Topological centres of group actions

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Banff — June 19, 2012
If $G$ is a group then recall that a (left) action of $G$ on the set $S$ is a mapping $\cdot : G \times S \to S$ satisfying the associative law

$$g(h \cdot s) = (gh) \cdot s$$

and for which the identity element $e_G$ satisfies

$$e_G \cdot s = s$$

and note the important examples of groups acting on themselves in the natural way.

If $X \subseteq S$ then $\{g \cdot x \mid x \in X\}$ will be denoted by $g \cdot X$.
If \( \cdot : \mathbb{G} \times S \to S \) is a group action, then an invariant mean for this action is a real valued measure \( \mu \) on \( S \) such that:

- \( \mu \) is finitely additive
- \( \mu(X) = \mu(g \cdot X) \) for each \( X \subseteq S \) and \( g \in \mathbb{G} \)
- \( \mu(S) = 1 \)

An action of a group is said to be amenable if it has an invariant mean. A group is said to be amenable if its natural action on itself is amenable.
While it is possible to define amenability for topological groups, topology will enter into this discussion via a different route. The first observation to make is that a mean $\mu$ on $S$ can also be thought of as an element $\ell^*_\infty(S)$ by integration:

$$\langle \mu, f \rangle = \int f(s) d\mu(s)$$

Recall that $\ell_\infty = \ell^*_1$. So $\mu$ will be thought of as an element of the double dual of $\ell_1$. The weak* topology on the double dual will play a key role.
Let \( \cdot : \mathbb{G} \times S \to S \) be a group action. Then extend this action to a map \( \circ : \ell_\infty^*(\mathbb{G}) \times \ell_\infty^*(S) \to \ell_\infty^*(S) \) by defining for \( m \in \ell_\infty^*(\mathbb{G}) \), \( n \in \ell_\infty^*(S) \), \( \psi \in \ell_\infty(S) \) and \( g \in \mathbb{G} \),

\[
\langle m \circ n, \psi \rangle = \langle m, n \psi \rangle
\]

\[
\langle n \psi, g \rangle = \langle n, g \psi \rangle
\]

\[
g \psi(s) = \psi(g \cdot s)
\]

and note that if \( g \in \mathbb{G} \subseteq \ell_\infty^*(\mathbb{G}) \) and \( s \in S \subseteq \ell_\infty^*(S) \) then \( g \cdot s = g \circ s \).
To see this, note that for any $\psi \in \ell_\infty(S)$ the following equalities hold:

$$\langle g \circ s, \psi \rangle = \langle g, s\psi \rangle = \langle s\psi, g \rangle = \langle s, g\psi \rangle = \langle g\psi, s \rangle = \psi(g \cdot s) = \langle g \cdot s, \psi \rangle$$

Henceforth, there is no confusion in denoting the action

$$\circ : \ell^*_\infty(G) \times \ell^*_\infty(S) \to \ell^*_\infty(S)$$

by $\cdot$ instead.

**Definition**

Define the topological centre, denoted by $\Lambda(\cdot)$, of the group action $\cdot$ to be the set of $m \in \ell^*_\infty(G)$ such that the mapping $\lambda_m$ defined by $\lambda_m(n) = m \cdot n$ is weak*-weak* continuous.
It should be noted that:

- The mapping $\rho_m$ defined by $\rho_m(n) = n \cdot m$ is always weak*–weak* continuous.
- The mapping $\lambda_m$ is always norm continuous.
- If $f \in \ell_1(G)$ then $\lambda_f$ is weak*–weak* continuous.

So two of the central question in studying topological centres of a group action $\cdot$ are:

- Is $\Lambda(\cdot) = \ell^*_\infty(G)$?
- Is $\Lambda(\cdot) = \ell_1(G)$?
For groups acting on themselves, the questions concerning topological centres have a nice resolution.

**Theorem (Lau)**

For any group \((G, \cdot)\) (now \(\cdot\) is the action of the group on itself)
\[\Lambda(\cdot) = \ell_1(G).\]

On the other hand,

**Lemma (Nuefang, Pachl & Steprāns — NPS)**

If \(G\) is an infinite amenable group and the action \(\cdot : G \times S \rightarrow S\) has a unique invariant mean then \(\Lambda(\cdot) \neq \ell_1(G)\).
The following simple example provides some of the hypotheses of the lemma.

**Example**

Let $\mathcal{U}$ be any non-principal ultrafilter on $\mathbb{N}$ and let $G_\mathcal{U}$ be the group of all permutations $\theta$ of $\mathbb{N}$ such that \{ $n \in \mathbb{N} \mid \theta(n) = n$ \} $\in \mathcal{U}$.

Then the natural action of $G_\mathcal{U}$ on $\mathbb{N}$ is amenable and this is witnessed by $\mathcal{U}$ thought of as a $\{0, 1\}$-valued measure on $\mathbb{N}$.
To see that $\mathcal{U}$ is unique, suppose that $\nu$ is some other measure on $\mathbb{N}$. Since $\mathcal{U}$ is maximal and $\{0, 1\}$-valued there must be some $A \notin \mathcal{U}$ such that $\nu(A) > 0$.

Let $k > 1/\nu(A)$ and choose $\{A_i\}_{i=1}^k$ disjoint and not belonging to $\mathcal{U}$. Then there is some $\pi \in \mathcal{G}_\mathcal{U}$ such that $\pi^i(A) = A_j$. The invariance of $\nu$ and the fact that $\nu(\mathbb{N}) = 1$ then implies that $\nu(A) \leq 1/k < \nu(A)$.
The mean $\mathcal{U}$ is $\{0, 1\}$-valued and this is impossible for means of groups acting on themselves.

The mean $\mathcal{U}$ is unique and this is also impossible for means of infinite groups acting on themselves.

The group $G_{\mathcal{U}}$ is not amenable itself for non-trivial ideals.
A more interesting example of a non-amenable group acting with an invariant mean was constructed by van Douwen.

**Definition**

The action \( \cdot : G \times S \to S \) is said to be **free** if \( g \cdot s \neq s \) for all \( g \in G \) other than \( e_G \) and for all \( s \in S \). The action is **almost free** if \( g \cdot s \neq s \) for all \( g \in G \) other than \( g = e_G \) and all but finitely many \( s \).

The action of a group on itself is always free. Moreover, if a group has any free action with an invariant mean then it is amenable.

**Theorem (van Douwen)**

There is an almost free action of \( \mathbb{F}_2 \) on a countable set.
However, to apply the lemma to find a non-trivial member of $\Lambda(\cdot)$ it is necessary to have an amenable group acting with a unique invariant mean. Answering a question of Rosenblatt, Foreman showed that it is consistent with set theory that there is a locally finite group of permutations of $\mathbb{N}$ — indeed, a subgroup of $\mathcal{G}_U$ for some ultrafilter $\mathcal{U}$ — whose natural action on $\mathbb{N}$ has a unique $\{0,1\}$-valued invariant mean.

Before proceeding with this it is worth remarking:

- Locally finite groups are amenable.
- A group of permutations of $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique $\{0,1\}$-valued invariant mean can not have a simple definition — in particular, it can not be analytic.
**Definition**

The least cardinal of a generating set for a free ultrafilter on \(\mathbb{N}\) is denoted by \(u\).

**Definition**

The least cardinal of a filter \(\mathcal{F}\) on \(\mathbb{N}\) such that there is no infinite \(X \subseteq \omega\) such that \(X \subseteq^* A\) — in other words, \(X \setminus A\) is finite — for all \(A \in \mathcal{F}\) is denoted by \(p\).

**Theorem (Foreman)**

If \(u = p\) then there is a locally finite subgroup of some \(\mathbb{G}_\mathcal{U}\) whose natural action on \(\mathbb{N}\) has a unique \(\{0, 1\}\)-valued invariant mean.
The consistency of the negation remains open.

**Theorem (Foreman)**

*In the model of set theory obtained by adding $\aleph_2$ Cohen reals there is no locally finite group of permutations of $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique $\{0, 1\}$-valued invariant mean.*

However the following question remains open.

**Question**

*Is it consistent with set theory that there is no amenable group of permutations of $\mathbb{N}$ whose natural action on $\mathbb{N}$ has a unique $\{0, 1\}$-valued invariant mean?*
In the context of this question it is of interest to know whether there are amenable groups, that are not locally finite (in some non-trivial sense) acting with unique invariant mean.

**Theorem (Raghavan & Steprāns)**

Assuming there is an ultrafilter on \( \mathbb{N} \) generated by a tower, there is a subgroup \( G \) of the full symmetric group on \( \mathbb{N} \) whose natural action on \( \mathbb{N} \) has a unique invariant mean and that has a generating set all of whose elements have infinite order. The group is a solvable extension of a locally finite group and, hence, amenable.
**Question**

Is there a locally solvable subgroup of the full symmetric group on \(\mathbb{N}\) whose natural action on \(\mathbb{N}\) has a unique invariant mean in the Cohen model?

**Question**

Is there a model where there is no locally solvable (or even locally nilpotent) subgroup of the full symmetric group on \(\mathbb{N}\) whose natural action on \(\mathbb{N}\) has a unique invariant mean?

Of course, neither of these addresses the main open question, but a solution to the following warm up question would be of interest.
**Question**

*Is there a construction, using nothing more than the Axiom of Choice, of a subgroup of the full symmetric group on \( \mathbb{N} \) whose natural action on \( \mathbb{N} \) has a unique invariant mean and which does not contain \( \mathbb{F}_2 \)?*

In looking for new constructions of amenable groups with unique means on actions, a solution to the following may well provide some insight:

**Question**

*Is it consistent (or even true) that there is an amenable group with an action that has a unique invariant mean, but such that this mean is a measure taking values in all of \([0, 1]\)?*
Recall the following lemma:

**Lemma (NPS)**

If $G$ is an infinite amenable group and the action $\cdot : G \times S \to S$ has a unique invariant mean then $\Lambda(\cdot) \neq \ell_1(G)$.

This implies that the topological centre of Foreman’s group $G$ contains more than just $\ell_1(G)$. But, is it the case, perhaps, that the action of Foreman’s group is all of $\ell^\ast_\infty$? To answer this a slight detour is needed.
Definition (Erdös & Shelah)

A completely separable maximal almost disjoint family is a collection $\mathcal{A}$ of subsets of $\mathbb{N}$ such that:

- if $A$ and $B$ are distinct elements of $\mathcal{A}$ then $A \cap B$ is finite
- if $X \subseteq \mathbb{N}$ is infinite then there is $A \in \mathcal{A}$ such that $|A \cap X| = \aleph_0$
- if $X \subseteq \mathbb{N}$ does not belong to the ideal generated by $\mathcal{A}$ then $\{A \in \mathcal{A} \mid |A \cap X| = \aleph_0\}$ has cardinality $2^{\aleph_0}$.

Theorem (Shelah)

If $2^{\aleph_0} < \aleph_\omega$ then a completely separable maximal almost disjoint family exists.
The hypotheses of Shelah’s theorem are actually weaker than stated, but involve some technically complicated pcf hypotheses. The following theorem eliminates some of these.

**Theorem (Mildenberger, Raghavan & Steprāns)**

*If $s \leq \alpha$ then a completely separable maximal almost disjoint family exists.*

Here $s$ is the least cardinal of a splitting family and $\alpha$ is the least cardinal of a maximal almost disjoint family.
Theorem (NPS)
If a completely separable maximal almost disjoint family exists and \( \mathcal{U} \) is a non-principal ultrafilter on \( \mathbb{N} \) then there is some \( m \in \ell^*(\mathcal{G}_\mathcal{U}) \) such that \( \lambda_m \) (defined from the group action of \( \mathcal{G}_\mathcal{U} \) on \( \mathbb{N} \)) is not weak*-weak* continuous on the linear space generated by \( \beta\mathbb{N} \setminus \mathbb{N} \) — indeed, not even continuous on \( (\beta\mathbb{N} \setminus \mathbb{N}) - (\beta\mathbb{N} \setminus \mathbb{N}) \).

- There is a natural embedding of \( \beta\mathbb{N} \setminus \mathbb{N} \) into \( c_0(\mathcal{G}_\mathcal{U})^\perp \subseteq \ell^\infty(\mathcal{G}_\mathcal{U}) \).
- The argument easily extends to the action of Foreman’s subgroup of \( \mathcal{G}_\mathcal{U} \).
- The linear functional \( m \) is weak*-weak* continuous on \( \beta\mathbb{N} \setminus \mathbb{N} \) itself.
**Question**

Is the hypothesis that a completely separable maximal almost disjoint family exists necessary to obtain $m \in \ell^*(\mathcal{G}_U)$ not weak*–weak* continuous on $c_0^\perp$ or the linear space generated by $\beta\mathbb{N} \setminus \mathbb{N}$?

The hypothesis of the existence of completely separable maximal almost disjoint family has other implications in functional analysis.

**Theorem (Shelah & Steprāns)**

If there is a strong completely separable maximal almost disjoint family then there is a masa in the Calkin algebra that does not lift from $B(H)$.

Anderson had shown this from CH. Raghavan has shown the consistency of the no existence of strong completely separable maximal almost disjoint families.
Theorem (NPS)

If $\text{Sym}_{\aleph_0}(X)$ denotes the group of permutations of $X$ that fix all but finitely many elements of $X$ with the natural action on $X$ then $\lambda_m$ is weak*–weak* continuous on $c_0^\perp(X)$ for any $m \in \ell^*(\text{Sym}_{\aleph_0}(X))$.

Observe that the topological centre of the action of $\text{Sym}_{\aleph_0}(X)$ is contained in that of $G_U$ so the topological centre is large. However, ...
the following question remains unanswered:

**Question**

Does the topological centre of the action of $G_U$ have cardinality $2^{2^\aleph_0}$? Does this depend on $U$? What about Foreman’s group?

Note that this sort of question is not interesting for group actions because of the theorem of Lau.

Nor is the far more ambitious goal of finding a characterization of the topological centres of various group actions. But some information is available...
If \( m \in \ell^* (X) \) and is thought of as a finitely additive, signed measure on \( X \) then \( |m| \) denotes the variation of \( m \).

**Definition**

*For any group \( G \) acting on \( X \) define \( \text{Fix}(x) = \{ g \in G \mid gx = x \} \) for any \( x \in X \). Define \( \Xi_m : X \to \mathbb{C} \) by*

\[
\Xi_m(x) = |m|(G \setminus \text{Fix}(x))
\]

*and note that \( \Xi_m \in \ell_\infty(X) \).*

**Theorem (NPS)**

*If the action of \( G \) on a countable set \( X \) is such that \( m \circ n = m(G)n \) for all for all \( m \in \ell^*_\infty(G) \) and \( n \in c_0^\perp(X) \) then for any \( m \in \ell^*_\infty(G) \) the following are equivalent:

- \( m \in \Lambda(\circ) \)
- \( \Xi_m \in c_0(X) \).*
Lemma (NPS)

Let $G$ be a group acting on the set $X$ and $m \in \ell_\infty^*(G)$ of norm 1. Suppose there exist $A \subseteq X$ and $C \subseteq G$ such that $m(C) \geq 2/3$ such that if

$$B = \{x \in X \mid m(\{g \in G \mid gx \notin A\}) \geq 2/3\}$$

then one of the following hold:

1. $m$ is positive and $\{g^{-1}A \cap B \mid g \in C\}$ generates a filter
2. $\{g^{-1}A \cap B \mid g \in C\} \cup \{B \setminus g^{-1}A \mid g \notin C\} \cup \{B \setminus F \mid F \in [B]^{<\aleph_0}\}$ generates a filter

Then $\lambda_m$ is not weak*-weak* continuous.
Lemma (NPS)

Let $G$ be a countable group acting on the set $X$ such that:

- the action is almost free
- for each $g \in G$ the mapping $x \mapsto gx$ is finite-to-one
- there are infinitely many $x \in X$ such that for all $y \in X$ there are at most finitely many $g \in G$ such that $gx = y$.

Then for any $m \in c_0^\perp(G)$ of norm 1 there exist $A \subseteq X$ and $C \subseteq G$ such that $m(C) \geq 2/3$ such that if

$$B = \{ x \in X \mid m(\{ g \in G \mid gx \notin A \}) \geq 2/3 \}$$

then

$$\{ g^{-1}A \cap B \mid g \in C \} \cup \{ B \setminus g^{-1}A \mid g \notin C \} \cup \{ B \setminus F \mid F \in [B]^{<\aleph_0} \}$$

generates a filter.
Corollary (Lau)

If $G$ is a group acting on itself (countable is not necessary here) then no non-trivial positive element of $\ell^*(G)$ is in the topological centre.