kappa$ -complete directed systems of algebras of density character $< \kappa$ and prove results analogous to those for σ -complete directed systems so that the latter coincide with $< \aleph_1$ -complete systems. Given this and Lemma 2.1, a straightforward extension of the proof of Theorem 1.2 gives the following.

Theorem 5.8. *If κ is a regular cardinal then there are 2^κ nonisomorphic AM algebras of density character κ .* \square

However, this method does not work for singular cardinals and we shall treat this case in the following section.

6. NON-CLASSIFICATION OF AM ALGEBRAS IN ALL DENSITY CHARACTERS

The proof of the present section relies on two components. The first is the non-structure theory as developed by Shelah in [27] and adapted to metric structures in [14], and the second is the order property of theories of C^* -algebras and II_1 factors proved in [11]. We shall define a functor from the category of linear orders to the category of AM algebras and argue that if in cardinality κ there are many sufficiently different linear orders then in density character κ there are many nonisomorphic AM algebras (see Lemma 6.4). Readers with background in model theory will notice that the algebras that we construct are EM-models generated by indiscernibles which witness that their theory has the order property.

Fix a total ordering Λ and let Λ^+ denote $\Lambda \times \mathbb{N}$ with the lexicographical ordering. We identify Λ with $\Lambda \times \{0\} \subseteq \Lambda^+$ and note that between any two elements $\xi < \eta$ of Λ there are infinitely many elements of $\Lambda^+ \setminus \Lambda$. For each $\xi \in \Lambda^+$, let A_ξ be the C^* -algebra generated by two self-adjoint unitaries v_ξ, w_ξ with $v_\xi w_\xi = -w_\xi v_\xi$. By [13, Lemma 4.1], A_ξ is isomorphic to $M_2(\mathbb{C})$.

We define a UHF algebra A_Λ by $A_\Lambda := \bigotimes_{\xi \in \Lambda^+} A_\xi \cong \bigotimes_{\Lambda^+} M_2(\mathbb{C})$. For $\xi \in \Lambda^+$ we write

$$\begin{aligned} [0, \xi) &:= \{\delta \in \Lambda^+ : \delta < \xi\} \\ [0, \xi] &:= \{\delta \in \Lambda^+ : \delta \leq \xi\}. \end{aligned}$$

For each $\delta \in \Lambda$, we define $\alpha_\delta \in \text{Aut}(A)$ by

$$\alpha_\delta = \bigotimes_{\xi \in [0, \delta)} \text{Ad } v_\xi.$$

Then we have $\alpha_\delta^2 = \text{id}$ and $\{\alpha_\delta\}_{\delta \in \Lambda}$ commute with each other. Let G_Λ be the discrete abelian group of all finite subsets of Λ as in [13, Definition 6.5]. Define an action α of G_Λ on A_Λ by $\alpha_F := \prod_{\delta \in F} \alpha_\delta$ for $F \in G_\Lambda$ and let

$$B_\Lambda := A_\Lambda \rtimes_\alpha G_\Lambda.$$

For each $\delta \in \Lambda$, the unitary implementing α_δ will be denoted by $u_\delta \in B$. For $S \subseteq \Lambda$ let $A_S := \bigotimes_{\xi \in S \times \mathbb{N}} A_\xi$ and consider it as a subalgebra of A_Λ .

Note that B_Λ is generated by the set $\{v_\xi, w_\xi, u_\xi : \xi \in \Lambda\}$. Moreover, the relation between these generators depends only on the order between their indices; for example, u_ξ and w_η commute if and only if $\xi \geq \eta$. It is not difficult to check that if Λ is a suborder of Λ' then B_Λ is a unital subalgebra of $B_{\Lambda'}$. We moreover have a functor $\Lambda \mapsto B_\Lambda$ from the category of linear orders into the category of AM algebras (see [18, 11.2] for the general setup).

If S is a subset of Λ define a subalgebra D_S of B_Λ by

$$D_S := C^*(A_S \cup \{u_\delta\}_{\delta \in S}).$$

Lemma 6.1. *For each uncountable total order Λ the algebra A_Λ is AM. Also, $\{D_S : S \in [\Lambda]^{\aleph_0}\}$ is a σ -complete directed family of subalgebras of A_Λ isomorphic to the CAR algebra with dense union.*

Proof. This proof is almost identical to the proof of Lemma 5.2. The assumption that λ is a limit ordinal used in the former proof is replaced by the fact that the generators are indexed by Λ^+ . \square

Our plan is to prove that ‘sufficiently different’ linear orders result in nonisomorphic algebras.

Proposition 6.2. *For every infinite cardinal κ and total ordering Λ of cardinality κ , B_Λ is a unital AM algebra of character density equal to κ with the same K_0 , K_1 , and the Cuntz semigroup as the CAR algebra.*

Proof. Since $\chi(A_\Lambda) = \kappa$ and $|G_\Lambda| = \kappa$, $\chi(B_\Lambda) = \kappa$. By Lemma 6.1 the algebra B_S is the direct limit of the σ -complete system D_S , $S \in [\Lambda]^{\aleph_0}$, of its separable subalgebras each of which is isomorphic to the CAR algebra. By Lemma 3.1 and [13, Lemma 2.6], B_S has the same K_0 , K_1 , and the Cuntz semigroup as the CAR algebra. \square

Assume $P(\vec{x}, \vec{y})$ is a *-polynomial in $2n$ variables. Then for every C^* -algebra A the expression $\phi(\vec{x}, \vec{y}) = \|P(\vec{x}, \vec{y})\|$ defines a uniformly continuous

map from A^{2n} into the nonnegative reals. Let $(A_{\leq 1})$ denote the unit ball of A and on $(A_{\leq 1})^n$ define a binary relation \prec_ϕ by letting $\vec{a} \prec_\phi \vec{b}$ if

$$\phi(\vec{a}, \vec{b}) = 1 \quad \text{and} \quad \phi(\vec{b}, \vec{a}) = 0.$$

Note that \prec_ϕ is not required to be an ordering. If Λ is a total ordering we shall say that an indexed set \vec{a}_λ , for $\lambda \in \Lambda$ is a ϕ -chain if $\vec{a}_\lambda \prec_\phi \vec{a}_{\lambda'}$ whenever $\lambda < \lambda'$. We write $\vec{a} \preceq_\phi \vec{b}$ if $\vec{a} = \vec{b}$ or $\vec{a} \prec_\phi \vec{b}$.

Here is an example of a formula ϕ and a ϕ -chain. Formula

$$\phi(x_1, x_2, y_1, y_2) = \frac{1}{2} \|[x_1, y_2]\|$$

defines a uniformly continuous function on A^4 for any C^* -algebra A . With Λ , B_Λ , u_ξ , and w_ξ as above, for all ξ and η in Λ we have

$$\phi(u_\xi, w_\xi, u_\eta, w_\eta) = \begin{cases} 0, & \text{if } \xi < \eta \\ 1, & \text{if } \xi \geq \eta, \end{cases}$$

and therefore (u_ξ, w_ξ) , for $\xi \in \Lambda$, is a ϕ -chain. Moreover, the algebra B_Λ is generated by this ϕ -chain, and this is exactly what the following definition is capturing.

Definition 6.3 ([14, Definition 3.1]). A ϕ -chain \mathcal{C} is *weakly (\aleph_1, ϕ) -skeleton like* inside A if for every $\vec{a} \in A^n$ there is a countable $\mathcal{C}_{\vec{a}} \subseteq \mathcal{C}$ such that for all \vec{b} and \vec{c} in \mathcal{C} for which we have $\vec{b} \preceq_\phi \vec{c}$ and no $\vec{d} \in \mathcal{C}_{\vec{a}}$ satisfies $\vec{b} \preceq_\phi \vec{d} \preceq_\phi \vec{c}$ we have

$$\phi(\vec{b}, \vec{a}) = \phi(\vec{c}, \vec{a}) \quad \text{and} \quad \phi(\vec{a}, \vec{b}) = \phi(\vec{a}, \vec{c}).$$

Behind the following lemma is the idea that sufficiently different linear orders Λ produce nonisomorphic C^* -algebras generated by weakly (\aleph_1, ϕ) -skeleton like chains.

Lemma 6.4. *Assume \mathcal{K} is a class of C^* -algebras, $\phi(\vec{x}, \vec{y})$ is as above, and κ is an uncountable cardinal. If for every linear ordering Λ of cardinality κ there is $B_\Lambda \in \mathcal{K}$ of density character κ such that the n -th power of the unit ball of B_Λ includes a ϕ -chain \mathcal{C} isomorphic to Λ which is weakly (\aleph_1, ϕ) -skeleton like, then \mathcal{K} contains 2^κ nonisomorphic algebras of density character κ .*

Proof. This is an immediate consequence of results from [14], but we sketch a proof for the convenience of the reader. By [14, Lemma 2.5] for every $m \in \mathbb{N}$ (actually $m = 3$ suffices) there are 2^κ total orderings of cardinality κ that have disjoint representing sequences of m, κ -invariants (in the sense of [14, §2.2]). For any such ordering Λ the algebra B_Λ has density character κ and therefore the m, κ -invariant of Λ belongs to $\text{INV}^{m, \kappa}(B_\Lambda)$, as defined in [14, Definition 3.8 and §6.2]. By [14, Lemma 6.4] for each C^* -algebra B of density character κ the set $\text{INV}^{m, \kappa}(B)$ has cardinality at most κ . Since 2^κ cannot be written as the supremum of κ smaller cardinals ([20, Corollary 10.41]), by a

counting argument there are 2^κ isomorphism classes among algebras B_Λ for a total ordering Λ of cardinality κ . \square

Proof of Theorem 1.2. As noted earlier, formula $\phi(x_1, x_2, y_1, y_2) = \frac{1}{2} \|[x_1, y_2]\|$ defines a uniformly continuous function on A^4 for any C*-algebra A , and for a linear ordering Λ with the notation from the first paragraph of §5 we have that (u_ξ, w_ξ) , for $\xi \in \Lambda$, is a ϕ -chain in B_Λ

Consider $S \subseteq \Lambda$. On the set $\Lambda \setminus S$ define an equivalence relation, $\xi \sim_S \eta$ if and only if no element of S is between ξ and η . Then for $\xi \sim_S \eta$ we have that the algebras $C^*(B_S \cup \{u_\xi, w_\xi\})$ and $C^*(B_S \cup \{u_\eta, w_\eta\})$ are isomorphic via an isomorphism that is an identity on B_S and sends u_ξ to u_η and w_ξ to w_η .

We claim that Λ is weakly (\aleph_1, ϕ) -skeleton like in B_S . First note that every finite set $F \subseteq B_\Lambda$ is included in D_S for some countable $S = S(F) \subseteq \Lambda$. For a_1 and a_2 in B_Λ fix a countable S such that $\{a_1, a_2\} \subseteq B_S$. Then let $\mathcal{C}_{\{a_1, a_2\}} = S$ and note that $\xi \sim_S \eta$ implies that $\phi(a_1, a_2, u_\xi, w_\xi) = \phi(a_1, a_2, u_\eta, w_\eta)$ and $\phi(u_\xi, w_\xi, a_1, a_2) = \phi(u_\eta, w_\eta, a_1, a_2)$.

Therefore our distinguished Λ -chain (u_ξ, w_ξ) , for $\xi \in \Lambda$, is (\aleph_1, ϕ) -skeleton like Lemma 6.4 applies to show that there are 2^κ isomorphism classes among algebras B_Λ for $|\Lambda| = \kappa$. By Proposition 6.2 these algebras have the same K -theory and Cuntz semigroup as the CAR algebra. \square

The assumption that we were dealing with C*-algebras in Lemma 6.4 was not crucial. This lemma applies to any class of models of logic of metric structures ([1], [10]), and in particular to II_1 factors. We shall now state the general form of Lemma 6.4. The definition of ‘metric structure’ and ‘formula’ is given in [1] (see also [10] for the case of C*-algebras and tracial von Neumann algebras). Although this lemma uses logic for metric structures, we note that class \mathcal{C} is not required to be axiomatizable. Indeed, neither AM algebras nor hyperfinite II_1 factors are axiomatizable (cf. the proof of [10, Proposition 6.1], but see also [5]). We state this lemma in the case of bounded metric structures, and the version for II_1 factors necessitates requiring that the chain be included in the n -power of the unit ball. The proof of Lemma 6.5 is identical to the proof of Lemma 6.4.

Lemma 6.5. *Assume \mathcal{C} is a class of bounded metric structures and $\phi(\vec{x}, \vec{y})$ is a $2n$ -ary formula. Assume that for every linear ordering Λ there is $A_\Lambda \in \mathcal{C}$ of density character $|\Lambda|$ such that A_Λ^r includes a ϕ -chain \mathcal{C} isomorphic to Λ which is weakly (\aleph_1, ϕ) -skeleton like. Then \mathcal{C} contains 2^κ nonisomorphic structures in every uncountable density character κ . \square*

Proof of Theorem 1.3. For each of the algebras B_Λ constructed in the proof of Theorem 1.2 consider the GNS representation corresponding to its unique trace and let R_Λ be the weak closure of the image of B_Λ . Then each R_Λ is a hyperfinite II_1 factor whose predual has density character $\kappa = |\Lambda|$. The formula

$$\psi(x_1, x_2, y_1, y_2) = \|[x_1, y_2]\|_2$$

defines a uniformly continuous with respect to the 2-norm function on R_Λ . Let A_Λ denote the operator norm unit ball of B_Λ . Then each A_Λ , equipped with the ℓ_2 -norm and function that evaluates ψ is a bounded metric structure and it suffices to check that Lemma 6.5 applies to this family. Again (u_ξ, w_ξ) , for $\xi \in \Lambda$, is a ϕ -chain that is weakly (\aleph_1, ϕ) -skeleton like. By Lemma 6.5 there are 2^κ nonisomorphic unit balls of II_1 -factors of the form B_Λ with the predual of density character κ . Therefore there are 2^κ nonisomorphic hyperfinite II_1 factors with the density character κ for every uncountable cardinal κ . \square

Note that the assumption that κ is uncountable is necessary in Lemma 6.4, since the hyperfinite II_1 factor with separable predual is unique.

The remainder of this section is aimed at logicians. A class of models is non-classifiable in a strong sense if it does not allow sequences of cardinal numbers as complete invariants (see [26]). The following proposition shows that AM algebras and hyperfinite II_1 factors are non-classifiable even in this strong sense.

Proposition 6.6. *There are AM algebras A and B of density character \aleph_1 and a forcing notion \mathbb{P} that does not collapse cardinals or add countable sequences of cardinals such that A and B are not isomorphic, but \mathbb{P} forces that A and B are isomorphic.*

There are also hyperfinite II_1 factors of density character \aleph_1 with the same property.

Proof. Let $S \subseteq \aleph_1$ a stationary set whose complement is also stationary. Let η denote the ordering of the rational numbers. Let $\Lambda(1)$ be the linear ordering obtained from \aleph_1 by replacing all points with a copy of η (i.e., the lexicographical ordering of $\aleph_1 \times \eta$). Let $\Lambda(2)$ be the linear ordering obtained from \aleph_1 by replacing points in S with a copy of η and leaving points in $\aleph_1 \setminus S$ unchanged.

Since $\aleph_1 \setminus S$ is stationary, the argument from the proof of Proposition 5.7 shows that the algebras $A := A_{\Lambda(1)}$ and $B := A_{\Lambda(2)}$ are not isomorphic. Let \mathbb{P} be Jensen's forcing for adding a club subset of S . Then \mathbb{P} is σ -distributive (see e.g., [20, VII.H18]) and therefore it does not collapse \aleph_1 and does not add new sequences of cardinals. Since \mathbb{P} has cardinality \aleph_1 , it does not collapse cardinals larger than \aleph_1 .

We claim that \mathbb{P} nevertheless forces A and B to be isomorphic. It clearly suffices to show that it forces $\Lambda(1)$ and $\Lambda(2)$ are isomorphic linear orderings. If $C \subseteq S$ is the club added by \mathbb{P} , then points in C separate $\Lambda(1)$ and $\Lambda(2)$ into \aleph_1 sequence of countable linear orderings without endpoints. Any two such orderings are isomorphic by Cantor's classical back-and-forth argument, and these isomorphisms together define an isomorphism between $\Lambda(1)$ and $\Lambda(2)$.

Construction of the required II_1 factors is analogous. \square

Proposition 6.6 shows that the classification problem of AM algebras of density character \aleph_1 is at least as complicated as the classification of subsets

of \aleph_1 modulo the nonstationary ideal. The latter problem is largely considered to be intractable. One reason for this is that, as demonstrated in the proof of Proposition 6.6, a function that associates a sequence of cardinals to a subset of \aleph_1 in an absolute way cannot be a complete invariant for the relation of equality modulo the nonstationary ideal.

7. CONCLUDING REMARKS

The number of AM algebras and UHF algebras in some density characters as well as the number of hyperfinite II_1 -factors whose predual has the same density character is given in the table below. We identify each cardinal with the least ordinal having it as a cardinality, write $\mathfrak{c} := 2^{\aleph_0}$, and \mathfrak{c}^+ denotes the least cardinal greater than \mathfrak{c} .

density character	\aleph_0	\aleph_1	\aleph_2	\dots	\aleph_ω	\dots	\aleph_{ω_1}	\dots	$\aleph_{\mathfrak{c}^+}$	\dots
the number of UHF algebras	\mathfrak{c}	\mathfrak{c}	\mathfrak{c}	\dots	\mathfrak{c}	\dots	\mathfrak{c}	\dots	\mathfrak{c}^+	\dots
the number of AM algebras	\mathfrak{c}	2^{\aleph_1}	2^{\aleph_2}	\dots	2^{\aleph_ω}	\dots	$2^{\aleph_{\omega_1}}$	\dots	$2^{\aleph_{\mathfrak{c}^+}}$	\dots
the number of hyperfinite II_1 factors	\mathfrak{c}	2^{\aleph_1}	2^{\aleph_2}	\dots	2^{\aleph_ω}	\dots	$2^{\aleph_{\omega_1}}$	\dots	$2^{\aleph_{\mathfrak{c}^+}}$	\dots

While the cardinals in this table resemble those predicted by Shelah's Main Gap Theorem for the number of models of classifiable and non-classifiable theories in uncountable cardinalities ([26]), it should be noted that all algebras appearing in our proofs are elementarily equivalent to the CAR algebra and that the class of UHF algebras does not seem to have a natural model-theoretic characterization. On the other hand, all AM (and even all LM) algebras are by [5], atomic models. It is not difficult to see that the methods of [5] also show that hyperfinite II_1 factors (of arbitrary density character) are atomic models.

For a unital C^* -algebra A , we define two generalized integers, $\kappa(A)$ and $\kappa'(A)$, associated to A as follows. Recall that \mathcal{P} denotes the set of all primes. If $A = \bigotimes_{p \in \mathcal{P}} \bigotimes_{\kappa_p} M_p(\mathbb{C})$, then let $\kappa(A)_p = \kappa_p$ for $p \in \mathcal{P}$.

$$\kappa'(A)_p := \sup \{ |X| : \text{there exists a unital homomorphism} \\ \text{from } \bigotimes_X M_p(\mathbb{C}) \text{ to } A \}$$

for each $p \in \mathcal{P}$. Clearly these two definitions coincide when A is separable and $\kappa(A) \leq \kappa'(A)$, pointwise.

Problem 7.1. If A is a UHF algebra, is $\kappa(A) = \kappa'(A)$?

Here is a version of Problem 7.1.

Problem 7.2. Assume A is UHF, $\kappa < \kappa'$ are cardinals and $\bigotimes_{\kappa'} M_2(\mathbb{C})$ unittally embeds into $\bigotimes_{\kappa} M_2(\mathbb{C}) \otimes A$. Can we conclude that there is a unital embedding of $\bigotimes_{\kappa} M_2(\mathbb{C})$ into A ?

We cannot even prove that in the above situation $M_2(\mathbb{C})$ unittally embeds into A . The most embarrassing version of Problem 7.2 is whether $\bigotimes_{\aleph_1} M_2(\mathbb{C})$ unittally embeds into $\bigotimes_{\aleph_0} M_2(\mathbb{C}) \otimes \bigotimes_{\aleph_1} M_3(\mathbb{C})$. Since any two unital copies of $M_n(\mathbb{C})$, for $n \in \mathbb{N}$, in a UHF algebra are conjugate, Problem 7.2 has a positive answer when κ is finite.

Standard results on classification of unital, separable, nuclear, simple C^* -algebras imply that if A is not UHF then the answer to Problem 7.2 is negative. We shall need the following well-known fact.

Lemma 7.3. *There are an abelian group G and a nonzero $g_0 \in G$ such that*

- (1) *no infinite nonzero sequence (f_n) in G satisfies $f_n = 2f_{n+1}$ for all n ,*
- (2) *for every n there is h_n satisfying $2^n h_n = g_0$, and*
- (3) *$h_1 \neq 2f$ for all $f \in G$.*

Proof. Let m be the generalized integer defined by $m_2 = \aleph_0$, $m_p = 0$ for $p \neq 2$ and let G be the subgroup of (recall that $\mathbb{Z}[1/m]$ was defined before Proposition 3.2) $\prod_{n=1}^{\infty} (\mathbb{Z}/2^n\mathbb{Z}) \times \mathbb{Z}[1/m]$ consisting of all $(x_n)_{n \leq \infty}$ such that

$$x_n = 2^n x_{\infty} \pmod{1}$$

for all but finitely many n . Since $x_{\infty} \in \mathbb{Z}[1/m]$, for a large enough n we will have that x_{∞} is equal to an element of $\mathbb{Z}/2^n\mathbb{Z}$ for all large enough n . Let $g_0 = (x_n)$ where $x_n = 0$ for all finite n and $x_{\infty} = 1$. Then $h_n = (x_j)$ where $x_j = 0$ for $j < n$, $x_j = 2^{j-n}$ for $j \geq n$ finite and $x_{\infty} = 2^{-n}$ satisfy $2^n h_n = g_0$ for each n . If $f = (y_n)$ is such that $2f = h_1$ then necessarily $2y_1 = 1$, but there is no such element in $\mathbb{Z}/2\mathbb{Z}$. This proves (3) and the proof of (1) is similar. \square

The following two propositions rely on the Kirchberg–Phillips classification of Kirchberg algebras A by its K -theoretic invariants

$$I(A) = (K_0(A), [1]_0, K_1(A))$$

and the fact that every pair of countable abelian groups with a distinguished element is an invariant of some Kirchberg algebra (see e.g., [21] or [24]).

Proposition 7.4. *There is a C^* -algebra A such that $\bigotimes_X M_2(\mathbb{C})$ unittally embeds into A if and only if X is finite. Moreover, $A = M_2(\mathbb{C}) \otimes B$ for some B such that $M_2(\mathbb{C})$ does not unittally embed into B .*

Proof. Let G and g_0 be as in Lemma 7.3 and let A be the Kirchberg algebra with $K_0(A) = G$ such that g_0 is equal to the class of the identity and with trivial $K_1(A)$. For $n \in \mathbb{N}$, $M_{2^n}(\mathbb{C})$ embeds unittally into A by K -theoretic consideration. Pick a projection q in A such that $[q] = g$, and let C be a unital copy of $M_2(\mathbb{C})$ in A with q as its matrix unit. Then $A \cong M_2(\mathbb{C}) \otimes B$, with $B = Z_A(C)$. By (3) and K -theoretic considerations B has no unital copy of $M_2(\mathbb{C})$ and the proof is complete. \square

Proposition 7.5. *There is no unital $*$ -homomorphism from $M_2(\mathbb{C})$ into the Cuntz algebra \mathcal{O}_3 , but there is a unital $*$ -homomorphism from the CAR algebra into $M_2(\mathcal{O}_3)$.*

Proof. Let A denote the CAR algebra. The algebras \mathcal{O}_3 and $M_2(\mathcal{O}_3)$ are Kirchberg algebras. Since $I(\mathcal{O}_3) = (\mathbb{Z}/2\mathbb{Z}, 1, 0)$ (see [24]), the identity in $K_0(\mathcal{O}_3)$ is not divisible by 2 and therefore $M_2(\mathbb{C})$ is not a unital subalgebra of \mathcal{O}_3 . Since $M_2(\mathcal{O}_3) \otimes \mathcal{K} \cong \mathcal{O}_3 \otimes \mathcal{K}$ we have $K_0(M_2(\mathcal{O}_3)) = \mathbb{Z}/2\mathbb{Z}$ but the class of the identity element is 0 and we have $I(M_2(\mathcal{O}_3)) = (\mathbb{Z}/2\mathbb{Z}, 0, 0)$.

Since $2 \times 0 = 0$ and $M_2(\mathcal{O}_3)$ is purely infinite, it contains a unital copy of \mathcal{O}_2 and therefore a unital copy of any other simple nuclear C^* -algebra—including the CAR algebra. \square

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