Ilijas Farah

Combinatorial set theory of C*-algebras

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Springer
To the memory of my mother Vera Gajić
and my uncle Radovan Gajić.

Chapter 17 is dedicated to Colette.
Another smart, generous, and witty soul gone too soon.
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Reader,

I Here put into thy Hands, what has been the diversion of some of my idle and heavy Hours: If it has the good luck to prove so of any of thine, and thou hast but half so much Pleasure in reading, as I had in writing it, thou wilt as little think thy Money, as I do my Pains, ill bestowed. Mistake not this, for a Commendation of my Work; nor conclude, because I was pleased with the doing of it, that therefore I am fondly taken with it now it is done.

John Locke, The Epistle to the Reader
An Essay Concerning Human Understanding, 1660

Well, that’s about it for tonight ladies and gentlemen, but remember if you’ve enjoyed watching the show just half as much as we’ve enjoyed doing it, then we’ve enjoyed it twice as much as you. Ha, ha, ha.

Preface

This book is shorter than *In Search of Lost Time*, easier to read than *Principia Mathematica*, and it has more mathematical content than *War and Peace*.\(^1\)\(^2\) It provides an introduction to set-theoretic methods in the field of C\(^\ast\)-algebras, functional analysis, and general large metric algebraic structures. The main objects of study are the two classes of C\(^\ast\)-algebras: (i) nonseparable but usually nuclear, and even approximately finite, C\(^\ast\)-algebras and (ii) properties of massive quotient C\(^\ast\)-algebras such as coronas, ultraproducts, and relative commutants of separable subalgebras of massive algebras.

While writing this book I had in mind four types of readers:

1. Graduate students who had already taken an introductory course in C\(^\ast\)-algebras and would like to learn set-theoretic methods.
2. Graduate students who had already taken an introductory course in combinatorial set theory and would like to apply their knowledge to C\(^\ast\)-algebras.
3. Graduate students who had taken a first course in functional analysis, and possibly a first course in mathematical logic (the latter can be replaced by ‘sufficient mathematical maturity’), and are interested in learning about set-theoretic methods in functional analysis, and C\(^\ast\)-algebras in particular.
4. Mature mathematicians interested in learning about applications of set theory to C\(^\ast\)-algebras.

This book can be used as a text for an advanced two-semester graduate course. Alternatively, one can use Chapters 1–8, §9.2, 10, and 11 for a one-semester course on constructions of nonseparable C\(^\ast\)-algebras.

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1 If you thought there wasn’t much mathematical content in “War and Peace”, then you haven’t made it as far as the second epilogue, where the following sentence can be found: “Arriving at infinitesimals, mathematics, the most exact of sciences, abandons the process of analysis and enters on the new process of the integration of unknown, infinitely small, quantities.” Tolstoy proceeded to speculate on applications of calculus to history. This was written in the 1860s, barely ten years after the birth of Riemann’s integral and full eighty years before Asimov’s ‘Foundation.’

2 . . . and some of the jokes were not stolen from Douglas Adams.
Another alternative is to use Chapters 1, 2, and all of Part III (except §12.5, which relies on Chapter 5), for a one-semester course on set-theoretic aspects of the Calkin algebra, other coronas, and ultraproducts.

Yet another possibility for a mini-course on representations of C*-algebras would be to use only chapters 1–5. This option involves no set theory, but it covers aspects of the representation theory of C*-algebras not covered elsewhere.

If a course is given to students with a solid background in set theory, then all of Chapter 6 and parts of Chapters 7 and 8 should be omitted.

In the dual situation, when the audience consists of students with a solid background in C*-algebras, Chapters 1–3 can be omitted.

Acknowledgments

First of all, I should thank Paul Szeptycki and Ray Jayawardhana for kindly arranging a half-course teaching reduction in the fall 2016 semester that greatly helped in preparation of this book. I would also like to thank Bruce Blackadar, Sarah L. Browne, George A. Elliott, Eusebio Gardella, Saeed Ghasemi, Bradd Hart, Se–Jin Sam Kim, Akitaka Kishimoto, Boriša Kuzeljević, Paul Larson, Mikkel Munkholm, Narutaka Ozawa, N. Christopher Philips, Assaf Rinot, Ralf Schindler, Hannes Thiel, Andrea Vaccaro, Alessandro Vignati, and Beatriz Zamora–Aviles for critical remarks on the early drafts. I am indebted to Bruce Blackadar, Se–Jin Sam Kim, and Narutaka Ozawa who provided a significant mathematical input, including simple proofs of Lemma 2.3.11, Example 2.4.5, and Lemma 3.1.13 (B.B.) and Lemma 1.10.7, Theorem 1.10.8, Lemma 3.2.10, and Theorem 3.2.9, as well as Lemma 3.4.3 and its proof (N.O.). Special thanks to Chris Schafhauser for the occasional illuminating remark. I am indebted to my two wonderful editors: Eugene Ha, who made me start this project (he is forgiven) and Elizabeth Loew for expertly and patiently navigating me throughout this endeavour. While we are at the editors, many thanks to Assaf Rinot for suggesting that I try using Texpad; it made writing the last few sections of this text feel even more drastically different than Richard Strauss’s writing ‘An Alpine Symphony.’ Most of this book had been written using certain well-known LaTeX editor that shall remain nameless. Last, but not least, I owe special thanks to my daughter Gala for impeccable and generous linguistic support.

Toronto,
July 4, 2019

Iljias Farah

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3 'This occasionally did feel like “a job that, when all’s said and done, amuses me even less than chasing cockroaches”—which is how Strauss described the process of writing the said piece.

4 I claim credit for the remaining mistakes, obscurities, and all missing or misplaced articles in particular.
Why Bother?

In his 1925 PhD thesis, John von Neumann defined the cumulative hierarchy by transfinite recursion on the ordinals. The 0th level is the empty set, the successor of a given level is its power set, and a limit level is equal to the union of all preceding levels (see §A.5). Virtually all of mathematics as we know it (and then some) takes place within the first ten infinite levels of this hierarchy.

The cumulative universe (not drawn to scale).

So why do we need all these sets?

The use of abstract set theory in mathematics can be compared to the analytic number theory, where analytic methods are applied to prove statements about natural numbers. Or think of the definition of cohomology, using uncountable free groups. The only difference is that the gap between the cardinalities of objects considered and tools used can be much larger, and that it is known that in many situations this is necessarily so. Here are a few examples.

Take \( n \geq 1 \) and a Borel subset \( X \) of \( \mathbb{R}^{n+2} \). Let \( Y \) be the projection of \( X \) to \( \mathbb{R}^{n+1} \). Let \( Z \) be the projection of \( \mathbb{R}^{n+1} \setminus Y \) to \( \mathbb{R}^n \). Can one prove that \( Z \) is Lebesgue measurable for every choice of the Borel set \( X \)? In Gödel’s constructible universe \( L \) the answer is negative, and with \( n = 2 \) one can even choose \( X \) so that \( Z \) is a well-ordering.
of the reals. However, the \textit{existence} of a very large cardinal called measurable cardinal\(^5\) implies a positive answer. This is the question of Lebesgue measurability of \(\Sigma^1_2\) sets of reals (see [149], [218]).

The assertion that there are uncountably many infinite cardinals cannot be proved in Zermelo’s original axiomatization of set theory \(Z\). This can be proved by using the Replacement Axiom, added by Fraenkel to obtain the Zermelo–Fraenkel Set Theory, \(ZF\). \textit{Borel Determinacy} asserts that natural two-player games of perfect information with a Borel payoff are determined. Borel Determinacy cannot be proved in \(Z\) ([110]). It was proved in \(ZF\) by Martin (see [152]) by using transfinite iteration of the power set operation. The interplay between the Axiom of Determinacy and large cardinals that dwarf measurables provides one of the most fascinating justifications of the higher set theory (see [149] and [265]).

Laver used one of the strongest large cardinal assumptions not known to lead to a contradiction (the existence of a nontrivial elementary embedding of a rank-initial segment of von Neumann’s universe into itself) to solve a problem about algebras satisfying the left-distributive law \(a(bc) = (ab)(ac)\). This assumption has been removed, but large cardinals provided a natural route towards the solution (see [53]).

Last, but not least, some questions about operator algebras on a (separable!) Hilbert space can be answered only by using abstract set theory. Read on.

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\(^5\) The extent of largeness of measurable cardinals is the subject of Exercise 16.8.32.
Introduction for Experts

Some papers should be seen as territorial claims, not instruments of instruction.

A.R.D. Mathias

A C*-algebra is an algebra of bounded linear operators on a complex Hilbert space closed under the formation of adjoints and the norm topology. A von Neumann algebra is a unital subalgebra of $\mathcal{B}(H)$, the algebra of all bounded linear operators on a complex Hilbert space $H$, which is closed in the weak operator topology. The study of von Neumann algebras, under the name of ‘rings of operators,’ was initiated in the 1930s and 1940s in the work of Murray and von Neumann. The study of C*-algebras began in the 1940s by a result of Gelfand and Naimark stating that a complex Banach algebra with an involution is isomorphic to a C*-algebra if and only if it satisfies the C*-equality $\|aa^*\| = \|a\|^2$. It has since expanded to touch much of modern mathematics, including number theory, geometry, ergodic theory, mathematical physics, and topology.

Set theory is an area of mathematical logic concerned with the foundational aspect of mathematics and to some extent (but by no means exclusively), independence results. Gödel’s Incompleteness Theorem implies that no consistent and recursive set of axioms that extends the theory of natural numbers can decide every statement expressible in its language. Therefore some statements of number theory can be neither proved nor refuted on the basis of the standard Zermelo–Fraenkel set theory with the Axiom of Choice, ZFC. Gödel’s statements code metamathematical statements and are unnatural from the point of view of number theory. In spite of Gödel’s theorem, parts of core mathematics are generally considered immune to set-theoretic independence, and such independent statements in the field of operator algebras were found only recently. The answers to some prominent and long-standing open problems on operator algebras are independent from ZFC, and this is one of the main themes of this text.

In the past century, the connection between logic and operator algebras was sparse albeit fruitful. In this text we present one aspect of this progress that brought two subjects closer together. Our main goal is to give a self-contained and as elementary as possible “instrument of instruction” for set-theoretic methods used in
the past fifteen years to resolve several long-standing problems in the theory of C*-algebras. Along the way we provide an introduction to several loosely related topics, such as the representation theory of C*-algebras, massive C*-algebras such as coronas, asymptotic sequence algebras (reduced products), and ultraproducts, as well as infinitary combinatorics and its applications to functional analysis.\textsuperscript{6}

The problems from the field of operator algebras that were resolved using set theory generally fall into one of the two following categories.

Nonseparable Examples

Problems in the first group ask whether all operator algebras with a certain property \( P \) also satisfy a related property, \( Q \). Two textbook examples of such problems are Dixmier’s 1967 problems, asking is every unital inductive limit of full matrix algebras isomorphic to a tensor product, and is every C*-algebra that locally looks like a full matrix algebra isomorphic to an inductive limit of full matrix algebras? (In technical terms: is every unital approximately matricial (AM) C*-algebra is uniformly hyperfinite (UHF) and is every unital locally matricial (LM, or matroid) C*-algebra approximately matricial?) Yet another example is Naimark’s problem, asking whether a C*-algebra all of whose irreducible representations are unitarily equivalent is isomorphic to the algebra of compact operators on some Hilbert space. A problem closely related to Naimark’s is whether Glimm’s dichotomy, asserting that the number of unitary equivalence classes of irreducible representations of a separable and simple C*-algebras are unitarily equivalent is either 1 or not smaller than \( c^7 \) holds for all simple C*-algebras. Another problem asks whether every amenable norm-closed algebra of operators on a Hilbert space is isomorphic to a C*-algebra. All of these problems, except possibly the last one, have positive solutions when restricted to separable C*-algebras.

In the simplest form of a set-theoretic resolution to a problem of this sort, an example is defined from concrete parameters such as cardinals or real numbers. Both Dixmier’s problems fall into this class, with counterexamples of minimal density characters\textsuperscript{8} equal to the first two uncountable cardinals, \( \text{\textcrl{aleph}}_1 \) and \( \text{\textcrl{aleph}}_2 \), respectively.

Some other problems in the first group are solved by a recursive construction of transfinite length. An example was provided by Weaver’s construction of a prime, but not primitive, C*-algebra. Crabb and Katsura later simplified Weaver’s construction and provided a definition of such an algebra, thus ‘upgrading’ (or perhaps downgrading?) the solution of this problem to the first class. The problem of the existence of an amenable norm-closed algebra of operators on a Hilbert space not isomorphic to a C*-algebra is resolved by defining a counterexample, but the definition uses

\textsuperscript{6} Students and non-experts, please proceed to “Annotated Contents” on page xvi or even straight to “Prerequisites and Appendices” on page xx and later use the earlier pages of this introduction as a reference.

\textsuperscript{7} \( c := 2^{\text{\textcrl{aleph}}_0} \), the cardinality of the real line.

\textsuperscript{8} The density character of a topological space is the smallest cardinality of a dense subset.
as a parameter a so-called Luzin family of subsets of \( \mathbb{N} \), constructed by transfinite recursion.

In the third subgroup of examples the algebras are again constructed by a transfinite recursive construction, but the construction is facilitated by a diagonalization principle independent from the standard ZFC axioms of set theory such as the Continuum Hypothesis or its strengthening, Jensen’s diamond \( \Diamond_1 \). The latter is used to construct counterexamples to Naimark’s problem and Glimm’s dichotomy for \( C^* \)-algebras of density character \( \aleph_1 \). More precisely, for every \( 1 \leq n \leq \aleph_0 \) we construct a simple \( C^* \)-algebra with exactly \( n \) irreducible representations (up to the unitary equivalence) that is not isomorphic to the algebra of compact operators on a Hilbert space. This algebra can be chosen to be nuclear, stably finite, and even approximately finite.

**Properties of Massive Quotients**

Another, even more exciting and more fundamental, group of questions is concerned with properties of familiar, canonical, examples of \( C^* \)-algebras. Explicitly defined \( C^* \)-algebras, some of whose essential properties may depend on set theory, are unlikely to be separable.\(^9\) The massive quotient algebras are the most likely candidates for having a ‘set-theoretically malleable’ theory. This unruliness is closely related to the fact observed in both noncommutative geometry and descriptive set theory, that the quotient spaces are frequently intractable. The simplest, and most established, example of a massive quotient \( C^* \)-algebra is the quotient of the algebra of bounded linear operators on \( \ell_2(\mathbb{N}) \) over the ideal of compact operators. This is the Calkin algebra, \( \mathcal{Q}(H) \).

Motivated by their work on extensions of separable \( C^* \)-algebras by the algebra of compact operators, Brown, Douglas, and Fillmore asked whether \( \mathcal{Q}(H) \) has an outer automorphism. By results of Phillips–Weaver and the author, the answer to this question is independent from ZFC.

**Separable \( C^* \)-algebras and the General Theory**

While the main goal of this text is to present the theory of nonseparable \( C^* \)-algebras, a substantial amount of space and effort is devoted to the theory of \( C^* \)-algebras that does not explicitly involve set theory. In addition, a fair portion of this theory applies only to separable \( C^* \)-algebras.

This choice was guided by two rationales. First, students may appreciate having all the required information in a single volume. The second one is more substantial. Many of the standard results needed in latter parts of this text cannot be easily found

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\(^9\) This is essentially a consequence of Shoenfield’s Absoluteness Theorem and standard descriptive set-theoretic coding arguments (see Theorem B.2.12).
elsewhere, and in at least one case a complete proof hasn’t been at all available until now. A list of these results follows.

The strong homogeneity of the pure state space of every separable C*-algebra is Theorem 5.6.1. A rough sketch of a proof of this theorem was given by Akemann and Weaver in [5], combining techniques of [162], [111], and [122]. We give a complete proof, including Kishimoto’s rather elementary proof ([160]) of the existence of an approximate diagonal of an irreducible representation of a C*-algebra (Theorem 5.1.2). The Wolf–Winter–Zacharias structure theorem for completely positive maps of order zero is Theorem 3.2.9. Theorem 12.3.2 is the Johnson–Parrott theorem about derivations of C*-algebras (or rather, its consequence that the image of any masa in \( \mathcal{B}(H) \) under the quotient map is a masa in the Calkin algebra).

The Akemann–Anderson–Pedersen theorem on excision of pure states is proved in Theorem 5.2.1. We use excision to provide a proof of Kirchberg’s Slice Lemma. Excision also forms the basis for the theory of noncommutative analogs of ultrafilters, known as maximal quantum filters (see §5.3). They are used to facilitate the use of set-theoretic methods in the study of pure states on C*-algebras. A proof of Ulam stability of ε-representations of compact groups and \( \varepsilon \)-*-homomorphisms between finite-dimensional C*-algebras is in §17.2.

Many of the proofs have been taken apart, have had all of their gears oiled, and were then reassembled. One example is Theorem 3.7.2, Glimm’s theorem that every non-type I C*-algebra has a subalgebra with a quotient isomorphic to the CAR algebra.12

**Annotated Contents**

Part I is about C*-algebras.

In Chapter 1 we introduce the abstract C*-algebras and work towards the Gelfand–Naimark–Segal Theorem (Theorem 1.10.1). Along the way we discuss abelian C*-algebras and Gelfand–Naimark and Stone dualities, continuous functional calculus, positivity in C*-algebras, approximate units, and quasi-central approximate units.

In Chapter 2 we give basic examples of constructions C*-algebras: direct sums and products, inductive limits, stabilization, suspension, cone, hereditary C*-subalgebras, quotients, tensor products, full and reduced group C*-algebras, and full and reduced crossed products. Finite-dimensional C*-algebras and *-homomorphisms between them are classified by Bratteli diagrams. Universal C*-algebras given by generators and relations are studied in some detail. After a discussion of automorphisms of C*-algebras, we conclude with a section on C*-algebras of real rank zero.

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10 As I was writing these lines, a more detailed exposition of this proof became available in [244].
11 This proof is adapted from [208], where a proof of the excision of pure states in the case of purely infinite and simple C*-algebras can be found.
12 This result can be found e.g., in the classic [194], but the proof given here is somewhat different.
Chapter 3 starts with a telegraphic introduction to von Neumann algebras. We also prove the Stinespring and Wolf–Winter–Zacharias structure theorems for completely positive maps and completely positive maps of order zero, respectively, and introduce averaging techniques and conditional expectations. We prove the Kadison Transitivity Theorem and its generalization due to Glimm–Kadison. After studying pure states and equivalence relations on the space of pure states of a C*-algebra (unitary/spatial equivalence and conjugacy by an automorphism), we conclude with a study of the second dual of a C*-algebra.

In Chapter 4 we adapt a technique, borrowed from the theory of II\textsubscript{1} factors, of juxtaposing the GNS Hilbert space structure associated with a tracial state and the C*-algebra structure to study reduced group C*-algebras. An emphasis is given to the C*-algebras associated to free products of groups. We give basic norm estimates for the elements of a group algebra, and present basics of Powers groups and criteria for simplicity of reduced group C*-algebras. The chapter concludes with a study of normalizers of diffuse masas.

Chapter 5 starts with Kishimoto’s construction of approximate diagonals. We then prove the Akemann–Anderson–Pedersen theorem on excision of pure states and apply it to prove Glimm’s Lemma and Kirchberg’s Slice Lemma. Excision is also used to exhibit a bijection between ‘maximal quantum filters’ and pure states of a C*-algebra. The maximal quantum filters are used to study extensions of pure states. The chapter concludes with a proof of the Kishiomoto–Ozawa–Sakai theorem on the homogeneity of the pure state space of separable C*-algebras.

In Part II we introduce set-theoretic tools and apply them to C*-algebras. Chapter 6 is devoted to infinitary combinatorics: Club filter, nonstationary ideal, the pressing down lemma, variants of the \(\Delta\)-system lemma, and Kueker’s structure theorem for clubs in \([X]^{\aleph_0}\).

In Chapter 7 we introduce continuous variants of the results from Chapter 6 in which \([X]^{\aleph_0}\), the family of countably infinite subsets of a fixed uncountable set \(X\), is replaced with Sep\((A)\), the family of all separable substructures of a nonseparable metric structure \(A\). The latter directed set is \(\sigma\)-directed, but not concretely represented, and in some cases only the approximate versions of results studied in Chapter 6 hold. These approximate versions are used to reflect properties of large C*-algebras to their separable subalgebras and prove that algebras indistinguishable by any of the standard K-theoretic invariants are not isomorphic. The spaces of models—both discrete and metric—are studied in \(\S\)7.1. Proposition 7.2.9 is a metric variant of the Pressing Down Lemma.

Chapter 8 is devoted to the additional set-theoretic axioms used in this text. They are literally treated as axioms—no attempt has been made to prove their relative consistency with ZFC. We start with rather standard, brief, and self-contained treatments of the Continuum Hypothesis and Jensen’s \(\diamondsuit_{K_1}\) and use \(\diamondsuit_{K_1}\) to construct a Suslin tree. The \(\sigma\)-complete directed systems of isomorphisms between separable substructures are studied in some detail. The axiom asserting that a Polish space cannot be covered by fewer than \(c\) meager subsets is used to construct a selective ultrafilter on \(\mathbb{N}\). The chapter ends with a discussion of the Ramseyan axiom OCA\(_T\) and some of its applications.
Several themes picked up later on in this text originate in Chapter 9. The first one is the structure of the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$ and related quotient structures. The interplay between separability of $\mathcal{P}(\mathbb{N})$ (identified with the Cantor space) and countable saturation of $\mathcal{P}(\mathbb{N})/\text{Fin}$ is used to construct several objects witnessing the incompactness of $\aleph_1$, such as the independent families, almost disjoint families, and gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$. In the latter sections this Boolean algebra is injected into massive corona $C^*$-algebras. This is used to construct subalgebras of $\mathcal{B}(H)$ with unexpected properties, such as an amenable norm-closed algebra of operators on a Hilbert space not isomorphic to a $C^*$-algebra ($\S$15.5), and Kadison–Kastler near, but not isomorphic, $C^*$-algebras ($\S$14.4). We introduce the Rudin–Keisler ordering on the ultrafilters and construct Rudin–Keisler incomparable nonprincipal ultrafilters on $\mathbb{N}$. Basics of the Tukey ordering of directed sets are presented in $\S$9.6. We prove that two directed sets are cofinally equivalent if and only if they are isomorphic to cofinal subsets of some directed set. We study the directed set $\mathbb{N}^{\mathbb{N}}$, the associated small cardinals $b$ and $\delta$, and two directed sets cofinally equivalent to $\mathbb{N}^{\mathbb{N}}$ used to stratify the Calkin algebra, $\text{Part}_N$ and $\text{Part}_\ell_2$. This chapter ends with a convenient structure result for comeager subsets of products of finite spaces.

In Chapter 10, infinitary combinatorics is applied to study graph CCR algebras. These ‘twisted’ reduced group $C^*$-algebras associated to a Boolean group and a cocycle given by a graph are AF (approximately finite), and even AM (approximately matricial) if they are simple (i.e., if they have no proper norm-closed, two-sided, ideals). After developing structure theory, we recast results of the author and Katsura and show that in spite of their simplicity (here ‘simplicity’ stands for ‘lack of complexity’), graph CCR algebras provide counterexamples to several conjectures about the structure of simple, nuclear $C^*$-algebras. We construct an AM $C^*$-algebras that is not UHF but it has a faithful representation on a separable Hilbert space. In every uncountable density character $\kappa$ there are $2^\kappa$ nonisomorphic graph CCR algebras with the same K-theoretic invariants as the CAR algebra. By using an independent family of subsets of $\mathbb{N}$, we construct a simple graph CCR algebra that has irreducible representations on both separable and nonseparable Hilbert spaces.

Other examples of nonseparable $C^*$-algebras are constructed in Chapter 11. We start with Akemann’s $C^*$-algebra with no abelian approximate unit. This is followed by a strengthening of a result of Akemann and Weaver due to the author and Hirshberg. It gives a counterexample to Glimm’s Dichotomy for nonseparable $C^*$-algebras: For every $n \leq \aleph_0$ there exists a simple $C^*$-algebra of density character $\aleph_1$ with exactly $n$ unitarily inequivalent irreducible representations. The case $n = 1$ is a counterexample to Naimark’s problem. These results use Jensen’s $\diamondsuit_{\aleph_1}$, and it is not known whether they can be proved in ZFC. The chapter concludes with a study of $C^*_r(F_\kappa)$, the reduced group algebra of the free group with $\kappa$ generators. For every $\kappa$, this $C^*$-algebra has only separable abelian $C^*$-subalgebras (Popa), and every two pure states are conjugate by an automorphism (Akemann–Wassermann–Weaver). Both results apply to $C^*_r(\Gamma)$, where $\Gamma$ is the free product of any family of nontrivial countable groups.

Part III of this text is devoted to the Calkin algebra and other massive quotient structures—coronas, ultraproducts, asymptotic sequence algebras, and relative com-
mutants of their separable \( C^* \)-subalgebras. A mix of set-theoretic, model-theoretic, and operator-algebraic techniques provides means for a unified treatment of these \( C^* \)-algebras.

In Chapter 12 we introduce the Calkin algebra \( \mathcal{Q}(H) \) and establish the parallel between the poset of its projections and the quotient Boolean algebra \( \mathcal{P}(\mathbb{N})/\text{Fin} \). We prove the Weyl-von Neumann-Berg-Sikonia theorem on the existence of diagonalized liftings of singly-generated abelian \( C^* \)-subalgebras of \( \mathcal{Q}(H) \). In the separable case the only obstructions to the existence of diagonalized liftings are K-theoretic. In the non-separable case examples can be provided by a simple counting argument or by a more profound Luzin-type construction of a ‘twist’ of projections in \( \mathcal{Q}(H) \). We prove the Johnson–Parrot Theorem, that the image of a masa in \( \mathcal{B}(H) \) under the quotient map is a masa in \( \mathcal{Q}(H) \). Pure states on \( \mathcal{Q}(H) \) are the noncommutative analogs of nonprincipal ultrafilters on \( \mathbb{N} \), and this connection is brought forth by the language of maximal quantum filters. A recursive construction of maximal quantum filters facilitated by the Continuum Hypothesis is used to construct non-diagonalizable pure states on \( \mathcal{Q}(H) \), refuting a conjecture of Anderson and giving a negative answer to a problem of Kadison and Singer; this is a theorem of Akemann and Weaver. We offer additional proofs of this theorem from two weakenings of the Continuum Hypothesis \( \text{cov}(\mathcal{M}) = \mathfrak{c} \) and \( d \leq \dagger \) that between them cover many ‘common’ models of ZFC. No ZFC construction of a nondiagonalizable pure state on \( \mathcal{Q}(H) \) is presently known.

In Chapter 13 we study multiplier algebras and coronas of non-unital \( C^* \)-algebras. These are the non-commutative analogs of the Čech–Stone compactification and the Čech–Stone remainder, respectively, of a locally compact Hausdorff space.

Gaps in coronas are studied in Chapter 14. We show that the rich and well-studied gap spectrum of \( \mathcal{P}(\mathbb{N})/\text{Fin} \) embeds into the corona of every \( \sigma \)-unital, non-unital, \( C^* \)-algebra. This is used to prove two incompactness results. The Choi–Christensen construction of Kadison–Kastler near, but non-isomorphic, \( C^* \)-algebras is recast in terms of gaps: every gap in the Calkin algebra can be used to produce a family of examples of this sort. Every uniformly bounded representation of a countable, amenable, group in the Calkin algebra is unitarizable. Using a Luzin family, one defines a uniformly bounded, non-unitarizable, representation of \( \bigoplus_{\mathfrak{r}} \mathbb{Z}/2\mathbb{Z} \) in the Calkin algebra. This example yields an amenable operator algebra not isomorphic to a \( C^* \)-algebra. This is a result of the author, Choi and Ozawa.

In Chapter 15 we study the overarching concept of countable degree-1 saturation, and prove a theorem of author and Hart that that all massive quotient \( C^* \)-algebras (coronas of \( \sigma \)-unital, non-unital \( C^* \)-algebras, ultraproducts associated with an nonprincipal ultrafilter on \( \mathbb{N} \), and relative commutants of separable \( C^* \)-subalgebras of countably degree-1 saturated \( C^* \)-algebras) have this property. Countable degree-1 saturation subsumes several separation properties of massive \( C^* \)-algebras with neat acronyms, such as Pedersen’s SAW*, CRISP, AA-CRISP, also sub-Stonean and Kirchberg’s \( \sigma \)-sub-Stonean \( C^* \)-algebras, and those \( C^* \)-algebras satisfying the conclusion of Kasparov’s Technical Lemma. Many of these properties are analogs of the absence of gaps with countable sides in \( \mathcal{P}(\mathbb{N})/\text{Fin} \). Among other applica-
tions, we prove that countably degree-1 saturated $C^*$-algebras are essentially non-factorizable (§15.4.3), that every uniformly bounded representation of a countable amenable group into such $C^*$-algebra is unitarizable (§15.4.1; this fails for uncountable groups, §14.5), and that such $C^*$-algebras admit a poor man’s version of Borel functional calculus that generalizes the Brown–Douglas–Fillmore ‘Second Splitting Lemma’ (§15.4.2).

In Chapter 16 we use continuous model theory to study ultraproducts and asymptotic sequence algebras (i.e., reduced products associated with the Fréchet filter). The Fundamental Theorem of Ultraproducts (Łoś’s Theorem) and the corresponding result for reduced products, Ghasemi’s Feferman–Vaught Theorem, are proved for arbitrary metric theories. These theorems are used to prove the countable saturation of ultraproducts associated with countably incomplete ultrafilters and the countable saturation of reduced products associated with the Fréchet filter. The $\sigma$-complete back-and-forth systems of partial isomorphisms between $C^*$-algebras (introduced in §8.2) are used in §16.7 to prove that the Continuum Hypothesis implies all ultrapowers and all relative commutants of a separable $C^*$-algebra associated with non-principal ultrafilters on $\mathbb{N}$ are isomorphic. The chapter ends with theorems due to Ge–Hadwin and the author, Hart, and Sherman, in which a large number of (outer) automorphisms of ultrapowers, asymptotic sequence algebras, and related massive $C^*$-algebras are constructed using the Continuum Hypothesis.

Chapter 17 begins with a rather elementary proof of the Phillips–Weaver Theorem: the Continuum Hypothesis implies that the Calkin algebra has outer automorphisms. The analogous result, due to Coskey and the author, is proved for the coronas of all stable, $\sigma$-unital $C^*$-algebras. This is followed by a proof of the Burger–Ozawa–Thom theorem on Ulam stability of $\varepsilon$-homomorphisms, used to prove an Ulam stability result for $\varepsilon$-$^*$-homomorphisms whose domain is a finite-dimensional $C^*$-algebra. The final sections of this chapter (and the text) are devoted to a complete proof that OCA$_T$ implies that all automorphisms of the Calkin algebra are inner.

**Prerequisites and Appendices**

The reader is assumed to have taken a standard one-semester first course in functional analysis. This subsumes some familiarity with the basic point-set topology: compactness, Hausdorffness, nets, Cauchy nets. Paracompactness is used exactly once in this text. The reader is also assumed to be familiar with rudimentary axiomatic and naive set theory. Some acquaintance with model theory and logic of metric structures is helpful, but not necessary (except in Chapter 16, but this chapter is an end in itself). To be specific, most uses of model theory outside of Chapter 16 can be summarized by three words: “Löwenheim–Skolem Theorem” and the readers who would rather call it “Blackadar’s method” are by all means welcome to do so. In addition, all uses of Łoś’s Theorem for ultraproducts relevant to us can be

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13 This assertion is equivalent to the Continuum Hypothesis. In order to keep the cardinality of the set of pages of this text within reasonable limits the proof of the converse is only outlined.
proved with one’s bare hands. However, one can also solve any cubic equation using Tartaglia’s original algorithm.

The appendices contain brief reviews of the axiomatic and naive set theory (§A), descriptive set theory (§B), functional analysis (§C), and model theory (§D).  

**Notation**

*Our notation is mostly standard* is what I wish I could say at this point. Alas! This text attempts to bridge the gap between two sophisticated areas of mathematics each of which has its own (often idiosyncratic) notation and terminology. The best that I can do is provide a list of notational conventions and conflicts.

The issue of choosing the right fonts and symbols cannot be overestimated—one of the most serious criticisms of [52] that I am aware of is that ‘the author used all the wrong fonts.’ I will mostly refrain from using ω (see Exercise 0.0.1 and Exercise 0.0.2), with two exceptions. The vector state associated to a unit vector ξ will be denoted ωξ. I will write N instead of ω almost everywhere; in transfinite constructions, ω will denote the least infinite ordinal.

Greek letters ξ, η, ζ, denote vectors in a Hilbert space except in the appendix, where they denote ordinals. Thus ωξ stands for the vector state associated with vector ξ throughout this text, except in §A (see Exercise 0.0.2). The letters ε and δ will stand for small but positive real numbers, with one exception. The standard δ symbol is defined by δxy = 1 if x = y and δxy = 0 otherwise.

The symbol π (sometimes embellished with subscripts) is used to denote representations of C*-algebras, projections from a Cartesian product to its components, various quotient maps, and, last but not least, the area of the unit disk. Symbols m, n, i, j, k, l, stand for natural numbers, with an occasional i = √−1.

The asterisk is even more overused than ω or π (at least in operator algebras). In addition to having a∗ denote the adjoint operation, X* denote the dual of a Banach space, and assorted C’s and W’s denoting self-adjoint operator algebras, I will use the standard set-theoretic notation X ⊆ Y for ‘the difference X \ Y is finite.’

We shall be using |a| only to denote the absolute value of a scalar, the absolute value of a function or an operator, and the cardinality of a set.  

Some of the rules for the assignment of fonts to data types are described in the following lines.

Ultrafilters are assumed to be nonprincipal (or free) ultrafilters on N, and they will be denoted ℵ,U, V, and W.

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14 An early draft of this text contained an extensive appendix on absoluteness. I am convinced that understanding absoluteness is necessary for understanding the role (and the limitations) of set theory as we presently know it. Nevertheless, the insane idea of cramming this sensitive material into an appendix to a 500 page book that already contained diverse, and sometimes technically demanding, material has been abandoned.

15 Indeed, N.C. Phillips pointed out that the absolute value signs are even more overused in mathematics than the asterisk.
Blackboard bold font is used for sets of numbers, \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{T}, \mathbb{D} \), where

\[ T := \{ z \in \mathbb{C} : |z| = 1 \} \text{ and } \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}. \]

The remaining symbols, \( \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \) and \( \mathbb{C} \), are I believe standard enough (but note that \( 0 \in \mathbb{N} \) and \( \omega = \mathbb{N} \) when convenient).

For abstract sets I use the sans-serif font: \( A, X, Y, s, t \). This font is also used to denote distinguished sets of operators or functionals associated to a \( \mathbb{C}^* \)-algebra \( A \):

- \( S(A) \) is the set of states on \( A \),
- \( P(A) \) is the space of pure states in \( A \), and
- \( U(A) \) is the unitary group of \( A \). In order to avoid confusion with the power set \( \mathcal{P}(A) \) and the set of pure states \( P(A) \), the poset of projections of a \( \mathbb{C}^* \)-algebra \( A \) is denoted \( \text{Proj}(A) \).

Capital letters \( A, B, C, D, \) will usually denote \( \mathbb{C}^* \)-algebras, and capital letters \( M \) and \( N \) will usually denote von Neumann algebras or multiplier algebras of non-unital \( \mathbb{C}^* \)-algebras. Operators are denoted by lower case letters; mostly \( a, b, c, d \) but in some of the more complex arguments we use up a fair portion of the alphabet. When venturing into operator theory and talking about concrete operators on a Hilbert space that do not belong to any given operator algebras, we denote them with capital Roman letters \( R, S, T, \ldots \). Capital letters in fraktur font are used to denote structures (in the model-theoretic sense), both discrete and metric: \( \mathfrak{A}, \mathfrak{B} \). The domains of these structures are denoted by the corresponding letters in sans serif font, so that the domain of \( \mathfrak{A} \) is denoted \( \mathfrak{A} \), the domain of \( \mathfrak{B} \) is denoted \( \mathfrak{B} \), and so on. This convention is used only until the distinction has been made very clear. Small fraktur font is reserved for small cardinals associated with the continuum.

And now, for the good news. The fruitful interaction between operator algebras and descriptive set theory is reflected in some common terminology. In both subjects analytic sets are continuous images of Borel sets and an equivalence relation is smooth if the quotient Borel space is standard. Unlike 'normal,' the word 'compact' has, to the best of my knowledge, the same meaning throughout all of mathematics. ‘Weakly compact’ will mean ‘compact in some weak topology’ or ‘compact in the weak operator topology’; weakly compact cardinals haven’t been used in the field of \( \mathbb{C}^* \)-algebras, yet.

The \( \varepsilon \)-ball centered at \( x \) in a metric or normed space is denoted

\[ B_\varepsilon(x) := \{ y : d(x,y) < \varepsilon \}. \]

If \( X \) and \( Y \) are subsets of a metric space I write \( X \subseteq_\varepsilon Y \) if \( \inf_{y \in Y} d(x,y) < \varepsilon \) for all \( x \in X \). For elements \( x \) and \( y \) of a metric space \( x \approx_\varepsilon y \) stands for \( d(x,y) < \varepsilon \). (In spite of the suggestive notation, this is certainly not an equivalence relation.)

We write \( F \in A \) for ‘\( F \) is a finite subset of \( A \’ \)

It will be convenient to use the following two quantifiers:

\[ (\forall^m n) \text{ stands for } (\exists m \in \mathbb{N})(\forall n \geq m), \text{ and } \]

\[ (\exists^m n) \text{ stands for } (\forall m \in \mathbb{N})(\exists n \geq m). \]

Following a convention going back to von Neumann, an ordinal is identified with the set of smaller ordinals and natural numbers are identified with finite ordinals:
0 := ∅, 1 := {∅}, and \( n = \{0, \ldots, n - 1\} \) for all \( n \in \mathbb{N} \). This is but one reason why it is important to distinguish between \( f(X) \) and 
\[
f[X] := \{f(x) : x \in X\}.
\]
The characteristic function of a set \( X \) (considered as a subset of some fixed set clear from the context) is denoted \( \chi_X \). (Some authors, and functional analysts in particular, use \( 1_X \), but in this text \( 1_A \) is reserved for the unit of a \( C^* \)-algebra \( A \).)

Symbols for index sets are omitted whenever this is convenient both for myself and—to the best of my knowledge—for the reader. I will interchangeably write \((b_j : j \in J)\), \((b_j)_j\), or even \((b_j)\) when the index-set is clear from the context. The same remark applies to standard abbreviations such as \( \prod \mathcal{A}_j \) for the ultraproducts, \( \bigotimes \mathcal{A}_n \) or \( \prod_n \mathcal{A}_n \) for products. \( \lim_n \) stands for \( \lim_{n \to \omega} \), and \( \lim_\lambda \) stands for \( \lim_{\lambda \to \Lambda} \) if \( \Lambda \) is a net clear from the context.

Apart from the hopefully innocuous conventions described in the previous paragraph, between redundancy and confusion I systematically choose redundancy.\(^{17}\)

Every ordered set is therefore either ‘linearly ordered’ (this is synonymous to ‘totally ordered’) or ‘partially ordered.’ A relation is a quasi-ordering if it is transitive but not necessarily antisymmetric.

### Exercises

As an ice-breaker I provide a multiple choice quiz.

**Exercise 0.0.1.** What does \( R^0 \) stand for?

1. An ultrapower of the hyperfinite \( \text{II}_1 \) factor \( R \) associated to a free ultrafilter \( \omega \) on \( \mathbb{N} \), or
2. the space of all sequences \((r_n : n \in \omega)\) of real numbers, where \( \omega \) denotes the least infinite ordinal, identified with \( \mathbb{N} \) (and yes, zero is a natural number).

**Exercise 0.0.2.** What does \( \omega_\xi \) stand for?

1. The vector state on \( \mathcal{B}(H) \) associated with a vector \( \xi \) in the Hilbert space \( H \), in symbols \( \omega_\xi(a) := \langle a\xi | \xi \rangle \).
2. The \( \xi \)th infinite cardinal, also denoted \( \aleph_\xi \), where \( \xi \) is an ordinal and counting starts at 0.

**Exercise 0.0.3.** What is the meaning of “\( \phi \) is a contraction”?

1. \( \| \phi(x) - \phi(y) \| \leq \| x - y \| \) for all \( x \) and \( y \) in the domain of \( \phi \).
2. \( \| \phi(x) - \phi(y) \| < \| x - y \| \) for all \( x \) and \( y \) in the domain of \( \phi \).
3. Well, it’s the shortening of a word or a group of words by omission of a sound or letter.

---

\(^{16}\) The line has to be drawn somewhere; I avoid writing \((\forall j \in n)\) in place of the equivalent \((\forall j < n)\).

\(^{17}\) With apologies to George Elliott.
Hint: Apparently there is not much use for Banach’s fixed point theorem in operator algebras. *Contraction* is a 1-Lipshitz function, i.e. function $f$ between metric spaces such that $d(x,y) \geq d(f(x), f(y))$ for all $x, y$. (A notable special case of a contraction is a linear operator of norm $\leq 1$.)

**Exercise 0.0.4.** What does $|A|$ stand for?
1. $(A^*A)^{1/2}$.
2. The smallest ordinal equinumerous with $A$, assuming the Axiom of Choice.
   Otherwise this is the equivalence class of all sets equinumerous with $A$.

**Exercise 0.0.5.** A $C^*$-algebra $A$ is finite if
1. For every partial isometry $v \in A$ such that the projections $p := vv^*$ and $q := v^*v$ satisfy $pq = p$ one has $p = q$.
2. $|A| < \aleph_0$.

**Exercise 0.0.6.** What are the elements of $L_2$?
1. Equivalence classes of square-integrable functions.
2. They are $\emptyset$ and $\{\emptyset\}$.

If you answered (1) to more than half of the questions, you are an operator algebraist. If you answered (2) to more than half of the questions, you are a set theorist.\(^{18}\) If you answered (3) then you may be a linguist . . . or a “Weird Al” Yankovich fan.

### Exercises

A fair number of the exercises form an integral part of the text. They are chosen to widen and deepen the material from the corresponding chapters. Some other exercises serve as a warmup for the latter chapters, either by preparing the technical grounds or by putting a bug into the reader’s ear. Every single instance of the dreaded ‘it is easy to see’ phrase has been (at least) repackaged as a timely exercise, invoked later on, sometimes several chapters later. Every exercise used in some proof has been clearly marked. Finally, entire subsections work as mini Moore-style courses, enticing and cajoling the reader to learn more about $C^*$-algebras and set theory.\(^{19}\)

### Silliness

Every effort has been made to relieve and reward the reader’s efforts. In addition to providing the best available proofs and unearthing analogies and connections

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\(^{18}\) Bonus question: What is $M_n$?

\(^{19}\) Or so I like to think.
previously unknown to humanity, the text contains a variety of quips of varying relevance (and, regrettably, of varying funnyness degree). All of them serve the purpose of putting the reader’s mind at ease before hitting them with complex (no pun intended) mathematics. Most quotations are related to the section they precedes in some, frequently unobvious, way.

**What has been omitted**

In order to maintain an elementary level and a (relatively) slow pace, it was necessary to omit numerous results. Descriptive set theory and abstract classification are mentioned only in passing, and model theory plays a nontrivial role only in Chapter 16 (cf. [84]). We do not prove the relative consistency of set-theoretic axioms, such as Jensen’s or OCA, with ZFC. The reader can either take this on faith or read any of the existing excellent presentations (e.g. [166, §III.7], also [258]).

The original (abandoned) plan for this text included a section on selective ultrafilters, but see [95].

The most recent general book on C*-algebras that omits K-theory that comes to my mind is the (recently reissued) forty year old [194]. The K-theory, K-homology, and Ext haven’t yet been applied to C*-algebras in conjunction with set theory, but this may be only a matter of time.

As I am writing these lines, exciting new developments are taking place in direct applications of forcing to C*-algebras and set-theoretic analysis of the uniform Roe algebras, but it is now too late to start another elephant (see [185, p. 155]).

All C*-algebras in Part I are separable unless otherwise specified. All C*-algebras in Part II and Part III are nonseparable unless they are obviously separable.
Part I

$C^*$-algebras
Chapter 1
C*-algebras, Abstract and Concrete

For a moment, nothing happened. Then, after a second or so, nothing continued to happen.
Douglas Adams, The Hitchhiker’s Guide to the Galaxy

In the first chapter we start with linear operators on a Hilbert space, define concrete and abstract C*-algebras, and prove the GNS representation theorem.

1.1 Operator Theory and C*-algebras

This section contains a fast-paced review of operator theory in Hilbert spaces, including the polar decomposition theorem.

Let $H$ denote a complex infinite-dimensional Hilbert space equipped with the inner product $(\xi | \eta)$ and norm $\|\xi\| := (\xi | \xi)^{1/2}$. The inner product is sesquilinear, i.e., linear in the first coordinate and conjugate linear in second. By the polarization identity

$$(\xi | \eta) = \frac{1}{4} \sum_{k=0}^{3} i^k (\xi + i^k \eta | \xi + i^k \eta)$$

the Banach space structure on a Hilbert space completely determines its inner product structure. All other properties of $H$ follow from these first principles. Moreover for every infinite cardinal $\kappa$ all infinite-dimensional Hilbert spaces of density character $\kappa$ are isometric to the sequence space

$$\ell_2(\kappa) := \{a \in C^{\kappa} : \sum_j |a_j|^2 < \infty\}$$

equipped with the inner product $(a|b) := \sum_j a_j \bar{b}_j$.

The Cauchy–Schwarz inequality (also known as the Cauchy–Bunyakowski–Schwarz, or CBS, inequality) states that

$$| (\xi | \eta) | \leq \| \xi \| \| \eta \|,$$

$^1$ The density character of a topological space is the minimal cardinality of a dense subset.
where the equality holds if and only if $\xi$ and $\eta$ are linearly dependent. By the Riesz–Fréchet Theorem for Hilbert space every bounded linear functional $\varphi$ on $H$ is implemented by a vector $\xi_\varphi$, via $\varphi(\cdot) = (\cdot | \xi_\varphi)$. The transformation $\varphi \mapsto \xi_\varphi$ is a conjugate linear isometry from $H^*$ onto $H$. This identification of $H$ with its Banach space dual equips $H$ with the weak* topology, usually referred to as the weak topology on $H$ (after all, $H$ is reflexive and this topology coincides with the weak topology on $H$ induced by $H^*$). A net of vectors in $H$ converges weakly if and only if the associated functionals converge pointwise. By the Banach–Alaoglu Theorem, the unit ball $H_{\leq 1}$ of $H$ is weakly compact\(^2\) and, since the unit ball of a Banach space is norm compact if and only if the space is finite-dimensional, the norm and weak topologies agree only if $H$ is finite-dimensional.

If $K$ is a linear subspace of $H$ then its orthogonal complement

$$K^\perp := \{ \eta : (\xi | \eta) = 0 \text{ for all } \xi \in K \}$$

is a norm-closed (see Exercise 3.10.1) subspace, and $(K^\perp)^\perp$ is equal to the norm-closure of $K$.

Let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on $H$. (There is no consensus on what font to use for $\mathcal{B}$ in $\mathcal{B}(H)$, and some authors use $L(H)$.) The operator norm

$$\|a\| := \sup_{\|\xi\| \leq 1} \|a \xi\|$$

is a Banach algebra norm on $\mathcal{B}(H)$. A sesquilinear form is a function $\alpha : H^2 \to \mathbb{C}$ which is linear in the first variable, conjugate linear in the second variable, with the norm defined by $\|\alpha\| := \sup_{\|\xi\| = \|\eta\| = 1} |\alpha(\xi, \eta)|$. Every $a \in \mathcal{B}(H)$ uniquely determines a bounded sesquilinear form $\alpha_a(\xi, \eta) := (a \xi | \eta)$ and (by the Riesz–Fréchet Representation Theorem for Hilbert space) for every bounded sesquilinear form $\alpha$ there is a unique $a \in \mathcal{B}(H)$ such that $\alpha = \alpha_a$. This correspondence between bounded linear operators and bounded sesquilinear forms is isometric. The adjoint of $a \in \mathcal{B}(H)$ is defined implicitly by its sesquilinear form

$$(a^* \xi | \eta) := (\xi | a^* \eta).$$

The association $a \mapsto a^*$ is a conjugate-linear isometry. Given $a \in \mathcal{B}(H)$ and $\|\xi\| \leq 1$ the Cauchy–Schwarz inequality implies

$$\|a \xi\|^2 = (a \xi | a \xi) = (a^* a \xi | \xi) \leq \|a^* a\| \|a \xi\| \leq \|a\|^2 \|a \xi\|$$

and therefore $\|a\|^2 \leq \|a^* a\|$. Since $\|a^* a\| \leq \|a\| \|a^*\| \leq \|a\|^2$, all operators in $\mathcal{B}(H)$ satisfy the $C^*$-equality

$$\|a\|^2 = \|a^* a\|.$$

An operator $a \in \mathcal{B}(H)$ has finite-rank if $a[H]$ is finite-dimensional; in this case the rank of $a$ is the dimension of $a[H]$. An operator $a \in \mathcal{B}(H)$ is compact if the

\(^2\) Any similarity with the weakly compact large cardinals ([149]) is accidental—‘weakly compact’ means ‘compact in the weak topology.’
There exists \( c \in \text{Supp} \). Suppose \( a \) and \( b \) are in Lemma 1.1.2. Compact operators are weak-norm continuous, which is why a sequence of finite-rank operators.

If \( \xi \) and \( \eta \) are vectors in \( H \) define operator \( \xi \odot \eta \) via

\[
(\xi \odot \eta)\xi := (\xi|\eta)\xi.
\]

This is the unique linear operator that sends \( \eta \) to \( \|\eta\|^2\xi \) whose kernel is equal to \( \{\eta\}^\perp \). Every rank-1 operator is of the form \( \xi \odot \eta \) for some \( \xi \) and \( \eta \), and every finite-rank operator is a linear combination of rank-1 operators. We have

\[
(\xi \odot \eta)^* = \eta \odot \xi,
\]

\[
(\xi \odot \eta)(\zeta \odot \theta) = (\xi|\eta)(\zeta \odot \theta).
\]

In particular, if \( \|\xi\| = \|\eta\| = 1 \) then \( (\xi \odot \eta)^*(\xi \odot \eta) \) is the orthogonal projection to \( \mathbb{C}\eta \) and \( (\xi \odot \eta)(\xi \odot \eta)^* \) is the orthogonal projection to \( \mathbb{C}\xi \). The norm-closure of the algebra of finite-rank operators is the algebra of compact operators, \( \mathcal{K}(H) \) ([189, §2.4]). (This does not necessarily hold for compact operators in other Banach spaces.)

**Example 1.1.1.** Perhaps the most important single operator (see [27, Examples 1.2.4.3]) is the **unilateral shift**, \( s \). It is defined relative to an orthonormal basis, \( \xi_n \), for \( n \in \mathbb{N} \), of a separable Hilbert space \( H \) by

\[
s(\xi_n) := \xi_{n+1}
\]

for \( n \in \mathbb{N} \). Its adjoint \( s^* \) is the ‘reverse’ shift, defined by \( s^*(\xi_{n+1}) = \xi_n \) for \( n \in \mathbb{N} \) and \( s^*(\xi_0) = 0 \).

2. An operator \( a \) is **normal** if \( aa^* = a^*a \). It is **self-adjoint** (sometimes called **Hermitian**) if \( a^* = a \).

3. Suppose \( H \) is presented as \( L_2(X, \mu) \) for a \( \sigma \)-finite measure space \( (X, \mu) \). The **multiplication operator**, \( M_f \), associated to \( f \in L_\infty(X, \mu) \) is defined as follows. If \( \eta \in L_2(X, \mu) \) and \( x \in X \), then \( M_f(\eta)(x) \) is the (a.e. equality class of) function

\[
M_f(\eta)(x) := f(x)\eta(x).
\]

Then \( M_f^* = M_{\bar{f}} \) (where \( \bar{f} \) is the pointwise conjugation of \( f \)), and \( M_f \) is normal because \( M_f M_f^* = M_{\bar{f}f} = M_{\bar{f}f} M_f \). Also, \( f \) is real-valued if and only if \( M_f \) is self-adjoint. Then \( \|M_f\| = \text{esssup}(f) \) and \( M_f \) is invertible if and only if \( f^{-1}(B_\varepsilon(0)) \) is null for a small enough \( \varepsilon > 0 \). In this case \( M_f^{-1} \) is \( M_g \), where \( g \in L_\infty(X, \mu) \) is any function that satisfies \( g(x) := 1/f(x) \) if \( x \geq \varepsilon \).

We give a sample of a construction that can be performed in \( \mathcal{B}(H) \) (but not in a \( C^* \)-algebra).

**Lemma 1.1.2.** Suppose \( a \) and \( b \) are in \( \mathcal{B}(H) \) and \( \|a\xi\| \geq \|b\xi\| \) for all \( \xi \in H \). Then there exists \( c \in \mathcal{B}(H) \) such that \( ca = b \) and \( \|c\| \leq 1 \).
extend a \subset \ast unitization of \ast are continuous.

In this section we introduce concrete and abstract \ast-algebras and construct the unitization of a \ast-algebra. We prove that the invertible elements form an open subset of a unital \ast-algebra, and that all \ast-homomorphisms between \ast-algebras are continuous.

A subalgebra of \mathcal{B}(H) is self-adjoint if it is closed under adjoint operation. A concrete \ast-\textit{algebra} is a norm-closed, self-adjoint, algebra of operators on some Hilbert space.

\textbf{Example 1.2.1.} 1. The most obvious examples of concrete \ast-algebras are \mathcal{B}(H) and its finite-diagonal instances, \(n \times n\) matrix algebras \(M_n(\mathbb{C}) = \mathcal{B}(\ell_2(n))\) for \(n \geq 1\).

2. The algebra of finite-rank operators is denoted \mathcal{B}_f(H). Its norm-closure is the algebra \mathcal{K}(H) of compact operators on \(H\) (see Theorem C.6.1). It is a concrete \ast-algebra and it is also a two-sided, self-adjoint ideal of \(\mathcal{B}(H)\) (Proposition C.6.2). The algebra of compact operators on \(\ell_2(\mathbb{N})\) will be denoted by \mathcal{K}.

3. If \((X, \mu)\) is a measure space then \(L_\infty(X, \mu)\) is isomorphic to a subalgebra of \(\mathcal{B}(L_2(X, \mu))\), via identifying \(f\) with the multiplication operator \(M_f\) as in Example 1.1.1 above.

\textbf{Proof.} On \(a[H]\) let \(c(a \xi) := b \xi\). Since our assumption implies \(\ker(a) \subseteq \ker(b)\), \(a \xi = a \eta\) implies \(b \xi = b \eta\) and \(c\) is well-defined. For \(\eta \in a[H]^\perp\) let \(c \eta = 0\). Linearly extend \(c\) to all of \(H\). Our assumption implies \(|c| = 1\) and clearly \(ca = b\). \(\square\)

An operator \(a\) on a Hilbert space is \textit{positive} if \((a \xi | \xi) \geq 0\) for all \(\xi \in H\), and \textit{projection} is an orthogonal projection to a closed subspace of \(H\). If \(K\) is a closed subspace of \(H\), then the orthogonal projection to \(K\) is denoted \(\text{proj}_K\). An operator \(v\) such that both \(v^*v\) and \(vv^*\) are projections is a \textit{partial isometry}. The following is the analog of polar decomposition of complex numbers, \(z = \text{re}^{i\theta}\).

\textbf{Theorem 1.1.3.} For every \(a\) in \(\mathcal{B}(H)\) there exists a partial isometry \(v \in \mathcal{B}(H)\) and positive operators \(|a|\) and \(|a^*|\) such that \(a = v|a| = |a^*|v\).

\textbf{Proof.} Once \(|a|\) is defined as \((a^* a)^{1/2}\) (as can be done in any \ast-algebra, see \S 1.4), this follows from Lemma 1.1.2. See e.g., [196, Theorem 3.2.17]. \(\square\)

Suppose \(H\) is presented as \(\ell_2(\mathbb{J})\) for an index-set \(\mathbb{J}\), with a distinguished orthonormal basis \(\delta_j\), for \(j \in \mathbb{J}\). Operators \(a \in \mathcal{B}(H)\) are in bijective correspondence with sesquilinear forms \(\alpha_\mathbb{J}\) on \(H\). By linearity a sesquilinear form can be identified with its restriction to the basis vectors. Such restriction is a \(K \times \kappa\) matrix’ whose entries are \((a \delta_i \delta_j)\), for \(i \in \mathbb{J}\) and \(j \in \mathbb{J}\). (Not every matrix corresponds to an element of \(\mathcal{B}(H)\).) A ‘diagonal’ matrix—i.e., one such that \((ae_i | e_j) = 0\) unless \(i = j\)—with uniformly bounded entries corresponds to an element of \(\ell_\infty(\mathbb{J})\). Therefore \(\mathcal{B}(H)\) can be thought of as the ‘noncommutative’ analog of \(\ell_\infty\).
4. If \( X \) is a compact Hausdorff space then
\[
C(X) := \{ f : X \to \mathbb{C} : f \text{ is continuous} \}
\]
is isometrically isomorphic to a concrete \( C^* \)-algebra: If \( \mu \) is a Radon measure on \( X \) such that \( \mu(U) > 0 \) for every nonempty open set \( U \), then \( C(X) \) is identified with a subalgebra of \( L_\infty(X, \mu) \). It is not difficult to see that this identification is an isometry. If \( X \) is a locally compact Hausdorff space then let
\[
C_0(X) := \{ f : X \to \mathbb{C} : f \text{ is continuous and it vanishes at infinity} \}.
\]
This \( C^* \)-algebra is identified with the maximal ideal of \( C(X \cup \{ \infty \}) \) consisting of all \( f \) such that \( f(\infty) = 0 \).

Thus in the case when \( X \) is compact the \( C^* \)-algebras \( C_0(X) \) and \( C(X) \) coincide. If \( X \) is a nonempty compact subset of \( \mathbb{C} \) and \( 0 \in X \), then some authors (notably, [27]) write \( C_0(X) \) for \( \{ f \in C(X) : f(0) = 0 \} \) and \( C_0(X) \) for \( C_0(X) \). The reader should therefore take a note that, for example, the notation \( C_0([0, 1]) \) in [27] has the same meaning as our \( C_0((0, 1]) \), but \( \text{not} \) the same meaning as our \( C_0((0, 1]) \).

**Abstract \( C^* \)-algebras**

**Definition 1.2.2.** An abstract \( C^* \)-algebra is a complex Banach algebra with an isometric involution that satisfies the \( C^* \)-equality, \( \|aa^*\| = \|a\|^2 \).

From now until the end of the proof of Theorem 1.10.1, all \( C^* \)-algebras are assumed to be abstract \( C^* \)-algebras.

A \( C^* \)-subalgebra of a \( C^* \)-algebra \( A \) is any \( B \subseteq A \) that is norm closed and closed under algebraic operations +, \( \cdot \), and \( * \).

A \( C^* \)-algebra \( A \) is unital if it has a multiplicative unit, denoted \( 1_A \) or 1. The unit of \( M_n(\mathbb{C}) \) will be denoted by \( 1_n \). A unital \( C^* \)-subalgebra of a unital \( C^* \)-algebra is a \( C^* \)-subalgebra that contains the unit.

In model theory a homomorphism is defined to be a morphism that preserves all algebraic and relational structure. In the theory of operator algebras, a homomorphism sometimes (but not always) refers to an algebra homomorphism—a map that preserves +, \( \cdot \), and multiplication by scalars but not necessarily the adjoint operation. In order to avoid ambiguity we follow the common practice and refer to a homomorphism that preserves the adjoint operation a \( * \)-homomorphism. We will see later (Lemma 1.2.10) that every \( * \)-homomorphism is automatically continuous. The only place in this book where we consider homomorphisms of operator algebras that are not necessarily \( * \)-homomorphisms is 

**Definition 1.2.3.** A \( * \)-polynomial in non-commuting variables \( x_0, \ldots, x_{n-1} \) is an expression of the form \( p(x_0, \ldots, x_{n-1}, x_0^*, \ldots, x_{n-1}^*) \) where \( p \) is a complex polynomial in non-commuting variables.
As in the case of other algebraic structures, C*-subalgebras generated by a given set can be characterized both ‘from above’ and ‘from below,’ and the proof of the following lemma is similar to the proof of any analogous statement.

**Lemma 1.2.4.** Suppose $A$ is a C*-algebra, $S \subseteq A$, and $B \subseteq A$. The following are equivalent.

1. The set $B$ is equal to the intersection of all C*-subalgebras of $A$ that include $S$.
2. The set $B$ is equal to the norm-closure of \( \{ p(\bar{s}) : \bar{s} \in S^n, p(\bar{x}) \text{ is a } \ast\text{-polynomial in } n \text{ non-commuting variables with zero constant term and } n \in \mathbb{N} \} \).

If $A$ is in addition unital then the following conditions are equivalent.

3. The set $B$ is equal to the intersection of all unital C*-subalgebras of $A$ that include $S$.
4. The set $B$ is equal to the norm-closure of \( \{ p(\bar{s}) : \bar{s} \in S^n, p(\bar{x}) \text{ is a } \ast\text{-polynomial in } n \text{ non-commuting variables and } n \in \mathbb{N} \} \).

\( \Box \)

The C*-subalgebra $B$ that satisfies either of the first two conditions in Lemma 1.2.4 is the C*-subalgebra generated by $S$, and it is denoted $C^* (S)$. If $S \subseteq A$ then $C^* (S)$ denotes the C*-subalgebra of $A$ generated by $S$.

If $A$ is a C*-algebra (unital or not) the unitization $A^\dagger$ of $A$ is the algebra defined on Banach space $A \oplus \mathbb{C}$ (identified with the algebra of formal sums $a + \lambda \cdot 1_A$ for $a \in A$ and $\lambda \in \mathbb{C}$) with the addition and adjoint operations defined pointwise and the multiplication defined by

\[
(a + \lambda \cdot 1_A)(b + \mu \cdot 1_A) := ab + \lambda b + \mu a + \lambda \mu \cdot 1_A.
\]

The elements of $A^\dagger$ are considered as left multiplication operators on $A$. When endowed with the corresponding operator norm,

\[
\| a + \lambda \cdot 1_A \| := \sup_{b \in A, \| b \| \leq 1} \| ab + \lambda b \|,
\]

it is a unital C*-algebra and $A$ is isomorphic to its maximal ideal $A \oplus \{ 0 \}$. The construction is functorial and a \( \ast\)-homomorphism between C*-algebras uniquely extends to a \( \ast\)-homomorphism between their unitizations.

**Definition 1.2.5.** For a C*-algebra $A$ we define $\tilde{A}$ to be $A$ if $A$ is unital and the unitization $A^\dagger$ of $A$ otherwise.

Let $\text{GL}(A)$ (some authors use $A^{-1}$) denote the group of invertible elements in a unital C*-algebra $A$.

**Lemma 1.2.6.** Suppose $A$ is a unital C*-algebra. If $\| a - u \| = \varepsilon < 1$ for some unitary $u$, then $a$ is invertible and $\| u^* - a^{-1} \| \leq \varepsilon/(1 - \varepsilon)$. Also, $\text{GL}(A)$ is open in $A$.

\( ^3 \) Here, and elsewhere, we use logician’s convention that $\bar{a}$ stands for a tuple $(a_0, \ldots, a_{n-1})$ of an unspecified length $n$. 
Proof. By replacing $a$ with $u^{-1}a$, we may assume $\|a-1\| < 1$. The geometric series $b := \sum_{n=0}^{\infty} (1-a)^n$ is absolutely convergent and since $a = 1 - (1-a)$, a telescoping argument shows $ba = ab = 1$. Clearly $\|1-b\| \leq \varepsilon/(1-\varepsilon)$. A computation shows that the ball of radius $\|e^{-1}\|^{-1}$ centered at an invertible $c$ is included in $\text{GL}(A)$. □

The spectrum of an element $a$ of a $C^*$-algebra $A$ (unital or not) is defined as

$$\text{sp}_A(a) = \{ \lambda \in \mathbb{C} : a - \lambda I \text{ is not invertible in } \tilde{A} \}.$$  

The spectrum of an operator $a$ in $\mathcal{B}(H)$ is defined analogously (Definition C.6.6).

Example 1.2.7. 1. If $A = M_n(\mathbb{C})$ then $\text{sp}_A(a)$ is the set of eigenvalues of $a$.

2. If $A = C(X)$ then $\text{sp}_A(a)$ is the range of $a$. If $X$ is not compact then $\text{sp}_A(a)$ is equal to the union of the range of $a$ and $\{0\}$ for $a \in C_0(X)$.

Every compact operator on an infinite-dimensional Hilbert space has $0$ in its spectrum. If $a \in A$ then $\lambda \mapsto a - \lambda I$ is a continuous map from $\mathbb{C}$ into $A$. Therefore Lemma 1.2.6 implies that $\text{sp}_A(a)$ is a closed subset of $\mathbb{C}$. A calculation shows that $\text{sp}_A(a)$ is included in the $\|a\|$-disk centred at $0$. Liouville’s theorem is used to prove that $\text{sp}_A(a)$ nonempty for every $a \in A$ (and this is why we consider $C^*$-algebras on complex Hilbert space; for a proof see e.g., [196, Theorem 4.1.13]). The proof of Lemma 1.2.6 uses a simple form of the holomorphic functional calculus. Another notable instance of the holomorphic functional calculus is the definition of the exponential function on $\tilde{A}$ by

$$\exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!}.$$  

We have $\exp(a + b) = \exp(a) \exp(b)$ if $a$ and $b$ are commuting (but not in general), and the range of $\exp(a)$ is included in $\text{GL}(\tilde{A})$ with $\exp(a)^{-1} = \exp(-a)$. Therefore if $a$ is self-adjoint then $\exp(ia)^* = \exp(-ia) = \exp(ia)^{-1}$, hence $\exp(ia)$ is a unitary. It is as good a moment as any to make a simple estimate.

Lemma 1.2.8. There is a universal constant $K < \infty$ such that for all $a \in \mathcal{B}(H)$, $r \in \mathbb{C}$, and $\xi \in H$ satisfying $\max(\|a\|, |r|, \|\xi\|) \leq 1$ we have

$$\|\exp(i\pi r)\xi - \exp(i\pi a)\xi\| \leq K\|r\xi - a\xi\|.$$  

Proof. Let $\delta := \|r\xi - a\xi\|$. Since $r$ is a scalar and max$(\|a\|, |r|, \|\xi\|) \leq 1$, by induction for $n \geq 1$ we have $\|a^n\xi - r^n\xi\| \leq n\delta$. Therefore $\|(i\pi a)^n\xi - (i\pi r)^n\xi\| \leq n\pi^n\delta$ and $\|\exp(i\pi a)\xi - \exp(i\pi r)\xi\| \leq \sum_{n=1}^{\infty} \pi^n/(n-1)! = \pi e^\delta$ and $K := \pi e^\pi$ is as required. □

Definition 1.2.9. The spectral radius of $a$ is $r(a) := \max\{|\lambda| : \lambda \in \text{sp}_A(a)\}$.

If $a$ is self-adjoint then the $C^*$-equality implies $\|a^2\| = \|a\|^2$, and by induction we have $\|a^{2n}\| = \|a\|^{2n}$ for all $n \geq 1$. Since another use of Complex Analysis gives $r(a) = \lim_n \|a^n\|^{1/n}$ (e.g., [189, Theorem 1.2.7]) we conclude that
\[ r(a) = \lim_n \|a^n\|^{2-n} = \|a\| \]

for a self-adjoint \(a\) (see also Exercise 1.11.13).

**Lemma 1.2.10.** *Every \(\ast\)-homomorphism between \(C^\ast\)-algebras is contractive.*

**Proof.** We have only to prove that the unital extension \(\tilde{\Phi}: \tilde{A} \to \tilde{B}\) of any \(\ast\)-homomorphism is contractive. Fix a self-adjoint \(a\). Then \(\|a\| = r(a)\) and \(\text{sp}(\tilde{\Phi}(a)) \subseteq \text{sp}(\Phi(a))\), and therefore \(r(\tilde{\Phi}(a)) \leq r(a)\). Because \(\Phi\) sends self-adjoint elements to self-adjoint elements, we conclude that \(\|\Phi(a)\| \leq \|a\|\) for a self-adjoint \(a\). For every \(b \in A\) we have \(\|b\|^2 = \|b^*b\| = r(b^*b)\), and the conclusion follows. \(\square\)

A norm on a complex algebra with an involution is a \(C^\ast\)-norm if it is a Banach algebra norm that satisfies the \(C^\ast\)-equality.

**Corollary 1.2.11.** *Every injective algebraic \(\ast\)-homomorphism \(\Phi\) between \(C^\ast\)-algebras is an isometry.*

**Proof.** Lemma 1.2.10 implies \(\|\Phi\| \leq 1\). This implies that the range of \(\Phi\) is complete, and therefore a \(C^\ast\)-algebra. Since \(\Phi\) is injective, Lemma 1.2.10 applies to \(\Phi^{-1}\) and therefore \(\|\Phi^{-1}\| = 1\). \(\square\)

We will prove a strengthening of Corollary 1.2.11 in Corollary 1.3.3.

**Example 1.2.12.** By Corollary 1.2.11, any algebraic \(\ast\)-isomorphism \(\Phi: A \to B\) between a \(C^\ast\)-algebra and a normed \(\ast\)-algebra that satisfies the \(C^\ast\)-equality is an isometry. We emphasize that the assumption that the norm on at least one of \(A\) or \(B\) be complete is necessary. Take for example the algebra \(\mathbb{C}[x]\) of all \(\ast\)-polynomials over \(\mathbb{C}\) in a single variable \(x\). For a nonempty compact subset \(K\) of \(\mathbb{C}\) take a normal operator \(a\) in \(B(H)\) with \(\text{sp}(a) = K\). Then \(\|p(x)\|_K := \|p(a)\|\) defines a \(C^\ast\)-norm on this algebra. The fact that the norms \(\|\cdot\|_K\) and \(\|\cdot\|_L\) differ if \(K \neq L\) can be checked for example by combining the Tietze extension theorem and the Stone–Weierstrass theorem.

Also, the assumption that \(\Phi\) is \(\ast\)-preserving is necessary. If \(A\) is a unital \(C^\ast\)-algebra and \(a\in \text{GL}(A)\) then \(x \mapsto axa^{-1}\) is an automorphism of \(A\). It is a \(\ast\)-automorphism if and only if \(a\) is a unitary. See also note (1.11) at the end of this section.

### 1.3 Abelian \(C^\ast\)-algebras

In this section we prove the Gelfand–Naimark theorem, that the category of abelian \(C^\ast\)-algebras is equivalent to the category to the locally compact Hausdorff spaces. We also prove that every injective \(\ast\)-homomorphism between \(C^\ast\)-algebras is contractive.

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4 Warning: In the theory of operator algebras ‘contractive’ is synonymous with ‘of norm \(\leq 1\).’
an isometry, and that every homomorphism from an abelian C*-algebra into any C*-algebra is continuous. The Stone duality between Boolean algebras and compact, totally disconnected Hausdorff spaces is also briefly discussed.

A character of a C*-algebra $A$ is a unital *-homomorphism from $A$ into $\mathbb{C}$. The spectrum of $A$, $\hat{A}$, is the space of all characters of $A$. It is considered with respect to the weak* topology. Lemma 1.2.6 implies that every maximal ideal in a unital C*-algebra is automatically norm-closed. Since the kernel of a character is a maximal ideal, every character of a unital C*-algebra is automatically continuous. Hence $\hat{A}$ is included in the Banach space dual $A^*$ of $A$. If $\varphi \in \hat{A}$ and $a \in A$ then $a - \varphi(a) \in \ker(\varphi)$, and therefore $\varphi(a) \in \text{sp}(a)$. Since $\text{sp}(a) \subseteq \{z \in \mathbb{C} : |z| \leq \|a\|\}$, this implies that $\|\varphi\| \leq 1$ and, since $\varphi$ is unital, $\|\varphi\| = 1$. Therefore $\hat{A}$ is a subset of the unit dual ball of $A$, and it is weak*-closed if $A$ is unital. (In the nonunital case the weak* closure of $\hat{A}$ in the dual unit ball is its one-point compactification, $\hat{A} \cup \{0\}$.)

If $A$ is unital the Gelfand transform $\Gamma : A \to C(\hat{A})$ given by $\Gamma(a)(\varphi) := \varphi(a)$. It is a *-homomorphism and, since $\hat{A}$ is included in the unit dual ball, it is contractive. The space $\hat{A}$ is called the Gelfand spectrum of $A$.

**Theorem 1.3.1 (Gelfand–Naimark).** Every abelian C*-algebra is isomorphic to $C_0(X)$ for some locally compact Hausdorff space $X$.

*Proof.* Suppose $A$ is an abelian C*-algebra. If it is unital, then $\hat{A}$ is weak*-compact and Hausdorff and the Gelfand transform is a *-homomorphism from $A$ of norm $\leq 1$ into $C(\hat{A})$.

**Claim.** Every $\varphi \in \hat{A}$ is self-adjoint, i.e. $\varphi(a^*) = \overline{\varphi(a)}$ for all $a \in A$.

*Proof.* It suffices to prove that $\varphi(a)$ is real if $a$ is self-adjoint. Then $u_t := \exp(it a)$ is a unitary for every $t \in \mathbb{R}$. We have

$$\exp(-t \Im(\varphi(a))) = \exp(\Re(it \varphi(a)) = |\exp(it \varphi(a))| = |\varphi(u_t)| = 1.$$  

Since $t \in \mathbb{R}$ is arbitrary, this implies $\Im(\varphi(a)) = 0$. \qed

In order to show that $\Gamma$ is an isometry, fix $a \in A$ and $\lambda \in \text{sp}(a)$. Then $a - \lambda$ generates a proper ideal in $A$. Use Zorn’s Lemma to extend this ideal to a maximal proper ideal $J$. Then $A/J \cong \mathbb{C}$ (by Exercise 1.11.1) and character $\varphi$ with kernel $J$ satisfies $\varphi(a) = \lambda$. We have proved that $\Gamma[A]$ is isometric to $A$. Since $\Gamma[A]$ is a self-adjoint, norm-closed algebra which separates points in $\hat{A}$, the Stone–Weierstrass Theorem implies the claimed surjectivity of $\Gamma$.

In the nonunital case, $\hat{A}$ is the one-point compactification of $\hat{A}$, and $\Gamma[A]$ consists of all $f \in C(\hat{A}) \cong C(\hat{A} \cup \{\infty\})$ vanishing at infinity. \qed

**Theorem 1.3.2.** The category of unital abelian C*-algebras is contravariantly equivalent to the category of compact Hausdorff spaces.

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5 Not to be confused with the spectrum of an operator!
Proof. By Theorem 1.3.1 the spectrum of a unital abelian $C^*$-algebra $A$ is a compact Hausdorff space. The map $F(A) := \hat{A}$ is a bijection between unital abelian $C^*$-algebras and compact Hausdorff spaces.

Morphisms in the category of $C^*$-algebras are $^*$-homomorphisms and morphisms in the category of topological spaces are continuous maps. Suppose $t : X \to Y$ is a continuous map between compact Hausdorff spaces. Define $t^* : C_0(Y) \to C_0(X)$ by $t^*(f) := f \circ t$; this is clearly a $^*$-homomorphism.

Conversely, suppose $\Phi : A \to B$ is a $^*$-homomorphism and consider the adjoint map between the dual spaces, $\Phi^* : B^* \to A^*$ (i.e., $\Phi^*(\psi) := \psi \circ \Phi$). Since the composition of $^*$-homomorphisms is a $^*$-homomorphism, $\Phi^*$ sends $B$ to $A$. As an adjoint map, $\Phi^*$ sends weak $^*$-continuous maps to weak $^*$-continuous maps. It is straightforward to check that $F(\Phi) := \Phi^*$ commutes with taking compositions and that it implements the equivalence of categories. \[\square\]

While abelian $C^*$-algebras correspond to locally compact Hausdorff spaces by Theorem 1.3.1, we do not have an equivalence of categories because the zero homomorphism does not correspond to a continuous map between the spectra. A more interesting example can be found in Exercise 1.11.10 and a remedy is given in Exercise 1.11.11. Additional information on the Gelfand–Naimark duality between the categories of compact spaces and unital abelian $C^*$-algebras is given in Exercise 1.11.8. An important consequence of Theorem 1.3.2 is the following strengthening of Corollary 1.2.11.

**Corollary 1.3.3.** Every injective $^*$-homomorphism between $C^*$-algebras is an isometry. In particular, if a complex algebra $A$ has a $C^*$-norm with respect to which it is complete, then this is its unique $C^*$-norm.

**Proof.** Suppose $\Phi : A \to B$ is an injective $^*$-homomorphism between $C^*$-algebras. By the $C^*$-equality it suffices to prove that $\|\Phi(a)\| = \|a\|$ for a self-adjoint $a$. Fix a self-adjoint $a$, restrict $\Phi$ to $C^*(a)$, and extend it to the unitization of $C^*(a)$. This is an abelian $C^*$-algebra, and Theorem 1.3.2 implies that $\text{sp}(a) = \text{sp}(\Phi(a))$, and accordingly $\|a\| = \|\Phi(a)\|$. The second part follows immediately. \[\square\]

One can also consider homomorphisms between $C^*$-algebras that are not necessarily $^*$-homomorphisms.

**Lemma 1.3.4.** Every homomorphism $\Phi : A \to C$ from an abelian $C^*$-algebra into a $C^*$-algebra is necessarily a $^*$-homomorphism.

The assumption that $A$ be abelian cannot be dropped from Lemma 1.3.4.
Example 1.3.5. Not every homomorphism between $C^*$-algebras is a $^*$-homomorphism. E.g., take the endomorphism $b \mapsto aba^{-1}$ of $M_2(\mathbb{C})$ for $a := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

In general, if $a$ is an invertible element of a $C^*$-algebra $A$ then by Lemma 1.4.4 there exists a unitary $u \in A$ such that $a = u |a|$. It can be proved that the endomorphism $b \mapsto aba^{-1}$ is a $^*$-homomorphism if and only if $|a|$ belongs to the center of $A$.

Proof (Lemma 1.3.4). Since $\Phi$ uniquely extends to a homomorphism from the unitization of $A$ into the unitization of $C$ by $\tilde{\Phi}(a + \lambda 1) = \Phi(a) + \lambda 1$, we may assume that it is a unital homomorphism between unital $C^*$-algebras. It clearly sends invertible elements to invertible elements. Since $\Phi(\lambda a) = \lambda \Phi(a)$ for all scalars $\lambda$ and all $a \in A$, $\Phi$ is equal to the identity on the scalars. Therefore $sp(a) \supseteq sp(\Phi(a))$ and in particular the spectral radius of $a$ (Definition 1.2.9) satisfies $r(a) \leq r(\Phi(a))$ for all $a \in A$. Since $A$ is abelian, each $a \in A$ is normal and therefore $r(a) = \|a\|$ (Exercise 1.11.13).

We have proved that $\Phi$ is a contraction. Therefore the images of a unitary $u$ and its adjoint $u^*$ both have norm $\leq 1$, and $\Phi(u)^{-1} = \Phi(u^*)$. But the continuous functional calculus implies that $sp(\Phi(u)^{-1}) = \{\lambda^{-1} : \lambda \in \Phi(u)\}$. Therefore the spectrum of $\Phi(u)$ is included in $\mathbb{T}$. Since $\Phi(u)$ is normal, it is a unitary, and $\Phi(u)^{-1} = \Phi(u)^*$. Since every element of $A$ is a linear combination of unitaries (Exercise 1.11.16), $\Phi(a^*) = \Phi(a)^*$ for all $a \in A$. Therefore $\Phi$ is a $^*$-homomorphism.

\[ \square \]

1.3.1 Stone Duality

The duality between compact, totally disconnected Hausdorff spaces and Boolean algebras ought to be mentioned at this point. The Stone space of a Boolean algebra $B$, denoted $\text{Stone}(B)$, is the space of all ultrafilters of $B$. The basis for its topology consists of sets of the form

$$ U_a := \{ \mathcal{U} \in \text{Stone}(B) : a \in \mathcal{U} \}, $$

for $a \in B \setminus \{0_B\}$. This space is clearly Hausdorff and since (assuming the Axiom of Choice) every filter on $B$ can be extended to an ultrafilter, it is compact. Compactness implies that the algebra of closed and open (known as clopen) subsets of $\text{Stone}(B)$ is isomorphic to $B$. Hence $\text{Stone}(B)$ is zero-dimensional, i.e., it has a basis consisting of clopen sets. Conversely, a compact Hausdorff space is a Stone space of a Boolean algebra if and only if it is zero-dimensional.

**Theorem 1.3.6.** The category of compact zero-dimensional Hausdorff spaces is contravariantly equivalent to the category of Boolean algebras.

**Proof.** We have already defined a bijection between the objects in these two categories. Fix a Boolean algebra homomorphism $\Phi : B_1 \rightarrow B_2$. For $\mathcal{U} \in \text{Stone}(B_2)$ let
This defines a continuous map $f_\Phi : \text{Stone}(\mathbb{B}_2) \to \text{Stone}(\mathbb{B}_1)$. Conversely, suppose $f : \text{Stone}(\mathbb{B}_2) \to \text{Stone}(\mathbb{B}_1)$ there exists $b \in \mathbb{B}_2$ such that $f^{-1}(U_a) = U_b$; then $\Phi_f(a) := b$ defines a Boolean algebra homomorphism. It is evident that

(i) the operations $f \mapsto \Phi_f$ and $\Phi \mapsto f_\Phi$ are inverses of one another,

(ii) $f$ is a surjection if and only if $\Phi_f$ is an injection, and

(iii) $f$ is an injection if and only if $\Phi_f$ is a surjection.

This completes our proof that the functor $F(\mathbb{B}) := \text{Stone}(\mathbb{B})$, $F(\Phi) := f_\Phi$ is an equivalence of categories.

Together with the Gelfand–Naimark duality (Theorem 1.3.2), Theorem 1.3.6 gives us the equivalence of three categories: Boolean algebras, Stone spaces, and C*-algebras of continuous functions on Stone spaces.

A Boolean algebra $\mathbb{B}$ is identified with a subset of $C(\text{Stone}(\mathbb{B}))$ whose elements are characteristic functions of clopen subsets of $\text{Stone}(\mathbb{B})$. An element $p$ in a C*-algebra is projection if it is a self-adjoint idempotent (i.e., $p^* = p$ and $p^2 = p$). The set of projections in a C*-algebra $A$ is denoted $\text{Proj}(A)$.

**Example 1.3.7.** Suppose $X$ is a compact, zero-dimensional, Hausdorff space. Then $X$ is equal to the Stone space of the algebra of all projections of $X$, $\text{Stone}(\text{Proj}(C(X)))$. (The Boolean algebra $\text{Proj}(C(X))$ is also naturally isomorphic to $\text{Clop}(X)$, the algebra of all clopen subsets of $X$.) Every element of $C(X)$ is a norm-limit of elements with finite spectrum and $\text{sp}(a)$ is finite if and only if $a \in C(X)$ is measurable with respect to the algebra $\text{Clop}(X)$.

### 1.4 Elements of C*-algebras. Continuous Functional Calculus

In this section we introduce most important types of elements of C*-algebras and the continuous functional calculus for normal elements. We also prove that the well-supported elements allow polar decomposition in the C*-algebra that they generate. The section ends with some soft numerical estimates.

The taxonomy of elements of a C*-algebra $A$ is imported from $\mathcal{B}(H)$.

**Definition 1.4.1.** Suppose $A$ is a C*-algebra. An element $a \in A$ is (assuming $A$ is unital in (4), (5) and (7))

1. normal if $aa^* = a^*a$.
2. self-adjoint (or Hermitian) if $a = a^*$;
3. projection if $a^2 = a^* = a$;
4. unitary if $aa^* = a^*a = 1$;
5. isometry if $a^*a = 1$;
6. partial isometry if both $aa^*$ and $a^*a$ are projections, called the range projection
   and the source projection, respectively, of $a$ (see Exercise 1.11.19);
7. coisometry if $aa^* = 1$;
8. contraction if $\|a\| \leq 1$.

Self-adjoint elements of a $C^*$-algebra $A$ form a real Banach subspace of $A$, denoted $A_{sa}$. If $A$ is unital, then the unitaries of $A$ form a multiplicative group, the unitary group of $A$, denoted $U(A)$. If $A$ is a concrete $C^*$-algebra containing $1_{\mathcal{H}}(\ell)$ then this terminology agrees with the standard terminology from Hilbert space operator theory.

If $B$ is a unital $C^*$-subalgebra of $A$ then $\text{sp}_A(a) = \text{sp}_B(a)$ for every $a \in B$ (Exercise 1.11.6). Consequently, from now on we will drop the subscript and write $\text{sp}(a)$.

**Theorem 1.4.2.** If $A$ is a $C^*$-algebra and $a \in A$ is normal then

$$C^*(a) \cong C_0(\text{sp}(a) \setminus \{0\})$$

and the natural isomorphism sends $\text{id}$ to $a$. If $A$ is unital, then $C^*(a,1) \cong C(\text{sp}(a))$.

**Proof.** Theorem 1.3.1 implies that $B := C^*(a)$ is isomorphic to $C_0(\hat{B})$ where $\hat{B}$ is the space of its characters equipped with the weak$^*$-topology. Every character $\varphi$ of $C^*(a)$ is uniquely determined by $\varphi(a)$, and necessarily $\varphi(a) \in \text{sp}(a)$. Conversely, every $\lambda \in \text{sp}(a) \setminus \{0\}$ defines a unital $^*$-homomorphism of $A$ into $\mathbb{C}$ that sends $a$ to $\lambda$. Consequently, $\hat{A} \cong \text{sp}(a) \setminus \{0\}$. The last sentence follows immediately. \hfill $\Box$

Theorem 1.4.2 provides us with a powerful tool.

If $a$ is a normal element of a $C^*$-algebra and $f \in C_0(\text{sp}(a) \setminus \{0\})$, then $f(a) \in A$ is uniquely defined by Theorem 1.4.2 as the image of $f$ under the isomorphism of $C_0(\text{sp}(a) \setminus \{0\})$ onto $C^*(a)$ that sends the identity function to $a$. This is the continuous functional calculus. It enables us for example to define $|a|$ for a self-adjoint element $a$ and write $a$ as a difference of its positive and negative parts, $a = a_+ - a_-$, where $a_+ := (a + |a|)/2$ and $a_- := (|a| - a)/2$. More generally, the absolute value of an element $a$ is

$$|a| := (a^*a)^{1/2}.$$ 

Note that $(aa^*)^{1/2} \neq (a^*a)^{1/2}$ unless $a$ is normal. If $a$ is self-adjoint, then the two definitions of $|a|$ agree. This notation agrees with the previously defined holomorphic functional calculus, in particular $P(a)$ (for a polynomial $P$) and $\exp(a)$.

In applications it will be convenient to use continuous functional calculus with piecewise linear functions. We introduce a convenient terminology for defining such functions.

**Definition 1.4.3.** A function $f: [0,1] \to [0,1]$ is a piecewise linear function with breakpoints $f(x_j) = y_j$, for $j < m$, if $0 = x_0 < x_1 < \cdots < x_m = 1$, the restriction
of \( f \) to each of the intervals \([x_j, x_{j+1}]\) is linear for \( j < m - 1 \), and \( f(x_j) = y_j \) for all \( j < m \) (see Fig. 1.1).

![Fig. 1.1 A piecewise linear function with breakpoints \( f(x_i) = y_i \), for \( 0 \leq i \leq 4 \)](image)

Every \( a \in \mathcal{B}(H) \) has a **polar decomposition** \( a = v|a| \), where \( v \) is a partial isometry. The situation in \( \mathcal{C}^* \)-algebras is less straightforward; in addition to Lemma 1.4.4 below see Exercise 1.11.17. An element \( a \) of a \( \mathcal{C}^* \)-algebra is well-supported if \( \text{sp}(aa^*) \) doesn’t contain 0 or has 0 as an isolated point (Exercise 1.11.4 implies \( \text{sp}(aa^*) \cup \{0\} = \text{sp}(a^*a) \cup \{0\} \)).

**Lemma 1.4.4.** If an element \( a \) of a \( \mathcal{C}^* \)-algebra is well-supported then there exists a partial isometry \( v \in \mathcal{C}^*(a) \) such that \( a = v|a| \). If \( a \) is invertible then \( v \) is a unitary.

**Proof.** We may assume \( a \neq 0 \). Let \( f : [0, \infty) \to [0, \infty) \) be defined by \( f(0) = 0 \) and \( f(t) = t^{-1/2} \) for \( t > 0 \). The restriction of \( f \) to \( \text{sp}(a^*a) \) is continuous, and consequently by the continuous functional calculus \( f(a^*a) \in \mathcal{C}^*(a) \). Let \( v := af(a^*a) \). By Theorem 1.4.2, \( \mathcal{C}^*(a^*a) \cong C_0(\text{sp}(a^*a)) \) and \( v^*v = f(a^*a)a^*af(a^*a) \). Suppose that \( \phi \) is a character of \( C_0(\text{sp}(a^*a)) \). Then \( \phi(v^*v) = 0 \) if \( \phi(a^*a) = 0 \) and if \( \phi(a^*a) = t > 0 \) then \( \phi(v^*v) = 1 \). Consequently, \( \text{sp}(v^*v) \subseteq \{0, 1\} \), and since \( v^*v \) is self-adjoint it is a projection. This implies that \( v \) is a partial isometry.

If \( a \) is invertible then so are \( a^*a \), \( (a^*a)^{-1/2} \), and \( u = a(a^*a)^{-1/2} \). But a partial isometry is invertible if and only if it is a unitary. \( \square \)

Quantitative estimates in the conclusion of Lemma 1.4.4 will be useful in §5.1 and in §17.2.

**Lemma 1.4.5.** Suppose \( a = v|a| \) is the polar decomposition of \( a \).

1. If \( \|1 - a\| < \varepsilon < 1 \) then \( \|1 - |a||\ < 2\varepsilon \) and \( \|1 - v|| < 3\varepsilon \).
2. If \( u \) is a unitary and \( \|u - a\| < \varepsilon \) then \( \|1 - |a||\ < 2\varepsilon \) and \( \|u - v|| < 3\varepsilon \).

**Proof.** (1) Lemma 1.4.4 implies \( v \) is a unitary, and therefore

\[ \varepsilon > \|1 - a\| = \|v^* - |a||\ = \|v - |a||. \]

Since \( t/(1 + t) \leq 1 \) and \( 1/(1 + t) \leq 1 \) for \( t \geq 0 \), continuous functional calculus implies \( |a|(1 + |a|)^{-1} \leq 1 \) and \( (1 + |a|)^{-1} \leq 1 \). Using
1 - |a|^2 = 1 - a + v|a| - |a|^2 = (1 - a) + (v - |a|)|a|,

we have

1 - |a| = (1 - |a|^2)(1 + |a|)^{-1} = (1 - a)(1 + |a|)^{-1} + (v - |a|)|a|(1 + |a|)^{-1},

and therefore \(|1 - |a|| \leq |1 - a| + |v - |a|| < 2\varepsilon.

For the second estimate, \(|1 - v| \leq |1 - |a|| + |a - v|| < 3\varepsilon.

(2) Assume \(|u - a| < \varepsilon\). Applying (1) to \(au^*\), we obtain \(|1 - |au^*|| < 2\varepsilon\) and \(|1 - vu^*| < 3\varepsilon\). Clearly \(|u - v| = |1 - vu^*| < 3\varepsilon\). By continuous functional calculus, \(|au^*| = (uau^*)^{1/2} = u|a|u^*\) hence \(|1 - |au^*|| = |1 - |a||\) < \(2\varepsilon\).

\[\text{Lemma 1.4.6. Suppose } \max(|a^*a - 1|, |aa^* - 1|) < \varepsilon < 1 \text{ and } a = u|a| \text{ is the polar decomposition of } a. \text{ Then } u \text{ is a unitary and } ||u - a|| < \varepsilon.\]

\[\text{Proof. } \text{Lemma 1.2.6 implies that both } a^*a \text{ and } aa^* \text{ are invertible. Therefore } a \text{ is invertible, } u \text{ is a unitary, and } ||a - u|| = ||u|a| - u|| = ||a - 1||. \text{ Since } a^*a = |a|^2, \text{ we have } \varepsilon > ||a|^2 - 1|| = \max_{\varepsilon \in \text{sp}(|a|)}|t^2 - 1| > \max_{\varepsilon \in \text{sp}(|a|)}|t - 1| = ||a - 1|| \text{ by the continuous functional calculus.}\]

The distance between a point \(x\) and a subset \(Y\) of a metric space is

\[\text{dist}(x, Y) := \inf_{y \in Y} d(x, y).\]

The Hausdorff distance between two compact sets \(K\) and \(L\) in a metric space is

\[d_H(K, L) := \max(\sup_{x \in K} \text{dist}(x, L), \sup_{y \in L} \text{dist}(y, K)).\]

\[\text{Lemma 1.4.7. Suppose } a \text{ and } b \text{ are normal (and not necessarily commuting). Then } d_H(\text{sp}(a), \text{sp}(b)) \leq ||a - b||.\]

\[\text{Proof. Let } \varepsilon := ||a - b|| \text{ and fix } \lambda \in \mathbb{C} \text{ such that } \text{dist}(\lambda, \text{sp}(a)) > \varepsilon. \text{ We will prove that } \lambda \notin \text{sp}(b). \text{ Let } c = (a - \lambda \cdot 1). \text{ Then } c \text{ is invertible and } ||c^{-1}|| < 1/\varepsilon (\text{Exercise 1.11.15}). \text{ Then } c^{-1}(b - \lambda \cdot 1) = c^{-1}(a - \lambda \cdot 1) - c^{-1}(a - b) = 1 - c^{-1}(a - b). \text{ The right-hand side is invertible by Lemma 1.2.6, and therefore } b - \lambda \cdot 1 \text{ is invertible as required. We have proved that sp(b) \subseteq_\varepsilon sp(a), and the proof of the converse inclusion is analogous. This completes the proof.}\]

It is not surprising that the issue of continuity is a bit more subtle in the noncommutative context. The commutator of elements \(a\) and \(b\) is defined as

\[[a, b] := ab - ba.\]

\[\text{Lemma 1.4.8. For every polynomial } f(x) \text{ there exists a constant } K_f < \infty \text{ such that for every } C^*\text{-algebra } A \text{ and all normal } a \in A \text{ with } ||a|| \leq 1 \text{ and all } b \in A \text{ we have } ||f(a, b)|| \leq K_f |||a, b||.\]

\[\text{Proof. First prove } ||a^n, b|| \leq n||a, b|| \text{ by induction on } n. \text{ If } f(x) = \sum_{j=0}^n \alpha_jx^j, \text{ this implies } ||f(a, b)|| \leq \sum_{j=1}^n j|\alpha_j|||a, b||, \text{ and } K_f := \sum_{j=1}^n j|\alpha_j| \text{ is as required.}\]
1.5 Projections

In this section we study projections in C∗-algebras and compare the Murray–von Neumann equivalence, unitary equivalence, and homotopy of projections.

The set of projections in a C∗-algebra A is denoted \( \text{Proj}(A) \). Projections are ‘quantized’ analogs of sets and Murray–von Neumann equivalence is a quantized analog of the equinumerosity relation of sets (see [255]). It is also a continuous analog of the dimension of closed subspaces of the Hilbert space (see Example 1.5.5).

For projections \( p \) and \( q \) we write \( p \preceq q \) if \( pq = p \). A simple algebraic manipulation proves that this is a partial ordering. It also yields the following lemma.

Lemma 1.5.1. Suppose \( p \) and \( q \) are projections in C∗-algebra \( A \). Then \( p \preceq q \) if and only if \( qp = p \), if and only if \( q - p \) is a projection. \( \square \)

Unless \( H \) is one-dimensional, there are projections \( p \) and \( q \) in \( \text{Proj}(B(H)) \) such that \( p \preceq q \) and \( q \preceq p \). An ordered set with this property is called a partially ordered set, or a poset. The poset \( \text{Proj}(B(H)) \) is a lattice, but for some C∗-algebras \( A \) the poset \( \text{Proj}(A) \) is not necessarily a lattice (see e.g., Proposition 13.3.3).

Lemma 1.5.2. If \( A \) is a C∗-algebra and \( p, q \) are projections in \( A \) then the following are equivalent.

1. \( pq = qp \).
2. \( pq \) is self-adjoint.
3. \( pq \) is a projection.

If any of these applies then \( pq \) is the maximal lower bound for \( p \) and \( q \) in \( \text{Proj}(A) \) and \( p + q - pq \) is the minimal upper bound for \( p \) and \( q \) in \( \text{Proj}(A) \).

Proof. If \( pq = qp \) then a short computation shows \( (pq)^* = pq \) and \( (pq)^2 = pq \). Every projection is self-adjoint and if \( pq \) is self-adjoint then \( qp = (pq)^* = pq \).

If \( pq \) is a projection then \( pq \leq p \) and \( pq \leq q \). If \( r \) is a projection satisfying \( rp = r \) and \( rq = r \) then \( rpq = r \) hence \( r \leq pq \). Therefore \( pq = p \wedge q \). If \( pq \) is a projection then so is \( (1 - p)(1 - q) \) and the above shows \( (1 - p)(1 - q) = (1 - p) \wedge (1 - q) \).

Since \( r \leftrightarrow 1 - r \) is an order-reversing involution of \( \text{Proj}(A) \) it follows that \( p + q - pq \) is the minimal upper bound for \( p \) and \( q \). \( \square \)

Definition 1.5.3. Projections \( p \) and \( q \) in a C∗-algebra \( A \) are Murray-von Neumann equivalent (in \( A \)) if there is a partial isometry \( v \in A \) such that \( vv^* = p \) and \( v^*v = q \). We write \( p \sim q \) and keep in mind that this relation depends on the ambient algebra \( A \).

Lemma 1.5.4. Projections \( p \) and \( q \) in a unital C∗-algebra \( A \) are unitarily equivalent if and only if \( p \sim q \) and \( 1 - p \sim 1 - q \).

Proof. Suppose \( p = uu^* \) for a unitary \( u \). Then \( v = uq \) is a partial isometry that satisfies \( vv^* = p \) and \( v^*v = q \) and \( w := u(1 - q) \) is a partial isometry that satisfies \( w^*w = 1 - q \) and \( ww^* = 1 - p \). Conversely, if we have such \( v \) and \( w \) then \( u := v + w \) is a unitary such that \( p = uu^* \). \( \square \)
In $M_n(\mathbb{C})$ two projections are Murray–von Neumann equivalent if and only if they have the same rank. In this case Murray–von Neumann equivalence coincides with the unitary equivalence of projections.

**Example 1.5.5.** If $A = \mathcal{B}(H)$ then $p \sim q$ if and only if the range of $p$ and the range of $q$ have the same dimension, where the dimension of a closed subspace of the Hilbert space is the cardinality of an orthonormal basis, because two complex Hilbert spaces with the same dimension are linearly isometric. In $\mathcal{B}(H)$ we can find projections such that $p \sim q$ but $1 - p \not\sim 1 - q$: e.g., if $p = 1$ and $q(H)$ is a subspace of a finite dimension. Subsequently, the Murray–von Neumann equivalence and unitary equivalence do not necessarily coincide. See however Exercise 1.11.30.

**Example 1.5.6.** Projections of $C(X, M_n(\mathbb{C}))$ are maps $f : X \to M_n(\mathbb{C})$ such that $f(x)$ is a projection for all $x \in X$. By identifying a projection in $M_n(\mathbb{C})$ with a subspace of $\mathbb{C}^n$ one sees that projections of $C(X, \mathcal{X})$ are vector bundles over $X$. Murray–von Neumann equivalence of these projections is the usual equivalence of vector bundles (see e.g., [133]).

Two projections are *homotopic* in a $C^*$-algebra $A$ if they belong to the same path-connected component of $\text{Proj}(A)$.

**Lemma 1.5.7.** Suppose that $p$ and $q$ are projections in a $C^*$-algebra $A$ such that $\varepsilon := \|p - q\| < 1$. Then the following hold.

1. The projections $p$ and $q$ are homotopic.
2. The projections $p$ and $q$ are unitarily equivalent in $A$.
3. Some self-adjoint unitary $u \in A$ satisfies $upu^* = q$ and $uqu = p$.
4. The unitary $u$ also satisfies $\|\{u, b\}\| \leq 6\varepsilon$ for all $b \in qAq$.

**Proof.** (1) For $0 \leq t \leq 1$ let $a_t := tp + (1 - t)q$. This is a continuous path of self-adjoint elements connecting $p$ and $q$. For every $t$ we have

$$\min(\|a_t - p\|, \|a_t - q\|) \leq \frac{1}{2}\|p - q\| < \frac{1}{2},$$

and since $\text{sp}(p) = \text{sp}(q) \subseteq \{0, 1\}$ Lemma 1.4.7 implies $1/2 \notin \text{sp}(a_t)$.

If $f : \mathbb{R} \to \{0, 1\}$ is a function that sends $(-\infty, 1/2)$ to 0 and $(1/2, \infty)$ to 1 then the restriction of $f$ to $\text{sp}(a_t)$ is continuous and $r_t := f(a_t)$ is a projection for all $t$. We also have $r_0 = q$, $r_1 = p$, and by Exercise 1.11.12 the path $r_t$, for $0 \leq t \leq 1$, is continuous.

(2) clearly follows from (3), and we proceed to prove the latter. Since the self-adjoint $a := p + q - 1$ is within $\varepsilon$ of the unitary $2p - 1$, it is invertible by Lemma 1.2.6. Since $a^2 = pq + qp - p - q + 1$ commutes both with $p$ and $q$, so do all elements of $C^*(a^2)$; $|a|$ and $|a|^{-1}$ in particular. Since $a$ is self-adjoint, $u := a|a|^{-1}$ is a self-adjoint unitary. A calculation shows that $qa^2 = qpq$ and $upu^* = qpq|a|^{-2}$, hence $upu^* = q$. An analogous calculation shows that $uqu^* = p$.

(4) With $a := p + q - 1$ as in the proof of (4) we have $\|(2q - 1) - a\| < \varepsilon$, and Lemma 1.4.5 (2) implies $\|(2q - 1) - u\| < 3\varepsilon$. If $b \in qAq$ then $\|2q - 1, b\| = 0$, hence $\|\{u, b\}\| \leq 6\varepsilon$. □
Corollary 1.5.8. Homotopic projections are unitarily equivalent.

Proof. ‘Discretize’ the path between homotopic projections $p$ and $q$ by finding a large enough $n$ and projections $p = p_0, p_1, \ldots, p_{n-1} = q$ such that $\|p_i - p_{i+1}\| < 2$ for all $i < n - 1$. By the second part of Lemma 1.5.7 there are unitaries $u_i$ such that $u_i^* p_i u_i = p_{i+1}$, and therefore $v := \prod_{i<n-2} u_i$ satisfies $v^* p v = q$. \hfill \Box

1.6 Positivity in $C^*$-algebras

The purpose of this section is to introduce positivity and order on the set of self-adjoint elements of a $C^*$-algebra. We give equivalent definitions of positivity and show that the positive elements of a $C^*$-algebra form a cone. After our first encounter with the approximate units, the section ends with applications of the continuous functional calculus to positive elements.

Definition 1.6.1. An element of a $C^*$-algebra $A$ is positive if it is self-adjoint and $\text{sp}(a) \subseteq [0, \infty)$. If $a$ is positive then we write $a \geq 0$ and define a translation-invariant partial order on $A_{sa}$ by $a \geq b$ if $a - b \geq 0$.

Lemma 1.6.2. If $a \in A_{sa}$ and $\|a\| \leq 2$ then $a \geq 0$ if and only if $\|1 - a\| \leq 1$. If $A$ is abelian, then $A_+$ is closed under addition and multiplication. \hfill \Box

A cone in a Banach space is a subset closed under addition and multiplication by positive scalars (Definition C.5.9). The $n$-ball in $A$ will be denoted by $A_{\leq n}$, or sometimes $A_n$. (There will be no danger of confusion since we will never need to refer to the $n$-sphere of a $C^*$-algebra.)

Lemma 1.6.3. Positive elements form a closed cone in a $C^*$-algebra $A$.

Proof. Since $\text{sp}(ta) = t \text{sp}(a), \mathbb{R}^+ A_+ \subseteq A_+$. Lemma 1.6.2 implies that $A_+ \cap A_{\leq 2}$ is equal to the intersection of three convex sets, $A_{sa} \cap A_{\leq 2} \cap \{a : \|1 - a\| \leq 1\}$, and is therefore convex itself. \hfill \Box

Lemma 1.6.4. An element $a$ of a $C^*$-algebra is positive if and only if $a = c^* c$ for some $c$.

Proof. If $a \geq 0$ then $c := a^{1/2}$ satisfies $a = c^* c$. The inequality $a^* a \geq 0$ is trivial for a normal $a$ since then $\text{sp}(a^* a) = \{ |\lambda|^2 : \lambda \in \text{sp}(a) \}$. It is true for every $a$, normal or not, but the proof requires a clever algebraic manipulation; see [27, II.3.1.3(ii)]. \hfill \Box

Corollary 1.6.5. For all $a, b, c$ in a $C^*$-algebra $A$ the following holds.
1. There exists \( d \in C^*(a, b) \) such that \( a^*a + b^*b = d^*d \).

2. If \( a \geq 0 \) then \( c^*ac \geq 0 \).

3. If \( a \geq b \) then \( c^*ac \geq c^*bc \).

4. If \( b \in A_{sa} \) then \( a^*ba \leq a^*a\|b\| \).

Proof. These are immediate consequences of Lemma 1.6.4. For (1) also use that the positive elements form a cone. \( \square \)

Example 1.6.6. Working out the computations in a few peculiar examples may help furnish the intuition.

1. There are \( 0 \leq a \leq b \) in a \( C^* \)-algebra \( A \) such that \( a^2 \not\leq b^2 \). Take e.g., \( A = M_2(\mathbb{C}) \),
\[
    a = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}.
\]

2. There are \( 0 \leq c \leq d \) and \( e \) in a \( C^* \)-algebra \( A \) such that \( \|ec\| > \|ed\| \).

   With \( A = M_2(\mathbb{C}) \) and \( a, b \) as in (1), take \( c = e = a \) and \( d = b \).

We are about to switch to a higher gear, but only after introducing a concept so fundamental that it appears two sections earlier than may be obvious from the table of contents.

A poset \( \Lambda \) is **directed** if every finite subset of \( \Lambda \) has an upper bound.

Definition 1.6.7. An **approximate unit** (or an **approximate identity**) in a \( C^* \)-algebra is a net \( (e_\lambda : \lambda \in \Lambda) \) of positive contractions indexed by a directed set \( \Lambda \) such that
\[
    \lim_{\lambda} \|a - e_\lambda a\| = 0
\]
for all \( a \in A \). If in addition \( \lambda \leq \mu \) implies \( e_\lambda \leq e_\mu \), the approximate unit is said to be **increasing**.

An approximate unit is **sequential** if it is indexed by \( \mathbb{N} \). A \( C^* \)-algebra is \( \sigma \)-**unital** if it has a countable approximate unit.

Not every countable approximate unit is sequential, but a \( C^* \)-algebra has a countable approximate unit if and only if it has a sequential approximate unit. This is because every countable directed set has a cofinal subset isomorphic to \( (\mathbb{N}, \leq) \).

Since \( A \) is self-adjoint, \( (e_\lambda) \) is an approximate unit if and only if
\[
    \lim_{\lambda} \|a - e_\lambda a\| = \lim_{\lambda} \|a - ae_\lambda\| = \lim_{\lambda} \|a - e_\lambda ae_\lambda\| = 0
\]
for all \( a \in A \). If \( A \) is unital then \( \{1\} \) is an approximate unit of \( A \), and \( A \) has a finite approximate unit if and only if it is unital.

Proposition 1.6.8. Every \( C^* \)-algebra \( A \) has an approximate unit. If \( A \) is separable then it has a sequential approximate unit.

Proof. We may assume \( A \) is nonunital. Let \( \Lambda := \{a \in A_+ : \|a\| < 1\} \). We will prove that \( \Lambda \) is directed with respect to \( \leq \).

If \( b \in A \) then \( 1 - b \) is invertible by Lemma 1.2.6. Exercise 1.11.38 implies that if \( b \leq c \) are in \( A \) then \( 1 \leq (1 - b)^{-1} \leq (1 - c)^{-1} \). For that reason,
\[
    \Phi(b) := (1 - b)^{-1} - 1
\]
defines an order-preserving map from $\Lambda$ into $A_+$. Conversely, $d \in A_+$ implies that $d + 1$ is invertible with $\|d\|^{-1} \leq (d + 1)^{-1} \leq 1$. Thus $1 - (d + 1)^{-1}$ is in $\Lambda$. By Exercise 1.11.38,

$$\Psi(d) := 1 - (d + 1)^{-1}$$

defines an order preserving map from $A_+$ to $\Lambda$. A calculation gives $\Phi \circ \Psi = \text{id}_{A_+}$ and $\Psi \circ \Phi = \text{id}_\Lambda$. Therefore $\Lambda$ is order-isomorphic to $A_+$. Since $A_+$ is directed, this proves our claim.

If $a \in A_+$ then $\|a^{1/n}a - a\| \to 0$. Since $a^{1/n}(1 - 1/n) \in \Lambda$, we have

$$\inf_{b \in \Lambda} \|ba - a\| = 0.$$ 

Every element of $\Lambda$ is a linear combination of positive elements (Exercise 1.11.16) and $\Lambda$ is directed; therefore $\Lambda$ is an approximate unit.

Suppose $\Lambda$ is separable and fix a countable dense subset of $\Lambda$. It is included in a countable directed subset, and every countable directed set has a cofinal subset of order type $\omega$. This set is easily checked to be the required sequential approximate unit. \hfill $\square$

Proposition 1.6.8 implies that every separable C*-algebra is $\sigma$-unital, but the converse is false (see Example 2.1.2). We will revisit approximate units in §1.8 and a curious example given in Theorem 11.1.2 and in important §1.9.

As hinted earlier, the remaining part of this section is somewhat technical. The readers not particularly fond of intricate estimates may want to skip ahead to the next section on the first reading. The following proposition is worth the trouble caused by parsing its statement.

**Proposition 1.6.9.** Suppose $b,c,d$ belong to a C*-algebra $A$, $f,g \in C([0,||b||])_+$, and $h(t) := f(t)g(t)t^{-1}$ continuously extends to $[0,||b||]$. If $0 \leq b$, $c^*c \leq f(b)^2$, and $dd^* \leq g(b)^2$ then the sequence$^6$ $a_n := c(b + 1/n)^{1/n}d$ norm converges to a limit $a$ with $\|a\| \leq \|h(b)\|$.

**Proof.** The functions $h_n(t) := f(t)(t + 1/n)^{-1}g(t)$ converge to their supremum, $h$, pointwise on $[0,||b||]$. Since $h_m(t) \leq h_n(t)$ for all $m \leq n$ and $t \geq 0$, the convergence is uniform by Dini’s Theorem.

Content advisory: Merciless pummelling of $\|a_m - a_n\|^2$ with the C*-equality ahead. Let $\Delta_{mn} := (b + 1/m)^{-1} - (b + 1/n)^{-1}$. Then

$$\|a_m - a_n\|^2 = \|c\Delta_{mn}dd^*\Delta_{mn}c^*\| \leq \|c\Delta_{mn}g(b)^2\Delta_{mn}c^*\| = \|g(b)\Delta_{mn}c^*c\Delta_{mn}g(b)\| \leq \|g(b)\Delta_{mn}f(b)^2\Delta_{mn}g(b)\| = \|(fg)(b)\Delta_{mn}\|^2 = \|h_m(b) - h_n(b)\|.$$ 

Since $h_m$ converge to $h$ uniformly on $[0,||b||]$, the sequence $(a_m)$ is Cauchy. Let $a := \lim_m a_m$. By replacing $\Delta_{mn}$ in the above computation with $(b + 1/m)^{-1}$, we obtain $\|a_m\| \leq \|h_m(b)\|$ for all $m$ and therefore $\|a\| \leq \|h(b)\|$. \hfill $\square$

Proposition 1.6.9 has some ‘obvious’ statements with tricky proofs as corollaries.

$^6$ That is $|b| + \frac{1}{2}$, certainly not $((|b| + 1)/n)!$
Corollary 1.6.10. Suppose $a$ and $e \geq 0$ are elements of a $C^*$-algebra and $a^*a \leq e$. Then there exists $c \in C^*(a,e)$ such that $\|c\| \leq \|e^{1/6}\|$ and $a = ce^{1/3}$.

**Proof.** The idea is to prove that the sequence $c_n := a(e^{2/3} + 1/n)^{-1}e^{1/3}$ is Cauchy and that its limit $c$ satisfies the requirements. Let $b := e^{2/3}$ and $d := e^{1/3}$. Then $a^*a \leq b^{3/2}$ and $d^2 \leq b$, therefore $f(t) := t^{3/4}$, $g(t) := t^{1/2}$, and $h(t) := t^{1/4}$ satisfy the assumptions of Proposition 1.6.9. This implies that $c := \lim_n c_n$ exists and satisfies $\|c\| \leq \|b^{1/4}\| = \|e^{1/6}\|$. In order to prove that $a = ce^{1/3}$, using $a^*a \leq e$ we have

$$\|a - c_n\|^2 = \|e^{1/3}(1 - e^{-2/3})^{-1}a^*a(1 - e^{2/3})^{-1}e^{1/3}\| \leq \|e^{5/3}(1 - e^{-2/3})^{-2}\|.$$  

Dini’s theorem implies that $t^{5/3}(1 - t^{2/3})^{-2} \to 0$ uniformly on $[0, \|e\|]$. By the continuous functional calculus, the right-hand side converges to 0. □

Corollary 1.6.11. If $a_n$ are elements of a $C^*$-algebra $A$ and $\|a_n\| \leq 2^{-n-1}$ for all $n \in \mathbb{N}$, then there exist contractions $b$ and $c_n$ in $A$ such that $a_n = c_nb$ for all $n$.

**Proof.** Let $e := \sum a_n^*a_n$. Then $\|e\| \leq 1$ and $a_n^*a_n \leq e$ for all $n$. By Corollary 1.6.10, with $b := e^{1/3}$ for every $n$ there exists a $c_n$ as required. □

Corollary 1.6.12. Suppose $a$ and $b_1$ belong to a $C^*$-algebra $A$. If $0 \leq a$ and $a \leq b_1^*b_1$ then $a \in \overline{b_1^*Ab_1}$.

**Proof.** Let $f(t) = t^{1/2}$, $g(t) = t$, $d = b = b_1^*b_1$, and $c = a^{1/2}$. The assumptions of Proposition 1.6.9 are satisfied, and the sequence $x_n := a((b_1^*b_1) + 1/n)^{-1}b_1^*b_1$ is Cauchy. Its limit is $a$, since this sequence clearly SOT-converges to $a$. As $x_n^*x_n \in \overline{b_1^*Ab_1}$ for all $n$, $a^* = \lim_n x_n^*x_n$ also belongs to $\overline{b_1^*Ab_1}$. Since this is a $C^*$-algebra, it also contains $a$. □

Corollary 1.6.13. Suppose $b = \nu|b|$ is the polar decomposition of $b$.

1. If $g \in C_0(\text{sp}(|b|))$ then $\nu g(|b|) \in C^*(b)$.
2. If $A$ is a $C^*$-algebra such that $b \in A$ and $c \in \overline{b^*Ab}$, then $\nu c \in A$.
3. If $a \leq b^*b$ then $\nu a \in C^*(a,b)$.

**Proof.** (1) Let $a_n := b(|b| + 1/n)^{-1}g(|b|)$. Then WOT-$\lim_n a_n = \nu f(|b|)$ Proposition 1.6.9, with $f(t) = t$, implies that $(a_n)$ is a Cauchy sequence. Therefore $\nu f(|b|)$ is its limit, and it belongs to $C^*(b)$.

(2) Since $|b|^{1/n}$, for $n \in \mathbb{N}$, is an approximate unit for $\overline{b^*Ab}$, we have

$$\lim_n \|\nu c - \nu|b|^{1/n}c\| = \lim_n \|c - |b|^{1/n}c\| = 0$$

Since (1) implies $\nu|b|^{1/n} \in C^*(b)$, $\nu c \in C^*(b,c) \subseteq \overline{b^*Ab}$.

(3) By Corollary 1.6.12, $a \in \overline{b^*Ab}$ hence this is a special case of (2). □

Proposition 1.6.14. If $a$ and $b$ are positive contractions such that $\|a - b\| < \varepsilon$ then there is $x \in C^*(a,b)$ such that $xbx^* = (a - \varepsilon)_+$ and $\|x\| \leq 1$. 

Proof. Since \( \lim_{\delta \to 0^+} \| b - b^{1+\delta} \| = 0 \), some \( \delta > 0 \) satisfies \( \varepsilon' := \| a - b^{1+\delta} \| < \varepsilon \). Since \( (a-\varepsilon)_+ \leq (a-\varepsilon')_+ \), a contraction \( h \in C^*(a) \) satisfies \( (a-\varepsilon)_+ = h(a-\varepsilon')h \), and therefore \( (a-\varepsilon)_+ \leq h b^{1+\delta} h \).

Let \( c := b^{\frac{1+\delta}{2}} h \), with polar decomposition \( c = v_c |c| \). Since \( (a-\varepsilon)_+ \leq e^* c \), Corollary 1.6.13 (3) applied to \((a-\varepsilon)_+)^{1/2}\) implies that \( d := v_c ((a-\varepsilon)_+)^{1/2} \) belongs to \( C^*(a,c) \). Then \( dd^* = v_c (a-\varepsilon)_+ v_c \leq v_c e^* c v_c = e c^* = b^{\frac{1+\delta}{2}} h^* h \leq b^{1+\delta} \).

Proposition 1.6.9 implies that \( x_n := d^* (b^{\frac{1+\delta}{2}} + 1/n)^{-1/2} b^{1/2} \) is a Cauchy sequence, \( x := \lim_n x_n \) belongs to \( C^*(a,b) \), and \( \| x \| \leq 1 \). By adding up the exponents one sees that \( xb^{1/2} = d^* d = (a-\varepsilon)_+ \).

\[\Box\]

1.7 Positive Linear Functionals

In this section we introduce positive functionals and states and prove that the linear span of positive functionals is equal to the dual space of a \( C^* \)-algebra. We also prove some variants of the fact that every state has a limited extent of multiplicativity on normal elements.

Definition 1.7.1. A linear functional \( \varphi \) on a \( C^* \)-algebra \( A \) is positive if \( \varphi(a) \geq 0 \) for every \( a \in A_+ \). It is a state if it additionally satisfies \( \| \varphi \| = 1 \).

The space of all states of \( C^* \)-algebra \( A \) is denoted by \( S(A) \). It is also called the state space of \( A \). Since the set of positive functionals (including the zero functional) is clearly weak*-closed and the unit ball of \( A^* \) is weak*-compact, if \( A \) is unital then \( S(A) \) is a weak*-compact subset of the dual unit sphere. If \( A \) is not unital, the weak* closure of \( S(A) \) contains 0, and it is equal to the convex closure of \( S(A) \cup \{0\} \).

Example 1.7.2. If \( A \) is an abelian \( C^* \)-algebra then \( A \cong C_0(\hat{A}) \) by the Gelfand–Naimark duality. The Riesz Representation Theorem (Theorem C.3.8) implies that the continuous linear functionals on \( A \) are precisely the integrals \( \int a d\mu \) with respect to finite (complex) Radon measures \( \mu \) on \( \hat{A} \). This correspondence is isometric: \( \| \varphi \| = \| \mu \| \), where \( \| \varphi \| \) is the norm \( A^* \) and \( \| \mu \| \) is the total variation of \( \mu \).

The following two sentences consist of three easy exercises, one of which is solved in the Appendix. Positive functionals correspond to positive measures, and states correspond to probability measures. The extreme points of the state space of \( A \) (called pure states; see Definition 3.6.1) correspond to the point mass (Dirac) measures (see Example C.5.4). In other words, pure states on \( C_0(X) \) are the evaluation functionals, \( f \mapsto f(x) \) for \( x \in X \). This implies that a state on an abelian \( C^* \)-algebra \( A \) is extremal if and only if it is multiplicative (and therefore a character, since states are self-adjoint by Lemma 1.7.4 below).

Assume, in addition, that \( A \) is unital, and therefore that \( A = C(X) \) for a compact Hausdorff space \( X \). Furthermore assumed that the spectrum \( X \) is zero-dimensional. Since every normal element with finite spectrum is a linear combination of projections, a state \( \varphi \) of \( C(X) \) is uniquely determined by its restriction to the Boolean algebra of projections, \( \text{Proj}(C(X)) \). Hence if \( X \) is zero-dimensional then the state
Lemma 1.7.4. Suppose \( a \) is a negative real. Write an arbitrary \( b \) on the \( \text{Clop}(S) \) space to the standard proof of the Cauchy–Schwarz inequality for the inner product. Since

\[
\|b\| \leq \|a\| \quad \text{and} \quad \|a\| \leq \|b\|
\]

is an ultrafilter of \( \text{Clop}(X) \).

Example 1.7.3. Fix a Hilbert space \( H \).

1. Given a unit vector \( \xi \in H \) define a functional \( \omega_{\xi} \) on \( \mathcal{B}(H) \) by

\[
\omega_{\xi}(a) = (a\xi | \xi).
\]

Then \( \omega_{\xi}(a) \geq 0 \) for a positive \( a \) and \( \omega_{\xi}(1) = 1 \); hence it is a state. A state of this form is a vector state.

2. For two unit vectors \( \xi \) and \( \eta \) the corresponding vector states satisfy \( \omega_{\xi} = \omega_{\eta} \) if and only if \( \xi = z\eta \) for some \( z \in \mathbb{T} \). Therefore, the space \( \mathcal{P}(M_n(\mathbb{C})) \) of pure states on \( M_n(\mathbb{C}) \) is homeomorphic to the space of one-dimensional linear subspaces of \( \mathbb{C}^n \) (the \( n \)-dimensional complex projective space).

3. The restriction of a state to a unital \( C^* \)-subalgebra is a state. In particular if \( A \) is a unital \( C^* \)-subalgebra of \( \mathcal{B}(H) \) then the restriction of \( \omega_{\xi} \) to \( A \) is a state.

Lemma 1.7.4. Suppose \( \varphi \) is a positive functional on a \( C^* \)-algebra \( A \).

1. (The Cauchy–Schwarz inequality) \( |\varphi(b^*a)|^2 \leq \varphi(b^*b)\varphi(a^*a) \) for all \( a \) and \( b \) in \( A \).

2. The functional \( \varphi \) is self-adjoint, i.e., it satisfies \( \varphi(b^*) = \overline{\varphi(b)} \) for all \( b \).

3. We have \( \|\varphi\| = \sup\{\varphi(a) : 0 \leq a, \|a\| \leq 1\} \).

Proof. Since \( (a, b) \mapsto \varphi(b^*a) \) is a sesquilinear form on \( A \), a proof of (1) is identical to the standard proof of the Cauchy–Schwarz inequality for the inner product.

(2) If \( a \in A_{\mathbb{R}} \) then \( \varphi(a) = \varphi(a_+) - \varphi(a_-) \) is a real as a difference of two non-negative reals. Write an arbitrary \( b \in A \) as a linear combination of two self-adjoints, \( a + ic \), and verify \( \varphi(b) = \varphi(b^*) \).

(3) In the unital case the Cauchy–Schwarz inequality implies

\[
\|\varphi\|^2 = \sup_{|b| \leq 1} |\varphi(b)|^2 \leq \varphi(1) \sup_{|b| \leq 1} \varphi(b^*b) \leq \|\varphi\| \sup_{0 \leq a \leq 1} \varphi(a)
\]

hence \( \|\varphi\| \leq \sup_{0 \leq a \leq 1} \varphi(a) \). The converse inequality is trivial.

The nonunital case is a straightforward modification of the above, using an approximate unit \( (e_\lambda) \) of \( A \), in which \( \varphi(1) \) is replaced by \( \sup_\lambda \varphi(e_\lambda) \).

Suppose \( A \) is an abelian \( C^* \)-algebra with zero-dimensional spectrum \( X \). By the last part of Example 1.7.2, pure states on \( A \) are in a bijective correspondence to ultrafilters on the Boolean algebra \( \text{Proj}(A) \) of its projections, and this algebra is isomorphic to the algebra \( \text{Clop}(X) \) of clopen subsets of \( X \). ‘Poor man’s projections’ in a projectionless \( C^* \)-algebras are positive contractions of norm 1. We define...
and note that Proj(A) ⊆ A_{+1}. Part (1) of Lemma 1.7.5 is one of the most valuable properties of states, and part (2) is its quantitative version. See Proposition 1.7.8 and Lemma 1.10.7 for some variations on the theme.

**Lemma 1.7.5.** Suppose ϕ is a state on A, a ∈ A_{+1}, and ε > 0. 

1. If ϕ(a) = 1 then ϕ(b) = ϕ(ab) = ϕ(ba) for all b ∈ A. 
2. If ϕ(a) ≥ 1 − ε then 
   a. |ϕ(ab) − ϕ(b)| ≤ ε\|b\|, 
   b. |ϕ(ba) − ϕ(b)| ≤ ε\|b\|, and 
   c. |ϕ(aba) − ϕ(b)| ≤ 2ε\|b\|.

**Proof.** This is a consequence of Lemma 1.7.4. Since (1) is (2) with ε = 0 it suffices to prove the latter.

(2) Since the inequalities are trivial when b = 0 and both sides of the inequality are homogeneous in b, by replacing b with b/\|b\| we may assume \|b\| = 1. By the Cauchy–Schwarz inequality, (1 − a)^2 ≤ 1 − a, and the positivity of ϕ we have

\[|ϕ((1 − a)b)| \leq \sqrt{(ϕ(1 − a))^2ϕ(b^*b)} \leq \sqrt{ϕ(1 − a)ϕ(b^*b)} ≤ \sqrt{ε}.\]

Also, ϕ(b) − ϕ(ab) = ϕ((1 − a)b) and therefore |ϕ(b) − ϕ(ab)| ≤ ε\|b\| as required. Since ϕ is self-adjoint we have |ϕ(b) − ϕ(ba)| = |ϕ(b^*) − ϕ(ab^*)| ≤ ε\|b\|. The third inequality follows because \|ab\| ≤ \|a\|\|b\| ≤ 1.

Since a ≥ a^2, the assumption of 1 is an immediate consequence of (2). \qed

Complete positivity of a linear functional is an elusive condition. The following lemma gives a practical reformulation that makes the task of extending a state to a larger C^*-algebra straightforward.

**Lemma 1.7.6.** Suppose A is a unital C^*-algebra.

1. Then S(A) = \{ϕ ∈ A^* : \|ϕ\| = 1 = sup_λ ϕ(e_λ)\}, where (e_λ) is any approximate unit in A. If (e_λ) is in addition increasing, then

   \[S(A) = \{ϕ ∈ A^* : \|ϕ\| = 1 = sup_λ ϕ(e_λ) = lim_λ ϕ(e_λ)\}.\]

2. If A is unital then S(A) = \{ϕ ∈ A^* : \|ϕ\| = 1 = ϕ(1)\}.

3. Every state on any C^*-subalgebra B of A can be extended to a state on A.

4. Every positive a ∈ A has a norming functional which is a state.

**Proof.** (1) For the direct inclusion, suppose that ϕ is a state. In order to prove that sup_λ ϕ(e_λ) = 1 fix ε > 0. By Lemma 1.7.4 there exists a ∈ A_{+1} such that ϕ(a) > 1 − ε. Lemma 1.7.5 implies that |ϕ(λ e_λ) − ϕ(e_λ)| < \sqrt{ε} for all λ. Since \lim_λ \|ϕ(a) − ϕ(λ e_λ)\| ≤ \lim_λ \|a − λ e_λ\| = 0 and ε > 0 was arbitrary, we conclude that sup_λ ϕ(e_λ) = 1.
For the converse suppose \( \varphi \in A^* \) satisfies \( \sup_{f \in K} \varphi(f) = 1 = \| \varphi \| \), fix \( a \in A_+ \), and suppose for contradiction \( \varphi(a) \notin [0, \infty) \). Let \( \mu \) be a finite regular Radon measure on \( \text{sp}(a) \) such that \( \varphi(f(a)) = \int f \, d\mu \) for all \( f \in C(\text{sp}(a)) \) (such \( \mu \) exists by Example 1.7.2 to \( C^*(a, 1) \)). Then \( \mu \) is not a positive measure and hence \( 1 < \| \mu \| = |\varphi| \); contradiction.

This proves (1) in the case of a general approximate unit \( (e_\lambda) \), and the case when \( (e_\lambda) \) is increasing follows immediately.

(2) is a special case of (1).

Clause (3) is a consequence of the first part and the Hahn–Banach extension theorem. For (4) fix \( a \geq 0 \). Since \( \text{sp}(a) \subseteq \mathbb{R} \) is compact we can let \( x := \max(\text{sp}(a)) \) and the point-evaluation at \( x \) is a character of \( C^*(a, 1) \) such that \( \varphi(a) = |a| \). By (3) this character can be extended to a state on \( A \).

Suppose \( A \) is a \( C^* \)-algebra and let \( A^* \) denote the Banach space dual of \( A \). Every linear functional \( \varphi \) on \( A \) is uniquely determined by its restriction to \( A_{sa} \), and \( \varphi \) is self-adjoint if and only if this restriction is real-valued. This correspondence is an isometry of the space of self-adjoint functionals on \( A \) with \( A_{sa}^* \), the dual of the real Banach space \( A_{sa} \). Succinctly stated, we have \( (A_{sa})^* = (A^*)_{sa} \) and with a modest abuse of notation one might want to write \( A_{sa}^* \) in place of \( (A^*)_{sa} \). We will do so, but only in Lemma 1.7.7 and its proof.

Every element of a \( C^* \)-algebra is a linear combination of four positive elements (Exercise 1.11.16) and a similar statement holds for linear functionals.

**Lemma 1.7.7.** Suppose \( A \) is a \( C^* \)-algebra.

1. Every self-adjoint functional on \( A \) is a difference of two positive functionals.
2. Every functional on \( A \) is a linear combination of four states.
3. The weak topology on \( A \) coincides with the weak topology induced by its states.

**Proof.** (1) We will prove that every \( \theta \in A_{sa}^* \) of norm \( \leq 1 \) is a convex combination of \( \varphi \) and \( -\varphi \) for states \( \varphi \) and \( \psi \). Consider \( A_{sa}^* \) with respect to the weak*-topology. The set \( \mathcal{S} := \{ r\varphi - (1 - r)\psi : \varphi, \psi \subseteq S(A), 0 \leq r \leq 1 \} \) is weak* compact and convex. If this were false then by the Hahn–Banach separation theorem (Corollary C.4.5) there would be \( a \in A_{sa} \), \( \theta \in (A_{sa}^*)_{\leq 1} \), and \( s \in \mathbb{R} \) such that

\[
|\varphi(a)| \leq s < \theta(a)
\]

for all \( \varphi \in \mathcal{S} \). Since \( \mathcal{S} \) is symmetric, we have \( \sup_{\varphi \in \mathcal{S}} \varphi(a) \leq s \leq \theta(a) \). But Lemma 1.7.6 implies \( ||a|| \leq s \) and therefore \( ||\theta|| > 1 \); contradiction.

If \( \varphi \in A^* \) then \( \theta_0(a) := \frac{1}{2}(\theta(a) + \overline{\theta(a^* d)}) \) and \( \theta_1(a) := \frac{1}{2}((\theta(a) - \overline{\theta(a^*)})) \) are self-adjoint functionals such that \( \theta = \theta_0 - i\theta_1 \). By (1), each \( \theta_j \) is a linear combination of two positive functionals and (2) follows; (3) is an immediate consequence. \( \square \)

We end this section with a variation on the first part of Lemma 1.7.5. It gives sufficient conditions under which states possess some degree of multiplicity.

**Proposition 1.7.8.** Assume \( \varphi \) is a state on a \( C^* \)-algebra \( A \) and \( a \) is a normal element such that \( \lambda = \varphi(a) \) is an extreme point of the convex closure of \( \text{sp}(a) \). Then for every \( f \in C(\text{sp}(a)) \) and every \( b \in A \) we have

...
1. $\varphi(f(a)) = f(\varphi(a)) = f(\lambda)$ and
2. $\varphi(f(a)b) = f(\varphi(a))\varphi(b) = f(\lambda)\varphi(b)$.

In particular,
3. If $a \in A$ then $\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(aba)$ for all $b \in A$.
4. If $a$ is a unitary and $\varphi(a) \in T$ then $\varphi(ab) = \varphi(ba)\varphi(b)$ for all $b \in A$.

Proof. (1) Since $\text{sp}(a)$ is compact, our assumption implies $\lambda \in \text{sp}(a)$. The restriction of $\varphi$ to $C^*(a,1)$ is a state on this algebra and by the Riesz Representation Theorem (Theorem C.3.8) there is a Borel probability measure $\mu$ on $\text{sp}(a)$ such that $\varphi(f(a)) = \int f \, d\mu$ for all $f \in C(\text{sp}(a))$ (see Example 1.7.2). In particular $\lambda = \int xd\mu(x)$. Since $\lambda$ is an extreme point of $\text{sp}(a)$, we have the following.

The following claim actually requires a proof.

Claim. The measure $\mu$ is the point mass measure concentrated at $\lambda$.

Proof. Let $M$ denote the space of all Borel probability measures on $\text{sp}(a)$. It is convex and weak$^*$-compact (we identify measures with states on $C^*(a,1)$). Then $\nu \mapsto \int xd\nu(x)$ is an affine homeomorphism that maps $M$ onto $Z := \text{conv}(\text{sp}(a))$.

Since $\lambda$ is an extreme point of $Z$, $F := \{\nu \in M : \int xd\nu(x) = \lambda\}$ is a face of $M$. For $g \in C(\text{sp}(a))$ such that $0 \leq g \leq 1$ and $r := \int g \, d\mu > 0$, let $\mu_r(A) := r^{-1}\int_A g \, d\mu$. We claim that $\mu_r \in F$. Since $\mu_r \leq \mu$, $r = 1$ implies $\mu = \mu_r$. If $r < 1$ then we have $\mu = r\mu_r + (1-r)\mu_{1-r}$. Since $F$ is a face, $\mu_r \in F$.

Suppose that the assertion is false. The set $Z \setminus \{\lambda\}$ is convex, relatively open, and of positive measure (identify $\mu$ with a measure on $Z$). Find a family $(K_n, L_n)$, for $n \in \mathbb{N}$, such that $K_n$ and $L_n$ are compact convex subsets of $Z \setminus \{\lambda\}$, $K_n$ is included in the interior of $L_n$, and $\bigcup_n K_n = Z \setminus \{\lambda\}$. By the countable additivity of $\mu$ there is $n$ such that $\mu(K_n) > 0$. Fix $g \in C(Z)$ such that $0 \leq g \leq 1$, $g(x) = 1$ for all $x \in K_n$, and $\text{supp}(g) \subseteq L_n$. Then $\int g \, d\mu \geq \mu(K_n) > 0$, hence $\mu_g \in F$ but $\int xd\mu_g(x) \in L_n$; contradiction.

Claim implies that the restriction of $\varphi$ to $C^*(a,1)$ is a point evaluation at $\lambda$, and therefore a character. Therefore $\varphi(f(a)) = f(\varphi(a)) = f(\lambda)$.

(2) Fix $b \in A$. In order to prove $\varphi(f(a)b) = f(\lambda)\varphi(b)$ we first consider the case when $f(\lambda) = 0$. By the first part and $f(\lambda) = 0$ we have $\varphi(f(a)^*f(a)) = 0$ and therefore the Cauchy–Schwarz inequality implies

$$|\varphi(f(a)b)|^2 \leq \varphi(f(a)^*f(a))\varphi(b^*b) = 0.$$ 

Hence $\varphi(f(a)b) = 0 = f(\lambda)\varphi(b)$ if $f(\lambda) = 0$. For the general case, consider $f_1(t) := f(t) - f(\lambda)$. By the above,

$$\varphi(f(a)b) = \varphi(f_1(a)b) + \varphi(f(\lambda)b) = f(\lambda)\varphi(b).$$

This concludes the proof of (2).

Both (3) and (4) are special cases of (1) and (2). 

A better insight into Proposition 1.7.8 is given in Exercise 1.11.46.
1.8 Approximate Units and Strictly Positive Elements

In this section we continue the discussion of approximate units (Definition 1.6.7) and introduce the notion of a strictly positive element of a C*-algebra.

A self-refinement of Proposition 1.6.8 will come in handy in the analysis of ideals (Lemma 2.5.1).

**Corollary 1.8.1.** If \( L \) is a left ideal in a C*-algebra \( A \) then there exists an increasing family \( \{ e_\lambda : \lambda \in \Lambda \} \) of positive contractions in \( L \) such that \( \lim_\lambda \|a - ae_\lambda\| = 0 \) for all \( a \in L \).

**Proof.** Since \( B := L \cap L^* \) is a C*-subalgebra of \( A \), by Proposition 1.6.8 it has an approximate unit \( \{ e_\lambda : \lambda \in \Lambda \} \). If \( a \in L \) then \( \|a(1 - e_\lambda)\| = \|a - (1 - e_\lambda)a^*a(1 - e_\lambda)\| \) and \( a^*a \in B \); hence \( \{ e_\lambda : \lambda \in \Lambda \} \) is as required. \( \square \)

In the following lemma we identify \( C_0(X \setminus 0) \) with \( \{ f \in C(X) : f(0) = 0 \} \).

**Lemma 1.8.2.** In a C*-algebra \( A \), for every \( h \in A_+ \) the following are equivalent.

1. \( hA \) is dense in \( A \).
2. \( Ah \) is dense in \( A \).
3. \( hAh \) is dense in \( A \).
4. \( \{ f_\lambda(h) : \lambda \in \Lambda \} \) is an approximate unit in \( A \) whenever \( \{ f_\lambda : \lambda \in \Lambda \} \) is an approximate unit in \( C_0(\text{sp}(h) \setminus \{0\}) \).
5. \( \varphi(h) > 0 \) for every state \( \varphi \) of \( A \).

**Proof.** Since both \( h \) and \( A \) are self-adjoint, (1)–(3) are easily equivalent.

We shall prove that (1) implies (4) and that (4) implies (5). We shall not need the fact that (5) implies other conditions, and it is omitted. See e.g., [194, Proposition 3.10.5].

Suppose (1) holds, and in order to prove (4) fix an approximate unit \( \{ f_\lambda : \lambda \in \Lambda \} \) in \( C_0(\text{sp}(h) \setminus \{0\}) \). It suffices to check that \( \|a - f_\lambda(h)\| \to 0 \) for all \( a \in A \). Fix \( \varepsilon > 0 \) and find \( b \in A \) such that \( \|a - b\| < \varepsilon \). If \( \lambda \) is such that \( \|hb - f_\lambda(h)hb\| < \varepsilon \) then

\[
a \approx_\varepsilon hb \approx_\varepsilon f_\lambda(h)hb \approx_\varepsilon f_\lambda(h)a
\]

and therefore \( \|a - f_\lambda(h)a\| < 3\varepsilon \). Since \( \varepsilon > 0 \) and \( a \in A \) were arbitrary, this completes the proof.

Now suppose that a supposedly weaker form of (4) holds, and \( \{ f_\lambda(h) : \lambda \in \Lambda \} \) is an approximate unit in \( A \) for some family of functions \( f_\lambda \in C_0(\text{sp}(h) \setminus \{0\})_{+,1} \). If \( \varphi \) is a state in \( A \) then Lemma 1.7.6 implies that \( \lim_\lambda \varphi(f_\lambda(h)) = 1 \). Therefore the restriction of \( \varphi \) to \( C^*_+(h) \cong C_0(\text{sp}(h) \setminus \{0\}) \) doesn’t vanish, and \( \varphi(h) > 0 \). Since \( \varphi \) was arbitrary, this proves (5). \( \square \)

A positive element \( h \) of a C*-algebra \( A \) is strictly positive any of the equivalent conditions in Lemma 1.8.2 holds.
Lemma 1.8.3. A C*-algebra A is σ-unital if and only if it has a strictly positive element.

If A is unital then h ∈ A is strictly positive if and only if 1_A ≤ mh for some m ∈ N.

Proof. If h is strictly positive in A then Lemma 1.8.2 implies that h^{1/n}, for n ≥ 1, is an approximate unit for A. Conversely, suppose e_n, for n ∈ N, is an approximate unit for A and let h := \sum_n 2^{-n}e_n. If \phi is a state on A then \lim_n \phi(e_n) = 1 and therefore \phi(h) > 0.

Now suppose A is unital. If mh ≥ 1_A then it is clearly strictly positive. For the converse, suppose that 0 ∈ sp(h). Since A is unital, h generates a proper ideal J of A. Any state on A/J lifts to a state on A that violates (5) of Lemma 1.8.2. □

The following lemma will play a role in the construction of quasi-central approximate units in §1.9.

Lemma 1.8.4. Suppose J is an ideal in A and (e_λ : λ ∈ Λ) is an approximate unit in J. For every a ∈ A the net (e_λa - ae_λ : λ ∈ Λ) weakly converges to 0.

Proof. Since the weak topology on a C*-algebra coincides with the weak topology induced by its states (Lemma 1.7.7) it suffices to verify that \lim_λ \phi(e_λa - ae_λ) = 0 for every state \phi and all a ∈ A.

Fix a state \phi on A and let π : A → B(H) be the corresponding GNS representation with cyclic vector \xi. Let K be the closure of the space π(J)\xi. Then π(e_λ) ≤ p_K for all λ and by Exercise 1.11.44 \{π(e_λ) : λ ∈ Λ\} is a net that converges to p_K in the strong topology. By Exercise 1.11.57 we have p_K ∈ π(A)' and therefore

\lim_λ \phi(e_λa - ae_λ) = \lim_λ (π( ae_λ e_λ a )\xi | \xi) = ((π(a)p_K - p_K π(a)\xi | \xi)) = 0.

Since a ∈ A was arbitrary, this completes the proof. □

We conclude this section with an application of approximate units.

Lemma 1.8.5. Suppose B is a hereditary C*-subalgebra of a C*-algebra A. Then every state \phi of B has a unique extension to a state on A. In particular, every state on A has a unique extension to a state on its unitization A^†.

Proof. Let e_λ, for λ ∈ Λ, be an approximate unit for B (Proposition 1.6.8). Fix a ∈ A and suppose ψ is a state on A that extends ϕ. Since sup_λ \phi(e_λ) = 1 by Lemma 1.7.6, the second part of Lemma 1.7.5 implies ψ(a) = \lim_λ ψ( e_λ a e_λ ). Since B is hereditary, e_λ a e_λ ∈ B for all λ and the conclusion follows. □

1.9 Quasi-central Approximate Units

In this section we construct quasi-central approximate units. These will be used in §15.1 to prove that coronas of σ-unital C*-algebras and other massive C*-algebras
are countably degree-1 saturated. The section ends with the so-called tri-diagonal construction of an element of $M_n(A)$ that “almost commutes” with the elements of a discretization of a continuous path of elements of $A$.

In a seminal paper [15] Arveson introduced a refinement, the so-called quasi-central approximate units, and used them to give simplified proofs of (then hot off the press) theorems of Choi–Effros and Voiculescu. Our interest in quasi-central approximate units, and used them to give simplified proofs of (then hot off the press) theorems of Choi–Effros and Voiculescu. Our interest in quasi-central approximate units, and used them to give simplified proofs of (then hot off

Definition 1.9.1. Suppose $A$ is an ideal in $M$. An approximate unit $(e_\lambda : \lambda \in \Lambda)$, for $n \in \mathbb{N}$, in $A$ is quasi-central if $\lim \|ae_\lambda - e_\lambda a\| = 0$ for every $a \in M$. If $X \subseteq M$ and this condition holds for all $a \in X$ then the approximate unit is $X$-quasi-central.

An approximate unit is idempotent: if $e_\lambda e_\mu = e_\lambda$ whenever $\lambda \leq \mu$.

The following doubles as a warmup for the proof of Proposition 1.9.3 and a lemma used in the proof of Theorem 5.1.2.

Lemma 1.9.2. Suppose $n \geq 1$, $a_0, \ldots, a_{n-1}$ belong to $\mathcal{B}(H)^n$ and $E$ is a finite-rank projection in $\mathcal{B}(H)$. Then for every $\varepsilon > 0$ there exists a finite-rank operator $T \in \mathcal{B}(H)$ such that $E \leq T \leq 1$ and $\| [a_j, T] \| < \varepsilon$ for all $j < n$.

Proof. The set $\mathcal{Z} := \{ (P, P, \ldots) : P$ is a finite-rank projection in $\mathcal{B}(H) \}$ with the coordinatewise ordering is an approximate unit for $\mathcal{B}(H)^n$. Lemma 1.8.4 implies that the net $\| [\alpha, Q] \|$, for $Q \in \mathcal{Z}$, weakly converges to 0. By the Hahn–Banach separation theorem there is a convex combination $T_0$ of elements of $\mathcal{Z}$ such that $\| [\alpha, T_0] \| < \varepsilon$. Then $T_0 = (T, T, \ldots, T)$ for some $T$, and $T$ is as required. \qed

Proposition 1.9.3. Suppose $\Lambda$ is $\sigma$-unital ideal in a $\mathcal{C}^*$-algebra $M$ and $X \subseteq M$ is separable. Then there exists an $X$-quasi-central, sequential, idempotent approximate unit $e_n$, for $n \in \mathbb{N}$, in $A$.

Proof. Let $a_0$, for $n \in \mathbb{N}$, be an enumeration of a norm-dense subset of $X$. Fix a strictly positive element $h \in A$. We will find $f_n \in C_0(\text{sp}(h))$ which satisfy the following for all $n$.

1. $0 \leq f_n \leq f_{n+1} \leq 1$.
2. $f_n f_{n+1} = f_n$.
3. There exists $\varepsilon_n > 0$ such that $f_n$ vanishes on $[0, \varepsilon_n]$.
4. $\| f_n(h) a_j - a_j f_n(h) \| \leq 2^{-n}$ for all $j < n$.

If $f_n$ satisfy conditions (1)–(4) then $e_n := f_n(h)$, for $n \in \mathbb{N}$, is the required $X$-quasi-central approximate unit.

To get the recursive construction of the sequence $f_n$, for $n \in \mathbb{N}$ going, choose $f_0$ to be identically 0. Suppose that $f_k$ and $e_k$, for $k \leq n$, satisfy (1)–(4). Let $M^{\oplus n}$ denote
the direct sum of \( n \) copies of \( M \). The direct sum \( A^{\oplus n} \) of \( n \) copies of \( A \) is an ideal of \( M^{\oplus n} \) and \((h, \ldots, h)\) is strictly positive in \( A^{\oplus n} \). With

\[
\mathcal{Z}_n := \left\{ f \in C(\text{sp}(h)) : f(t) = 1 \text{ for all } t \geq \varepsilon_n \right\}
\]

and \( f \) vanishes on \([0, \varepsilon]\) for some \( \varepsilon > 0 \)

we have \( ff_n = f_n \) for every \( f \in \mathcal{Z}_n \).

By Lemma 1.8.2 the set \( \{(f(h), \ldots, f(h)) : f \in \mathcal{Z}_n\} \) is an approximate unit of \( A^{\oplus n} \) with respect to the natural order on positive elements.

Consider \( b := (a_0, \ldots, a_{n-1}) \) and \( h' := (h, \ldots, h) \) in \( A^{\oplus n} \). By the Hahn–Banach theorem and Lemma 1.8.4 we can find a convex combination \( f_{n+1} \) of elements of \( \mathcal{Z}_n \) such that \( \|b f_{n+1}(h') - f_{n+1}(h') b\| \leq 2^{-n} \). Then \( f_n \leq f_{n+1} \leq 1, f_n f_{n+1} = f_n, \) and \( e_{n+1} := f_{n+1}(h) \) satisfies \( \|a_0 e_{n+1} - e_{n+1}\| \leq 2^{-n} \) for all \( j < n \). Since \( f_{n+1} \) is a convex combination of elements of \( \mathcal{Z}_n \) it vanishes on \([0, \varepsilon_{n+1}]\) for some \( \varepsilon_{n+1} > 0 \).

This describes the recursive construction and concludes the proof. \( \Box \)

Discretization of homotopy paths leads to \( k \)-tuples \( \bar{a}(j) \in A^k \) as in the following lemma. They are naturally identified with diagonal elements of \( M_k(A) \), and therefore with elements of \( \mathcal{B}(C^* \otimes H) \) if \( A \subseteq \mathcal{B}(H) \) is a concrete \( C^* \)-algebra.

**Lemma 1.9.4.** Suppose \( A \) is a \( C^* \)-subalgebra of \( \mathcal{B}(H) \), \( \varepsilon > 0, n \geq 1, k \geq 1, \) and \( \bar{a}(j) = (a(j)_i : i < k), \) for \( j < n \), in \( A^k \) satisfy

\[
\max_{j < n, i < k-1} \|a(j)_i - a(j)_{i+1}\| < \varepsilon.
\]

Then for every projection \( E \) in \( \mathcal{B}(H) \) of finite-rank there exists a projection \( F \) of finite-rank in \( \mathcal{B}(C^* \otimes H) \) such that \( E \oplus 0 \oplus \cdots \oplus 0 \leq F \) and \( \max_{j, n} \|\bar{a}(j), F\| < \varepsilon. \)

**Proof.** Let \( \varepsilon_0 = \max_{j \leq n} \|a(j)_i - a(j)_{i+1}\| \) and \( \delta_0 = \varepsilon - \varepsilon_0. \) Using Lemma 1.4.8 fix \( \delta \in (0, \delta_0/4) \) such that if \( a \) and \( b \) satisfy \( \|a, b\| < \delta, 0 \leq a \leq 1, \) and \( \|b\| < 1 \) then \( \|a(1-a)^{1/2}, b\| < \delta_0/4. \) Using Lemma 1.9.2 find finite-rank operators \( F_j, \) for \( 1 \leq j < k, \) in \( \mathcal{B}(H) \) such that with \( E_j \) denoting the support projection of \( F_j \) for \( j \geq 1 \) we have \( 0 = F_0 \leq E_0 \leq F_1 \leq \cdots \leq F_{k-1}, \|F_j, a(i)_{j+1}\| < \delta, \) and \( \|F_j, a(i)_j\| < \delta \) for all \( j < k. \) Then consider the tridiagonal matrix

\[
F := \begin{pmatrix}
F_1 - F_0 & (F_1(1-F_1))^{1/2} & 0 & \cdots & 0 \\
(F_1(1-F_1))^{1/2} & F_2 - F_1 & (F_2(1-F_2))^{1/2} & \cdots & 0 \\
0 & (F_2(1-F_2))^{1/2} & F_3 - F_2 & \cdots & 0 \\
0 & 0 & (F_3(1-F_3))^{1/2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & F_{k-1} - F_k
\end{pmatrix}
\]

Clearly \( F^* = F \) and (almost as clearly) \( F^2 = F. \) Therefore \( F \) is a projection. Since \( F_1 - F_0 \geq E_0, \) we have \( F \geq E \oplus 0 \oplus \cdots \oplus 0. \) An exercise in multiplying matrices

---

\(^7\) For those readers who may prefer a precise definition of \( F = (F_{ij}) \) by its matrix entries:

\[
F_{ij} := F_{i+1} - F_i, F_{i+1, j+1} = F_{i+1, j} := (F_i(1-F_i))^{1/2}, \text{ and } F_{ij} := 0 \text{ if } |i-j| \geq 2.
\]
shows that
\[ \| [F, \bar{a}(j)] \| \leq \varepsilon_0 + \max_{i<k} \|[a(j), (F_i - F_{i-1})]\| + 2\max_{i<k} \|[a(j), (F_i(1-F_i))^{1/2}]\| < \varepsilon, \]
and \( F \) is as required. \( \square \)

1.10 The GNS construction

I had always thought of mathematics as being much more straightforward: a formula is a formula, and an algebra is an algebra, but Gel'fand found hedgehogs lurking in the rows of his spectral sequences!

Dusa McDuff

In this section we prove that every abstract \( C^\ast \)-algebra is isomorphic to a concrete \( C^\ast \)-algebra. The proof uses the simple but ingenious GNS construction, whose discovery marked the beginning of the theory of \( C^\ast \)-algebras.\(^8\) A consequence of this result is the axiomatic characterization of \( C^\ast \)-algebras as complex Banach algebras with the involution that satisfy the \( C^\ast \)-equality. We also give a variant of the GNS construction for an arbitrary functional on a \( C^\ast \)-algebra. Finally, we briefly discuss cyclic representations and prove that GNS representations are the building blocks of the representation theory of a \( C^\ast \)-algebra.

The definitions of concrete and abstract \( C^\ast \)-algebras were given in §1.2.

**Theorem 1.10.1 (Gelfand–Naimark–Segal).** Every abstract \( C^\ast \)-algebra \( A \) is isomorphic to a concrete \( C^\ast \)-algebra.

**Definition 1.10.2.** A representation of a \( C^\ast \)-algebra \( A \) is a \( \ast \)-homomorphism
\[ \pi : A \to \mathcal{B}(H) \]
for some Hilbert space \( H \). It is faithful if \( \ker(\pi) = \{0\} \) (equivalently, if it is isometric—Lemma 1.2.10). A unit vector \( \xi \in H \) is cyclic for \( \pi \) if the orbit \( \pi[A]\xi \) is dense in \( H \). A representation that has a cyclic vector is said to be cyclic. A cyclic vector, if one exists, is not unique; some representations even have the property that every nonzero vector is cyclic. (These representations are so important that the entire Chapter §5 is devoted to them.) The left kernel of a state \( \varphi \) is
\[ L_{\varphi} := \{a \in A : \varphi(a^*a) = 0\} \]
(denoted \( N_{\varphi} \) in [27]).

**Proposition 1.10.3 (The GNS construction).** Suppose \( \varphi \) is a state on \( A \). Then there exist a Hilbert space \( H_{\varphi} \) and cyclic representation \( \pi_{\varphi} : A \to \mathcal{B}(H_{\varphi}) \) with a cyclic vector \( \xi_{\varphi} \) such that \( \varphi(a) = \langle \pi_{\varphi}(a)\xi_{\varphi}, \xi_{\varphi} \rangle_{\varphi} \) for all \( a \in A \).

\(^8\) See however footnote 2 in §12.
We refer to \((\pi_\varphi, H_\varphi, \xi_\varphi)\) as the GNS triplet associated with \(\varphi\).

**Proof.** The formula
\[
(a|b)_\varphi := \varphi(b^*a)
\]
defines a sesquilinear form on \(A\) of norm 1 and \((\cdot|\cdot)_\varphi\) drops to an inner product on \(A/L_\varphi\.\) Let \(H_\varphi\) denote the completion of this pre-Hilbert space. For \(b \in A\) we have
\[
\|b + L_\varphi\| = \varphi(b^*b)^{1/2}.
\]
Moreover, for every \(a \in A\) let \(\pi_\varphi(a)\) be the left multiplication operator on \(A/L_\varphi\.\) Since
\[
b^*a^*ab \leq b^*b\|a^*a\|,
\]
\[
\|\pi_\varphi(a)\| \leq \|a\|\]
and \(\pi_\varphi(a)\) therefore extends to a bounded linear operator on \(H_\varphi\) and we have a representation \(\pi_\varphi\) of \(A\) on \(H_\varphi\). If \(A\) is unital, then \(\xi_\varphi := 1 + L_\varphi\) is clearly cyclic and satisfies
\[
\varphi(a) = (\pi_\varphi(a)\xi_\varphi|\xi_\varphi)_\varphi
\]
for all \(a \in A\). If \(A\) is not unital, let \(\bar{\varphi}\) be the unique state extension of \(\varphi\) to \(\bar{A}\) (Lemma 1.8.5). Then 
\[
\bar{\varphi}((1 - e_\lambda)^2) \leq \bar{\varphi}(1 - e_\lambda) \to 0,
\]
and therefore \(e_\lambda + L_\varphi\) is a Cauchy sequence in \(H_\varphi\) converging to \(1 + L_\varphi\) and we have \(H_\varphi = H_\bar{\varphi}\). □

**Proof (Theorem 1.10.1).** By Lemma 1.7.6 for every \(a\) in \(A\) there exists a state \(\varphi\) such that \(\varphi(a^*a) = \|a^*a\|\). Therefore
\[
\|\pi_\varphi(a)\|^2 \geq \|\pi_\varphi(a)\xi_\varphi\|^2 = (\pi_\varphi(a)\xi_\varphi|\pi_\varphi(a)\xi_\varphi) = (\pi_\varphi(a^*a)\xi_\varphi|\xi_\varphi) = \varphi(a^*a)
\]
which is by the choice of \(\varphi\) and the \(C^\ast\)-equality equal to \(\|a\|^2\). Therefore the representation \(\bigoplus_{\varphi \in \mathcal{S}(A)} \pi_\varphi\) provided by Proposition 1.10.3 is faithful. □

**Corollary 1.10.4.** Every separable abstract \(C^\ast\)-algebra can be faithfully represented on a separable Hilbert space. More generally, for every cardinal \(\kappa\) an abstract \(C^\ast\)-algebra of density character \(\kappa\) can be faithfully represented on a Hilbert space of density at most \(\kappa\).

**Proof.** It suffices to prove the second, more general, statement. Suppose \(A\) is a \(C^\ast\)-algebra of density character \(\kappa\). If \(\varphi \in \mathcal{S}(A)\) then \(\|\varphi\| = 1\) and hence \((a|a)_\varphi \leq \|a\|^2\) for all \(a \in A\). Therefore the density character of \(H_\varphi\) is not greater than \(\kappa\).

Fix a dense \(D \subset A_{\mathcal{S}}\) of cardinality \(\kappa\). As in the proof of Theorem 1.10.1, for every \(a \in D\) there exists \(\varphi_a \in \mathcal{S}(A)\) such that \(\|\pi_\varphi(a)\| = \|a\|\). Therefore \(\bigoplus_{a \in D} \pi_\varphi_a\) is a faithful representation of \(A\) on a Hilbert space of density at most \(\kappa\). □

**Corollary 1.10.5.** If \(A\) is a \(C^\ast\)-algebra then the \(\ast\)-algebra \(M_n(A)\) of all \(n \times n\) matrices over \(A\) has a unique \(C^\ast\)-algebra norm such that the diagonal embedding of \(A\) into \(M_n(A)\) is isometric.

**Proof.** By Theorem 1.10.1 we can identify \(A\) with a \(C^\ast\)-subalgebra of \(\mathcal{B}(H)\) for some Hilbert space \(H\). Then \(M_n(A)\) is identified with \(\mathcal{B}(H^n)\). The operator norm of

a matrix is not smaller than the absolute value of any of its entries. It is not difficult
to see that this implies \( M_n(A) \) is norm-closed in \( \mathcal{B}(H^n) \). Uniqueness of the C*-norm
on \( M_n(A) \) follows from Lemma 1.2.10.

A relative of Lemma 1.7.5 for bounded linear functionals that are not necessarily
positive (Lemma 1.10.7) is proved after a preliminary result.

Fix \( m \geq 1 \) and \( n \geq 1 \) and suppose \((b_{ij})\) is an \( m \times n \) matrix over \( A \). If \( A \subseteq \mathcal{B}(H) \)
then \((b_{ij})\) can be identified with an element of \( \mathcal{B}(H^n,H^m) \) and equipped with an
operator norm. We can also identify \((b_{ij})\) with an element of \( M_{\max(m,n)}(A) \) obtained
by adding zero rows or columns. This identification is clearly norm-preserving and
Corollary 1.10.5 implies that the operator norm on of \( m \times n \) matrices over \( A \)
is uniquely determined.

**Lemma 1.10.6.** Suppose \( A \) is a C*-algebra and \( a \in A_{\leq 1} \). Then the operator norm of
the \( 1 \times 2 \) matrix \[ 1 - aa^* \] is at most 1.

**Proof.** Working inside \( M_2(A) \), we have
\[
\| [1 - aa^* a] \|^2 = \| [1 - aa^* a] [1 - aa^* a]^* \|.
\]
Furthermore, \([1 - aa^* a] [1 - aa^* a]^* = (1 - aa^*)^2 + aa^* = 1 - aa^* + (aa^*)^2 \). Since
\( \|a\| \leq 1 \), we have \( 1 \geq aa^* \geq (aa^*)^2 \geq 0 \) and \( 0 \leq 1 - aa^* + (aa^*)^2 \leq 1 \). This implies
the required estimate. \( \square \)

**Lemma 1.10.7.** Suppose a linear functional \( \phi \) on a C*-algebra \( A \) and \( a \in A \) are
such that \( \phi(a) = \|a\| = \|\phi\| = 1 \). Then \( \phi(b) = \phi(aa^* b) \) for all \( b \in A \).

**Proof.** For any \( \lambda \in \mathbb{C} \) the following holds (using submultiplicativity of the norm
and Lemma 1.10.6)
\[
\| (1 - aa^*)b + \lambda a \| \leq \| [1 - aa^* a] \| \left\| \begin{bmatrix} b \\ \lambda \end{bmatrix} \right\| \leq (\|b\|^2 + |\lambda|^2)^{1/2}.
\]
Thus
\[
\|b\|^2 + |\lambda|^2 \geq |\phi((1 - aa^*)b + \lambda a)|^2 \\
= |\phi((1 - aa^*)b)|^2 + 2\Re(\bar{\lambda} \phi((1 - aa^*)b)) + |\lambda|^2,
\]
and \( \|b\|^2 \geq |\phi((1 - aa^*)b)|^2 + 2\Re(\bar{\lambda} \phi(b - aa^* b)) \). Since \( \lambda \in \mathbb{C} \) was arbitrary, this
is possible only when \( \phi(b - aa^* b) = 0 \). \( \square \)

**Theorem 1.10.8.** Suppose \( \phi \) is a bounded linear functional on a C*-algebra \( A \). Then
there are a *-representation \( \pi : A \to \mathcal{B}(H) \) and \( \xi, \eta \in H \) such that
\( \|\phi\| = \|\pi(a)\| \) and \( \phi(b) = (\pi(b)\xi|\eta) \) for all \( b \in A \). Furthermore \( \pi \) can be chosen to be cyclic.

**Proof.** We may assume that \( \phi \) is nonzero and, by replacing \( \phi \) with \( ||\phi||^{-1} \phi \), that
\( ||\phi|| = 1 \). Fix a Banach limit (Theorem C.3.12) \( \text{Lim} : \ell_\infty(\mathbb{N}) \to \mathbb{R} \) and define \( \phi \) on
\( \ell_\infty(A) \) by \( \phi((x_n)_n) := \text{Lim}_n \phi(x_n) \).
Choose $a_0 \in A$ such that $\|a_n\| = 1$ and let $a := (a_n)_n$ and let $\psi(b) := \tilde{\psi}(ab)$. If $(e_j)$ is an approximate unit of $\ell_\infty(A)$ then $\|\psi\| = \lim_j \psi(e_j) = 1$ and $\psi$ is a state by Lemma 1.7.6. Let $(\pi, H, \xi)$ be the GNS triplet associated with $\psi$. Let $\eta := \pi(a)\xi$ and view $A$ as a C*-subalgebra of $\ell_\infty(A)$ via the diagonal embedding.

Lemma 1.10.7 implies $(\pi(b)\xi \mid \eta) = (\pi(a^*b)\xi \mid \eta) = \psi(a^*b) = \psi(b)$ for all $b \in A$.

It remains to show that $\pi$ can be chosen to be cyclic. Let $K$ be the norm-closure of $\pi[A]\xi$ and let $p$ be the projection of $H$ to $K$. Then $p \in \pi[A]'$ and $\pi'(a) := p\pi(a)p$ is the required cyclic representation.

We conclude with a brief introduction of a concept that will play a pivotal role in §3.8 and subsequent sections. Two representations $\pi_i : A \to \mathcal{B}(H_i)$ (for $i < 2$) are spatially equivalent (denoted $\pi_0 \sim \pi_1$) if there exists a unitary operator $u : H_0 \to H_1$ such that the diagram 1.2 commutes.

![Fig. 1.2 Spatially equivalent representations, $\pi_0 \sim \pi_1$.](image)

**Lemma 1.10.9.** A representation of a C*-algebra is cyclic if and only if it is spatially equivalent to a GNS representation.

**Proof.** Since the GNS construction produces a cyclic representation, only the forward direction requires a proof. If $\pi : A \to \mathcal{B}(H)$ is a representation and $\xi \in H$ is a cyclic unit vector then $\phi_\xi(a) := (\pi(a)\xi \mid \xi)$ is a state. Recall that $L_\phi$ is the left kernel of $\phi$ and that $H_\phi$ is the completion of $A/L_\phi$ with respect to the inner product $(a|b)_\phi = \phi(b^*a)$. Then $a + L_\phi \mapsto a\xi$ is a linear isometry from $A/L_\phi$ into $H$. Since $\xi$ is a cyclic vector for $\pi$, this isometry has dense range and therefore extends to a unitary $u : H_\phi \to H$. A routine computation gives $\pi(a) = u\pi(a)u^*$ for all $a \in A$. \hfill $\square$

Suppose $\pi_j : A \to \mathcal{B}(H_j)$, for $j \in J$, are representations of C*-algebra $A$. Their direct sum is the naturally defined representation $\bigoplus_{j \in J} \pi_j$ of $A$ on the direct sum of Hilbert spaces $\bigoplus_{j \in J} H_j$.

**Proposition 1.10.10.** Every representation $\pi : A \to \mathcal{B}(H)$ is a direct sum of cyclic representations. Every representation $\pi : A \to \mathcal{B}(H)$ of a C*-algebra is spatially isomorphic to a direct sum of GNS representations.

**Proof.** Use Zorn’s Lemma to find a maximal set of unit vectors $\mathcal{X}$ such that for distinct $\xi$ and $\eta$ in $\mathcal{X}$ spaces $\pi[A]\xi$ and $\pi[A]\eta$ are orthogonal. Since every cyclic representation is spatially equivalent to a GNS representation by Lemma 1.10.9, the second claim follows. \hfill $\square$
1.11 Exercises

The following extensive list can be used as a Moore-style course within a course for students with no background in C*-algebras. Some of the exercises are very easy but very important.

**Exercise 1.11.1.** Prove that $\mathbb{C}$ is the only C*-algebra in which all nonzero elements are invertible.\(^9\)

**Exercise 1.11.2.** Prove that in the definition of an abstract C*-algebra the requirement that the involution $\ast$ be isometric is redundant.

**Exercise 1.11.3.** Suppose $A$ is a C*-algebra and $a \in A$.

1. If $a$ is normal, prove that $\text{sp}_A(P(a)) = P(\text{sp}_A(a))$ for every $*$-polynomial $P$ over $\mathbb{C}$.
2. Prove $\text{sp}_A(\exp(a)) = \exp[\text{sp}_A(a)]$.

The following few exercises are purely algebraic.

**Exercise 1.11.4.** Suppose $A$ is a C*-algebra and $a, b \in A$.

1. Prove that $1 - ab$ is invertible if and only if $1 - ba$ is invertible.
2. Use this to prove that $\text{sp}_A(ab) \cup \{0\} = \text{sp}_A(ba) \cup \{0\}$.

**Exercise 1.11.5.** 1. If $A$ is unital, the following are equivalent for $b \in A$.

   a. $b$ is invertible.
   b. Both $b^*b$ and $bb^*$ are invertible.
   c. $b^*$ is invertible.

2. Give an example of an element $b$ of a unital C*-algebra such that $b^*b$ is invertible but $b$ is not.

**Exercise 1.11.6.** Suppose $B$ is a C*-subalgebra of a C*-algebra $A$ and $b \in B$. Prove that $\text{sp}_A(b) = \text{sp}_B(b)$ or $\text{sp}_A(b) = \text{sp}_B(b) \cup \{0\}$, and that $\text{sp}_B(b) = \text{sp}_A(b)$ if $B$ is a unital C*-subalgebra of $A$.

Hint: In the unital case, if $b \in B_{sa}$ then use the second part of Lemma 1.2.6 to prove $\text{sp}_B(b) \supseteq \text{sp}_A(b)$ and $\partial \text{sp}_B(b) \subseteq \partial \text{sp}_A(b)$ ($\partial Z$ denotes the topological boundary of $Z \subseteq \mathbb{C}$). For the general case use Exercise 1.11.5.

A projection $p$ in a C*-algebra $A$ is nontrivial if it equal to neither 0 nor 1 and it is scalar if $pAp \cong \mathbb{C}$.

**Exercise 1.11.7.** Prove that a simple C*-algebra has a scalar projection if and only if it is isomorphic to the algebra of compact operators on some Hilbert space.

---

\(^9\) Used in the proof of Theorem 1.3.1.
Exercise 1.11.8. Suppose $X$ and $Y$ are compact Hausdorff spaces $\iota: X \to Y$ is a continuous map, and $\iota^*: C(Y) \to C(X)$ is the associated $^*$-homomorphism (see the proof of Theorem 1.3.2).

1. Prove that $\iota^*$ is a surjection if and only if $\iota$ is an injection.
2. Prove that $\iota^*$ is an injection if and only if $\iota$ is a surjection.

The **weight** of a topological space is the minimal cardinality of its basis.

Exercise 1.11.9. Suppose $X$ is an infinite compact Hausdorff space. Prove that the density character of $C(X)$ is equal to the weight of $X$.

Exercise 1.11.10. Find a $^*$-endomorphisms of $C_0(\mathbb{R})$ that is not of the form $f \mapsto f \circ g$ for a continuous $g: \mathbb{R} \to \mathbb{R}$.

Exercise 1.11.11. Suppose $a$ is a normal element of a $C^*$-algebra. Prove that every $\lambda \notin \text{sp}(a)$ satisfies $\| (a - \lambda)^{-1} \|^{-1} = \text{dist}(\lambda, \text{sp}(a))$.

Exercise 1.11.12. For a compact $K \subseteq \mathbb{C}$ and $C^*$-algebra $A$ let $\Omega^A_K$ be the set of all normal elements $a \in A$ such that $\text{sp}(a) \subseteq K$. Prove that the function that sends $a$ to $f(a)$ is continuous on $\Omega^A_K$ for every $f \in C(K)$.

Exercise 1.11.13. Suppose $a \in \mathcal{B}(H)$. Prove that $r(a) = \|a\|$ if $a$ is normal. Find $a$ such that $r(a) < \|a\|$.

Exercise 1.11.14. Prove that every separable abelian $C^*$-algebra is isomorphic to a $C^*$-subalgebra of $C(\{0, 1\})$.

**Hint:** In addition to what you learned here, you need to know that every compact metrizable space is a continuous image of the Cantor space.

Exercise 1.11.15. Suppose $a$ is a normal element of a $C^*$-algebra. Prove that every $\lambda \notin \text{sp}(a)$ satisfies $\| (a - \lambda)^{-1} \|^{-1} = \text{dist}(\lambda, \text{sp}(a))$.

It is time for the easy exercises promised earlier.

Exercise 1.11.16. Suppose $A$ is a $C^*$-algebra.

1. For every $b \in A$ there are $b_0$ and $b_1$ in $A_{sa}$ of norm $\leq \|b\|$ such that $b = b_0 + ib_1$.
2. An element $b$ of $A$ is normal if and only if its ‘real’ and ‘imaginary’ parts as in (1) above commute.

---

10 Used in the proof of Lemma 1.5.7.
11 Used in the proof of Lemma 1.3.4.
12 Used in the proof of Lemma 1.4.7.
13 Parts of this exercise will be used throughout this book.
3. Prove that every $a \in A_{sa}$ can be written as a difference of two positive elements of norm $\leq \|a\|$. Therefore by (1) every $a \in A$ can be written as a linear combination of four positive elements of norm $\leq \|a\|$.

4. Every $-1 \leq a \leq 1$ can be written as a convex combination of two unitaries in $\bar{A}$.

5. Every $a \in A_{\leq 1}$ is a convex combination of four unitaries in $\bar{A}$.

6. The density characters of $A, A_{sa},$ and $U(\bar{A})$ are equal.

Exercise 1.11.17. Prove that $a \in \mathcal{H}(H)$ has a polar decomposition $a = v|a|$ with $v \in \mathcal{H}(H)$ if and only if it has finite rank.

Exercise 1.11.18. Suppose $A$ is a $C^*$-algebra and $a \in A$.

1. The following are equivalent.
   a. $a$ is a projection.
   b. $a$ is self-adjoint and $a - a^2 = 0$.
   c. $a$ is normal and $\text{sp}(a) \subseteq \{0, 1\}$.

2. If $A$ is unital and $a \in A_{sa}$ then $\exp(ia)$ is a unitary.

3. If $A$ is unital then $a$ is a unitary if and only if it is normal and $\text{sp}(a) \subseteq \mathbb{T}$.

4. If $v$ is a partial isometry, then $\|v\|^2 = \|v^*v\|$ and therefore $\|v\| \in \{0, 1\}$

5. If $A$ is unital and $a \in \text{GL}(A)$ then $\text{sp}(a^{-1}) = \{\lambda^{-1} : \lambda \in \text{sp}(a)\}$.

6. If $A$ is unital and $u$ is a unitary, then $\|u\|^2 = \|u^*u\| = 1$, hence $\text{sp}(u) \subseteq \mathbb{D}$ and
   $\text{sp}(u^*) \subseteq \mathbb{D}$. Therefore $\text{sp}(u) \subseteq \mathbb{T}$.

Exercise 1.11.19. Suppose $A$ is a $C^*$-algebra and $v \in A$. Prove that the following are equivalent.

1. Both $v^*v$ and $vv^*$ are projections (i.e., $v$ is a partial isometry).
2. $v = vv^*v$.
3. $v^*v$ is a projection.
4. $vv^*$ is a projection.

Exercise 1.11.20. Prove that the following are equivalent for every element $u$ of a unital $C^*$-algebra.

1. $u$ is both self-adjoint and a unitary.
2. $u = 1 - 2p$ for some projection $p$.
3. $u$ is a unitary involution: $uu^* = u^*u = u^2 = 1$.

Exercise 1.11.21. Prove that $a \in \text{GL}(A)$ is a unitary if and only if $\|a\| = \|a^{-1}\| = 1$.

Exercise 1.11.22. Suppose $n \geq 1$ and $\bar{a} := (a_j : j < n)$ is an $n$-tuple of commuting normal elements of a $C^*$-algebra. The joint spectrum, $\text{jsp}_A(\bar{a})$, is defined as the set of all $\lambda \in \mathbb{C}^n$ such that $\{\lambda_1 - a_1, \lambda_2 - a_2, \ldots, \lambda_n - a_n\}$ generates a proper ideal in $A$. Prove that $C^*(\bar{a}) \cong C_0(\text{jsp}(\bar{a})) \setminus \{(0, 0, \ldots, 0)\}$ and $C^*(\bar{a}, 1) \cong C(\text{jsp}(\bar{a}))$. 

In 1967 Kadison asked whether every simple and separable C*-algebra is singly generated. This problem is still open. Every separable C*-algebra is isomorphic to a subalgebra of a singly-generated C*-algebra; see [238] for recent results. The assertion that every separable and simple C*-algebra is singly generated is \( \Sigma_2^1 \)-statement and therefore absolute between transitive models of a large enough fragment ZFC that contain all countable ordinals (this is an exercise for the readers sufficiently familiar with Theorem B.2.12 and the paragraph preceding it).

**Exercise 1.11.23.** Prove that for every \( n \) there exists an abelian C*-algebra generated by \( n + 1 \) elements, but not by \( n \) elements.

**Exercise 1.11.24.** Prove that \( C([0, 1]) \) has an isomorphic copy of \( C([0, 1]^n) \) as a unital C*-subalgebra for all \( 1 \leq n \leq \aleph_0 \).

**Exercise 1.11.25.** Prove that every separable abelian C*-algebra is isomorphic to a subalgebra of a singly-generated abelian C*-algebra.

**Exercise 1.11.26.** Suppose \( u \) is a unitary and \( \text{sp}(u) \neq \mathbb{T} \). Find an \( a \in C^*(u)_{\text{sa}} \) such that \( \|a\| \leq 2\pi \) and \( u = \exp(ia) \).

**Exercise 1.11.27.** Prove that any two unitaries \( u \) and \( v \) such that \( \|u - v\| < 2 \) are homotopic.

**Exercise 1.11.28.** Suppose \( a, b, \) and \( c \) are elements of a unital C*-algebra \( A \) such that \( a \) and \( b \) are self-adjoint, \( c \) is invertible, and \( a =cbc^{-1} \). Prove that \( a = ubu^* \) for a unitary \( u \).

Two unitaries \( u_0 \) and \( u_1 \) in a C*-algebra \( A \) are homotopic if there exists a continuous \( f : [0, 1] \to U(A) \) such that \( f(0) = u_0 \) and \( f(1) = u_1 \).

**Exercise 1.11.31.** Prove that there exists \( \varepsilon > 0 \) such that for all C*-algebras \( A \) and projections \( p \) and \( q \) in \( A \) we have \( p \sim q \) if and only if there exists \( \varepsilon > 0 \) such that \( \|aa^* - p\| < \varepsilon \) and \( \|a^*a - q\| < \varepsilon \).

**Exercise 1.11.32.** Suppose \( n \geq 1 \) and \( p_1, \ldots, p_n \) are projections in \( B(H) \). Prove that the following are equivalent.

1. \( \|p_1 \cdots p_n\| = 1 \).
2. For every \( \varepsilon > 0 \) there exists a unit vector \( \xi \) such that \( \|p_j \xi\| \geq 1 - \varepsilon \) for all \( j \).

**Exercise 1.11.33.** 1. If an abelian C*-algebra has no nontrivial projections, then it has a C*-subalgebra isomorphic to \( C_0(\mathbb{R}) \).
2. If an abelian C*-algebra has no scalar projections, then it has a C*-subalgebra isomorphic to $C_0(\mathbb{R})$.

*Hint:* First reduce the problem to the case when $A$ is unital and separable. In addition to Theorem 1.3.2 and Theorem 1.4.2 in (2) you will need a standard fact from general topology: Every uncountable compact metrizable space maps continuously onto $[0,1]$.

Positive elements of a C*-algebra are *orthogonal* if their product is 0.

**Exercise 1.11.34.** Suppose $A$ is a C*-algebra, $n \geq 1$, $a_j \in A$, and $\lambda_j \in \mathbb{C}$, for $j < n$.

1. If $a_j$ are orthogonal positive elements, prove that $\|\sum_j \lambda_j a_j\| = \max_j |\lambda_j|\|a_j\|$.
2. Prove the same conclusion, $\|\sum_j \lambda_j a_j\| = \max_j |\lambda_j|\|a_j\|$, but assuming only that $a_j a_i^* = a_i^* a_j = 0$ for all $i \neq j$.
3. If $a_j$ are isometries with orthogonal range projections, prove that

$$\|\sum_j \lambda_j a_j\| = (\sum_j (|\lambda_j|\|a_j\|)^2)^{1/2}.$$  

**Exercise 1.11.35.** Suppose that $a \geq 0$ and $b \geq 0$. Prove that $\|a + b\| \geq \max(\|a\|, \|b\|)$.

**Exercise 1.11.36.** Suppose $0 \leq c \leq d$.

1. If in addition $0 \leq c' \leq d'$ prove that $\|c'bc\| \leq \|d'bd\|$ for all $b$.
2. If $u$ is a unitary, prove that $\|au\| = \|a\|$ for all $a$.

**Exercise 1.11.37.** Prove that $0 \leq a \leq b$ implies $0 \leq a^{1/2} \leq b^{1/2}$.

**Exercise 1.11.38.** Prove that $0 \leq a \leq b$ and $a \in \text{GL}(A)$ together imply $b \in \text{GL}(A)$ and $0 \leq b^{-1} \leq a^{-1}$.

**Exercise 1.11.39.** Suppose $H$ is a Hilbert space and $K$ is a closed subspace of $H$ such that $\dim(K) \leq \dim(K^\perp)$. Prove that for every $a \in \mathcal{B}(K)$ such that $0 \leq a \leq 1$ there exists a projection $q \in \mathcal{B}(H)$ such that $a = \text{proj}_K$. Such $q$ is the dilation of $a$.

*Hint:* If $K$ and $K^\perp \cap H$ are isometric try $\left(\frac{a}{\sqrt{a-a^2}}, \frac{\sqrt{a-a^2}}{1-a}\right)$.

**Exercise 1.11.40.** Suppose $H$ is a Hilbert space and $K$ is a closed subspace of $H$ such that $\dim(K) \leq \dim(K^\perp)$. Prove that for every $a \in \mathcal{B}(K)$ such that $\|a\| \leq 1$ there exists a unitary $u \in \mathcal{B}(H)$ such that $a = \text{proj}_K q \text{proj}_K$. Such $q$ is the unitary dilation of $a$.

*Hint:* If $K$ and $K^\perp$ are isometric try $\left(\frac{a}{\sqrt{a-a^2}}, \frac{\sqrt{a-a^2}a}{-a^*}\right)$.

**Exercise 1.11.41.** Prove that the elements of every C*-algebra satisfy the following.

1. The parallelogram identity: $(a+b)^*(a+b) + (a-b)^*(a-b) = 2(a^*a + b^*b)$.
2. The polarization identity: \( a^*b = \frac{1}{2} \sum_{j=0}^{3} i^j (b + i^j a)^*(b + i^j a) \).

3. \((a+b)^*(a+b) \leq 2(a^*a + b^*b)\).\(^{17}\)

4. \( \langle \sum_{j<n} a_j \rangle^*(\sum_{j<n} a_j) \leq n \sum_{j<n} a_j^* a_j \) for all \( n \geq 1 \).

5. If \( a_i a_j = 0 \) for \( i \neq j \) then \( \| \sum_{j<n} a_j b_j \| \| \sum_{j<n} a_j \| \leq \sum_{j<n} \| a_j b_j \| \) (Bessel’s inequality).

**Exercise 1.11.42.** Suppose \( A \) is unital, \( a \in A \), and \( b \in A \). Then

1. \[
\begin{pmatrix}
1 & a \\
\quad & 1
\end{pmatrix}
\]
   is non-negative in \( M_2(A) \) if and only if \( \|a\| \leq 1 \).

2. \[
\begin{pmatrix}
1 & a \\
\quad & 1
\end{pmatrix}
\]
   is non-negative in \( M_2(A) \) if and only if \( a^*a \leq b \).

The following exercise can be vastly generalized; it comes handy in some constructions of counterexamples (see e.g., Exercise 5.7.9).

**Exercise 1.11.43.** In \( M_2(\mathbb{C}) \), suppose \( p \) and \( q \) are distinct rank-1 projections and \( a \leq 1 \) is such that \( p \leq a \) and \( q \leq a \). Prove that \( a = 1_2 \).

**Exercise 1.11.44.** Suppose \( J \) is an ideal in \( A \) and \((e_\lambda : \lambda \in \Lambda)\) is an approximate unit in \( J \). Suppose that \( \pi : A \rightarrow \mathcal{B}(H) \) is a representation of \( A \) and \( K \) is the closure of \( \{\pi(a)e_\lambda : a \in J, e_\lambda \in H\} \). Prove that \( \pi(e_\lambda) \) converges to \( p_K \) in the strong operator topology.\(^{18}\)

**Exercise 1.11.45.** Prove that for every \( \sigma \)-unital ideal \( A \) of a \( C^* \)-algebra \( M \) and every \( X \subseteq M \) there exists an \( X \)-quasi-central approximate unit in \( A \) of cardinality \( |X| \).

**Exercise 1.11.46.** The **multiplicative domain** of a state \( \varphi \) on a \( C^* \)-algebra \( A \) is

\[
\mathcal{M}_\varphi := \{a \in A : \varphi(a^*a) = \varphi(a)^*\varphi(a)\}.
\]

Prove the following.

1. \( \mathcal{M}_\varphi = \{a \in A : \varphi(ab) = \varphi(a)\varphi(b) \text{ for all } b \in A\} \).

2. \( \mathcal{M}_\varphi \) is a \( C^* \)-subalgebra of \( A \).

3. \( \{a \in \mathcal{M}_\varphi : \varphi(a) = 1\} = \{a \in A : \varphi((1-a)^* (1-a)) = 0\} \).

**Exercise 1.11.47.** If \( A \) is an abelian \( C^* \)-algebra generated by its projections then a state \( \varphi \) of \( A \) is pure if and only if \( \varphi(p) \in \{0, 1\} \) for every projection \( p \) in \( A \).

**Exercise 1.11.48.** Suppose \( \pi : A \rightarrow \mathcal{B}(H) \) is a GNS representation of \( A \). Prove that the density character of \( H \) is not greater than the density character of \( A \).

**Exercise 1.11.49.** Suppose that \( B \) is a \( C^* \)-subalgebra of a \( C^* \)-algebra \( A \) and \( \pi : B \rightarrow \mathcal{B}(H) \) is a representation. Prove that there exist a Hilbert space \( K \supseteq H \) and a representation \( \sigma : A \rightarrow \mathcal{B}(K) \) such that the projection \( p_H \) of \( K \) onto \( H \) satisfies \( \pi(b) = p_H \sigma(b) p_H \).\(^{19}\)

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\(^{17}\) Used in the proof of Proposition 14.1.6.

\(^{18}\) Used in the proof of Lemma 1.8.4.

\(^{19}\) Used in the proof of Corollary 3.2.6.
1.11 Exercises

Exercise 1.11.50. For a Hilbert space $H$ classify all representations of $\mathcal{K}(H)$ on $\mathcal{B}(H)$ up to the spatial equivalence. More precisely, prove that the rank of $\pi(p)$, where $p \in \mathcal{K}(H)$ is a rank one projection, is a complete invariant for the spatial equivalence.

Exercise 1.11.51. Prove that every state $\phi$ on a C$^*$-algebra $A$ is uniquely determined by $\ker(\phi)$, and even by $\ker(\phi) \cap A_{sa}$.\footnote{Used in the proofs of Proposition 3.6.5 and Lemma 5.3.6.}

Hint: The abelian case is an exercise in measure theory (see Example 1.7.2). For the general case apply Exercise 1.11.16.

Exercise 1.11.52. Prove that projections $p$ and $q$ in a C$^*$-algebra $A$ satisfy $pq \neq 0$ if and only if $\phi(p) = 1$ and $\phi(q) > 0$ for some pure state $\phi$ of $A$.

Exercise 1.11.53. Prove the following for every C$^*$-algebra $A$, every state $\phi$ on $A$, and every $a \in A$.

1. $\|\pi\phi(a)\| = \sup \{ \phi(b^*a^*ab) : b \in A, \|b\| = 1 \}$.
2. If $\phi$ is GNS-faithful, then $\|a\|^2 = \sup \{ \phi(b^*a^*ab) : b \in A, \|b\| = 1 \}$.\footnote{Used in the proof of Lemma 4.3.2.}
3. $\|a\|^2 = \sup \psi \sup b \psi(b^*a^*ab)$, where the supremum is taken over all states $\psi$ and all $b \in A$ of norm 1.

Hint: Unravel the GNS.

Exercise 1.11.54. Prove that every normal element of a C$^*$-algebra has a norming functional which is a state. Then find a C$^*$-algebra $A$ and $a \in A$ without a norming functional which is a state.

Exercise 1.11.55. Suppose $J$ is a two-sided, norm-closed ideal of a C$^*$-algebra $A$ and $\phi$ is a state on $A$ which vanishes on $J$. Prove that $J \subseteq \ker \pi\phi$.

Exercise 1.11.56. If $\pi : A \to \mathcal{B}(H)$ is a representation and $p$ is a projection in $\mathcal{B}(H)$ then $p \in \pi(A)'$ if and only if $a \mapsto p\pi(a)p$ is a $^*$-homomorphism.

Exercise 1.11.57. Suppose $J$ is an ideal in a C$^*$-algebra $A$, $\pi : A \to \mathcal{B}(H)$ is a representation, and $\xi \in H$. Prove that the projection to the norm-closure of the orbit $\pi[J]\xi$ belongs to $\pi(A)'$.\footnote{Used in the proof of Lemma 1.8.4.}

Exercise 1.11.58. Suppose $A$ is a C$^*$-subalgebra of $\mathcal{B}(H)$. A vector $\xi \in H$ is cyclic if the orbit $A\xi$ is dense in $H$. A vector $\xi \in H$ is separating if $a\xi = 0$ if and only if $a = 0$ for $a \in A$. Prove that $\xi$ is a cyclic vector for $A$ if and only if it is a separating vector for $A'$. 
Notes for Chapter 1

§1.2 Example 1.2.12 (replacing the original, more involved, example, that used the free group algebra \( \mathbb{C}F_2 \) in the place of \( \mathbb{C}[x] \)) was suggested by N.C. Phillips.

The Gelfand transform can be defined for arbitrary complex Banach algebras; see e.g., [16, §1.9]. The conclusion of Exercise 1.11.1 is true for an arbitrary complex Banach algebra; this is the Gelfand–Mazur Theorem.

Lemma 1.3.4 shows that a homomorphism from an abelian \( \mathrm{C}^* \)-algebra into a \( \mathrm{C}^* \)-algebra is always continuous. It is not known whether it follows from ZFC that every homomorphism between \( \mathrm{C}^* \)-algebras is continuous (see the introduction to [190]). It is relatively consistent with ZFC that all Banach algebra homomorphisms are continuous ([51]) and that there exists a discontinuous homomorphism from a \( \mathrm{C}^* \)-algebra into a Banach algebra ([213]).

§1.5 The not-so-well-known proof of the well-known Lemma 1.5.7 (2) due to Halmos has been rescued by way of a MathOverflow post and [25].

§1.6 Proposition 1.6.9 is [194, Lemma 1.4.4] in a tuxedo. Proposition 1.6.14 is [158, Lemma 2.2].

§1.7 The notion of a multiplicative domain is standard and important (see [193]). The second part of Exercise 1.11.46 was taken from [167, Lemma 3.4].

§1.9 Quasi-central approximate units were introduced in [15]. Lemma 1.9.4 is [161, Lemma 4.1], and the tri-diagonal trick in its proof goes back to [252]; see also [35].

§1.10 The simple proofs of Lemma 1.10.7 and Theorem 1.10.8 were kindly provided by Narutaka Ozawa.
Chapter 2
Examples and Constructions of C*-algebras

Having the GNS construction under our belt, we proceed to study various ways in which the abstract C*-algebras can be put together to build new C*-algebras.

2.1 Putting the Building Blocks Together

In this section we introduce basic constructions of C*-algebras: direct sums and products, continuous fields, stabilization, suspension, cone, and hereditary C*-subalgebras.

Having proved that every abstract C*-algebra is isomorphic to a concrete C*-algebra, from this point on we will no longer be emphasizing the distinction between concrete and abstract C*-algebras.

2.1.1 Direct Sums

The Hilbert space sum (or the \(\ell_2\)-sum) of a family of Hilbert spaces \(H_j\), for \(j \in J\), is the vector space

\[
\bigoplus_{j \in J} H_j := \{ \xi \in \prod_{j \in J} : \sum_{j \in J} \|\xi(j)\|^2 < \infty \}.
\]

It is a Hilbert space with respect to the inner product \((\xi | \eta) := \sum_{j \in J} (\xi(j) | \eta(j))\). We will be using the obvious notational variations: \(H \oplus K\) and \(H^\oplus J\), if \(H = H_j\) for all \(j\).

A direct sum of C*-algebras \(A\) and \(B\) is the algebra of all pairs \((a,b)\) with the pointwise defined operations and norm defined by \(\|(a,b)\| = \max\{\|a\|,\|b\|\}\). If \(A\) and \(B\) are represented on Hilbert spaces \(H\) and \(K\), respectively, then \(A \oplus B\) has a natural representation on \(H \oplus K\) in which \((a,b)\) is identified with the block matrix

\[
\begin{pmatrix}
a & 0 \\
0 & b
\end{pmatrix}.
\]

The standard notation \(a + b\) for \((a,b)\) agrees with this representation.
when $a$ and $b$ are identified with the appropriate block matrices. A finite direct sum $\bigoplus_{j \in J} A_j$ of $C^*$-algebras is defined analogously. In general, the direct sum of $C^*$-algebras $A_i$, for $i \in J$, is

$$\bigoplus_{i \in J} A_i \doteq \{ (a_i : i \in J) : a_i \in A_i \text{ for all } i, \text{ and } \{ i : \|a_i\| > 1/n \} \text{ is finite for all } n \}.$$  

Operations are defined pointwise and norm is the supremum norm. if $A_j$ is represented on $H_j$ then $\bigoplus_{j \in J}$ is naturally represented on the $\ell_2$-direct sum $\bigoplus_{j \in J} H_j$. Alternatively, one can check that the direct sum satisfies the $C^*$-equality.

2.1.2 Direct products.

Given a family of $C^*$-algebras $A_i$, for $i \in J$, one defines

$$\prod_{i \in J} A_i = \{ (a_i : i \in J) : \sup_i \|a_i\| < \infty \}.$$  

The operations and the norm are defined pointwise. If the index-set $J$ is finite then $\prod_{i \in J} A_i = \bigoplus_{i \in J} A_i$. In general, if $A_j$ is represented on $H_j$ then $\prod_{i \in J} A_j$ is naturally represented on $\bigoplus_{j \in J} H_j$. It has $\bigoplus_{i \in J} A_i$ as a proper ideal if $J$ is infinite.

2.1.3 Inductive (direct) limits

An inductive system of $C^*$-algebras indexed by a directed set $\Lambda$ is a family $A_\lambda$, for $\lambda \in \Lambda$, of $C^*$-algebras together with is a commuting family of *-homomorphisms $f_{\lambda \eta} : A_\lambda \to A_\eta$, for $\lambda < \eta$ in $\Lambda$. Since each $f_{\lambda \eta}$ is a contraction (Lemma 1.2.10), for every $a \in A_\lambda$ the net $\|f_{\lambda \eta}(a)\|$, for $\eta \geq \lambda$, is non-increasing. Its limit defines a seminorm on the algebraic direct limit. It satisfies the $C^*$-equality and the completion of the quotient is the direct limit (or inductive limit) $C^*$-algebra, $\lim_\lambda A_\lambda$. If the connecting maps are injective then this seminorm is a norm.

2.1.4 $C^*$-algebras of continuous fields

Given a locally compact Hausdorff space $X$ and a $C^*$-algebra $A$, $C_0(X,A)$ is the algebra of all continuous functions $f : X \to A$ vanishing at the infinity with pointwise operations and the supremum norm. Clearly $C_0(X,\mathbb{C})$ is $C_0(X)$ and $C_0(X,M_\eta(\mathbb{C}))$ is easily seen to be isomorphic to $M_\eta(C_0(X))$ (see §2.1.5).

This is a very special case of a continuous field of $C^*$-algebras over a locally compact Hausdorff space (see [27, IV.1.6]).
2.1.5 Stabilization, suspension, and cone.

Given a $C^*$-algebra $A$ let $M_n(A)$ denote the algebra of $n \times n$ matrices over $A$, with the naturally defined algebraic operations. If $A$ is represented on Hilbert space $H$ then $M_n(A)$ can be identified with the $C^*$-algebra of operators on $\mathcal{B}(H^\oplus n)$ corresponding to $n \times n$ block matrices with entries in $A$. Corollary 1.10.5 implies that $M_n(A)$ has a unique $C^*$-norm.

Every $C^*$-algebra is canonically embedded into a unital $C^*$-algebra $\tilde{A}$ (Definition 1.2.5). We will describe three important canonical nonunital algebras containing $A$.

For $1 \leq m < n$ in $\mathbb{N}$ let $f_{mn}: M_m(A) \to M_n(A)$ denote the map $f_{mn}(a) := \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, where on the right-hand side we have a block matrix of size $n \times n$. The stabilization of $A$ is the inductive limit of the inductive system $M_n(A)$, for $n \geq 1$ with the connecting maps $f_{mn}$, for $1 \leq m < n$. Denoted $A \otimes K$ ($K$ stands for $K(\ell_2(\mathbb{N}))$, and for more on $\otimes$ see §2.4 below), it is nonunital and non-commutative.\footnote{This is a special case of the tensor product, defined in §2.4 below.}

If $A \cong M_k(\mathbb{C})$ for some $k \geq 1$ then $M_n(A) \cong M_{nk}(\mathbb{C})$ and $A \otimes \mathcal{K} \cong \mathcal{K}$. A moment of reflection shows that $\mathcal{K} \otimes \mathcal{K} \cong \mathcal{K}$; see also Exercise 2.8.6.

Definition 2.1.1. A $C^*$-algebra $A$ is stable if it is isomorphic to $A \otimes \mathcal{K}$.

Since $\mathcal{K}$ itself is stable, $A$ is stable if and only if it is isomorphic to $B \otimes \mathcal{K}$ for some $B$.

Example 2.1.2. 1. If $A$ is a nonseparable unital $C^*$-algebra then its stabilization is nonseparable, yet $\sigma$-unital: If $e_n$, for $n \in \mathbb{N}$, is an approximate unit of $\mathcal{K}$, then $1_A \otimes e_n$, for $n \in \mathbb{N}$, is an approximate unit of $A \otimes \mathcal{K}$.

2. An example of a stable, non-$\sigma$-unital $C^*$-algebra is $\mathcal{K}(\ell_2(\mathfrak{F}))$ (see Exercise 2.8.6).

Definition 2.1.3. The suspension of $A$ is

$$SA := C_0([0,1), A);$$

it is sometimes convenient to identify it with $C_0((0,1), A)$. The cone over $A$ is

$$CA := C_0((0,1], A).$$

Remark 2.1.4. Suppose $A$ is a $C^*$-algebra. Then $A$ is a quotient of $SA$ and of $CA$, for example via the $^*$-homomorphism $\Phi(f) := f(1/2)$. Both $CA$ and $SA$ are projectionless (i.e., have no nonzero projections) since every $\{0,1\}$-valued function on $(0,1)$ (or on $(0,1]$) vanishes. Therefore a $C^*$-algebra is rarely isomorphic to a $C^*$-subalgebra of its cone or its suspension.
2.1.6 Hereditary C*-subalgebras and corners

A C*-subalgebra $B$ of $A$ is hereditary if for every $b \in B_+$ and $a \leq b$ in $A_+$ we have $a \in B$. By her($\mathcal{X}$) we denote the hereditary C*-subalgebra of $A$ generated by $\mathcal{X} \subseteq A$. This is the intersection of all hereditary C*-subalgebras of $A$ containing $\mathcal{X}$.

A hereditary C*-subalgebra generated by $h \in A_+$, is the norm closure of $hAh$. If $h = p$ is a projection, this algebra is called a corner and it is equal to $pAp$. For example, $M_n(\mathbb{C})$ is a corner of $\mathcal{K}(H)$ for all $n$, and every corner of $\mathcal{K}(H)$ as defined in the previous sentence\(^2\) is isomorphic to $M_n(\mathbb{C})$ for some $n \geq 1$. A corner of the form $pAp$ for a projection $p$ in $A$ is a unital C*-algebra in its own right, but it is not a unital C*-subalgebra of $A$ unless it is equal to $A$.

2.2 Finite-Dimensional C*-algebras, AF algebras, and UHF algebras

In this section we classify the finite-dimensional C*-algebras and, using Bratteli diagrams, *-homomorphisms between them. This section concludes with a brief discussion of the inductive limits of finite-dimensional C*-algebras, UHF algebras and AF algebras.

We will give a complete classification of finite-dimensional C*-algebras. Then we will classify all *-homomorphism between finite-dimensional C*-algebras up to the relation of unitary equivalence. Concrete C*-algebra of all operators on the $n$-dimensional Hilbert space $\ell_2(n)$ is isomorphic to the algebra $M_n(\mathbb{C})$ of all $n \times n$ complex matrices. We can define this algebra abstractly as follows.

**Example 2.2.1.** Full matrix algebras are algebras $M_n(\mathbb{C})$, for $n \in \mathbb{N}$. The matrix units in $M_n(\mathbb{C})$ are any $e_{ij}$, for $i < n$ and $j < n$ that satisfy

\[ \|e_{ij}\| = 1, \quad e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad \text{and} \quad e_{ij}^* = e_{ji} \]

for all $i, j, k$, and $l$.\(^3\)

Suppose that an $n^2$-tuple of elements of a C*-algebra $A$ satisfies these relations; with a slight abuse of notation (and somewhat prematurely) we denote these elements by $e_{ij}$, for $i < n$ and $j < n$, and call them matrix units. Then every product of matrix units is equal to a matrix unit, and therefore $C^*(e_{ij} : i < n, j < n)$ is isomorphic to the linear span of these matrix units. This algebra is therefore isomorphic to $M_n(\mathbb{C})$ (and even isometrically isomorphic to $M_n(\mathbb{C})$ by Lemma 1.2.10). We conclude that $M_n(\mathbb{C})$ is, up to the isomorphism, the unique C*-algebra generated by elements satisfying these relations. A set of elements satisfying $(M_n)$ is a set of matrix units of type $M_n$.

\(^2\) The general definition of a corner will be given once we introduce the multiplier algebra (Definition 13.1.7). According to this definition, $\mathcal{K}(K)$ is a corner of $\mathcal{K}(H)$ for every closed subspace $K$ of $H$.

\(^3\) We will discuss relations, and universal algebras generated by relations, at length in §2.3.
Lemma 2.2.2. Every simple finite dimensional C*-algebra A is isomorphic to the full matrix algebra $M_n(\mathbb{C})$ for some $n$.

Proof. If $D$ is a maximal abelian C*-subalgebra of $A$ then Theorem 1.3.1 implies $D \cong \mathbb{C}^n$ for some $n$. If $p_j$, for $j < n$, denote the minimal nonzero projections in $D$ then $p_jA p_j \cong \mathbb{C}$ for all $j$ by the maximality of $D$. Since $A$ is simple we have $A_0A = A$ and therefore $p_0A p_j$ contains a nonzero element. Pick $v_j \in p_0A p_j$ such that $\|v_j\| = 1$. Then $v_j^*v_j$ is a positive element of norm 1 in $p_0A p_0$ and therefore $v_j^*v_j = p_0$. An analogous argument shows that $v_j^*v_j = p_j$ and $p_0A p_j = \{ \lambda v_j : \lambda \in \mathbb{C} \}$. The elements $e_{ij} := v_i^*v_j$ for $i < n$ and $j < n$ satisfy the matrix unit relations for $M_n(\mathbb{C})$. Since $\sum_{j<n} p_j = 1_A$, $C^*(e_{ij} : i, j < n)$ is a unital C*-subalgebra of $A$ isomorphic to $M_n(\mathbb{C})$. Any $a \in A$ satisfies $a = \sum_{j<n} p_j a \sum_{j<n} p_j = \sum_{i<n, j<n} p_i a p_j$ and is therefore in the linear span of the $e_{ij}$'s; therefore $A \cong M_n(\mathbb{C})$. \hfill $\square$

Lemma 2.2.3. Every finite-dimensional C*-algebra $A$ is isomorphic to a direct sum of full matrix algebras.

Proof. One could prove that C*-algebras are semisimple and apply the Artin–Wedderburn theorem, but we include a self-contained proof. The proof starts exactly like the proof of Lemma 2.2.2. Let $D$ be a maximal abelian C*-subalgebra of $A$. Then Theorem 1.3.1 implies that $D \cong \mathbb{C}^n$ for some $n$. Let $\mathcal{P}$ denote the set of minimal nonzero projections in $D$. The relation $E$ on $\mathcal{P}$ defined by $pE q$ if $pA q \neq \{0\}$ is an equivalence relation. Let $\mathcal{P}_i$, for $i < k$, be an enumeration of the equivalence classes. If $n(i) := |\mathcal{P}_i|$ then the algebra $B_i$ generated by $\mathcal{P}_i$ is isomorphic to $M_{n(i)}(\mathbb{C})$, as in the proof of Lemma 2.2.2. Projections $q_i := \sum_{p \in \mathcal{P}_i} p$ are orthogonal and $q_i A q_j = \{0\}$ for $i \neq j$. Therefore each $q_i$ belongs to the center of $A$, and the linear map $a \mapsto \sum_{i<k} q_i a q_i$ is an isomorphism of $A$ onto $\bigoplus_{i<k} M_{n(i)}(\mathbb{C})$; this completes the proof. \hfill $\square$

Lemma 2.2.4. There is a unital *-homomorphism from $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$ if and only if $m$ divides $n$. All unital *-homomorphisms from $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$ are unitarily equivalent.

Proof. Suppose $m \geq 1$, $n \geq 1$ and $m$ divides $n$. Define $\Phi_0 : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ by

$$\Phi_0(a) := \text{diag}(a, \ldots, a),$$

where $\text{diag}(a, \ldots, a)$ denotes the block-diagonal matrix with $k = n/m$ blocks of size $m \times m$. It is straightforward to check that $\Phi_0$ is a unital *-homomorphism.

Suppose $\Phi : M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital *-homomorphism. Then $\Phi[M_m(\mathbb{C})]$ is the algebra generated by $f_{ij} := \Phi(e_{ij})$, for $i, j < m$. Each $f_{ij}$ is a partial isometry, and projections $p_i := f_{ii} f_{ii}^*$ have the same rank and satisfy $\sum_{i<m} p_i = 1$. Therefore the rank of each $p_i$ is $n/m$, and $m$ divides $n$. The algebra $B := p_0 M_n(\mathbb{C}) p_0$ is isomorphic to $M_{m/n}(\mathbb{C})$, and

$$\Psi(b) := \sum_{i<m} f_{ii} b f_{ii}^*$$

defines a unital *-homomorphism $\Psi : B \to M_n(\mathbb{C})$. All elements in $\Psi[B]$ clearly commute with each $f_{ii}$, and $f_{ij} = f_{ii} f_{jj}^*$ for all $i$ and $j$. Hence $\Phi[M_m(\mathbb{C})]$ and $\Psi[B]$
are commuting unital C*-subalgebras of $M_n(\mathbb{C})$ isomorphic to $M_m(\mathbb{C})$ and $M_{m/n}(\mathbb{C})$, respectively. Also, $C^*(\Phi[M_m(\mathbb{C})],\Psi[B])$ is an $n^2$-dimensional C*-subalgebra of $M_n(\mathbb{C})$, and therefore equal to it.

Therefore $M_n(\mathbb{C})$ can be identified with $M_n(\mathbb{C}) \odot M_{m/n}(\mathbb{C})$. By Corollary 1.3.3, the latter algebra has a unique C*-norm. We have proved that for every unital $^*$-homomorphism of $M_m(\mathbb{C})$ into $M_n(\mathbb{C})$ the image of $M_m(\mathbb{C})$ is a tensor factor of $M_n(\mathbb{C})$. Since every automorphism of $M_n(\mathbb{C})$ is inner, the conclusion follows. □

**Lemma 2.2.5.** Every (unital or not) $^*$-homomorphism $\Phi: M_m(\mathbb{C}) \to M_n(\mathbb{C})$ is uniquely determined up to the unitary equivalence by its multiplicity, defined to be the rank of $\Phi(e_{11})$.

**Proof.** Let $p := \Phi(1_m)$. By Lemma 2.2.4 the rank of $p$, denoted $n'$, is divisible by $n$ and unital $^*$-homomorphism $\Phi: M_n(\mathbb{C}) \to pM_n(\mathbb{C})p$ is unique up to the unitary equivalence. Since any two projections of the same rank in $M_n(\mathbb{C})$ are unitarily equivalent, this concludes the proof. □

By Lemma 2.2.3 every finite dimensional C*-algebra $A$ is isomorphic to one of the form $\bigoplus_{i \in I} M_{m(i)}(\mathbb{C})$, and therefore uniquely determined by $l$ and the $l$-tuple $(n(0), \ldots, n(l-1))$.

**Lemma 2.2.6.** Every $^*$-homomorphism (unital or not) between finite-dimensional C*-algebras $\Phi: \bigoplus_{j \in J} M_{m(j)}(\mathbb{C}) \to \bigoplus_{j \in J} M_{n(j)}(\mathbb{C})$ is uniquely determined, up to the unitary equivalence, by a matrix $k(i,j)$, for $i < l'$ and $j < l$, of natural numbers.

**Proof.** We first consider the case when $l' = 1$. By Lemma 2.2.5, a $^*$-homomorphism $\Phi: M_{m(0)}(\mathbb{C}) \to \bigoplus_{j \in J} M_{m(j)}(\mathbb{C})$ is uniquely determined up to the unitary equivalence by the $l$-tuple $(k(0), \ldots, k(l-1))$, where $k(j)$ is multiplicity of the composition of $\Phi$ with the projection to $M_{n(j)}(\mathbb{C})$. For $i < l'$ let $\Phi_i: M_{m(i)}(\mathbb{C}) \to \bigoplus_{j \in J} M_{n(j)}(\mathbb{C})$ be the restriction of $\Phi$ to $M_{m(i)}(\mathbb{C})$. By the first paragraph $\Phi_i$ is uniquely determined by the $l$-tuple of multiplicities $(k(i,0), \ldots, k(i,l-1))$. Since the ranges of $\Phi_i$ for $0 \leq i < l'$ are mutually orthogonal, the conclusion follows. □

A $^*$-homomorphism between finite-dimensional C*-algebras can be visualized by a Bratteli diagram. A **Bratteli diagram** of a $^*$-homomorphism

$$\Phi: \bigoplus_{j \in J} M_{m(j)}(\mathbb{C}) \to \bigoplus_{j \in J} M_{n(j)}(\mathbb{C})$$

is a bipartite graph whose vertices $v_0, \ldots, v_{l'-1}$ on the left correspond to the direct summands of $\bigoplus_{j \in J} M_{m(j)}(\mathbb{C})$ and whose vertices $w_0, \ldots, w_{l-1}$ on the right correspond to the direct summands of $\bigoplus_{j \in J} M_{n(j)}(\mathbb{C})$. The vertices $v_i$ and $w_j$ are connected by $k(i,j)$ edges, with $k(i,j)$ denoting the multiplicity as in Lemma 2.2.6. Alternatively, if $k(i,j) > 0$ then there is a single edge between $v_i$ and $w_j$ labelled $k(i,j)$. The vertices may also be labelled by the sizes of the corresponding algebras; in our example $v_i$ would be labelled by $m(i)$ and $w_j$ would be labelled by $n(j)$, as in the following diagram (for simplicity of the diagram we are assuming that $k(i,j) = 0$ for many pairs $i, j$).

The proof of the following lemma is an easy computation.
Lemma 2.2.7. The bipartite graph described in the previous paragraph is a Bratteli diagram of a \(\ast\)-homomorphism if and only if \(\sum_{i<l}^{} m(i) k(i, j) \leq n(j)\) for all \(j < l\).

It is unital if and only if \(\sum_{i<l}^{} m(i) k(i, j) = n(j)\) for all \(j < l\). \(\square\)

2.2.1 UHF Algebras and AF Algebras

A \(C\ast\)-algebra is **approximately finite** (or AF) if it is an inductive limit of finite-dimensional \(C\ast\)-algebras. A separable AF algebra is an inductive limit of the form \(A = \lim_{n\in\mathbb{N}} A_n\). The **Bratteli diagram** of \(A\) is the \(\aleph_0\)-partite diagram with vertex set \(\bigcup_n V_n\) such that the vertices in \(V_n\) correspond to the direct summands of \(A_n\) and the induced graph on \(V_n \cup V_{n+1}\) is the Bratteli diagram of the connecting map \(\Phi_n: A_n \to A_{n+1}\). Since the label of a vertex corresponds to the size of the matrix algebra attached to it, Lemma 2.2.7 implies that if \(\Phi_n\) is unital then the labels of the vertices in \(V_n\) uniquely determine the labels of the vertices in \(V_{n+1}\). If \(A\) is unital then we may assume all \(\Phi_n\) are unital and let \(A_0 = \mathbb{C}\). Therefore in this case the label of every vertex in \(V_n\) is uniquely determined by the multiplicities of the maps.

Consider the Bratteli diagram of a separable, unital, AF algebra in Figure 2.1.

![Fig. 2.1 The Bratteli diagram of \(M_2\ast\)](image)

It describes a unital inductive system \(M_2(\mathbb{C}) \to M_4(\mathbb{C}) \to M_8(\mathbb{C}) \ldots\) whose limit is denoted \(M_2\ast\). As evident from Theorem 3.7.2 and from the diversity of the names under which it is known: the CAR (Canonical Anticommutation Relation) algebra, or the Fermion algebra, this is one of the most important \(C\ast\)-algebras.

Figure 2.2 gives another example of a Bratteli diagram.

It describes the unital inductive system \(M_3(\mathbb{C}) \to M_9(\mathbb{C}) \to M_{27}(\mathbb{C}) \ldots\) whose limit is denoted \(M_3\ast\). A \(C\ast\)-algebra is **approximately matricial** (or AM) if it is a unital inductive limit of full matrix \(C\ast\)-algebras. Unlike ‘AF algebras’ and ‘UHF algebras’ defined in the following paragraph, this terminology is uncommon.
Fig. 2.2 The Bratteli diagram of $M_3^\infty$.

A C*-algebra is uniformly hyperfinite (shortly UHF) if it is a tensor product of infinitely many (possibly uncountably many) full matrix algebras. The CAR algebra is, being isomorphic to $\bigotimes_{n=1}^\infty M_2(\mathbb{C})$, UHF and so is $M_3^\infty \cong \bigotimes_{n=1}^\infty M_3(\mathbb{C})$. Along with these examples, $M_n^\infty$ for $n \geq 5$ (clearly $M_2^\infty \cong M_4^\infty$), a notable example is the rational UHF algebra $\bigotimes_{n=1}^\infty M_n(\mathbb{C})$. Some authors denote it by $\mathcal{D}$ or $Q$.

Let $D_n$ denote the C*-subalgebra consisting of all diagonal matrices in $M_n(\mathbb{C})$. Then $D_n$ is a maximal abelian C*-subalgebra of $M_n(\mathbb{C})$ isomorphic to $\mathbb{C}^n$. In the CAR algebra, $\lim_n M_2^n(\mathbb{C})$, the inductive limit of $D_2^n$, is isomorphic to $C(\{0,1\}^\mathbb{N})$. This inductive limit is a maximal abelian C*-subalgebra (Exercise 12.6.17), and the CAR algebra can be considered as a noncommutative analog of the Cantor space. The analogously defined C*-subalgebra of $M_3^\infty$ is isomorphic to $C(\{0,1,2\}^\mathbb{N})$. Since every compact, zero-dimensional, metrizable space is homeomorphic to the Cantor space, the algebras $C(\{0,1,2\}^\mathbb{N})$ and $C(\{0,1\}^\mathbb{N})$ are isomorphic. However, $M_2^\infty$ and $M_3^\infty$ are not isomorphic (see Exercise 2.8.13).

### 2.3 Universal C*-Algebras Defined by Bounded Relations

In this section we introduce C*-algebras given by generators and relations, and give several examples: Finite-dimensional C*-algebras, algebras generated by a positive contraction, Cuntz algebras $\mathcal{O}_n$, and the algebras generated by a nilpotent of degree two. We introduce the notion of weak stability. This is also the section in which logic makes its first (implicit) appearance and our exposition starts to diverge from the standard one, if only slightly.

Numerous important examples of C*-algebras are universal C*-algebras given by generators and relations. ‘Generators,’ denoted $\mathcal{G}$, comprise a set of variables, and there doesn’t seem to be a consensus on what ‘relations,’ denoted $\mathcal{R}$, between the generators are exactly (see [171, §3], [27, II.8.3.3]). Our definition of the universal C*-algebra given by generators and relations is naturally influenced by model theory of metric structures. The readers familiar with it will notice that relations defined in (C) below are quantifier-free closed conditions of the language of C*-algebras (Definition 15.2.1, [87, §2.2, §4.1]). This definition is also closely related to degree-1 conditions (Definition 15.1.1).

For definiteness, we will adopt the following convention. Generators are variables, each of which is provided with a bound on its norm. In other words, to every generator $x$ we attach a relation of the form $\|x\| \leq r$ for some constant $r < \infty$. We will write $\bar{x}$ for a tuple of generators of an unspecified length $n$, $\bar{x} = (x_0, \ldots, x_{n-1})$. Terms are expressions of the form $p(\bar{x})$ where $p$ is a complex $^*$-polynomial in non-commuting generators $\bar{x}$ (recall that the adjoints of the generators are non-
commuting as well, Definition 1.2.3). Relations (or conditions) in generators \( \mathcal{G} \) are expressions of the following form.

(C) \( f(\|t_0\|, \ldots, \|t_{n-1}\|) \in K \), where \( t_0, \ldots, t_{n-1} \) is an \( n \)-tuple of terms for some \( n \geq 1 \), \( f : \mathbb{R}^n \to [0, \infty) \) is continuous, and \( K \subseteq [0, \infty) \) is compact.

We introduce some handy abbreviations.

**Lemma 2.3.1.** Each of the following is equivalent to a condition as defined in (C).

1. \( t_0 = t_1 \), where \( t_0 \) and \( t_1 \) are arbitrary terms.
2. \( x \) is self-adjoint (normal, isometry, partial isometry, coisometry, unitary, projection, . . . ),
3. \( x = x^* \) and \( x \geq 0 \).
4. \( xx^* = x^*x \) and \( \text{sp}(x) \subseteq K \), for compact \( K \subseteq \mathbb{C} \).

**Proof.** (1) The condition \( \|t_0 - t_1\| = 0 \) is equivalent to \( t_0 = t_1 \).

(2) A special case of (1), given the definitions from Definition 1.4.1.

The condition (3) is an abbreviation for \( x = y^*y \) where \( y \) is a new generator (a generator is ‘new’ if it does not appear in the set \( \mathcal{G} \) ). Lemma 1.6.4 implies that \( x \geq 0 \) is equivalent to ‘there exists \( y \) such that \( x = y^*y \).’ Therefore if \( y \notin \mathcal{G} \) and \( x \in \mathcal{G} \) then the universal \( \mathcal{C}^* \)-algebra given by \( \mathcal{G} \cup \{ y \} \) and \( \mathcal{R} \cup \{ x = y^*y, \|y\| \leq r^{1/2} \} \) is the same as the universal \( \mathcal{C}^* \)-algebra given by \( \mathcal{G} \) and \( \mathcal{R} \cup \{ x = x^*, x \geq 0, \|x\| \leq r \} \).

To prove that condition (4) is also an abbreviation, we will use the following fact: If \( A \) is a unital \( \mathcal{C}^* \)-algebra, \( a \in A \) is normal, and \( \lambda \in \mathbb{C} \setminus \text{sp}(a) \), then (see Exercise 1.11.15)

\[
\|(a - \lambda)^{-1}\| = \text{dist}(\lambda, \text{sp}(a)).
\]

Cover \( \mathbb{C} \setminus K \) with open disks \( D_n \), for \( n \in \mathbb{N} \). For each \( n \in \mathbb{N} \) add a new generator \( y_n \) to \( \mathcal{G} \). If \( D_n \) is the disk with center \( \lambda_n \) and radius \( r_n \), then add relations \( y_n(x - \lambda_n) = 1 \) and \( \|y_n\| \leq r_n^{-1} \). Therefore if \( y_n \notin \mathcal{G} \) for all \( n \in \mathbb{N} \) and \( x \in \mathcal{G} \) then the universal \( \mathcal{C}^* \)-algebra given by \( \mathcal{G} \cup \{ y_n : n \in \mathbb{N} \} \) and \( \mathcal{R} \cup \{ y_n(x - \lambda_n) = 1, \|y_n\| \leq r_n \} \) is the same as the universal \( \mathcal{C}^* \)-algebra given by \( \mathcal{G} \) and \( \mathcal{R} \cup \{ xx^* = x^*x, \text{sp}(x) \subseteq K \} \). \( \square \)

The condition (4) cannot be weakened by dropping the requirement that \( x \) be normal while retaining the conclusion of Lemma 2.3.1 (see the example in [171, p. 24]).

In some references (e.g., in [171]) separate definitions are given for universal \( \mathcal{C}^* \)-algebras in unital and nonunital categories. We instead introduce the convention that if the set of relations involves constant symbol 1 then the following definition is interpreted in the category of unital \( \mathcal{C}^* \)-algebras and that it is interpreted in the category of all \( \mathcal{C}^* \)-algebras otherwise.

**Definition 2.3.2.** The *universal \( \mathcal{C}^* \)-algebra* given by generators \( \mathcal{G} \) and relations \( \mathcal{R} \) is the \( \mathcal{C}^* \)-algebra \( \mathcal{C}^*(\mathcal{G}|\mathcal{R}) \) such that there exists an interpretation \( \tau : \mathcal{G} \to A \) satisfying the following.

5. Every relation \( R(\bar{x}) \) in \( \mathcal{R} \) is satisfied in \( A \) by \( \tau(\bar{x}) \); in symbols, \( R^A(\tau(\bar{x})) \) for all \( R(\bar{x}) \in \mathcal{R} \), or \( \mathcal{R}^A(\tau(\bar{x})) \).
6. \(A = C^* (t[\mathcal{G}])\).
7. If \(C^*-\)algebra \(B\) and \(t' : \mathcal{G} \rightarrow B\) satisfy (5) and (6) then there exists a (necessarily surjective) \(^*\)-homomorphism \(\Phi : A \rightarrow B\) such that \(\Phi \circ t = t'\).

The algebra satisfying (5) and (6), if it exists, is by the universality unique up to an isomorphism. It is called the \textit{universal algebra given by generators} \(\mathcal{G}\) \textit{and relations} \(\mathcal{R}\).

Every \(C^*-\)algebra \(A\) is the universal \(C^*-\)algebra given by some generators and relations: Introduce a generator \(x_a\) for every \(a \in A\), and conditions determining \(\|p(x_{\bar{a}})\|\) for all \(^*\)-polynomials \(p\) in non-commuting variables and all tuples \(\bar{a}\) in \(A\) of the appropriate length. More relevantly, numerous important \(C^*-\)algebras are given by a finite set of generators and a finite and natural set of relations between them.

Example 2.3.3. 1. For \(n \geq 1\), \(\mathbb{C}^n\) is the universal unital \(C^*-\)algebra generated by \(n\) orthogonal projections whose sum is 1 (the obvious relations are omitted).
2. Let \(\mathcal{G} := \{e_{ij} : 0 < i < n, 0 < j < n\}\) and
\[
\mathcal{R} := \{\|e_{ij}\| = 1, e_{ij}e_{kl} = \delta_{jk}e_{il}, e_{lj}^* = e_{li}^* : \text{for all } i, j, k, l \text{ in } \{0, \ldots, n - 1\}\}.
\]
Then \(C^* (\mathcal{G} | \mathcal{R}) = M_n (\mathbb{C})\) since \(M_n (\mathbb{C})\) is the unique \(C^*-\)algebra whose generators satisfy these relations. As in Example 2.2.1, we say that these generators are matrix units of type \(M_n\).
3. Every finite-dimensional \(C^*-\)algebra \(D\) is isomorphic to a direct sum of full matrix algebras (Lemma 2.2.2), \(D = \bigoplus_{k \leq m} M_{n(k)} (\mathbb{C})\). Fix \(k < m\) and let \(e_{ij}^k\), for \(i < n(k)\) and \(j < n(k)\), be the matrix units in \(M_{n(k)} (\mathbb{C})\). Then \(\{e_{ij}^k\}\) are a set of matrix units of type \(M_{n(k)}\). To the union of these relations over all \(k < m\) add the relation \(e_{ij}^k e_{i' j'}^k = 0\) for all \(k \neq l\) and all \(i, j, i', j'\). Generators satisfying these relations are a \textit{set of matrix units of type} \(D\). The universal \(C^*-\)algebra given by these generators and relations is isomorphic to \(D\).

Definition 2.3.4. A nonzero element \(x\) of a \(C^*-\)algebra is \textit{nilpotent} if \(x^n = 0\) for some \(n\). The minimal such \(n\) is the \textit{degree} of \(x\).

Since \(M_2 (\mathbb{C})\) will be used as the building block for some of our examples of non-separable \(C^*-\)algebras, we provide two additional representations of it as a universal \(C^*-\)algebra.

Example 2.3.5. The universal unital \(C^*-\)algebra given by a generator \(v\) which satisfies the relations \(vv^* + v^*v = 1\) and \((v^*v)^2 = v^*v\) is isomorphic to \(M_2 (\mathbb{C})\). Clearly \(t (v) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\) satisfies the relations and it generates \(M_2 (\mathbb{C})\).

Suppose that a \(C^*-\)algebra \(B\) is generated by \(v\) that satisfies the relations. Then \(v^*v\) is a projection, and \(v\) is a partial isometry which satisfies \(vv^* + v^*v = 1\). Since \(vv^*\) and \(v^*v\) are projections whose sum is a projection, they are orthogonal. Therefore \(\|v^2\| \leq \|v^*v\|^2 = 0\) and \(v\) is a nilpotent of degree 2. This implies that \(B\) is spanned by \(vv^*, v, v^*, v^*v\) (the four nonzero words in the alphabet \(\{v, v^*\}\)). These elements
satisfy the same relations as the matrix units $e_{11}, e_{12}, e_{21}$, and $e_{22}$. Therefore their linear span is a $C^*$-algebra isomorphic to $M_2(\mathbb{C})$.

**Example 2.3.6.** The universal unital $C^*$-algebra given by self-adjoint unitaries $u$ and $v$ and relations $uv = -vu$ (such $u$ and $v$ are said to be *anticommuting*). Then the unitaries $t(u) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $t(v) := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfy the relations and generate $M_2(\mathbb{C})$. Conversely, any algebra $B$ generated by anticommuting self-adjoint unitaries $u$ and $v$ is the linear span of $u, v, \overline{uv}$ and $u^2 = v^2 = 1$, and therefore isomorphic to $M_2(\mathbb{C})$.

We have seen examples in which $C^*(\mathcal{G}|\mathcal{R})$ is the unique $C^*$-algebra whose generators satisfy the relations $\mathcal{R}$. This is not the case for the $C^*$-algebra in the following example (cf. Exercise 2.8.22).

**Example 2.3.7.** The universal $C^*$-algebra defined by a positive contraction, i.e., $x$ satisfying $0 \leq x \leq 1$ is $C_0((0, 1])$. Clearly the identity function on $(0, 1]$ generates $C_0((0, 1])$ and satisfies the relations. Now suppose that $B$ is generated by a positive contraction $y$. Then $\text{sp}(y) \subseteq [0, 1]$ and the restriction of $f \in C_0((0, 1])$ to $\text{sp}(y)$ is a surjective *-homomorphism from $C_0((0, 1])$ onto $B$.

In order to describe the universal $C^*$-algebra given by a nilpotent of degree two we will need the following lemma.

**Lemma 2.3.8.** If $w$ is nilpotent of degree two then

$$C^*(w) \cong C_0(\text{sp}(w^*w) \setminus \{0\}, M_2(\mathbb{C})).$$

**Proof.** Let $a := w^*w$. The algebra $A := C^*(w)$ is the closed linear span of elements of the form $a^{m+1}, wa^m, a^mw^*$, and $wa^mw^*$ for $m \geq 0$. Theorem 1.4.2 implies that $C^*(a)$ is isomorphic to $C_0(\text{sp}(a) \setminus \{0\})$ and the isomorphism sends $a$ to $b := \text{id}_{C_0(\text{sp}(w^*w))}$.

Define $f : C^*(w) \to C_0(\text{sp}(w^*w) \setminus \{0\}, M_2(\mathbb{C}))$ by the images of the generators by $f(a^{m+1}) := \begin{pmatrix} b^{m+1} & 0 \\ 0 & 0 \end{pmatrix}$, $f(a^mw^*) := \begin{pmatrix} 0 & b^{m+1/2} \\ 0 & 0 \end{pmatrix}$, $f(wa^m) := \begin{pmatrix} 0 & 0 \\ 0 & b^{m+1/2} \end{pmatrix}$, and $f(wa^mw^*) := \begin{pmatrix} 0 & 0 \\ 0 & b^{m+1} \end{pmatrix}$ for all $m$. A computation shows that this defines a *-homomorphism from $A$ onto $C_0(\text{sp}(w^*w) \setminus \{0\}, M_2(\mathbb{C}))$. $\square$

**Example 2.3.9.** The universal $C^*$-algebra defined by a nilpotent contraction of degree two. Let $\mathcal{G} := \{x\}$ and $\mathcal{R} := \{\|x\| = 1, \|x^2\| = 0\}$. By Lemma 2.3.8, $C^*(\mathcal{G}|\mathcal{R})$ is isomorphic to $C_0((0, 1], M_2(\mathbb{C}))$.

**Example 2.3.10.** For $n \geq 2$ the Cuntz algebra $\Theta_n$ is the universal unital $C^*$-algebra generated by $n$ isometries $s_j$, for $j < n$, such that $\sum_{j<n} s_j s_j^* = 1$. Remarkably, $\Theta_n$ is the unique $C^*$-algebra generated by isometries satisfying these relations, and in particular it is simple (see [49]).

Not every set of relations corresponds to a $C^*$-algebra. One example is given by $\{\|x^4\| = 1/2, \|x\| = 1\}$. A more intriguing (and more famous) example is

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4 These relations can be realized only by unbounded self-adjoint operators; see e.g., [126, p. 63].
\[ \{ \|x\| \leq 1, \|y\| \leq 1, x = x^*, y = y^*, xy - yx = i \} \]

Borrowing the terminology from model theory, we say that a set of relations \( \mathcal{R} \) in generators \( \mathcal{G} \) is **satisfiable** if there exists a C*-algebra \( A \) and \( t: \mathcal{G} \to A \) such that \( \mathcal{R}(t(\vec{x})) \) holds for all \( R(\vec{x}) \in \mathcal{R} \). If such an \( A \) exists we may assume \( A = C^*(t[\mathcal{G}]) \), in which case we say that the pair \((t, A)\) is an *interpretation* of \( \mathcal{R} \).

**Lemma 2.3.11.** Suppose \( \mathcal{G} \) and \( \mathcal{R} \) are generators and relations. The following are equivalent.

1. \( C^*(\mathcal{G}|\mathcal{R}) \) exists.
2. \( \mathcal{R} \) is satisfiable.
3. Every finite subset of \( \mathcal{R} \) is satisfiable.

The reader should be warned that the equivalence of (2) and (3) is a feature of our (nonstandard) requirement that every generator \( x \) be equipped with an upper bound on its norm.

**Proof.** Obviously (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3).

(3) \( \Rightarrow \) (1): Suppose that (3) holds. Therefore every \( \mathcal{P} \in \mathcal{R} \) is satisfiable and \( A_{\mathcal{P}} := C^*(\mathcal{G}|\mathcal{P}) \) exists. Consider the set \( [\mathcal{R}]^{<\aleph_0} \) of all finite subsets of \( \mathcal{R} \). By the universality, if \( \mathcal{P} \subseteq \mathcal{L} \) then there is a *-homomorphism \( \Phi_{\mathcal{P}}: A_{\mathcal{P}} \to A_{\mathcal{L}} \) that sends generators to generators. These *-homomorphisms commute, and by universality the inductive limit of this directed system is \( C^*(\mathcal{G}|\mathcal{R}) \). □

We end this section with an important property of relations; it will be discussed in Exercises 2.8.10–2.8.12.

**Definition 2.3.12.** We now define when a set of relations \( \mathcal{R} \) in generators \( \mathcal{G} \) is **weakly stable**. To a relation \( R(\vec{x}) \) and \( \varepsilon > 0 \) one first associates the approximate relation \( R(\vec{x})_\varepsilon \). If \( R(\vec{x}) \) is \( \|r(\vec{x})\| \in K \) then the corresponding approximate relation is \( \text{dist}(\|t(\vec{x})\|, K) < \varepsilon \), and if \( R(\vec{x}) = f(t(\vec{x})) \in K \) then the corresponding approximate relation is \( \text{dist}(f(t(\vec{x})), K) < \varepsilon \).

A relation \( R(\vec{x}) \) is **weakly stable** if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that in every C*-algebra \( A \) the following holds. If \( R(\vec{x})_\delta(\vec{a}) \) holds for some \( \vec{a} \) then there exists \( \vec{b} \in A \) such that \( \text{dist}(\vec{b}, \vec{a}) < \varepsilon \) and \( R(\vec{x})_\varepsilon(\vec{b}) \) holds. A set of relations \( \mathcal{R} \) is weakly stable if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that in every C*-algebra \( A \) the following holds. If \( R(\vec{x})_\delta(\vec{a}) \) holds for some \( \vec{a} \) and all \( R \in \mathcal{R} \) then there exists \( \vec{b} \in A \) such that \( \text{dist}(\vec{b}, \vec{a}) < \varepsilon \) and \( R(\vec{x})_\varepsilon(\vec{b}) \) holds.

Continuous functional calculus (Theorem 1.4.2) implies that if \( a \) is a normal element of a C*-algebra and \( f \in C(\text{sp}(a)) \) then \( \|f(a)\| = \sup_{t \in \text{sp}(a)} \|f(t)\| \). This can be used to prove weak stability of some relations.

The standard reference for weak stability and its variants is [171]. Weak stability is closely related to definability in logic of metric structures (see [22] and [87]).
2.4 Tensor Products, Group Algebras, and Crossed Products

In addition to the three constructions mentioned in the title, in this section we introduce nuclear C*-algebras and prove that the AF algebras are nuclear.

Tensor Products

The theory of tensor products in the category of C*-algebras is a rich source of examples, ideas, and sometimes (because of its high complexity) frustrations. The introduction to tensor products provided here is certainly insufficient for a novice; an excellent source is [35, Chapter 3]. The algebraic tensor product $A \otimes B$ of C*-algebras over $\mathbb{C}$ is equipped with a *-algebra structure by defining multiplication and adjoint operations on the elementary tensors by

$$(a \otimes b)(a' \otimes b') := aa' \otimes bb'$$

and

$$(a \otimes b)^* := a^* \otimes b^*.$$ 

Example 2.4.1. 1. For every C*-algebra $A$, the algebraic tensor product $A \otimes M_n(\mathbb{C})$ is *-isomorphic to $M_n(A)$ (see Corollary 1.10.5). It is therefore equipped with a natural C*-norm. Since $M_n(\mathbb{C})$ is finite-dimensional, $A \otimes M_n(\mathbb{C})$ is complete in this norm and by Corollary 1.10.5 this is the unique C*-norm on $A \otimes M_n(\mathbb{C})$.

2. The algebraic tensor product $B(H) \otimes B(K)$ can be identified with a subalgebra of $B(H \otimes K)$ by Exercise 2.8.20. This identification provides $B(H) \otimes B(K)$ with a tensor product norm. This norm is not unique ([143]).

Suppose $A$ and $B$ are C*-algebras, and $\pi : A \rightarrow B(H)$ and $\sigma : B \rightarrow B(K)$ are representations. The representation $\pi \otimes 1$ of $A$ on $B(H \otimes K)$ defined by

$$(\pi \otimes 1)(a) := \pi(a) \otimes 1_{B(K)}$$

is the amplification of $\pi$ by $K$. The amplification $1 \otimes \sigma : B \rightarrow B(H \otimes K)$ of $\sigma$ by $H$ is defined analogously. Suppose $\pi$ and $\sigma$ are faithful and consider the C*-algebra generated by the images of the elementary tensors, $(\pi \otimes 1)(a)(1 \otimes \sigma)(b)$. This algebra is generated by commuting copies of $A$ and $B$. This provides the algebraic tensor product $A \otimes B$, with a tensor product norm. Remarkably, this norm does not depend on the choice of faithful representations of $A$ and $B$ ([35, Proposition 3.3.11] also Exercise 5.7.10). It is the spatial tensor product of $A$ and $B$, and it will be denoted by $A \otimes B$ throughout.

A construction analogous to that of the universal C*-algebra given by generators and relations in Lemma 2.3.11 proves the existence of a maximal tensor product norm on $A \otimes B$. The corresponding C*-algebra is called the maximal tensor product of $A$ and $B$ and denoted $A \otimes_{\text{max}} B$. By the universality, every other C*-algebra completion of $A \otimes B$ (there can be quite a variety of them; see [192]) is a quotient of $A \otimes_{\text{max}} B$. A bit deeper is the fact that the spatial tensor product of $A$ and $B$ is a...
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quotient of any other C*-algebra completion of $A \otimes_B B$. Because of this, the spatial tensor product is sometimes called the minimal tensor product and denoted $\otimes_{\text{min}}$.

If a C*-algebra $A$ is unital then any C*-algebra $B$ can be identified with the C*-subalgebra $1_A \otimes B$ of $A \otimes B$. By induction one proves that the spatial tensor product norm of finitely many C*-algebras is uniquely defined and associative. If $A_j$, for $j \in J$, is a family of unital C*-algebras then we can define the tensor product $\bigotimes_{j \in J} A_j$ as the inductive limit of $\bigotimes_{j \in F} A_j$ for $F \subseteq J$.

The CAR algebra defined in §2.2.1 is isomorphic to the infinite tensor product $M_2^\infty := \bigotimes_{N} M_2(C)$.

This follows from the uniqueness of the inductive limit.

Definition 2.4.2. A C*-algebra $A$ is nuclear if for every C*-algebra $B$ there is a unique C*-norm on $A \otimes B$.

Nuclear algebras form what is arguably the most important and most studied class of C*-algebras. The following lemma is a far cry from the current state of the art in the subject, but it is easy to prove and all that we need.

Lemma 2.4.3. 1. Finite-dimensional C*-algebras are nuclear.
2. All abelian C*-algebras are nuclear.
3. An inductive limit of nuclear C*-algebras is nuclear.
4. Every AF algebra is nuclear.

Proof. (1) Suppose $A$ is finite-dimensional and $B$ is an arbitrary C*-algebra. Then $A$ is isomorphic to a direct sum of full matrix algebras. Therefore $A \otimes B$ is isomorphic to a direct sum of algebras $M_n(C) \otimes B \cong M_n(B)$ for $n \in \mathbb{N}$. Since $M_n(B)$ has a natural complete C*-norm (see §2.1.5), $A$ is nuclear by Corollary 1.3.3.

(2) See e.g., [27, II.9.4.4].

(3) Suppose $A = \lim_{\lambda} A_\lambda$ and $A_\lambda$ is nuclear for all $\lambda$. Fix a C*-norm $\otimes_\alpha$ on $A \otimes B$. It suffices to prove that $\otimes_\alpha$ is the spatial tensor product norm. Since linear combinations of the elementary tensors are dense in $A \otimes_\alpha B$, the union of all $A_\lambda \otimes B$ is dense in $A \otimes_\alpha B$. Since the restriction of $\otimes_\alpha$ to $A_\lambda \otimes B$ agrees with the spatial tensor product norm for all $\lambda$, $\otimes_\alpha$ is the spatial tensor product norm.

(4) is a consequence of (1) and (3). ☐

Lemma 2.4.4. Suppose $M$ is a unital C*-subalgebra of a C*-algebra $A$ isomorphic to $M_n(C)$ for some $n$. Then $A \cong M_n(C) \otimes pA$, where $p \in M$ is a minimal projection.

Proof. Let $f_{ij}$, for $i < n$ and $j < n$ be matrix units in $M$ such that $f_{11} = p$. Then $1_A = \sum_{i<n} f_{ii}$ and therefore for $a \in A$ we have

$$a = \sum_{i<n} \sum_{j<n} f_{ii} a f_{jj} = \sum_{i<n} \sum_{j<n} f_{ii} f_{jj} f_{ij} f_{ij}.$$

5 This is Takesaki’s Theorem, a reasonably simple proof of which uses the excision of pure states studied in §5.2; see Exercise 5.7.10 or [35, Theorem 3.4.8].
Then $a_{ij} := f_{ij}af_{j1}$ belongs to $pAp$ for all $i$ and $j$. Define $\Phi : A \to M_n(\mathbb{C}) \otimes pAp$ by $\Phi(a) := \sum_{i,j} \sum_{a,b} a_{ij}b$. A computation shows that this is an isomorphism between $A$ and $M_n(\mathbb{C}) \otimes pAp$. 

An intuition-honing counterexample is always good to have.

Example 2.4.5. In general, $M_2(B) \cong M_2(C)$ does not imply $B \cong C$. Take for example $B := \ell(H)$ and $C := M_2(\ell(H))$ (the Cuntz algebra $\ell(H)$ was defined in Example 2.3.10). Then $M_2(\ell(H)) \cong M_4(\ell(H))$ by Exercise 2.8.15. On the other hand, $\ell(H)$ and $M_2(\ell(H))$ have different scaled $K_0$ groups ([210]) and are therefore not isomorphic.

This is also a simple consequence of the Kirchberg–Phillips classification theorem ([208]); see [94, Proposition 7.4].

We conclude our discussion of tensor products by pointing out to a minor pitfall and its correct relative.

Example 2.4.6. It is not true that if $A$ is a unital $C^*$-algebra, $B$ and $C$ are unital $C^*$-subalgebras of $A$, and $bc = cb$ for all $b \in B$ and $c \in C$ then $C^*(B,C) \cong B \otimes_a C$ for some $C^*$-norm $\| \cdot \|_a$ on $B \otimes C$. Take for example $A = C(T)$, and let $a$ denote the multiplication operator by the identity function on $A$. Let $B := C^*(a + a^*, 1)$ and $C := C^*((a - a^*), 1)$. Since both $a + a^*$ and $i(a - a^*)$ are self-adjoint with spectra homeomorphic to $[0, 1]$, we have $B \cong C \cong C([0, 1])$ and $B \otimes C \cong C([0, 1]^2)$. However, $C^*(B,C) = A \cong C(T)$.

The example in Example 2.4.6 is that the joint spectrum $\text{jsp}(a + a^*, i(a - a^*))$ (see Exercise 1.11.22) is not even homeomorphic to $\text{sp}(a + a^*) \times \text{sp}(i(a - a^*))$. The following immediate consequence of the universal property of $\otimes_{\text{max}}$ is all that we can say in general, but see also Lemma 5.2.13.

Lemma 2.4.7. Suppose $A$ is a unital $C^*$-algebra, $B$ and $C$ are unital $C^*$-subalgebras of $A$, and $bc = cb$ for all $b \in B$ and all $c \in C$. Then $C^*(B,C)$ is a quotient of $B \otimes_{\text{max}} C$ via a map that sends $b \otimes 1_C$ to $b$ and $1_B \otimes c$ to $c$ for all $b \in B$ and all $c \in C$.

### 2.4.1 Full and Reduced Group $C^*$-algebras

Suppose $\Gamma$ is a discrete group. A representation of $\Gamma$ on a Hilbert space $H$ is a homomorphism of $\Gamma$ into $\text{GL} (\mathcal{H}(H))$. It is unitary if its range is a subset of the unitary group.

The elements of the group algebra $\mathbb{C}[\Gamma]$ are finite linear combinations of group elements, $\sum_{\gamma} \alpha_{\gamma}g$ (it is understood that $\alpha_{\gamma} = 0$ for all but finitely many $g$). Multiplication on $\mathbb{C}[\Gamma]$ is given by convolution,

$$(\sum_{\gamma} \alpha_{\gamma}g)(\sum_{\delta} b_{\delta}h) := \sum_{\gamma} \sum_{\delta} \alpha_{\gamma}b_{\delta}h_{\gamma^{-1}g}.$$ 

With multiplication defined in this way, every unitary representation of $\Gamma$ naturally extends to a unitary representation of $\mathbb{C}[\Gamma]$ on a Hilbert space. Conversely, the restriction of a unital $^*$-representation of $\mathbb{C}[\Gamma]$ to $\Gamma$ is necessarily a unitary representation.
Given a $C^*$-norm on $\mathbb{C}[\Gamma]$, the completion is a $C^*$-algebra associated with $\Gamma$. To define a representation of $\Gamma$, consider the Hilbert space $\ell_2(\Gamma)$ with the orthonormal basis $\{\delta_g : g \in \Gamma\}$. Note that $\mathbb{C}[\Gamma]$ is naturally identified with all finitely supported vectors in $\ell_2(\Gamma)$. The left regular representation of $\lambda$ of $\Gamma$ on $\ell_2(\Gamma)$ is defined by its action on the elements of the basis,

$$\lambda(g)(\delta_h) := \delta_{gh}.$$ 

Then $u_g := \lambda(g)$ is a permutation unitary with respect to the basis $\{\delta_g : g \in \Gamma\}$. We have $(u_g)^* = u_{g^{-1}}$ and $u_g u_h = u_{gh}$. Extend $\lambda$ to a representation of $\mathbb{C}[\Gamma]$,

$$\lambda(\sum \alpha_g g) := \sum \alpha_g \lambda(g)$$

and defined a $C^*$-norm on $\mathbb{C}[\Gamma]$ by $\|a\| = \|\lambda(a)\|$. The completion of $\mathbb{C}[\Gamma]$ with respect to this norm is the reduced group $C^*$-algebra of the group $\Gamma$, denoted $C^r_\Gamma(\Gamma)$. Some authors write $C^r_\Gamma(\Gamma)$ for $C^r_\Gamma(\Gamma)$ (where ‘$\lambda$’ stands for ‘left’; cf. $C^r_\nu(\Gamma)$ defined below).

The left-regular representation is faithful on $\mathbb{C}[\Gamma]$. To see this, note that the evaluation of $\lambda(\sum \alpha_g g)$ at $\delta_e$ is equal to $\sum \alpha_g \delta_g$, and therefore for $a \in \mathbb{C}[\Gamma]$ (identified with a finitely supported vector in $\ell_2(\Gamma)$) we have $\|\lambda(a)\| \geq |a|_2$.

We shall follow the standard (and mostly harmless) practice and consider $C^r_\Gamma(\Gamma)$ as both an abstract $C^*$-algebra and a concrete subalgebra of $B(\ell_2(\Gamma))$.

**Example 2.4.8.**  
1. If $\Gamma$ is the finite cyclic group of order $n$ then $C^r_\Gamma(\Gamma)$ is isomorphic to the $C^*$-subalgebra of $M_n(\mathbb{C})$ consisting of all matrices 'constant down the diagonal.' It is isomorphic to $\mathbb{C}^n$.
2. A finite group $\Gamma$ of order $n$ is isomorphic to a subgroup of $S_n$, the group of permutations of $n = \{0, \ldots, n-1\}$. Therefore $\lambda[\Gamma]$ consists of $n \times n$ permutation matrices, and $C^r_\Gamma(\Gamma)$ is isomorphic to a $C^*$-subalgebra of $M_n(\mathbb{C})$. By Lemma 2.2.3 it is isomorphic to a direct sum of matrix algebras They correspond to the irreducible representations of $\Gamma$.
3. The $C^*$-algebra $C^r_\Gamma(\mathbb{Z})$ is isomorphic to $C(\mathbb{T})$ via the Fourier transform, which sends $u_n \in C^r_\Gamma(\mathbb{Z})$ to the function $f(z) = z^n$ in $C(\mathbb{T})$ (see also Exercise 2.8.25).

The right regular representation of $\Gamma$ on $\ell_2(\Gamma)$ is defined by

$$\rho(g)(\delta_h) := \delta_{h^{-1}g}.$$ 

The linear extension of $\rho$ to $\mathbb{C}[\Gamma]$ is the right regular representation of $\mathbb{C}[\Gamma]$. The concrete $C^*$-algebra associated with $\rho$ is denoted $C^r_\rho(\Gamma)$.

For a $C^*$-algebra $A \subseteq B(H)$, a vector $\xi \in H$ is separating if every nonzero $a \in A$ satisfies $a^* \xi \neq 0$. A state $\tau$ on a $C^*$-algebra $A$ is tracial if it satisfies $\tau(ab) = \tau(ba)$ for all $a$ and $b$ in $A$. We will study tracial states in more detail in §4.1.

**Lemma 2.4.9.** Suppose $\Gamma$ is a discrete group.

1. The left and right regular representations of $\Gamma$ are unitarily equivalent.
2. The $C^*$-algebras $C^r_\lambda(\Gamma)$ and $C^r_\rho(\Gamma)$ are isomorphic.
3. Every \( a \in C^*_\delta(\Gamma) \) commutes with every \( b \in C^*_{\rho}(\Gamma) \).

4. For all \( g \in \Gamma \), the vector \( \delta_g \) is both cyclic and separating for \( C^*_\delta(\Gamma) \) and \( C^*_{\rho}(\Gamma) \).

5. The restriction of the vector state \( \omega_\delta(a) := (a\delta_k|\delta_k) \) to \( C^*_\delta(\Gamma) \), denoted \( \tau \), is faithful and tracial.

Proof. (1) and (2) Define a permutation unitary \( u \) in \( B(\ell^2(\Gamma)) \) by \( u(\delta_k) := \delta_{g^{-1}} \).

As for \( h \in \Gamma \) we have \( u(\lambda(h)|u^{-1} = \rho(h) \), \( \lambda \) and \( \rho \) are unitarily equivalent and \( \text{Ad } u \) implements an isomorphism between \( C^*_{\rho}(\Gamma) \) and \( C^*_\delta(\Gamma) \).

(3) Since \( \lambda(g)\rho(h)\delta_f = \delta_{gfh^{-1}} = \rho(h)\lambda(g)\delta_f \), every \( a \in C^*_\delta(\Gamma) \) commutes with every \( b \in C^*_{\rho}(\Gamma) \).

(4) Fix \( g \in \Gamma \). Suppose \( a \in C^*_\delta(\Gamma) \) and \( a\delta_g = 0 \). By (2), for every \( h \in \Gamma \) we have \( a\delta_h = a\rho(g^{-1}h)\delta_h = \rho(g^{-1}h)\delta_g = 0 \). Since \( \ker(a) \) includes all basis vectors, \( a = 0 \) and \( \delta_g \) is a separating for \( C^*_\delta(\Gamma) \). The analogous statement for \( C^*_{\rho}(\Gamma) \) is proved similarly, or by composing with \( \text{Ad } u \). A proof that \( \delta_g \) is cyclic is similar.

(5) To prove that \( \tau \) is faithful, note that (4) implies that \( \tau(a^*a) = \|a\delta_g\|^2 \) is zero if and only if \( a^*a = 0 \). We have \( \tau(\lambda(g)\lambda(h)) = 1 \) if \( g = h^{-1} \) and zero otherwise. Therefore \( \tau(\lambda(g)\lambda(h)) = \tau(\lambda(h)\lambda(g)) \) for all \( g \) and \( h \) in \( \Gamma \). Since \( C[\Gamma] \) is dense in \( C^*_\delta(\Gamma) \), we have \( \tau(ab) = \tau(ba) \) for all \( a \) and \( b \).

Let \( B_\Gamma := \{ u_g : g \in \Gamma \} \) and \( B_\Gamma := \{ u_g u_h = u_h u_g , u_g = 1 , u_g u_h = u_{gh} : g \in \Gamma , h \in \Gamma \} \).

This is a set of relations as defined in \( \S 2.3 \) and unitary representations of \( \Gamma \) in \( B(H) \) correspond to interpretations of \( B_\Gamma \) in \( \text{U}(H) \).

Since the left regular representation of \( \Gamma \) witnesses that \( B_\Gamma \) it is satisfiable, Lemma 2.3.11 implies that the universal algebra \( C^*(B_\Gamma|B_\Gamma) \) exists; this is the full group \( C^*-algebra \), denoted \( C^*(\Gamma) \). This is particularly appealing if \( \Gamma \) is finitely presented. For example, in the case of the free group \( F_2 \) we have

\[
C^*(F_2) = C^*(u,v : uu^* = u^*u = vv^* = v^*v = 1).
\]

### 2.4.2 Full and Reduced Crossed Products

Suppose \( A \) is a \( C^*-algebra \), \( \Gamma \) is a discrete group, and

\[
\Gamma \ni g \mapsto \alpha_g \in \text{Aut}(A)
\]

is a homomorphism from \( \Gamma \) into \( \text{Aut}(A) \). Such homomorphism is usually referred to as an action of \( \Gamma \) on \( A \). The triplet \( (A, \Gamma, \alpha) \) is called a covariant system or a \( C^*-dynamical \) system.

A crossed product of a covariant system \( (A, \Gamma, \alpha) \) is the \( C^*-algebra \) generated by \( A \) and canonical unitaries \( u_\gamma \), for \( g \in \Gamma \), which implement \( \alpha \), as \( \alpha \text{Ad } u_\gamma \big| A = \alpha_g \). Just like the tensor products, crossed products can be minimal (called reduced) or maximal (called full), with many unnamed varieties in between.

A brief outline of a concrete crossed product construction, resembling semidirect product of groups, follows; the details can be found in [35, \S 4.1]. Fix any faithful
representation $\pi: A \to \mathcal{B}(H)$. Define a representation $\pi'$ of $A$ on $H \otimes \ell_2(\Gamma)$ by

$$\pi'(a)(\xi \otimes \delta_g) := \pi(\alpha_g^{-1}(a))(\xi) \otimes \delta_g.$$  

The left regular representation of $\Gamma$ on $\ell_2(\Gamma)$ naturally extends to a representation of $\Gamma$ on $H \otimes \ell_2(\Gamma)$ by $g \mapsto 1 \otimes \lambda(g)$. A calculation shows that for all $g \in \Gamma$ and $a \in A$ the conjugation by the unitary $1 \otimes \lambda(g)$ sends $\pi(a)$ to $\pi(\alpha_g(a))$. The algebra

$$A \rtimes_{\alpha, \Gamma} := C^*(\pi[A], \lambda[\Gamma])$$

is called the reduced crossed product of $(A, \alpha, \Gamma)$.

If $\Gamma$ is a discrete group we can identify $A$ with $\pi[A]$, and if $A$ is unital we can identify $\Gamma$ with $1 \otimes \lambda[\Gamma]$. If $A = \mathbb{C}$ then $C^*_{\alpha, \Gamma}$ is isomorphic to $C^*_{\alpha}(\Gamma)$.

**Lemma 2.4.10.** Suppose $A$ is a $C^*$-algebra and $\alpha$ is an action of a discrete group $\Gamma$ on $A$.

1. The algebra $B_0 := \{\sum_{g \in F} a_g u_g : F \subseteq \Gamma, a_g \in A \text{ for } g \in F\}$ is dense in $A \rtimes_{\alpha, \Gamma}$.

2. For $\sum_{g \in G} a_g u_g$ and $\sum_{h \in H} a_h u_h$ in $B_0$ we have

$$\begin{align*}
(\sum_{g \in G} a_g u_g)^* &= \sum_{g \in G} \alpha_g^{-1}(a_g) u_g^{-1}, \\
(\sum_{g \in G} a_g u_g)(\sum_{h \in H} a_h u_h) &= \sum_{g \in G} \sum_{h \in H} a_g \alpha_g(a_h) u_{gh} \\
&= \sum_{f \in G \cdot H} \sum_{g \in G} a_f \alpha_{g^{-1}f}(a_f) u_f.
\end{align*}$$

3. $\|\sum_{g \in F} a_g u_g\| \geq \max_{g \in F} \|a_g\|$.

**Proof.** (1) and (2) are left as an exercise (see $35$, §4.1 for the details).

(3) Let $\pi: A \to \mathcal{B}(H)$ be the faithful representation used to define the crossed product and identify $A$ with $\pi[A]$. Fix $h \in F$, $\varepsilon > 0$, and a unit vector $\xi \in H$ such that $\|\alpha_{h^{-1}}(a_h)\xi\| > \|a_h\| - \varepsilon$. Then

$$\|\sum_{g \in F} a_g u_g\| \geq \|\sum_{g \in F} a_g u_g(\xi \otimes \delta_g)\| \geq (\sum_{g \in F} \|a_g\| \|\alpha_g^{-1}(a_g)\| \xi \otimes \delta_g\|^2)^{1/2} \geq \|a_g\xi\|,$$

proving our claim.

A universal construction as in Lemma 2.3.11 proves the existence of the full crossed product $A \rtimes_{\alpha} \Gamma$. This is the universal $C^*$-algebra given by generators $A \cup \{u_g : g \in \Gamma\}$ and the relations $u_g a u_g^* = \alpha_g(a)$ (in addition to the relations coming from $A$).

**Example 2.4.11.** 1. Any action of a group $\Gamma$ on $\mathbb{C}$ is necessarily trivial, i.e., $\alpha_g = \text{id}_{\mathbb{C}}$ for all $g \in \Gamma$. In this case the reduced crossed product of $(\mathbb{C}, \alpha, \Gamma)$ is isomorphic to $C^*_{\alpha}(\Gamma)$ and the full crossed product is isomorphic to $C^*(\Gamma)$.

2. Suppose that a group $\Gamma$ acts on a $C^*$-algebra $A$ trivially. The reduced crossed product is naturally isomorphic to $A \otimes C^*_{\alpha}(\Gamma)$, where $\otimes$ denotes the spatial tensor product.
2.5 Quotients and Lifts

3. Suppose $\alpha$ is a rotation of $\mathbb{T}$ by the angle $\theta$. Suppose moreover that $\theta / \pi$ is irrational, so that $\alpha^n \neq \text{id}_\mathbb{T}$ for all $n \in \mathbb{N}$. The crossed product $C(\mathbb{T}) \rtimes_{\alpha} \mathbb{Z}$ is the 

irrational rotation algebra.

4. If $\Gamma$ is a finite group, then $A \rtimes_{\alpha, r} \Gamma$ is a finite-dimensional module over $A$ and therefore complete in every $C^*$-norm. By Corollary 1.3.3, a crossed product associated with a finite group has a unique $C^*$-norm.

5. If $\Gamma$ is a finite group of order $n$ then $A \rtimes_{\alpha, r} \Gamma = \{ \sum_{g \in \Gamma} a_g u_g : a_g \in A \}$ can be identified with a $C^*$-subalgebra of $M_n(A)$. Each unitary $u_g$ is identified with an $n \times n$ permutation matrix and $a \in A$ is identified with $a \otimes 1_n$. By (4), $A \rtimes_{\alpha, r} \Gamma$ is isomorphic to a $C^*$-subalgebra of $M_n(A)$.

2.5 Quotients and Lifts

In this section we introduce quotients and lifts. It is proved, using approximate units, that any quotient of a $C^*$-algebra modulo a self-adjoint, two-sided, norm-closed ideal is a $C^*$-algebra. We also consider the lifting problem for self-adjoint elements. The Calkin algebra and the reduced product $C^*$-algebras (prototypes for both ultraproduct and asymptotic sequence algebra constructions) make their first appearance. As we will see later, set theory has quite a bit to say about the Calkin algebra and other quotients.

All ideals in a $C^*$-algebra are assumed to be two-sided, norm-closed and self-adjoint, unless otherwise specified. The last one of these conditions is redundant.

**Lemma 2.5.1.** Any two-sided, norm-closed, ideal $J$ in a $C^*$-algebra is self-adjoint.

**Proof.** Corollary 1.8.1 implies there is a ‘right approximate unit’ for $J$, $(e_\lambda : \lambda \in \Lambda)$. If $x \in J$ then $\|x^* - e_\lambda x^*\| = \|x - xe_\lambda\|$ can be made arbitrarily small. Hence $x^*$ is in the closure of $J$ for every $x \in J$, and $J$ is self-adjoint.

If $J$ is a norm-closed ideal of $A$ then $A/J$ is the space of cosets $a + J$, for $a \in A$. The quotient norm is defined by $\|a + J\| = \inf_{b \in J} \|a + b\|$.

**Lemma 2.5.2.** Every quotient of a $C^*$-algebra is a $C^*$-algebra.

**Proof.** Suppose $A$ is a $C^*$-algebra and $J$ is a two-sided, norm-closed, ideal of $A$. It is by Lemma 2.5.1 self-adjoint, and therefore $A/J$ is a Banach algebra with an isometric involution $^*$. We need to show that the $C^*$-equality holds in $A/J$. Fix $a \in A$ and an approximate unit $(e_\lambda)$ for $A$. For $b \in J$ and $\lambda \in \Lambda$ we have

$$\|a - e_\lambda a\| \leq \|(1 - e_\lambda)(a - b)\| + \|e_\lambda b - b\|$$

and therefore $\inf_\lambda \|a - e_\lambda a\| \leq \|a + J\|$. Since the other inequality is trivial, we have $\inf_\lambda \|a - e_\lambda a\| = \|a + J\|$. By the $C^*$-equality applied in $A$ we have $\|(1 - e_\lambda)a\|^2 = \|(1 - e_\lambda)aa^*(1 - e_\lambda)\|$. Taking the infimum over $\lambda \in \Lambda$ we obtain $\|a + J\|^2 = \|aa^* + J\|$ as required.  \[ \square \]
A C*-algebra is simple if it has no non-trivial ideals.

**Proposition 2.5.3.** Suppose \( A = \lim_j A_j \) and \( J \) is a two-sided, norm-closed ideal of \( A \). Then \( J = \lim_j (J \cap A_j) \), where the inductive limit is taken with the restriction of the original connecting maps.

**Proof.** Let \( J_1 := \lim_j (J \cap A_j) \). This is a two-sided, norm-closed ideal of \( A \) included in \( J \). Let \( \Phi: A/J \to A/J \) be the quotient map. By Lemma 1.6.13 (1), lifts \( \Phi \) then \( \Phi \) is any lift such that \( \Phi(a) \subseteq \{ r, s \} \). With \( a \subseteq [r, s] \) and \( \pi(a) = g(\pi(a_0)) = g(b) = b \).

If \( a \subseteq [r, s] \) and \( \pi(a) = g(\pi(a_0)) = g(b) = b \).

Let \( \pi \) denote the quotient map. Choose any \( c \subseteq A \) such that \( \pi(c) = b \). Then \( a_0 := \frac{1}{2}(c + c^*) \) is self-adjoint and \( \pi(a_0) = \frac{1}{2}(\pi(c) + \pi(c)^*) = b \). If \( \pi(b) \subseteq [r, s] \) then let \( a := g(a_0) \), where \( g(t) = r \) if \( t \leq r \), and \( g(t) = s \) if \( t \geq s \), and \( g(t) = t \) for \( r \leq t \leq s \). Then \( \pi(a) \subseteq [r, s] \) and \( \pi(a) = g(\pi(a_0)) = g(b) = b \).

**Lemma 2.5.4.** Suppose \( A \) is a C*-algebra and \( J \) is an ideal in \( A \).

1. If \( b \subseteq A/J \) is self-adjoint then it has a self-adjoint lift \( a \subseteq A \). If in addition \( \text{sp}(b) \subseteq [r, s] \) for some \( r \leq s \) then a lift \( a \subseteq A \) can be chosen so that \( \|a\| = \|b\| \).
2. Every \( b \subseteq A/J \) has a lift \( a \subseteq A \) such that \( \|a\| = \|b\| \).

**Proof.** (1) Let \( \pi \) denote the quotient map. Choose any \( c \subseteq A \) such that \( \pi(c) = b \). Then \( a_0 := \frac{1}{2}(c + c^*) \) is self-adjoint and \( \pi(a_0) = \frac{1}{2}(\pi(c) + \pi(c)^*) = b \). If \( \pi(b) \subseteq [r, s] \) then let \( a := g(a_0) \), where \( g(t) = r \) if \( t \leq r \), and \( g(t) = s \) if \( t \geq s \), and \( g(t) = t \) for \( r \leq t \leq s \). Then \( \pi(a) \subseteq [r, s] \) and \( \pi(a) = g(\pi(a_0)) = g(b) = b \).

**Definition 2.5.5.** A (two-sided, self-adjoint) ideal \( J \) in a C*-algebra \( A \) is essential if for every \( a \subseteq A \setminus \{ 0 \} \) we have \( aJ \neq \{ 0 \} \). Since \( J = J^* \), this is equivalent to \( Ja \neq \{ 0 \} \) for all \( a \subseteq A \setminus \{ 0 \} \).

**Lemma 2.5.6.** Suppose \( J \) is an essential ideal in \( A \) and \( \Phi_j: A \to B \) is a *-homomorphism such that \( \Phi_j(J) \) is an essential ideal of \( B \) for \( j = 1, 2 \). If \( \Phi_1 \) and \( \Phi_2 \) agree on \( J \), then \( \Phi_1 = \Phi_2 \).

**Proof.** Suppose the contrary and fix \( a \subseteq A \) such that \( \Phi_1(a) \neq \Phi_2(a) \). Let \( c \subseteq J \) be such that \( \Phi_1(c)(\Phi_1(a) - \Phi_2(a)) \neq 0 \). Then \( \Phi_1(c) \neq \Phi_2(c) \); contradiction.

A proof of the following is left as an exercise.

**Lemma 2.5.7.** If \( A \) is a unital C*-algebra and \( J \) is a two-sided, norm-closed, ideal of \( A \) and \( \pi: A \to A/J \) is the quotient map, then \( \text{sp}(\pi(a)) \subseteq \text{sp}(a) \) for all \( a \subseteq A \). □

Liftings will be revisited at greater depth in §12.4.
2.6 Automorphisms of $C^*$-algebras

The Calkin Algebra

It is about time that we defined the oldest truly abstract $C^*$-algebra (see the introduction to §12). Suppose that $H$ is an infinite-dimensional Hilbert space and recall that $\mathcal{K}(H)$ denotes the ideal of compact operators in $\mathcal{B}(H)$. The quotient

$$\mathcal{Q}(H) = \mathcal{B}(H)/\mathcal{K}(H)$$

is the Calkin algebra. The most interesting case is when $H = \ell_2(\mathbb{N})$. This algebra provides the setting for analytic $K$-homology, a rich source of counterexamples, and a set-theoretically malleable object comparable to the Čech–Stone remainder of $\mathbb{N}$, $\beta\mathbb{N}\setminus\mathbb{N}$ (or its Stone dual, $\mathcal{P}(\mathbb{N})/\text{Fin}$ see 1.3.1). We will revisit it in Chapter 12.

Reduced Products, Asymptotic Sequence Algebras

Given a sequence $B_j$, for $j \in \mathbb{N}$, of $C^*$-algebras the algebra $\prod_j B_j$ has $\bigoplus_j B_j$ as an ideal. The quotient $\prod_j B_j/\bigoplus_j B_j$ is the reduced product of these algebras. In the special case when all $B_j$ are isomorphic to a fixed $C^*$-algebra $B$, the reduced product is equal to $\ell_\infty(B)/c_0(B)$, the so-called asymptotic sequence algebra.

This construction can be generalized using the notion of an ideal on an abstract set. An ideal $\mathcal{J}$ on a set $X$ is a family of subsets of $X$ closed under taking finite unions and subsets of its elements (we will return to ideals in §9.1). Given an ideal $\mathcal{J}$ on $\mathbb{J}$ and a bounded sequence of real numbers $r_j$, for $j \in \mathbb{J}$, let

$$\limsup_{j \to \mathcal{J}} r_j := \inf_{X \in \mathcal{J}} \sup_{j \in \mathbb{J} \setminus X} r_j.$$

Definition 2.5.8. Given a family $\{B_j : j \in \mathbb{J}\}$ of $C^*$-algebras and an ideal $\mathcal{J}$, let

$$\bigoplus_{\mathcal{J}} B_j := \{b \in \prod_{j \in \mathbb{J}} B_j : \limsup_{j \to \mathcal{J}} \|b_j\| = 0\}.$$

A straightforward computation shows that this is a norm-closed, self-adjoint, ideal of $\prod_{j \in \mathbb{J}} B_j$. The quotients $\prod_{j \in \mathbb{J}} B_j/\bigoplus_{\mathcal{J}} B_j$ will be studied in depth in §15.

2.6 Automorphisms of $C^*$-algebras

In this section we consider classes of automorphisms of $C^*$-algebras: inner, asymptotically inner, and approximately inner automorphisms. This section is technically undemanding; its deepest result is a characterization of when a sequence of unitaries, or a path of unitaries, in a $C^*$-algebra defines an automorphism.

An automorphism of a $C^*$-algebra $A$ is always assumed to be a $^*$-automorphism. Given this convention, Lemma 1.2.10 implies that every automorphism of a $C^*$-algebra is an isometry. A non-exhaustive list of prominent normal subgroups of Aut$(A)$ is in order.
Definition 2.6.1. If \( u \in \tilde{A} \) is a unitary then \( (\text{Ad}u)(a) := uau^* \) defines an automorphism of \( A \). An automorphism of this form is \textit{inner}. An endomorphism of the form \( \text{Ad}u \) is said to be \textit{implemented by a unitary} \( u \) if \( u \) belongs to a \( C^* \)-algebra that includes \( A \). An automorphism of \( A \) is \textit{multiplier inner} if it is implemented by a unitary \( u \) in the multiplier algebra \( \mathcal{M}(A) \) of \( A \) (see §13.2).

Every automorphism of every \( C^* \)-algebra is implemented by a unitary in an appropriate crossed product.

Example 2.6.2. Since \( \mathcal{M}(\mathcal{K}(H)) \cong \mathcal{B}(H) \) and every automorphism of \( \mathcal{B}(H) \) is inner (Exercise 2.8.29), every automorphism of \( \mathcal{K}(H) \) is multiplier inner. However, \( \mathcal{K}(H) \) has an ample supply of outer automorphisms (Exercise 2.8.30).

The definition of an inner automorphism as in Definition 2.6.1 is widely accepted in the literature (see e.g., the authoritative [27, II.5.5.12]). However in [199, Definition 3.3] (see also the discussion in the second paragraph of [260, §7.3]) an automorphism is defined to be inner if it is implemented by a unitary in the multiplier algebra.

The map \( U(\tilde{A}) \ni u \mapsto \text{Ad}u \in \text{Aut}(A) \) is a group homomorphism. Its range, the group of \textit{inner automorphisms}, is denoted \( \text{Inn}(A) \). An automorphism \( \Phi \) is \textit{approximately inner} if there exists a net of unitaries \( u_\lambda \), for \( \lambda \in \Lambda \), in \( \tilde{A} \) such that \( \Phi(a) = \lim_\lambda \text{Ad}u_\lambda(a) \) for all \( a \in A \), i.e., \( \Phi \) is a point-norm limit of the net \( (\text{Ad}u_\lambda) \). Thus the approximately inner automorphisms of \( A \) form a subgroup, denoted \( \overline{\text{Inn}}(A) \), of \( \text{Aut}(A) \) which is the point-norm closure of \( \text{Inn}(A) \).

Lemma 2.6.3. Suppose \( A \) is a \( C^* \)-algebra. For a net of unitaries \( u_\lambda \), for \( \lambda \in \Lambda \), in \( \tilde{A} \) the following are equivalent.

1. \( \Phi(a) := \lim_\lambda \text{Ad}u_\lambda(a) \) defines an automorphism of \( A \).
2. For every \( a \in A \) both limits \( \lim_\lambda \text{Ad}u_\lambda(a) \) and \( \lim_\lambda \text{Ad}u_\lambda^*(a) \) exist.
3. For every \( F \subseteq A \) and \( \varepsilon > 0 \) there exists \( \tilde{\lambda} \in \Lambda \) such that for all \( \lambda > \tilde{\lambda} \) we have \( \|u_\lambda^*u_\lambda a\| < \varepsilon \) and \( \|u_\lambda u_\lambda^* a\| < \varepsilon \) for all \( a \in F \).

Proof. (1) clearly implies (2).

If \( \Phi(a) := \lim_\lambda \text{Ad}u_\lambda(a) \) exists for all \( a \in A \), then \( \Phi \) is an endomorphism of \( A \) by uniform continuity. It is an automorphism if in addition \( \lim_\lambda \text{Ad}u_\lambda^*(a) \) exists for all \( a \in A \). This proves that (2) implies (1).

(3) is a Cauchy-type criterion for convergence of the nets \( (u_\lambda^*au_\lambda) \) and \( (u_\lambda au_\lambda^*) \) for all \( a \in A \). It is therefore equivalent to (2). \( \square \)

An automorphism \( \Phi \) is \textit{asymptotically inner} if there exists a continuous path of unitaries \( u_t \), for \( 0 \leq t < \infty \), in \( \tilde{A} \) such that \( \Phi(a) = \lim_{t \to \infty} \text{Ad}u_t(a) \) for all \( a \in A \). The inverse of \( \Phi \) is an asymptotically inner automorphism corresponding to the continuous path \( u_t^* \), for \( 0 \leq t < \infty \), and a composition of two asymptotically inner automorphisms is obtained by multiplying unitaries in the corresponding paths pointwise. Thus the asymptotically inner automorphisms of \( A \) form a subgroup of \( \text{Aut}(A) \), denoted \( \text{AInn}(A) \).
Lemma 2.6.4 gives a characterization of continuous paths that give rise to asymptotically inner automorphisms. Its proof, being analogous to the proof of Lemma 2.6.3, is omitted.

Lemma 2.6.4. Suppose $A$ is a $C^*$-algebra. For a continuous path of unitaries $u_t$, for $0 \leq t < \infty$, in $\tilde{A}$ the following are equivalent.

1. $\Phi(a) := \lim_{t \to \infty} \text{Ad} u_t(a)$ defines an automorphism of $A$.
2. For every $a \in A$ both limits $\lim_{t \to \infty} \text{Ad} u_t(a)$ and $\lim_{t \to \infty} \text{Ad} u_t^*(a)$ exist.
3. For every $F \subset A$ and $\varepsilon > 0$ there exists $\bar{t} \geq 0$ such that for all $t > \bar{t}$ we have $\|u_t^* u_t a\| < \varepsilon$ and $\|u_t u_t^* a\| < \varepsilon$ for all $a \in F$.

Definition 2.6.5. Suppose that a $C^*$-algebra $A$ is unital. Two unitaries $u$ and $v$ in $U(A)$ are said to be homotopic in $A$ if there is a continuous map $f : [0, 1] \to U(A)$ such that $f(0) = u$ and $f(1) = v$. The connected component of the identity in $U(A)$, denoted $U_0(A)$, is a normal subgroup of $U(A)$.

Example 2.6.6. 1. The unitary group $U(\mathcal{B}(H))$ (usually denoted $U(H)$) is connected. This is because by the Borel functional calculus every unitary in $\mathcal{B}(H)$ is of the form $u = \exp(ia)$ for a self-adjoint $a$, and therefore homotopic to 1 via the path $u_t := \exp(iat)$, for $0 \leq t \leq 1$.
2. If $X$ is a compact Hausdorff space, the spectral characterization of unitaries (Exercise 1.11.18) implies that $U(C(X))$ is equal to $C(X, T)$. This implies for example that $U(C(T))$ is not connected, since $\text{id}_T$ is not homotopic to 1. More generally, $U_0(C(T))$ is equal to the subgroup of functions with zero winding number, and the quotient group $U(C(T))/U_0(C(T))$ is isomorphic to $\mathbb{Z}$.
3. The unitary group of the Calkin algebra ($\S 2.5$, $\S 12$) is not connected. The image of the unilateral shift in $\mathcal{B}(H)$ is a unitary not homotopic to 1 (Proposition 12.4.3, Proposition C.6.5).
4. To $a \in A_{sa}$ one associates the one-parameter group of unitaries, $u_t := \exp(iat)$, for $t \in \mathbb{R}$. Then $t \mapsto u_t$ is a continuous group homomorphism from $\mathbb{R}$ into $U_0(A)$.

2.7 Real Rank Zero

In this section we define the real rank of a $C^*$-algebra and prove that a $C^*$-algebra has real zero if and only if it satisfies a certain separation property for disjoint open subsets of spectra of its normal elements.

It is generally agreed that $C^*$-algebras are the noncommutative analogs of compact, Hausdorff, topological spaces, and we will now introduce the property of $C^*$-algebras that is the noncommutative analog of 0-dimensionality. A $C^*$-algebra that satisfies any of the equivalent conditions listed in Theorem 2.7.1 is said to have real rank zero.

Theorem 2.7.1. Suppose $A$ is a $C^*$-algebra. Then the following are equivalent.
1. Every hereditary C*-subalgebra of A has an approximate unit consisting of projections.
2. The set of self-adjoint elements of A invertible in \( \hat{A} \) is dense in \( A_{\text{sa}} \).
3. The set of self-adjoint elements with finite spectrum is dense in \( A_{\text{sa}} \).

Proof. See [27, Theorem V.3.2.8].

Corollary 2.7.2. Suppose that A is a C*-algebra of real rank zero.

1. Every hereditary C*-subalgebra of A has real rank zero, and in particular every two-sided, norm-closed, ideal of A has real rank zero.
2. Every quotient of A has real rank zero.

Proof. Condition (1) of Theorem 2.7.1 is inherited by hereditary C*-subalgebras, and the condition (3) of Theorem 2.7.1 is preserved by taking quotients.

Example 2.7.3. 1. An abelian C*-algebra \( C_0(X) \) has real rank zero if and only if its spectrum X is zero-dimensional.
2. Every finite-dimensional C*-algebra has real rank zero since every self-adjoint element has finite spectrum. (2) of Theorem 2.7.1 is preserved under inductive limits, and therefore all AF algebras have real rank zero.

A less standard equivalent reformulation of having real rank zero is given in Proposition 2.7.5 and Exercise 2.8.34 below. Recall that \( A_{+,1} = \{ a \in A_+ : \|a\| = 1 \} \). If \( a \) and \( b \) belong to \( A_{+,1} \) we write \( a \ll b \) if \( ab = a \). This is equivalent to asserting that \( cb = c \) for all \( c \in C^*(a) \), i.e., that \( b \) acts as a unit for \( C^*(a) \).

Example 2.7.4. In a C*-algebra with real rank zero \( a \ll b \) does not necessarily imply that there exists a projection \( p \) such that \( a \ll p \) and \( p \ll b \). Fix \( X \subseteq \{0,1\}^\mathbb{N} \) which is closed but not open. If \( a \) and \( b \) in \( C(\{0,1\}^\mathbb{N})_{+,1} \) are such that \( \text{supp}(a) = X \) and the zero-set of \( 1 - b \) is \( X \), then no projection \( p \) in \( C(\{0,1\}^\mathbb{N}) \) satisfies \( a \ll p \ll b \).

The statement refuted in Example 2.7.4 has a poor, albeit correct and helpful, relative.

Proposition 2.7.5. If a C*-algebra A has real rank zero and \( a, b, c \) in \( A_{+,1} \) are such that \( a \ll c \ll b \) then there exists a projection \( p \) in A such that \( a \ll p \ll b \).

Proof. Since Theorem 2.7.1 implies that A has real rank zero if and only if its unitization has real rank zero, we may assume A is unital. Fix \( f \) and \( g \) in \( C(\{0,1\})_{+,1} \) such that \( g \ll f \), \( f(0) = g(0) = 0 \), and \( f(1) = g(1) = 1 \); take e.g., piecewise linear functions with breakpoints \( f(0) = 0 \), \( f(1/3) = f(1) = 1 \), and \( g(0) = g(1/3) = 0 \), \( g(2/3) = g(1) = 1 \).

With \( d := g(c) \) and \( e := f(c) \) we have \( a \ll d \ll e \ll b \). Fix \( e \in (0,1/3) \). By Theorem 2.7.1, in the hereditary subalgebra \( (1 - d)\hat{A}(1 - d) \) of \( A \) there is a projection \( q \) such that \( q(1 - d) \approx_\varepsilon 1 - d \). Since \( a \ll d \), we have \( ay = 0 \) for all \( y \in (1 - d)\hat{A}(1 - d) \), in particular \( aq = 0 \) and \( r := 1 - q \) satisfies \( a \ll r \). Also, \( (1 - d)q \approx_\varepsilon (1 - d)(q + r) \) implies \( (1 - d)r \approx_\varepsilon 0 \). Since \( d \ll e \), we have \( re \approx_\varepsilon r \) and \( rer \approx_\varepsilon r \).
Let $x := e^{1/2}r$. Then $x^*x = rer$ is invertible in the unital $C^*$-algebra $rAr$ by Lemma 1.2.6. Hence $x$ is well-supported, and by Lemma 1.4.4 it has a polar decomposition $x = v|x|$ in $C^*(r,e) \subseteq eAr$. Then $v^*v = r$ and $p := vv^*$ is a projection. It belongs to $eAr$ and since $e \ll b$ we have $p \ll b$.

It remains to check that $a \ll p$. Since $a \ll e$ and $a \ll r$, $a$ commutes with all elements of $C^*(r,e)$ and multiplication by any element of $C^*(r,e)$ annihilates $1 - a$. Since $v \in C^*(r,e)$, we have $av = va = a$ and—finally!—$a \ll p$. \hfill \square

**Corollary 2.7.6.** Suppose $A$ has real rank zero, $c$ is a normal element of $A$, and $U$ and $V$ are open subsets of $sp\{c\}$ with disjoint closures. Then some projection $p$ in $A$ satisfies

1. $f(c) \ll p$ for every $f \in C(sp\{c\})$ with $supp(f) \subseteq V$ and
2. $f(c) \ll 1 - p$ for every $f \in C(sp\{c\})$ with $supp(f) \subseteq U$.

In particular, if $c \in A_{+,1}$ and $0 < \varepsilon < 1$ then there exists a projection $p \in A$ such that $c \leq p + (1 - \varepsilon)(1 - p)$ and $p \leq a + \varepsilon$.

**Proof.** (1) We may assume that both $U$ and $V$ are nonempty. By the Tietze Extension Theorem find $f, g$, and $h$ in $C(sp\{c\})_+$ such that $f \ll g \ll h$, $f(t) = 1$ for all $t \in U$, and $h(t) = 0$ for all $t \in V$. Then $a := f(c)$, $c := g(c)$, and $b := h(c)$ satisfy $a \ll c \ll b$ and by Proposition 2.7.5 there exists a projection $p \in A$ such that $a \ll p \ll b$. Clearly $p$ satisfies the requirements.

(2) If $c \in A_{+1}$ then $U := sp\{c\} \cap [0, 1 - \varepsilon)$ and $V := sp\{c\} \cap [1 - \varepsilon/2, 1]$ are nonempty open subsets of $sp\{c\}$ with disjoint closures for all $0 < \varepsilon < 1$. Any $p \in Proj(A)$ as guaranteed by (2) is as required. \hfill \square

### 2.8 Exercises

A projection $p$ in a $C^*$-algebra $A$ is said to be **scalar** if $pAp \cong C$.

**Exercise 2.8.1.** Prove that if a $C^*$-algebra $A$ does not have scalar projections then every hereditary $C^*$-subalgebra of $A$ contains an isomorphic copy of $C_0(\mathbb{R})$.

**Exercise 2.8.2.** Suppose that $A$ is a simple $C^*$-algebra and $D$ is a masa in $A$. Prove that $D$ contains a scalar projection if and only if $A$ is isomorphic to $\mathcal{K}(H)$ for some Hilbert space $H$.

A nonzero projection $p$ in a $C^*$-algebra is **minimal** if the only projections $q$ such that $q \leq p$ are $0$ and $p$.

**Exercise 2.8.3.** Prove that if a nontrivial $C^*$-algebra $A$ has no minimal projections then every hereditary $C^*$-subalgebra of $A$ contains a copy of $C([0, 1])$.

**Exercise 2.8.4.** Suppose $C$ is a $C^*$-algebra and $A$ is a $C^*$-subalgebra of $C$. Prove that the hereditary $C^*$-subalgebra of $C$ generated by $A$ is $\{ c \in C : \lim_\lambda \parallel c - e_\lambda ce_\lambda \parallel = 0 \}$ where $e_\lambda$, for $\lambda \in \Lambda$, is any approximate unit of $A$. 

Exercise 2.8.5. Suppose $A_j$, for $j \in J$, are $C^*$-algebras. Prove that the density character of $\prod_{j \in J} A_j$ is at least $2^{|J|}$; in particular the direct product of infinitely many nontrivial $C^*$-algebras is nonseparable. In general, prove that the density character of this algebra is given by the formula $\chi(\prod_{j \in J} A_j) = \prod_{j \in J} \chi(A_j)$.

Exercise 2.8.6. Prove that $K(\ell_2(\kappa))$ is stable for every infinite cardinal $\kappa$. More generally, $K(\ell_2(\kappa)) \otimes K(\ell_2(\lambda)) \cong K(\ell_2(\lambda))$ for all infinite cardinals $\kappa \leq \lambda$.

Exercise 2.8.7. Prove that every simple hereditary $C^*$-subalgebra of the algebra of compact operators on a Hilbert space is isomorphic to the algebra of compact operators on some Hilbert space.

A maximal abelian self-adjoint subalgebra of a $C^*$-algebra $A$ is called a masa.

Exercise 2.8.8. Let $A$ be a $C^*$-algebra. Prove that the following are equivalent.

1. $A$ is finite-dimensional.
2. Every masa $C^*$-subalgebra of $A$ is finite-dimensional.
3. Some masa in $A$ is finite-dimensional.

Hint: It suffices to prove that (3) implies (1). If $D$ is a finite-dimensional masa then it is isomorphic to $\mathbb{C}^n$ for some $n$. Consider space $pAq$ for minimal projections $p$ and $q$ in $D$.

The natural generalization of Exercise 2.8.8 to higher cardinals is false, as there exist a nonseparable $C^*$-algebra all of whose masas are separable. One example is given in Theorem 10.4.3.

Exercise 2.8.9. Assume $A$ is an infinite-dimensional $C^*$-algebra. Prove that $A_{+,1}$ contains orthogonal elements $a_n$, for $n \in \mathbb{N}$.

Exercise 2.8.10. Prove that each of the following relations is weakly stable.

1. $\|a - a^*\| = 0$.
2. $\max(\|a - a^*\|, \|a - a^2\|) = 0$.
3. $\|aa^* - (aa^*)^2\| = 0$.
4. (In a unital $C^*$-algebra) $\|aa^* - 1\| = 0$.
5. (In a unital $C^*$-algebra) $\max(\|aa^* - 1\|, \|a^*a - 1\|) = 0$.

Hint: In addition to the continuous functional calculus one needs to adapt the proof that the well-supported elements have a polar decomposition (Lemma 1.4.4).

Exercise 2.8.11. Use Exercise 2.8.10 (5) to prove that if $A = \lim A_\lambda$ is a unital inductive limit of $C^*$-algebras, then $U(A) = \lim U(A_\lambda)$.

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6 In this exercise $\{0\}$ is not considered to be a $C^*$-algebra.
7 Used in the proof of Proposition 4.4.2.
8 Used in the proof of Theorem 15.4.5.
9 Used in the proof of Proposition 17.5.4.
10 Used in the proof of Proposition 17.5.4.
11 Used in the proof of Theorem 5.5.4.
Exercise 2.8.12. Prove that the defining relations for the matrix units of type $M_n$ (Example 2.2.1) are weakly stable.\textsuperscript{12}

Exercise 2.8.13. Prove that $M_2$- and $M_3$- are not isomorphic. Characterize the isomorphism relation of separable UHF algebras.

Cuntz algebras $\mathcal{O}_n$ were defined in Example 2.3.10.

Exercise 2.8.14. Suppose that $\mathcal{O}_2$ unitaly embeds into a $C^*$-algebra $A$. Prove that $M_n(A) \cong A$ for all $n \geq 1$.

Exercise 2.8.15. Prove that $\mathcal{O}_2$ embeds unitaly into $M_n(\mathcal{O}_{n+1})$ for all $n \geq 1$. Then prove that $M_{mn}(\mathcal{O}_{n+1}) \cong M_n(\mathcal{O}_{n+1})$ for all $m \geq 1$ and $n \geq 1$.\textsuperscript{13}

Exercise 2.8.16. Suppose $\tau$ is a state on a $C^*$-algebra $A$. Prove that the following are equivalent

1. $\tau$ is tracial.
2. $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$.
3. $\tau(u^*au) = \tau(a)$ for all unitaries $u \in \tilde{A}$.

A $C^*$-algebra $A$ is locally finite (or LF) if for every $\varepsilon > 0$ and every $F \subseteq A$ there exists a finite-dimensional $C^*$-algebra $M$ and a $^*$-homomorphism $\Phi: M \to A$ such that $F \subseteq \varepsilon \Phi[M]$.

Exercise 2.8.17. Prove that every separable LF algebra is AF.

Exercise 2.8.18. Suppose $A$ is an LF algebra. Prove that all of its hereditary $C^*$-subalgebras are LF.

Exercise 2.8.19. 1. Suppose $X$ and $Y$ are locally compact Hausdorff spaces. Prove that $C_0(X) \otimes C_0(Y) \cong C_0(X \times Y)$.
2. If $X_j$, for $j \in \mathbb{J}$, are compact Hausdorff spaces, prove $\bigotimes_j C(X_j) \cong C(\prod_j X_j)$.

Exercise 2.8.20. Suppose $H$ and $K$ are Hilbert spaces. Then any $a \in \mathcal{B}(H)$ can be identified with $a \otimes 1_K$ in $\mathcal{B}(H \otimes K)$. Hence $\mathcal{B}(H)$ can be identified with a $C^*$-subalgebra of $\mathcal{B}(H \otimes K)$, and similarly $\mathcal{B}(K)$ can be identified with a $C^*$-subalgebra of $\mathcal{B}(H \otimes K)$. Therefore $\mathcal{B}(H) \otimes \mathcal{B}(K)$ can be identified with a $C^*$-subalgebra of $\mathcal{B}(H \otimes K)$. Prove that this subalgebra is equal to $\mathcal{B}(H \otimes K)$ if and only if at least one of $H$ and $K$ is finite-dimensional.

Exercise 2.8.21. Suppose $X$ is a locally compact Hausdorff space and $A$ is a $C^*$-algebra. Prove that $C_0(X) \otimes A$ is isomorphic to the $C^*$-algebra of all norm-continuous functions from $X$ into $A$ vanishing at infinity.

Exercise 2.8.22. Prove that there are $\epsilon$ nonisomorphic $C^*$-algebras generated by $x$ such that $0 \leq x \leq 1$ (cf. Example 2.3.7).

\textsuperscript{12} Used in the proof of Theorem 6.1.3.

\textsuperscript{13} K-theoretic methods ([210]) can be used to prove e.g., that $M_n(\mathcal{O}_3) \cong \mathcal{O}_3$ if and only if $n$ is odd.
Exercise 2.8.23. Describe the universal unital C*-algebra generated by a nilpotent contraction of degree \( n \geq 3 \). (If \( n = 2 \) then this algebra is \( M_2(\mathbb{C}) \) by Example 2.3.5.)

For an infinite cardinal \( \kappa \) let \( F_\kappa \) denote the free group with \( \kappa \) generators.

Exercise 2.8.24. Prove that every C*-algebra of density character \( \kappa \) is isomorphic to a quotient of the full group algebra \( C^*_r(F_\kappa) \).

Exercise 2.8.25. A character of a discrete abelian group \( \Gamma \) is a continuous homomorphism of \( \Gamma \) into the circle group \( \mathbb{T} \). The space of all characters of \( \Gamma \) is the Pontryagin dual, denoted \( \hat{\Gamma} \), of \( \Gamma \). It is a closed, and therefore compact, subspace of \( \mathbb{T}^\Gamma \). Prove that every discrete abelian group \( \Gamma \) satisfies \( C^*_r(\Gamma) \cong C(\hat{\Gamma}) \).

Exercise 2.8.26. With \( \pi : B(H) \to D(H) \) denoting the quotient map, prove that \( \text{sp}(\pi(a)) = \text{sp}(a) \) for \( a \in B(H) \).

Exercise 2.8.27. Give an example of a projection \( q \) in a quotient C*-algebra \( A/J \) which is not an image of a projection in \( A \). Conclude that \( q \) does not have a lift \( a \) such that \( \text{sp}(a) = \text{sp}(q) \).

Exercise 2.8.28. Suppose \( A \) is a C*-algebra, \( J \) is a two-sided and norm-closed ideal of \( A \), \( \pi : A \to A/J \) is the quotient map, and \( u \) is a unitary in \( A/J \).

1. If \( \text{sp}(u) \neq \mathbb{T} \) prove that \( u \) has a lift \( v \) which is a unitary such that \( \text{sp}(v) \neq \mathbb{T} \).
2. Give an example of \( A, J, \) and \( u \in A/J \) showing that in (1) one cannot always choose lift \( v \) so that \( \text{sp}(v) = \text{sp}(u) \).
3. Give an example of \( A, J, \) and \( u \in A/J \) that cannot be lifted to a unitary.

Exercise 2.8.29. Prove that every automorphism \( \Phi \) of \( B(H) \) is inner.

Exercise 2.8.30. 1. Prove that the group of inner automorphisms of \( K(H) \) is separable with respect to the uniform topology.
2. Prove that every automorphism of \( K(H) \) is multiplier inner.
3. Prove that the group of multiplier inner automorphisms of \( K(H) \) is nonseparable with respect to the uniform topology.

Exercise 2.8.31. Suppose \( A \) is a UHF algebra. Prove that all automorphisms of \( A \) are asymptotically inner. Then prove that \( A \) has an outer automorphism.

Hint: For every \( F \in A \) and every \( \epsilon > 0 \) there exist \( u \in U(A) \) and \( b \in A_{\geq 1} \) such that \( \|uaa^* - a\| < \epsilon \) for all \( a \in F \) but \( \|uab^* - b\| \geq 1 \).

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14 All this works for locally compact abelian groups, but we defined \( C^*_r(\Gamma) \) for discrete \( \Gamma \) only.

15 Used in the proof of Theorem 14.4.7.
Exercise 2.8.32. An algebra automorphism of a C*-algebra $A$ is a bijection $\Phi: A \to A$ that preserves the addition, multiplication, and multiplication by scalars. Prove that every non-abelian C*-algebra $A$ other than $\mathbb{C}$ has an algebra automorphism $\Phi$ that is neither a *-homomorphism nor an isometry. (Cf. Lemma 1.3.4.)

Exercise 2.8.33. If $A$ is a C*-subalgebra of $B(H)$ then $\alpha \in \text{Aut}(A)$ is implemented by a unitary $u \in B(H)$ if $(\text{Ad}u) | A = \alpha$. Suppose $\alpha \in \text{Aut}(A)$ and $\varphi \in S(A)$, and identify $A$ with $\pi_\varphi[A]$. Prove that $\alpha$ is implemented by a unitary $u \in B(H_\varphi)$ if and only if $\varphi(\alpha(a^*a)) = \varphi(a^*a)$ for all $a \in A$.

Exercise 2.8.34. Prove that the following are equivalent for every C*-algebra $A$.

1. $A$ has real rank zero.
2. $A$ satisfies the conclusion of Proposition 2.7.5: If $a, b, c$ belong to $A_+, 1$ and $a \ll c \ll b$, then there exists a projection $p$ in $A$ such that $a \ll p \ll b$.
3. $A$ satisfies the conclusion of Corollary 2.7.6: If $c \in A$ is normal and $U$ and $V$ are open subsets of $\text{sp}(c)$ with disjoint closures, then there exists a projection $p$ in $C$ such that $f(c) \ll p$ for all $f \in C(\text{sp}(c))$ such that $\text{supp}(f) \subseteq U$ and $f(c) \ll 1 - p$ for all $f \in C(\text{sp}(c))$ such that $\text{supp}(f) \subseteq V$.

Exercise 2.8.35. Suppose $A$ is a unital C*-algebra. Prove that the set of invertible self-adjoint elements is dense in $A_{sa}$ if and only if the set of self-adjoint elements with a finite spectrum is dense in $A_{sa}$.

Exercise 2.8.36. Prove that every AF algebra $A$ has the property that the normal elements with a finite spectrum are dense in the set of normal elements of $A$. Then prove that the Calkin algebra does not have the latter property, and conclude that this property is strictly stronger than real rank zero.

Exercise 2.8.37. Construct a simple C*-algebra that is neither unital nor stable.

Hint: Find a nonunital inductive limit of full matrix algebras with a tracial state.

Exercise 2.8.38. Consider the example of a Bratteli diagram in Figure 2.3. Its $n$th level corresponds to the algebra $M_{F(n)}(\mathbb{C}) \oplus M_{F(n+1)}(\mathbb{C})$, where $F(n)$ is the $n$th Fibonacci number. Prove that this so-called Fibonacci algebra is a simple, unital AF algebra with a unique trace that is not a UHF algebra.

![Fig. 2.3 The Bratteli diagram of the Fibonacci algebra](image)

The following easy exercise gives a limiting example for Lemma 5.2.10.

Exercise 2.8.39. Find C*-algebras $A$ and $B$ and a hereditary C*-subalgebra $D$ of $A \otimes B$ that does not contain any elementary tensors.
Notes for Chapter 2

§2.2 Using Bratteli’s results, G.A. Elliott proved that the category of separable AF is equivalent to a subcategory of ordered abelian groups, the so-called scaled dimension groups (see e.g., [208] or [52]).

§2.4 Nuclear C*-algebras are undoubtedly the most studied class of C*-algebras. It is therefore somewhat curious that, in a technical set-theoretic sense, very few C*-subalgebras of \( \mathcal{B}(H) \) are nuclear. If \( H \) is an infinite-dimensional Hilbert space then there is a finite-dimensional subspace \( F \) of \( \mathcal{B}(H) \) such that no C*-subalgebra \( A \) of \( \mathcal{B}(H) \) that includes \( F \) is nuclear ([143]). Therefore the set of all separable nuclear C*-subalgebras of \( \mathcal{B}(H) \) is, in the language introduced in Definition 6.5.1, nonstationary. However, nearly every separable C*-algebra occurring in the literature or in the applications is nuclear, or at least isomorphic to a C*-subalgebra of a nuclear C*-algebra. Among separable C*-algebras, being isomorphic to a C*-subalgebra of a nuclear C*-algebra is, by a deep result of Kirchberg, equivalent to being exact; see [35, §3.9] for the definition of exactness. Every separable C*-subalgebra of a nuclear C*-algebra, separable or not, is exact. It is not known whether every exact C*-algebra is isomorphic to a subalgebra of a nuclear C*-algebra.

Lemma 2.4.3 is the only aspect of the deep theory of nuclear C*-algebras used in this book. However, nearly all of the counterexamples produced in the latter sections are nuclear, and many of them are even AF.

If a discrete group \( \Gamma \) is amenable then \( C_r^*(\Gamma) \) and \( C^*(\Gamma) \) are isomorphic ([35, Theorem 2.6.8]). On the opposite end of the spectrum, the reduced free group algebra of the free group with infinitely many generators, \( F_\infty \), is simple (Lemma 4.3.5) while \( C^*(F_\infty) \) has every separable C*-algebra as a quotient (Exercise 2.8.24).

By a theorem of Elliott, \( \mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2 \). The importance of this result can hardly be overstated (see [208]).

§2.7 Proposition 2.7.5 and its proof are taken from [27, Proposition V.3.2.17].

Although we will not need to consider higher values of real rank in this book, the definition is included for completeness. A C*-algebra \( A \) has real rank \( \leq n \) for some \( n \in \mathbb{N} \) if the set of \( n \)-tuples of elements in \( A_{sa} \) that do not generate a proper left ideal is dense in \( A_n \). Its real rank is infinite if there is no such \( n \). It is not difficult to see that the real rank of \( C(X) \) is equal to the covering dimension of a compact Hausdorff space \( X \).

Another important non-commutative analog of dimension is Rieffel’s stable rank (see [27]).

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16 Warning: The operator algebraist’s definition of an ordered abelian group differs from model theorist’s definition of an ordered abelian group, as in the former the ordering is not required to be linear.
Chapter 3

Representations of $C^*$-algebras

Is representation everything? Of course not.

H. R. Clinton, What Happened

Representation theory played a prominent role in the early theory of $C^*$-algebras (see e.g., [58] or [14]), and some of the material of the present chapter may bring the ’70s to mind. The representation theory of any abelian $C^*$-algebra, or any algebra of compact operators on a Hilbert space is fairly tractable. One common feature of representations $\pi: A \to \mathcal{B}(H)$ of such algebras is that the closure of $\pi[A]$ in the weak operator topology is a von Neumann algebra of type I. Kaplansky defined a $C^*$-algebra $A$ to be of type I if all of their representations have this property. This is equivalent to the assertion that the image of $A$ under any irreducible representation includes the algebra of compact operators on the target Hilbert space. Type I $C^*$-algebras have well-behaved representation theory. $C^*$-algebras that are not type I are called non-type I.\(^1\) Glimm proved that the representation theory of separable non-type I $C^*$-algebras is very complicated: a separable non-type I $C^*$-algebra has $c$ spatially inequivalent irreducible representations and factorial representations of type II and type III ([118]). Sakai extended parts of Glimm’s theorem to the not necessarily separable case ([215]). After this, there was little progress on remaining fundamental problems in representation theory of nonseparable non-type I $C^*$-algebras, such as Naimark’s problem: Is every $C^*$-algebra all of whose irreducible representations are unitarily equivalent isomorphic to the algebra of compact operators on some Hilbert space? The study of $C^*$-algebras moved on to other topics. Only in [5] Ake- mann and Weaver proved that it is relatively consistent with ZFC that Naimark’s problem has a negative solution, initiating the thread to which this text belongs.

We will start the present chapter with a brief review of the theory of von Neumann algebras. We will prove Kadison’s Transitivity Theorem for irreducible representations of $C^*$-algebras and its extension by Glimm and Kadison to direct sums of inequivalent irreducible representations. Other highlights include the proof of (a part of) Glimm’s theorem stated above and study of the second dual of a $C^*$-algebra. All

\(^1\) Von Neumann algebras are rarely type I $C^*$-algebras, even if they are of type I as von Neumann algebras. See the discussion of terminology in the preface to [14].
averaging techniques needed in this book are treated in a separate section. We also prove the structure theorems for completely positive maps (Stinespring’s Theorem) and completely positive maps of order zero.

3.1 Topologies on $\mathcal{B}(H)$ and von Neumann algebras

Besides the (operator) norm topology there are at least six weak topologies [on $\mathcal{B}(H)$] that are relevant for operator algebra theory.

G.K. Pedersen, Analysis NOW

In this section we cover the bare minimum of the theory of von Neumann algebras that will be needed in the following sections. This includes Kaplansky Density Theorem, order-completeness, the Bicommutant Theorem, Borel functional calculus, the, and the existence of preduals. Many proofs are omitted or barely sketched, and the readers unfamiliar with the material and not willing to take these results on faith can consult any of the many excellent sources. We also briefly discuss factors and have our first encounter with masas (maximal abelian self-adjoint subalgebras).

3.1.1 Topologies on $\mathcal{B}(H)$

It is impossible to understand the representation theory of C*-algebras without considering von Neumann algebras. The (concrete) C*-algebras are the self-adjoint operator algebras closed under operator norm topology on $\mathcal{B}(H)$, von Neumann algebras are the self-adjoint operator algebras closed under any or all of the other important weak topologies on $\mathcal{B}(H)$. This section concludes with a brief discussion of masas (maximal abelian subalgebras).

While a Hilbert space $H$ is endowed with two important topologies, norm and weak, the algebra $\mathcal{B}(H)$ has a vector space topology for every occasion. In this book we will consider only a few of them. Pointwise convergence on $H$, when $H$ is considered with respect to the norm topology, is the strong operator topology, abbreviated SOT. Pointwise convergence on $H$, when $H$ is considered with respect to the weak topology, is the weak operator topology, abbreviated WOT. For a bounded net $a_\lambda$, for $\lambda \in \Lambda$, in $\mathcal{B}(H)$ and $a \in \mathcal{B}(H)$ we have WOT-$\lim_\lambda a_\lambda = a$ if and only if $\lim_\lambda (a_\lambda \xi | \eta) = (a \xi | \eta)$ for all $\eta$ and $\xi$ in $H$. Therefore WOT-convergence of bounded nets coincides with the pointwise convergence of the associated sesquilinear forms.

The norm, strong, and weak closures of $Z \subseteq \mathcal{B}(H)$ are ordered as follows

$$Z^\| \subseteq Z^{\text{SOT}} \subseteq Z^{\text{WOT}}.$$
For an infinite-dimensional Hilbert space $H$ these three topologies differ even on the unit ball $B(H) \leq 1$ of $B(H)$.

**Example 3.1.1.** Let $s$ be the unilateral shift of a fixed basis on $H$ (Example 1.1.1).

1. Then $\lim_n (s^n \xi | \eta) = 0$ for all $\xi, \eta \in H$ and therefore WOT-$\lim_n s^n = 0$. On the other hand, $s^n$ is an isometry for all $n$ and the sequence $s^n$, for $n \in \mathbb{N}$, is not SOT-convergent.
2. WOT-$\lim_n (s^n)^* = $ SOT-$\lim_n (s^*)^n = 0$. Since $(s^n)^* s^n = 1$, multiplication is not jointly WOT-continuous on the unit ball $B(H) \leq 1$.
3. Both addition and scalar multiplication are continuous with respect to each of SOT, WOT, and the norm topologies. Multiplication is jointly continuous in norm topology, jointly continuous on bounded sets in SOT, and by (2) only separately continuous (even on bounded sets) in WOT.
4. Every SOT-neighbourhood of 0 in $B(H)$ contains $T$ such that $T^2 = 1$ (this is an exercise), showing that the multiplication is not jointly SOT-continuous on $B(H)$.

**Definition 3.1.2.** A von Neumann algebra is a strongly closed, unital C*-subalgebra of $B(H)$. A W*-algebra is a C*-algebra which is isomorphic to a von Neumann algebra.

**Lemma 3.1.3.** Suppose $M$ is a von Neumann algebra and $a_\lambda$, for $\lambda \in \Lambda$, is an increasing net in $M_+$ which is bounded above by some $b \in M_+$. Then there exists $a \in M_+$ such that SOT-$\lim_\lambda a_\lambda = \sup_\lambda a_\lambda = a$.

**Proof.** Since $M$ is closed in the strong operator topology it suffices to prove the lemma in case when $M = B(H)$. Let $a_\lambda$ be the sesquilinear form corresponding to $a_\lambda$. We define sesquilinear form $\alpha : H^2 \to \mathbb{C}$ in two steps. First, for $\xi \in H$ let

$$\alpha(\xi, \xi) := \lim_\lambda \alpha_\lambda(\xi, \xi).$$

This limit exists and it is equal to $\sup_\lambda \alpha_\lambda(\xi, \xi)$ since the net is increasing and bounded above by $(b \xi | \xi)$.

Second, let

$$\alpha(\xi, \eta) := \frac{1}{4} \sum_{k=0}^{3} i^k \alpha(\xi + i^k \eta, \xi + i^k \eta).$$

Then $\alpha(\xi, \eta) = \lim_\lambda \alpha_\lambda(\xi, \eta)$ for all $\xi$ and $\eta$, and $\alpha$ is a sesquilinear form corresponding to a positive operator of norm $\leq \|b\|$. By the definition $a_\lambda \leq a$ for all $\lambda$. If $a' \leq a$ and $a' \neq a$ then for some $\xi \in H$ we have $(a' \xi | \xi) < (a \xi | \xi)$, but this implies that $a_\lambda \not\leq a'$ for some $\lambda$. Therefore $a = \sup_\lambda a_\lambda$. 

**3.1.2 Commutants and Bicommutants**

The commutant of $X \subseteq B(H)$ is $X' := \{ b \in B(H) : [a, b] = 0 \text{ for all } a \in X \}$. 
Lemma 3.1.4. The commutant of a subset $X$ of $\mathcal{B}(H)$ is an algebra, and it is self-adjoint if $X$ is. If $a$ is normal, then $\{a\}'$ is a self-adjoint algebra.

Proof. The first statement is straightforward. The second follows by (or rather, is) Fuglede’s Theorem (Theorem C.6.12).

A $C^*$-subalgebra $A$ of $\mathcal{B}(H)$ acts nondegenerately if $A\xi = \{0\}$ implies $\xi = 0$ for all $\xi \in H$. We also say that $A$ is a nondegenerate $C^*$-subalgebra of $\mathcal{B}(H)$.

A proof of Lemma 3.1.5 uses a trick to ‘fold’ $n$ copies of a Hilbert space into a single Hilbert space. The amplification of a Hilbert space $H$ by $n$ is the Hilbert space $H^n$. An $a \in \mathcal{B}(H^n)$ is identified with an $M_n(\mathcal{B}(H))$, and the diagonal $^*$-homomorphism $\Phi : \mathcal{B}(H) \to \mathcal{B}(H^n)$ is defined by $\Phi(a) := \text{diag}(a,a,\ldots,a)$.

Lemma 3.1.5. If $A$ is a nondegenerate unital self-adjoint $C^*$-subalgebra of $\mathcal{B}(H)$ then $A$ is SOT-dense in its bicommutant $A''$.

Proof. Fix $b \in A''$. If $\xi \in H$ then the space $H_\xi := \{a(\xi) | a \in A\}$ is an $a$-invariant closed subspace of $H$ for all $a \in A$. Since $A$ is self-adjoint, $H_\xi$ is invariant for $a^*$ for all $a \in A$ and therefore the projection $p_\xi$ to $H_\xi$ belongs to $A'$. Hence $bp_\xi = p_\xi b$ and $\inf_{a \in A\cap A^*} \|a - b\|_\xi = 0$.

Consider the amplification of $H$ by $n$ and let $\Phi$ denote the diagonal $^*$-homomorphism. Then $\Phi[A]$ is a $^*$-$C^*$-subalgebra of $\mathcal{B}(H^n) \cong M_n(\mathcal{B}(H))$ acting nondegenerately. Fix $c \in \Phi[A]''$ and write it in the matrix form $c = (c_{ij})$. Then $\Phi(a)c = c\Phi(a)$ implies $ac_{ij} = c_{ij}a$ for all $i,j$. Therefore all matrix entries of $c$ belong to $A'$ and $[b,c] = 0$. Since $c$ was arbitrary, we conclude that $\Phi(b) \in \Phi[A]''$.

Fix a SOT-neighbourhood $U := \{a \in \mathcal{B}(H) : \max_{j<n} \|b\xi_j - a\xi_j\| < \varepsilon\}$ of $b$. By applying the above argument to $\Phi[A]$ and $\xi = (\xi_0,\ldots,\xi_{n-1}) \in H^n$ we conclude that $\inf_{a \in A\cap A^*} \|\Phi(a) - \Phi(b)\|_\xi = 0$; equivalently, $\inf_{a \in A} \max_{j<n} \|(a-b)\xi_j\| = 0$. Hence $A$ intersects every SOT-neighbourhood of $b$ nontrivially and $b \in \overline{A}_{\text{SOT}}$. \hfill \Box

Theorem 3.1.6 (von Neumann’s Bicommutant Theorem). If $A$ is a nondegenerate self-adjoint subalgebra of $\mathcal{B}(H)$ then $A'' = \overline{A}_{\text{SOT}} = \overline{A}_{\text{WOT}}$.

Proof. For all $Z \subseteq \mathcal{B}(H)$ we have $Z_{\text{SOT}} \subseteq Z_{\text{WOT}}$ and $Z'$ is WOT-closed, therefore $\overline{A}_{\text{WOT}} \subseteq A''$. Finally, Lemma 3.1.5 implies $A'' = \overline{A}_{\text{SOT}}$. \hfill \Box

Further iterations of the commutant relation do not result in anything new. Since $A \subseteq B$ implies $A' \supseteq B'$, we have $A'' = A'$ for any self-adjoint $A \subseteq \mathcal{B}(H)$.

If $Z \subseteq \mathcal{B}(H)$ then $W^*(Z)$ denotes the von Neumann algebra generated by $Z$. All nontrivial implications in the following analog of Lemma 1.2.4 are consequences of Theorem 3.1.6.\(^2\)

Lemma 3.1.7. Suppose $Z \subseteq \mathcal{B}(H)$. Then $M = W^*(Z)$ if and only if any of the following (a posteriori, equivalent) conditions applies.

\(^2\) Keep in mind that every von Neumann algebra is a unital subalgebra of $\mathcal{B}(H)$—this is why we consider $^*$-polynomials with nonzero constant terms.
3.1 Topologies on $\mathcal{B}(H)$ and von Neumann algebras

1. The set $M$ is equal to the intersection of all von Neumann subalgebras of $\mathcal{B}(H)$ that include $Z$.
2. The set $M$ is equal to the WOT-closure of $\{ p(\tilde{x}) : \tilde{x} \in Z^n, p(\tilde{x}) \text{ is a } \ast\text{-polynomial in } n \text{ non-commuting variables with } n \in \mathbb{N} \}$.
3. The set $M$ is equal to the SOT-closure of $\{ p(\tilde{x}) : \tilde{x} \in Z^n, p(\tilde{x}) \text{ is a } \ast\text{-polynomial in } n \text{ non-commuting variables with } n \in \mathbb{N} \}$.
4. $M = (Z \cup Z')''$. \hfill $\square$

Example 3.1.8. 1. Every von Neumann algebra is a $C^*$-algebra, but this is about as illuminating as saying that every continuous $f : [0, 1] \to [0, 1]$ is Borel-measurable.
2. The most obvious examples of von Neumann algebras are $\mathcal{B}(H)$ and $M_n(\mathbb{C})$ for $n \geq 1$. The most obvious example of a $C^*$-algebra that is not a von Neumann algebra is the algebra $C(H)$ of compact operators on an infinite-dimensional Hilbert space $H$. It is not a von Neumann algebra because it is WOT-dense in $\mathcal{B}(H)$. Since $C(H)$ is not unital, it is not even isomorphic to a von Neumann algebra.
3. If $(X, \mu)$ is a measure space then $L_\infty(X, \mu)$ is an abelian von Neumann algebra in $\mathcal{B}(L_2(X, \mu))$. Every abelian von Neumann algebra can be extended to a maximal abelian von Neumann algebra. Maximal abelian von Neumann subalgebras of $\mathcal{B}(H)$ are isomorphic—algebraically and topologically—to one of the form $L_\infty(X, \mu)$ (Theorem C.6.11, see also Example 3.1.20 and Example 3.1.21).
4. Continuing the discussion of (3), by the Gelfand–Naimark theorem $L_\infty(X, \mu)$ is isomorphic to $C(Y)$ for a compact Hausdorff space $Y$. This is the Stone space (see §1.3.1) of the quotient of the $\sigma$-algebra of $\mu$-measurable sets modulo the ideal of $\mu$-null sets.
5. Since for a $C^*$-subalgebra $B$ of $\mathcal{B}(H)$ we have $B' = B''$, Theorem 3.1.6 implies that $\pi[A]'$ is a von Neumann algebra for every representation of a $C^*$-algebra $A$—and that’s why we are here.

An important example is held back until Chapter 4 (Definition 4.4.5).

If $a \in \mathcal{B}(H)$ is normal, then by Theorem 3.1.6 and Theorem 1.4.2 we have $W^*(a) = \{a, a^\ast, 1\}''$. If $a$ is not necessarily normal and $a = v|a|$ is its polar decomposition (Theorem 1.1.3), then $v \in W^*(a)$ (and of course $|a| \in C^*(a) \subseteq W^*(a)$). Therefore every element of a von Neumann algebra $M$ has a polar decomposition in $M$ (cf. Exercise 1.11.17).

Theorem 3.1.9 (Kaplansky’s Density Theorem). Assume $A \subseteq \mathcal{B}(H)$ is a $C^*$-algebra. Then

1. $A_{sa}$ is SOT-dense in $(A'')_w$.
2. The unit ball of $A_{sa}$ is SOT-dense in the unit ball of $(A'')_w$.
3. The unit ball $A_{\leq 1}$ of $A$ is SOT-dense in the unit ball of $A''$.
4. If $A$ is unital, then $U(A)$ is SOT-dense in $U(A'')$.

Proof. (1) Theorem 3.1.6 implies that the SOT-closure of $A$ is $A''$. A functional on $\mathcal{B}(H)$ is strongly continuous if and only if it is weakly continuous (§194, Theorem 2.1.5]). Since the adjoint operation is weakly continuous, $A_{sa} = (A'')_w$.\hfill $\square$
Proposition 3.1.11. Every von Neumann algebra has real rank zero.

Example 3.1.12. Suppose $M$ is a von Neumann algebra and $a \in M$ is normal. Then $W^*(a)$ is isomorphic to $L_\infty(sp(a), \mu)$ for some Radon probability measure $\mu$ on $sp(a)$. The isomorphism sends $a$ to the equivalence class of the identity function and $f$ to $f(a)$.

The following may be the most rudimentary application of the Borel functional calculus.

Proposition 3.1.11. Every von Neumann algebra has real rank zero.

Proof. We will prove a stronger result: Every normal element $a$ of a von Neumann algebra $M$ can be approximated by normal elements with finite spectrum. Theorem 3.1.10 implies that $W^*(a)$ is isomorphic to $L_\infty(sp(a), \mu)$ for some Radon probability measure $\mu$ on $sp(a)$. Every function in $L_\infty$ can be uniformly approximated by step functions, and the conclusion follows.

Example 3.1.12. Suppose $M$ is a von Neumann algebra.

1. By the Borel functional calculus, every unitary $u \in M$ is of the form $u = \exp(ia)$ for $0 \leq a \leq 2\pi$ in $M$. Since $u$ is normal and $sp(u) \subseteq \mathbb{T}$, $a$ can be chosen as $f(u)$ where $f(e^{it}) = t$ for $0 \leq t < 2\pi$.
2. If $a \in M$ is normal and $E$ is a Borel subset of $sp(a)$ then the characteristic function of $E$ belongs to $W^*(a)$. It is the spectral projection of $a$, denoted $1_E(a)$.
3. Define $f : [0, \infty) \to \{0, 1\}$ by $f(0) = 0$ and $f(t) = 1$ for $t \neq 0$. For $a \in M$ both projections $p := f(|a|)$ and $q := f(|a'|)$ belong to $W^*(a)$. Since $|a|p = |a|$ and (by the polar decomposition) $a = v|a|$, we have $ap = a$. In addition, for every nonzero projection $r \leq p$ we have $|a|r \neq 0$ and therefore $ar \neq 0$. Similarly, $qa = a$ and for every nonzero projection $r \leq q$ we have $ra \neq 0$. The projections $p$ and $q$ are the domain projection and the range projection, respectively, of $a$. 

3.1.3 Borel Functional Calculus

Separate SOT-continuity of multiplication in $B(H)$ implies that the SOT-closure of any self-adjoint abelian subalgebra of $B(H)$ is abelian. Therefore for every normal element $a$ in a von Neumann algebra $M$ the isomorphism between $C(sp(a))$ and $C^*(a, 1)$ (the continuous functional calculus, Theorem 1.4.2) can be extended to a $^*$-homomorphism from $L^\infty(sp(a))$ into $W^*(a)$ by using the standard methods from measure theory and Lemma 3.1.3. This gives rise to the Borel functional calculus in every von Neumann algebra $M$.

Theorem 3.1.10. Suppose $M$ is a von Neumann algebra and $a \in M$ is normal. Then $W^*(a)$ is isomorphic to $L_\infty(sp(a), \mu)$ for some Radon probability measure $\mu$ on $sp(a)$. The isomorphism sends $a$ to the equivalence class of the identity function and $f$ to $f(a)$.

The following may be the most rudimentary application of the Borel functional calculus.
3.1 Topologies on $\mathcal{B}(H)$ and von Neumann algebras

Because of the Borel functional calculus, every von Neumann algebra has an abundance of projections. Suppose $M \subseteq \mathcal{B}(H)$ is a von Neumann algebra and $p$ and $q$ are projections in $M$. Then $p \leq q$ (as defined in §1.5) if and only if $p[H] \subseteq q[H]$. Also, all joins and meets exist in $M$ and are given by

$$p \wedge q := \text{the projection onto } p[H] \cap q[H]$$

$$p \vee q := \text{the projection onto } p[H] \cup q[H].$$

That is, $\text{Proj}(M)$ is a lattice. It is in fact a complete lattice, because the definitions of joins and meets naturally generalize to infinite joins and meets (see however Proposition 13.3.3).

**Lemma 3.1.13.** Suppose $A$ is a $C^*$-algebra of real rank zero and $J$ is a two-sided, norm-closed, ideal in $A$. Then every projection $q \in A/J$ lifts to a projection $p \in A$.

**Proof.** Let $\pi$ denote the quotient map and fix $a \in A$ such that $\pi(a) = p$. The $\sigma$-unital hereditary subalgebra $a^*Aa$ of $A$ has a sequential approximate unit consisting of projections, $p_n$, for $n \in \mathbb{N}$. Then $q_n := \pi(p_n)$ is an increasing sequence of projections such that $q_n \leq q$ for all $n$ and $\lim_n \|q_n - q\| = 0$. Therefore $q_n = q$ for a large enough $n$, and $p_n$ is as required. $\Box$

I should define the ultraweak topology on a $W^*$-algebra (see [27, §I.3.1.3]). If $M \subseteq \mathcal{B}(H)$ is a $W^*$-algebra, consider the amplification $\bigoplus \mathbb{N} : M \rightarrow \mathcal{B}(\bigoplus \mathbb{N} H)$ and identify $M$ with its range. The ultraweak topology (also known as the $\sigma$-weak topology) on $M$ is the restriction of the weak operator topology on $\mathcal{B}(\bigoplus \mathbb{N} H)$ to $M$. The ultraweak topology coincides with the weak topology on bounded subsets of $M$. Unlike WOT, the ultraweak topology on a $W^*$-algebra is intrinsic to the algebra and does not depend on its representation on a Hilbert space.

### 3.1.4 Preduals

We start with a noncommutative version of a basic fact about the classical Banach spaces. With $\ast$ denoting the Banach space dual (some operator algebraists use $\prime$ instead of $\ast$), we have $c_0^* \cong \ell_1$ and $\ell_1^* \cong \ell_\infty$, with the dualities implemented by the bilinear functional

$$(x, y) \mapsto \sum_j x(j)y(j).$$

Let $H := \ell_2(\mathbb{J})$ for some set $\mathbb{J}$. The algebra $\mathcal{B}(H)$ is the noncommutative analog of $\ell_\infty$ and its ideal $\mathcal{K}(H)$ is the noncommutative analog of $c_0$. An operator $a \in \mathcal{B}(\ell_2(\mathbb{J}))$ is a trace class operator if $\sum_{j \in \mathbb{J}} |\langle a | \delta_j, \delta_j \rangle| < \infty$. The trace of a trace class operator $a$ is

$$\text{tr}(a) := \sum_{j \in \mathbb{J}} \langle a | \delta_j, \delta_j \rangle.$$ 

As in the finite-dimensional case, the value of $\text{tr}(a)$ does not depend on the choice of the basis. The ideal of trace class operators on $H$ is denoted $\mathcal{B}_1(H)$. 
Theorem 3.1.14. Fix a Hilbert space $H$ and consider $\mathcal{K}(H)$, $\mathcal{B}_1(H)$, and $\mathcal{B}(H)$ as Banach spaces. Then $\mathcal{K}(H)^* \cong \mathcal{B}_1(H)$ and $\mathcal{B}_1(H)^* \cong \mathcal{B}(H)$, with dualities implemented by the bilinear map $(a,b) \mapsto \text{tr}(ab)$.

**Proof.** See for example [196, Theorem 3.4.13].

Theorem 3.1.14 implies that every WOT-closed $C^*$-subalgebra $M$ of $\mathcal{B}(H)$ is the dual space of a quotient of $\mathcal{B}_1(H)$ by the subspace $\{a \in \mathcal{B}_1(H) : (a,m) = 0 \text{ for all } m \in M\}$. We will describe this duality in some detail.

**Definition 3.1.15.** A functional $\varphi$ on a von Neumann algebra $M$ is normal if it is order-continuous; this means that if $a_\lambda$, for $\lambda \in \Lambda$, is a bounded increasing net of positive operators in $M$ then $\lim \varphi(a_\lambda) = \varphi(\lim a_\lambda)$ (Lemma 3.1.3 implies that the right-hand side is well-defined). Normal states are normal functionals that are also states.

Every vector state is normal. More generally, a functional on $\mathcal{B}(H)$ is normal if and only if it is of the form $b \mapsto \text{tr}(ab)$ for a trace-class operator $a$ (Exercise 3.10.2). Normal functionals on $M$ form a Banach subspace of its Banach space dual $M^*$; this subspace is denoted $M_*$ and called the predual of $M$. In general, we have an extension of Theorem 3.1.14.

**Theorem 3.1.16 (Sakai’s Theorem).** Every von Neumann algebra $M$ is isomorphic to the dual space of a unique Banach space $M_*$. 

**Proof.** We will prove only the (much easier) existence. For the uniqueness see e.g., [215]. Since $M \subseteq \mathcal{B}(H)$ and $\mathcal{B}(H) \cong \mathcal{B}_1(H)^*$, Proposition C.3.9 implies that $M$ is isomorphic to the Banach space dual of $\mathcal{B}_1(H)/M^\perp$. 

### 3.1.5 Tensor Products and Factors

The tensor product operation is better behaved in the categories of von Neumann algebras and $W^*$-algebras than it is in the theory of $C^*$-algebras. Given von Neumann algebras $M$ and $N$, the tensor product $M \otimes N$ is defined as the SOT-closure of the algebraic tensor product $M \circ N$. If $M$ and $N$ are $W^*$-algebras, it can be proved that this definition does not depend on the representations of $M$ and $N$ as von Neumann algebras.

If a Hilbert space is a tensor product of two Hilbert spaces, $H = K \otimes L$, then 

$$\mathcal{B}(H) = \mathcal{B}(K) \otimes \mathcal{B}(L).$$

Thus $\mathcal{B}(K)$ and $\mathcal{B}(L)$ can be considered as ‘factors’ of $\mathcal{B}(H)$. This observation and von Neumann’s Bicommutant Theorem motivate the abstract definition of von Neumann factors.

The center of an algebra $A$ is the set $Z(A) := \{a \in A : [a,b] = 0 \text{ for all } b \in A\}$. 

If $M$ is a von Neumann algebra then its center satisfies $Z(M) = M \cap M' = Z(M')$, hence $Z(M) = \mathbb{C}$ if and only if $Z(M') = \mathbb{C}$. Von Neumann algebras with trivial center are called factors.

Murray and von Neumann distinguished three types of factors. Every $M_n(\mathbb{C})$ is a type I$_n$ factor and $\mathcal{B}(H)$ is a type I$_\infty$ factor of $H$ is infinite-dimensional.\footnote{A set theorist may distinguish factors of type I$_\kappa$ for different infinite cardinals $\kappa$.} A factor is of type III if every nonzero projection is infinite, and of type II if it is of neither type I nor of type III. It can be proved that in a separably represented type III factor all nonzero projections are Murray–von Neumann equivalent. Murray and von Neumann proved that type II factors come in two varieties: A factor has type II$_1$ if it has a tracial state. A state $\tau$ is tracial if it satisfies $\tau(ab) = \tau(ba)$ for all $a, b$. We shall return to tracial states in §4.1. A factor has type II$_\infty$ if it has a corner isomorphic to a type II$_1$ factor and no tracial state. It is not difficult to prove that this is equivalent to having $M \cong M_0 \overline{\otimes} \mathcal{B}(H)$ where $M_0$ is a II$_1$ factor isomorphic to a corner of $M$ and $H$ is an infinite-dimensional Hilbert space.

3.1.6 Maximal Abelian C$^*$-subalgebras of $\mathcal{B}(H)$

Noncommutative measure spaces, along with all noncommutative topological spaces\footnote{That is, von Neumann algebras and C$^*$-algebras.} contain large commutative subspaces. It is about time that we introduced them.

**Definition 3.1.17.** Masa of a C$^*$-algebra $A$ is a maximal abelian C$^*$-subalgebra of $A$. The acronym stands for ‘Maximal Abelian C$^*$-subalgebra’ or ‘MAximal Self-Adjoint C$^*$-subalgebra.’

The Axiom of Choice (in the form of Zorn’s Lemma) implies that every self-adjoint abelian C$^*$-subalgebra of a C$^*$-algebra is included in a masa.

Two C$^*$-subalgebras $A$ and $B$ of $\mathcal{B}(H)$ are unitarily equivalent if there exists a unitary $u$ such that $uAu^* = B$.

**Example 3.1.18.** The diagonal matrices, $D_n$, comprise a masa in $M_n(\mathbb{C})$. Since every finite-dimensional abelian C$^*$-algebra is isomorphic to $\mathbb{C}^m$ for some $m$, every masa in $M_n(\mathbb{C})$ is isomorphic to $D_n$. Also, every two masas in $M_n(\mathbb{C})$ are unitarily equivalent. This applies to any two masas in any finite-dimensional C$^*$-algebra.

If $A = \lim_j M_n(j)(\mathbb{C})$ is a UHF algebra then algebras $D_n(j)$ form a inductive system and their limit $D$ is the diagonal masa in $A$. Although not difficult, the fact that $D$ is a masa in $A$ is not automatic.

**Definition 3.1.19.** If $B$ is a subset of a C$^*$-algebra $A$, the relative commutant of $B$ in $A$ is $B' \cap A := \{a \in A : ab = ba \text{ for all } b \in B\}$.

The relative commutant is a subalgebra, and it is a C$^*$-subalgebra if $B$ is self-adjoint.

There are two basic examples of masas in $\mathcal{B}(H)$.
Example 3.1.20. Suppose \((X, \mu)\) is an atomic measure space. With \(J\) denoting the set of all atoms of \((X, \mu)\), we clearly have \(L_2(X, \mu) \cong \ell_2(J)\) via the isomorphism that sends \(f \in L_2(X, \mu)\) to \(\sum_j f(j) \mu\{j\}^{-1/2}\). In this case \(L_\infty(X, \mu)\) is an atomic masa. Let \(e_j\), for \(j \in J\), denote the standard basis of \(\ell_2(J)\). Then the atomic masa is isomorphic to \(\ell_\infty(J) \cong C(\beta J)\), the Čech–Stone compactification of \(J\) taken with the discrete topology.

For \(X \subseteq J\) let \(p_X := \text{proj}_{\ell_2(J)}\{e_j; f \in J\}\). Every such projection belongs to \(L_\infty(X, \mu)\). Since every projection in \(L_\infty(X, \mu)\) has each \(e_j\) as an eigenvector, the map \(X \mapsto p_X\) is a Boolean algebra isomorphism between \(\mathcal{P}(J)\) and \(\mathcal{P}(L_\infty(X, \mu))\).

Example 3.1.21. Suppose \((X, \mu)\) is any atomless probability measure space. Then \(H \cong L_2(X, \mu)\) and \(L_\infty(X, \mu) \subseteq \mathcal{B}(H)\). In this case \(L_\infty(X, \mu)\) is an atomless masa. The projections in \(L_\infty(X)\) are the characteristic functions of measurable sets (modulo the \(\mu\)-null sets). Therefore the lattice of projections of an atomless masa is isomorphic to the measure algebra of \((X, \mu)\).

### 3.2 Completely Positive Maps

In this section we introduce positivity and complete positivity of linear maps. We prove Stinespring’s representation theorem for completely positive maps between \(C^*\)-algebras and a special case of Arveson’s extension theorem. We then prove the representation theorem for completely positive maps of order zero. The latter is used in our proof of Glimm’s Theorem (Theorem 3.7.2).

Richness of the theory of \(C^*\)-algebras is reflected in the variety of natural morphisms (in addition to *-homomorphisms and homomorphisms) between them. Suppose \(A\) and \(B\) are \(C^*\)-algebras and \(\varphi: A \to B\) is a bounded linear map. We say that \(\varphi\) is positive if it sends the positive cone of \(A\) into the positive cone of \(B\).

**Lemma 3.2.1.** Every positive linear map between \(C^*\)-algebras is bounded.

**Proof.** The triangle inequality implies \(\|a + b\| \leq \|a\| + \|b\|\) for positive \(a\) and \(b\). Assume \(\varphi\) is positive and unbounded. Fix \(a_n \geq 0\) such that \(\|a_n\| \leq 2^{-n}\) and \(\|\varphi(a_n)\| = 2^n\). Then \(a = \sum_n a_n\) is bounded but \(\|\varphi(a)\| \geq 2^n\) for all \(n\).

Given \(\varphi: A \to B\), for \(n \geq 1\) the map

\[
\varphi \otimes \text{id}_n: M_n(A) \to M_n(B)
\]

(sometimes denoted \(\varphi^{(n)}\)) is defined by applying \(\varphi\) entrywise.

**Definition 3.2.2.** A linear map \(\varphi\) between operator algebras is said to be completely positive if \(\varphi \otimes \text{id}_n\) is positive for all \(n \geq 1\).

We introduce two common abbreviations. A map is contractive and completely positive (or \(c.p.c.,\) or \(c.c.p.\)) if it is completely positive and of norm \(\leq 1\). A map is \(u.c.p.\) if it is unital and completely positive.
Example 3.2.3. 1. If \( \Phi \) is a \( * \)-homomorphism then it is completely positive. Since \( b \geq 0 \) if and only if \( b = a^*a \) for some \( a \), and \( \Phi(a^*a) = \Phi(a)^*\Phi(a) \), \( \Phi \) is positive. Since \( \Phi \otimes \text{id}_n \) is a \( * \)-homomorphism for all \( n \geq 1 \), it is completely positive.

2. Suppose \( \Phi: A \rightarrow B \) is a \( * \)-homomorphism and \( v \in B \). Then \( a \mapsto v\Phi(a)v^* \) defines a compression of \( \Phi \) by \( v \). Compressions provide simple examples of completely positive maps that are not necessarily \( * \)-homomorphisms.

3. Every state is completely positive. This is because a state \( \phi \) is a compression of the corresponding GNS representation to the 1-dimensional space spanned by the cyclic vector \( \xi_\phi \) (as constructed in Proposition 1.10.3).

And now, for some counterexamples.

Example 3.2.4. 1. An example of a positive map that is not completely positive is the transpose map \( f: M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C}) \). It is clearly positive, but it satisfies (we naturally identify \( M_2(M_2(\mathbb{C})) \) with \( M_4(\mathbb{C}) \))

\[
f \otimes \text{id}_2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

The matrix on the left-hand side is positive (being equal to \( 2p \) for a projection \( p \)), but the matrix \( a \) on the right-hand side isn’t: it has \( -1 \) as an eigenvalue.

2. A u.c.p. bijection from a \( C^* \)-algebra onto itself whose inverse is continuous, but not positive. Define \( \psi: \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) by \( \psi(x, y) = (x, \frac{1}{2}(x + y)) \). Both coordinate maps are states and therefore completely positive. This easily implies that \( \psi \) is completely positive. The inverse of \( \psi \) is a continuous linear map that sends \((1, 0)\) to \((1, -1)\).

Compressions of \( * \)-homomorphisms are essentially the only examples of completely positive maps.

**Theorem 3.2.5 (Stinespring).** If \( A \) is a \( C^* \)-algebra then \( \varphi: A \rightarrow \mathcal{B}(H) \) is completely positive if and only if there are a representation \( \pi: A \rightarrow \mathcal{B}(K) \) on a Hilbert space \( K \) and \( v \in \mathcal{B}(K, H) \) such that \( \varphi(a) = v\pi(a)v^* \) for all \( a \in A \).

Also, \( \| \varphi \| = \sup_{0 \leq a \leq 1} \| \varphi(a) \| \).

**Proof.** We sketch a proof in the case when \( A \) is unital. This suffices since a completely positive map from a nonunital \( C^* \)-algebra can be extended to a completely positive map between the unitizations. (This however requires more work than in the case of \( * \)-homomorphisms—see [35, Proposition 2.2.1].)

On the algebraic tensor product \( A \otimes H \) define a bilinear map \( (\cdot|\cdot)_\varphi \) by

\[
(a \otimes \xi | b \otimes \eta)_\varphi := (\varphi(b^*a)\xi | \eta).
\]

A routine GNS-style computation shows that this is a pre-inner product. The Hilbert space \( K \) is obtained by quotienting and completing, and \( \pi \) is the left multiplication,
\( \pi(b)(a \otimes \xi) = (ba \otimes \xi) \). Finally, \( v \in \mathcal{B}(K,H) \) is defined as \( v\xi := 1 \otimes \xi \). Then \( \varphi(a) = v\pi(a)v^* \). All of the (routine) calculations are left to the reader.

The norm estimate is proved using the Cauchy–Schwarz inequality exactly as in the proof of Lemma 1.7.4. \( \square \)

The triplet \( (\pi, K, v) \) is the Stinespring dilation of \( \varphi \). The following special case of Arveson’s extension theorem is an easy application of Stinespring’s Theorem.

**Corollary 3.2.6.** Suppose \( A \) is a C*-algebra and \( B \) is a C*-subalgebra of \( A \). Then every c.p.c. map \( \varphi: B \to \mathcal{B}(H) \) can be extended to a c.p.c. map \( \psi: A \to \mathcal{B}(H) \).

**Proof.** First we show that a *-homomorphism \( \pi: B \to \mathcal{B}(H) \) can be dilated to a *-homomorphism \( \sigma: A \to \mathcal{B}(K) \) for some \( K \supseteq H \). A *-homomorphism \( \pi: B \to \mathcal{B}(H) \) is a direct sum of cyclic representations by Proposition 1.10.10. A cyclic representation of \( B \) can be dilated to a cyclic representation of \( A \) by extending the state associated with it and comparing the GNS representations (Exercise 1.11.49). A dilation of \( \pi \) is the direct sum of dilations of the summands of \( \pi \).

Let \( (\pi, K, \nu) \) be a Stinespring dilation of \( \varphi \). By the first paragraph, we can find a Hilbert space \( L \supseteq K \) and a *-homomorphism \( \sigma: A \to \mathcal{B}(L) \) such that (with \( p_K \) denoting the projection of \( L \) to \( K \)) \( \pi(b) = p_K \sigma(b) p_K \).

Extend \( \nu \) to \( w \in \mathcal{B}(L, H) \) so that \( \ker(w) \supseteq K^\perp \). Then \( \psi(a) := w\sigma(a)w^* \) is the required c.p.c. map from \( A \) into \( \mathcal{B}(H) \). \( \square \)

### 3.2.1 Completely Positive Maps of Order Zero

The order zero maps have recently played a key role in the Elliott classification programme for nuclear C*-algebras. We will use them to prove (and rephrase) a 1960 theorem of Glimm’s (Theorem 3.7.2).

**Definition 3.2.7.** A completely positive map \( \varphi: A \to B \) has order zero if it preserves orthogonality: If \( a, b \) are in \( A_+ \) and \( ab = 0 \) then \( \varphi(a)\varphi(b) = 0 \).

**Example 3.2.8.** Suppose \( A \) and \( B \) are C*-algebras.

1. If \( A \) is unital, \( h \in A_{+1} \), then \( \varphi: B \to A \otimes B \) defined by \( h(b) = h \otimes b \) is a c.p.c. order zero map.
2. If \( \Phi: A \to B \) is a *-homomorphism and \( h \in (\Phi[A]^\prime \cap B)_+ \) is a contraction then \( \varphi(a) := h\Phi(a) = h^{1/2}\Phi(a)h^{1/2} \) is c.p.c. It has order zero because \( ab = 0 \) implies \( \varphi(a)\varphi(b) = h^2 \Phi(ab) = 0 \).

We will prove a structure theorem for c.p.c. order zero maps \( \varphi \). Although we will need only the case when the range of \( \varphi \) is included in a unital C*-algebra, for completeness we provide a proof in the general case. This requires a definition. The idealizer of a nondegenerate C*-subalgebra \( B \) of \( \mathcal{B}(H) \) is

\[ \mathcal{M}(B) := \{ c \in \mathcal{B}(H) : cB \subseteq B \text{ and } Bc \subseteq B \} \].
This is the multiplier algebra of $B$. The multiplier algebra $\mathcal{M}(B)$ of an abstract C*-algebra $B$ will be defined in Definition 13.1.6 (see also Corollary 13.2.2).

**Theorem 3.2.9.** Suppose that $\varphi : A \rightarrow B$ is a c.p.c. map between C*-algebras and $B = C^*(\varphi[A])$. Then the following are equivalent.

1. The map $\varphi$ has order zero.
2. There are a $^*$-homomorphism $\pi : A \rightarrow \mathcal{M}(B)$ and $h \in (\mathcal{M}(B) \cap \pi[A])^+ \setminus \{0\}$ such that $\varphi(a) = h\pi(a)$ for all $a \in A$.

Our proof of Theorem 3.2.9 makes a detour via von Neumann algebras. If $a$ and $b$ are positive elements, we write $a \perp b$ if $ab = 0$ and say that $a$ and $b$ are orthogonal.

**Lemma 3.2.10.** Suppose $A \subset \mathcal{B}(H)$ is a nondegenerate $C^*$-algebra and $T \in \mathcal{B}(H)$. The following are equivalent.

1. $TaTb = 0$ for all pairs of orthogonal elements $a$ and $b$ in $A^+$.
2. $T \in A'$.

**Proof.** If $T \in A'$ then clearly $TaTb = T^3ab$ and (1) follows. Assume (1). To prove (2) it suffices to show that $[c, T] = 0$ for every positive contraction $c$ in $A$. Since such $c$ can be approximated by convex combinations of its spectral projections of the form $1_{[r,1]}(c)$ via $\|c - \frac{1}{n} \sum_{k=1}^{n} 1_{[k/n,1]}(c)\| \leq 1/n$ for every $n$, it is enough to show $[p, T] = 0$ for every projection $p$ of the form $1_{[r,1]}(c)$ with $r \in (0, 1)$.

Let $f_0, f_n, g_n$ for $n > 2/r$ be piecewise linear functions in $C([0,1])$ with the following breakpoints (see Definition 1.4.3): $f_0(0) = 1, f_0(r/2) = 0, f_0(1) = 0, f_n(0) = 0, f_n(r/2) = 1, f_n(r-1/n) = 1, f_n(r-1/n) = 0, f_n(1) = 0, g_n(0) = 0, g_n(r-1/n) = 0, g_n(1) = 1$ (see Figure 3.1).

![Fig. 3.1 The functions $f_0$, $f_n$, and $g_n$.](image)

We list some properties of these functions. For $h \in \{f_0, f_n, g_n : n \geq r/2\}$ we have $0 \leq h \leq 1$. In addition, $(f_0 + f_n)g_n = 0$, the sequence $f_n$ is monotonically increasing, and the sequence $g_n$ is monotonically decreasing. Let $(e_n)_{\lambda}$ be an approximate unit for $A$. Since $f_0(T) \in A$ and SOT-$\lim_\lambda e_{\lambda} = 1_{\mathcal{B}(H)}$ by nondegeneracy (Exercise 3.10.3), we have $e_{\lambda} := f_0(c)^{1/2}e_{\lambda}f_0(c)^{1/2} \in A$ and SOT-$\lim_\lambda e_{\lambda} = f_0(c)$. All of $a_n := e_{\lambda}^* + f_n(c)$ and $b_n := g_n(c)$ are elements of $A$. We have $a_n b_n = 0$, SOT-$\lim_\lambda b_n \setminus 1_{[r,1]}(c) = p$, and SOT-$\lim_\lambda a_n \lambda = 1 - p$. Our assumption and joint SOT-continuity of multiplication on the unit ball imply
\[ T(1 - p)TpT = \text{SOT-lim}_{n, \lambda} Ta_{n, \lambda} Tb_{n}T = 0. \]

Therefore \( T^{2}pT = T(p + (1 - p))TpT = TpTpT \), and by taking the adjoints \( TpTpT = TpT^{2} \). Since \( [p, T]T = pT^{2} - TpT \), we have \( ([p, T]T^{*}) = ([p, T]T) = 0 \) and therefore \( [p, T]T = 0 \). This implies \( pT^{2} = TpT = T^{2}p \). Since \( T \geq 0 \) it belongs to \( C^{*}(T^{2}) \) and we conclude that \( pT = Tp \), concluding the proof of (2). \( \square \)

**Proof (Theorem 3.2.9).** If (2) holds, fix a \( ^{\ast} \)-homomorphism \( \pi : A \to \mathcal{M}(B) \) and \( h \in (\mathcal{M}(B) \cap \pi[A]')_{+1} \) so that \( \varphi(a) = h\pi(a) \) for all \( a \in A \). Then \( \varphi(a) = h^{1/2}\pi(a)h^{1/2} \) and it is therefore, being the compression of a \( ^{\ast} \)-homomorphism by a positive contraction, c.p.c. Since \( h \in \pi[A]' \), \( \varphi \) has order zero and (1) follows.

Suppose (1) holds. Identify \( B \) with a nondegenerate \( C^{\ast} \)-subalgebra of some \( \mathcal{B}(H) \). Let \( (\pi, K, \nu) \) be the Stinespring dilation of \( \varphi : A \to \mathcal{B}(H) \), so that

\[ \varphi(a) = v\pi(a)v^{\ast} \]

for all \( a \in A \). Since \( B = \varphi[A] \) we may assume that \( \pi \) is nondegenerate. By replacing \( A \) with \( A/\ker(\pi) \), we may assume that \( \pi \) is injective. Therefore for all \( a \) and \( b \) in \( A \) we have that \( \pi(a)\pi(b) = 0 \) implies \( ab = 0 \), and since \( \varphi \) has order zero we have

\[ T\pi(a)\pi(b)T = v^{\ast}\varphi(a)\varphi(b)v = 0. \]

Lemma 3.2.10 now implies that \( T := v^{\ast}v \) belongs to \( \pi[A]' \). Therefore \( \ker T \) is a \( \pi[A] \)-invariant subspace. If \( p \) denotes the projection to \( (\ker T)^{\perp} \), then \( \pi'(a) := p\pi(a)p \) is a \( ^{\ast} \)-homomorphism. The polar decomposition of \( v \) gives an isometry between \( (\ker T)^{\perp} \) and \( H \), which we can use to identify \( (\ker T)^{\perp} \) with \( H \) and replace \( \pi' \) with a \( ^{\ast} \)-homomorphism into \( \mathcal{B}(H) \). This operation identifies \( v \) with \( T^{1/2} \pi(a)T^{1/2} \) and \( \varphi(a) = T^{1/2}\pi(a)T^{1/2} \) for every \( a \in A \). Since \( T^{1/2} \in \pi[A]' \), we have \( \pi(a)\varphi(b) = \varphi(ab) \in B \) for all \( a \) and \( b \) in \( A \). This and \( B = \mathcal{C}^{\ast}(\varphi[A]) \) together imply \( \pi(a) \in \mathcal{M}(B) \) for all \( a \in A \). If \( (e_{\lambda})_{\lambda} \) is an approximate unit for \( A \) then

\[ T\varphi(b) = \lim_{\lambda} T\varphi(e_{\lambda}b) = \lim_{\lambda} TT^{1/2}\pi(e_{\lambda}b)T^{1/2} = \lim_{\lambda} \varphi(e_{\lambda})\varphi(b) \]

belongs to \( B \), and therefore \( T \in \mathcal{M}(B) \). \( \square \)

With the help of a lemma proved in §5.2 without using Proposition 3.2.11 or any of its consequences, one can recover a simple \( C^{\ast} \)-algebra from its image under a c.p.c. order zero map with a trivial kernel (see also Exercise 3.10.12).

**Proposition 3.2.11.** Suppose \( A \) is a simple unital \( C^{\ast} \)-algebra and \( \varphi : A \to \mathcal{B}(H) \) is a c.p.c. map of order zero. If \( \varphi[A] \neq \{0\} \) then there exist a \( C^{\ast} \)-algebra \( C \) and a completely positive map \( \psi : \varphi[A] \to C \) such that \( \psi \circ \varphi : A \to C \) is a \( ^{\ast} \)-homomorphism. In particular, \( \psi \circ \varphi[A] \) is isomorphic to \( A \).

**Proof.** Theorem 3.2.9 implies that there are a positive contraction \( h \in \mathcal{B}(H) \) and a \( ^{\ast} \)-homomorphism \( \pi : A \to \{h\}' \) such that \( \varphi(a) = h\pi(a) \). We will have \( C := \pi[A] \). Since \( C \) is abelian, it is nuclear (Lemma 2.4.3). Since \( A \) is simple and clearly \( \pi[A] \)
includes an approximate unit for $C^*(h, \pi[A])$, by Lemma 5.2.13 we can identify $C^*(h, \pi[A])$ with $C^*(h) \otimes \pi[A]$.

Fix $r \in \text{sp}(h) \setminus \{0\}$ and let $\varphi$ be the point-evaluation at $r$ in $C^*(h)$. This is a pure state on $C^*(h)$ and Example 3.3.5 (1) implies that $\theta := \varphi \otimes \text{id}_{\pi[A]}$ is a conditional expectation of $C^*(h) \otimes \pi[A]$ to $\pi[A]$. Therefore $\psi := r^{-1} \theta$ is a completely positive map. For $a \in A$ we have $\psi \circ \varphi(a) = r^{-1} (\varphi \otimes \text{id}_{\pi[A]}) (h \otimes \pi(a)) = \pi(a)$ and $\psi \circ \varphi = \pi$ is a unital *-homomorphism on $A$. Since $A$ is simple, it is an isomorphism. □

3.3 Averaging and Conditional Expectation

In this section we introduce averaging in $C^*$-algebras and von Neumann algebras, and use it to construct conditional expectations.

Suppose $A$ is a $C^*$-algebra, $\mu$ is a probability measure or a mean (see below) on a set $X$, and $f : X \to A$. We will explore two situations in which the integral

$$\int_X f(x) \, d\mu(x) \in A$$

is well-defined and well-behaved.

Example 3.3.1. Suppose $X$ is a compact metrizable space and $\mu$ is a Borel probability measure on $X$. If $A$ is a Banach space and $f : X \to A$ is continuous then the Riemann sums corresponding to the integral

$$\int f(x) \, d\mu(x)$$

form a Cauchy net in $A$, and the limit of this net is defined to be the integral.

Example 3.3.2. Suppose $\Gamma$ is an amenable group. Consider the von Neumann algebra $\ell_\infty(\Gamma)$ of all bounded, complex-valued functions on $\Gamma$ with respect to the sup norm. The group $\Gamma$ acts on $\ell_\infty(\Gamma)$ by the left translation,

$$g.a(h) := a(gh)$$

for $a \in \ell_\infty(\Gamma)$, $g \in \Gamma$, and $h \in \Gamma$. If $\Gamma$ is amenable then there exists an invariant mean, i.e., a functional $\mu : \ell_\infty(\Gamma) \to \mathbb{C}$ such that

1. $\mu$ is a state, and
2. $\mu(a) = \mu(ga)$ for all $g$ and $a$.

Now suppose $A$ is the dual of a Banach space $B$, and denote the bilinear functional implementing the corresponding duality by $(\cdot, \cdot)$. If $f : \Gamma \to A$ is uniformly continuous then we can define $\int_{\ell_\infty(\Gamma)} f(x) \, d\mu(x)$ as follows. For $b \in B$ the function $\theta_b : \Gamma \to \mathbb{C}$ defined by $\theta_b(g) := (f(g), b)$ is in $\ell_\infty(\Gamma)$ (it is bounded by $\|f\| \|b\|$). Then $\varphi(b) := \mu(\theta_b)$ defines a bounded linear functional on $B$. Since $A = B^*$, there exists $a \in A$ such that $(a, \cdot) = \varphi(\cdot)$. Define $\int_{\ell_\infty(\Gamma)} f(x) \, d\mu(x)$ to be $a$. 
Definition 3.3.3. Suppose $B$ is a $C^*$-subalgebra of $C^*$-algebra $A$. A conditional expectation from $A$ to $B$ is a completely positive contraction $\Theta : A \to B$ which satisfies the following bimodule condition

$$\Theta(bac) = b\Theta(a)c$$

for all $a \in A$ and all $b, c \in B$ (see [27, II.6.10]).

For a proof of the following theorem see e.g., [27, Theorem II.6.2.10] or [35, Theorem 1.5.10].

Theorem 3.3.4 (Tomiyama). Suppose $B$ is a $C^*$-subalgebra of a $C^*$-algebra $A$. Every projection $\Theta : A \to B$ of norm 1 (as a linear operator between Banach spaces) is a conditional expectation.

Example 3.3.5. 1. Suppose $A = B \otimes C$. If $\varphi$ is a state on $C$ then $\text{id} \otimes \varphi$ is a conditional expectation of $A$ to $B$ (see Definition 3.1.19). A proof of the following is analogous to that of Lemma 3.1.4.

Lemma 3.3.6. Suppose $F$ is a finite-dimensional $C^*$-subalgebra of a $C^*$-algebra $A$. Then there exists a conditional expectation $\theta : A \to F' \cap A$.

Proof. The unitary group $U(F)$ is a closed subset of the unit ball of a finite-dimensional Banach space $F$, and therefore compact in the norm topology. Denoting the Haar measure on $U(F)$ by $\mu$, for every $a \in A$ the integral (see Example 3.3.1)

$$\theta(a) := \int_{U(A)} uau^* d\mu(u)$$

is well-defined. The map $\theta$ is linear and of norm 1. If $a \in F' \cap A$ then $ua = a$ for all $u \in U(F)$, hence $\theta(a) = a$ for $a \in F' \cap A$.

For $v \in U(F)$ and $a \in A$ by substitution of variables we have

$$v\theta(a) = \int vuau^* d\mu(u) = (\int vuau^* v^* d\mu(u))v = \theta(a)v.$$
To recap, \( \Theta : A \to F' \cap A, \) \( \Theta(a) = a \) for \( a \in F' \cap A, \) and \( \Theta \) is a linear map of norm 1. By Theorem 3.3.4, \( \Theta \) is a conditional expectation of \( A \) onto \( F' \cap A. \)

**Corollary 3.3.8.** Suppose \( A \) is a finite-dimensional \( C^* \)-algebra and \( v \in A \) is a unitary. Then (\( Z(A) \) denotes the center of \( A) \)

\[
\text{dist}(v, Z(A)) \leq \sup_{a \in A, \|a\| \leq 1} \|v - uvu^*\| = \sup_{a \in U(A)} \|vu - uv\|
\]

which gives a slightly more precise result.

**Proof.** By applying the triangle inequality, we get

\[
\|v - \Theta(v)\| \leq \sup_{a \in U(A)} \|v - uvu^*\| = \sup_{a \in U(A)} \|vu - uv\|
\]

Theorem 3.3.4 implies that \( \Theta \) is a conditional expectation.

**Proposition 3.3.10.** Suppose \( M \) is a von Neumann algebra and \( D \) is an abelian von Neumann \( C^* \)-subalgebra of \( M \). Then there exists a conditional expectation

\[
\Theta : M \to D' \cap M.
\]

**Proof.** The proof is analogous to the proof of Lemma 3.3.7. The unitary group \( U(D) \) is abelian and we therefore carries an invariant mean \( \mu \). As in Example 3.3.2,

\[
\Theta(b) := \int ubu^* d\mu(u)
\]

is well-defined and as in Lemma 3.3.7 it is a conditional expectation of \( M \) to \( D. \)

**3.4 Transitivity Theorems, I: The Kadison Transitivity Theorem**

In this section we prove the Kadison Transitivity Theorem and provide some limiting examples. Towards its proof we introduce the Matrix Completion Trick.
Definition 3.4.1. A representation $\pi: A \to \mathcal{B}(H)$ is algebraically irreducible if the orbit $\pi[a]|\xi\rangle$ of every nonzero vector $\xi \in H$ is equal to $H$. It is topologically irreducible if the orbit $\pi[a]|\xi\rangle$ of every nonzero vector $\xi \in H$ is dense in $H$.

For a $C^*$-algebra $A$ the two definitions are equivalent; this is the Kadison Transitivity Theorem. The nontrivial implication is an immediate consequence of Theorem 3.4.5 below.

The following (nonstandard) property of a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is used in conjunction with the Kaplansky Density Theorem to prove the Kadison Transitivity Theorem ($Z(M)$ denotes the center of a von Neumann algebra $M$).

(*) For every finite-rank projection $p$ in $Z(M)'$, every $c \in M$, and every $\delta > 0$, there exists $b \in M$ such that $(c - b)p = 0$ and $\|b\| \leq \|cp\| + \delta$.

If $c$ is self-adjoint then $b$ can be chosen to be self-adjoint.

We will discuss the extent of (*) in Lemma 3.4.4 and Example 3.4.7.

Theorem 3.4.2. Suppose $A$ is a $C^*$-subalgebra of $\mathcal{B}(H)$ such that $M := A''$ has property (*). Let $p \in Z(M)'$ be a finite-rank projection and fix $c \in M$. Then the following hold for every $\varepsilon > 0$.

1. There exists $a \in A$ such that $(a - c)p = 0$ and $\|a\| \leq \|c\| + \varepsilon$.
2. If $c$ is in addition self-adjoint then we can choose $a$ to be self-adjoint.
3. If $c$ is in addition a unitary and $cp = pc$ then we can choose $a$ to be a unitary and satisfy $\|1 - a\| \leq \|1 - c\| + \varepsilon$.

Proof. (1) Fix $A, p, c$, and $\varepsilon$ as in the statement. The proof proceeds by building a convergent series $(a_n)$ in $A$ using (*) and the Kaplansky Density Theorem. Since all vector space topologies on a finite-dimensional space are equivalent, the set

$$\{a \in \mathcal{B}(H) : \|(a - d)p\| < \delta\}$$

is strongly open for all $d \in \mathcal{B}(H)$ and all $\delta > 0$.

We may assume $\|c\| = 1$. Let $\varepsilon_n := 2^{-2n-1}\varepsilon$ for $n \geq 0$.

By (*) choose $b_0 \in M$ such that $\|b_0\| < \|cp\| + \varepsilon_0 = 1 + \varepsilon_0$ and $(b_0 - c)p = 0$.

By the Kaplansky Density Theorem choose $a_0 \in A$ such that $\|(a_0 - b_0)p\| < \varepsilon_0$ and $\|a_0\| \leq \|b_0\| < 1 + \varepsilon_0$. Let $c_1 := c - a_0$. Then $\|c_1p\| = \|(c - a_0)p\| < \varepsilon_0$ and $c_1 \in M$.

By (*) choose $b_1 \in M$ such that $\|b_1\| < \|c_1p\| + \varepsilon_1 < \varepsilon_0 + \varepsilon_1$ and $(b_1 - c_1)p = 0$.

By the Kaplansky Density Theorem choose $a_1 \in A$ such that $\|(a_1 - b_1)p\| < \varepsilon_1$ and $\|a_1\| \leq \|b_1\| < \varepsilon_0 + \varepsilon_1$. Let $c_2 := c - a_1$. Then $\|c_2p\| = \|(c - (a_0 + a_1))p\| < \varepsilon_1$ and $c_2 \in M$.

By (*) choose $b_2 \in M$ such that $\|b_2\| < \|c_2p\| + \varepsilon_2 < \varepsilon_1 + \varepsilon_2$ and $(b_2 - c_2)p = 0$.

As before, choose $a_2 \in A$ such that $\|(a_2 - b_2)p\| < \varepsilon_1$ and $\|a_2\| \leq \|b_2\| < \varepsilon_1 + \varepsilon_2$.

Let $c_3 := c - a_1 - a_2$. Then $\|c_3p\| = \|(c - (a_0 + a_1 + a_2))p\| < \varepsilon_2$ and $c_3 \in M$.

Proceeding in this manner, for all $n \geq 1$ we find $c_n \in M, b_n \in M$, and $a_n \in A$ such that the following hold.

1. $\|c_n p\| = \|(c - \sum_{i=0}^{n-1} a_j)p\| < \varepsilon_{n-1}$.
3.4 Transitivity Theorems, I: The Kadison Transitivity Theorem

2. \( \|b_n\| < \varepsilon_{n-1} + \varepsilon_n \) and \((b_n - \varepsilon_n)p = 0\).

3. \( \| (a_n - b_n)p \| < \varepsilon_{n-1} \) and \( \| a_n \| < \varepsilon_{n-1} + \varepsilon_n \).

Since \( \varepsilon_{n-1} + \varepsilon_n < 2^{-n} \), the sum \( a := \sum_n a_n \) belongs to \( A \). It satisfies

\[
\|a\| < \|a_0\| + \varepsilon < 1 + \varepsilon_0 + \varepsilon
\]

and \((c - a)p = (c - \sum_{j=0}^\infty a_j)p = 0\).

(2) Suppose that \( c \) is self-adjoint. By (*) and the Kaplansky Density Theorem \( a_n \) as in the proof of (1) can be chosen to be self-adjoint, and by induction \( c - \sum_{j=0}^n a_j \) is self-adjoint for all \( n \). With this modification the proof of (1) gives a proof of (2).

(3) Now suppose \( c \) is a unitary such that \( pc = cp \). Then \((cp)^*cp = p = cp(cp)^*\) and \( cp \) is a unitary in \( \mathcal{B}(p[H]) \). Since \( p[H] \) is finite-dimensional, it has an orthonormal basis \( \{\xi_j : j < k\} \) consisting of eigenvectors of \( cp \). Since \( pc = cp \), each \( \xi_j \) is an eigenvector of \( c \). Let \( \lambda_j \) be the eigenvalue corresponding to \( \xi_j \), so that \( c\xi_j = \lambda_j\xi_j \).

By the Borel functional calculus there exists a self-adjoint \( d \in W^*(c) \) such that \( \{\lambda_j : j < k\} \subseteq \{e^t : -|d| \leq t \leq |d|\} \). Since \( d \in W^*(c) \) we have \( |d, p| = 0 \). By (2) for every \( \delta > 0 \) there exists \( a' \in A_{\varepsilon_0} \) such that \( sp(a') \subseteq [-r - \delta, r + \delta] \) and \( a'p = bp \). Since \( a' \) and \( b \) are self-adjoint and \( bp = pb \), we have \( a'p = pa' \). Then \( a := \exp(ia') \) is a unitary in \( A \). Because each \( \xi_j \) is an eigenvector of both \( a'p \) and \( ap, \) \( a \) is as required.

Since \( \delta > 0 \) can be arbitrarily small we can assure \( \| 1 - a \| \leq \| 1 - c \| + \varepsilon \). \( \square \)

Our proof of (*) for \( \mathcal{B}(H) \) and products of copies of \( \mathcal{B}(H) \) requires a trick well-known under the name of matrix completion (but see Exercise 3.10.16). Given a projection \( p \) in \( \mathcal{B}(H) \), an operator \( a \in \mathcal{B}(H) \) can be represented as an operator matrix

\[
\begin{pmatrix}
  pap & pa(1-p) \\
(1-p)ap & (1-p)a(1-p)
\end{pmatrix}.
\]

**Lemma 3.4.3.** Suppose \( A, B, \) and \( C \) in \( \mathcal{B}(H) \) are such that, with \( \| \cdot \| \) denoting the operator norm, \( \| (A B) \| \leq 1 \) and \( \| (A C) \| \leq 1 \). Then there exists \( X \in \mathcal{B}(H) \) such that \( \| (A B C X) \| \leq 1 \).

**Proof.** Since \( AA^* + BB^* = (A B) (A B)^* \), the \( C^* \)-equality implies \( AA^* + BB^* \leq 1 \). Therefore \( BB^* \leq 1 - AA^* \) and for all \( \xi \in H \) we have

\[
\| B^* \xi \|^2 = \| BB^* \xi \| \leq \| (1 - AA^*) \xi \| = \| (1 - AA^*)^{1/2} \xi \|^2.
\]

By Lemma 1.1.2 applied to \( a = B^* \) and \( b = (1 - AA^*)^{1/2} \) there exists a contraction \( B_0 \) such that \( B^* = B_0(1 - AA^*)^{1/2} \) and \( B = (1 - AA^*)^{1/2} B_0 \).

By \( \| (A C) \| \leq 1 \) we have \( C^* C + A^* A \leq 1 \) and by an argument similar to the above there exists a contraction \( C_0 \) such that \( C = C_0(1 - A^* A)^{1/2} \).

Let \( X := -C_0 A^* B_0 \). A computation shows

\[
\begin{pmatrix}
  A B \\
  C X
\end{pmatrix} = \begin{pmatrix}
  1 & 0 \\
  0 & C_0
\end{pmatrix} \begin{pmatrix}
  A \\
  (1 - A^* A)^{1/2}
\end{pmatrix} \begin{pmatrix}
  (1 - AA^*)^{1/2} \\
  -A^*
\end{pmatrix} \begin{pmatrix}
  1 & 0 \\
  0 & B_0
\end{pmatrix}.
\]
It remains to bound the norms of the three matrices on the right-hand side. Each of the two diagonal matrices is a contraction since \( B_0 \) and \( C_0 \) are contractions, and the middle matrix \( U \) satisfies \( U U^* = U^* U = 1 \). This concludes the proof. \( \square \)

**Lemma 3.4.4.** The following von Neumann algebras have property (*).

1. \( \mathcal{B}(H) \), as a subalgebra of itself, for any Hilbert space \( H \).
2. \( \prod_{j \in \mathbb{J}} \mathcal{B}(H_j) \), as a subalgebra of \( \mathcal{B}(\bigoplus_j H_j) \), for every family of Hilbert spaces \( H_j \), for \( j \in \mathbb{J} \).

**Proof.** We will see that in both cases \( b \) as required can be chosen with \( \delta = 0 \).

Since \( cp \in \mathcal{B}(H) \), only the case of (1) when \( c \) is self-adjoint requires a proof.

We need \( b \) of the form \( \begin{pmatrix} pc & pc(1-p) \\ (1-p)cp & X \end{pmatrix} \) for some \( X \in (1-p)\mathcal{B}(H)(1-p) \).

Since \( pc = (pc (1-p)) \) and \( (pc)^* = cp = \begin{pmatrix} pc \\ (1-p)cp \end{pmatrix} \), the assumptions of Lemma 3.4.3 are satisfied and we can find \( X \) and \( b \) as required.

For (2), fix a finite-rank-projection \( p \in Z(\mathcal{B}(\bigoplus_j H_j)) \) and \( c \in \mathcal{B}(\bigoplus_j H_j) \). Since \( p \) has finite rank, we may assume \( j = m \) for some \( m \in \mathbb{N} \). For every \( j \) the projection \( q_j \) to \( H_j \) is in the center of \( \mathcal{B}(\bigoplus_j H_j) \) and therefore \( c = \sum_{j<m} q_j c q_j \). Since \( p \in Z(M)' \), it commutes with \( q_j \) and therefore \( p_j := q_j p \) is a projection for all \( j \). Hence we have \( c p = \sum_{j<m} c p_j \). By applying (*) to \( \mathcal{B}(H_j) \), find \( b_j \in \mathcal{B}(H_j) \) such that \( (b_j - c) p_j = 0 \) and \( \| b_j \| \leq \| c p_j \| \) for all \( j < m \). Then \( b := \sum_{j<m} b_j \) belongs to \( M \) and it satisfies \( (b - c)p = 0 \) and \( \| b \| = \max_{j<m} \| b p_j \| \) \( \leq \max_{j<m} \| c p_j \| = \| c p \| \).

If \( c \) is self-adjoint then so is \( c q_j \) for all \( j \), hence each \( b_j \) can be chosen to be self-adjoint in which case \( b \) is self-adjoint as well. \( \square \)

If \( F \) is a subset of the domain of a function or an operator \( a \) we write \( a \upharpoonright F \) for the restriction of \( a \) to \( F \).

**Theorem 3.4.5.** Suppose \( A \) is a WOT-dense \( C^* \)-subalgebra of \( \mathcal{B}(H) \), \( F \subset H \), and \( c \in \mathcal{B}(H) \). Then for every \( \varepsilon > 0 \) we have the following.

1. There exists \( a \in A \) such that \( a \upharpoonright F = c \upharpoonright F \) and \( \| a \| \leq \| c \| + \varepsilon \)
2. If \( c \) is self-adjoint then there is \( a \in A_{sa} \) such that \( a \upharpoonright F = c \upharpoonright F \) and \( \| a \| \leq \| c \| + \varepsilon \).
3. If \( c \) is unitary and \( A \) is unital, then we can choose a unitary of the form \( \exp(ia) \)
   for \( a \in A_{sa} \) so that \( \exp(ia) \upharpoonright F = c \upharpoonright F \) and \( \| 1 - \exp(ia) \| \leq \| 1 - c \| \).

**Proof.** (1) and (2) follow by Theorem 3.4.2 and Lemma 3.4.4.

To prove (3), let \( K \) denote the span of \( F \cup c[F] \) and let \( p' := \text{proj}_K \). The partial isometry \( c \upharpoonright F \) can be extended to a unitary in \( \mathcal{B}(K) \), and then to a unitary \( c' \in \mathcal{B}(K) \), while maintaining \( \| 1 - c \| = \| 1 - c' \| \). Since \( p' c' = c' p' \), the assertion follows from (3) of Theorem 3.4.2. \( \square \)

Continuous functional calculus can be used to improve (2) and (3) of Theorem 3.4.5; see Exercise 3.10.17.

Corollary 3.4.6 below will be used in our proof of the main ingredient of Glimm Dichotomy (Corollary 5.5.8) in §3.7.
Corollary 3.4.6. Suppose $A$ is a WOT-dense $C^*$-subalgebra of $ℬ(H)$ that satisfies $A ∩ ℳ(H) = \{0\}$. If $a ∊ A_{+, 1}$ then the hereditary $C^*$-subalgebra $aAa$ is WOT-dense in $ℬ(a[H])$.

Proof. We need to prove that for every $F ∊ a[H]$, every $b : F → F$ can be uniformly approximated by elements of $aAa$. By Theorem 3.4.5, for fixed $F$ and $b$ we can find $d ∊ A$ such that $d \upharpoonright F = b \upharpoonright F$. Since $C^*(a)_+$ has an approximate unit, there exists $c ∊ C^*(a)_+$ such that $\sup_{ξ ∊ F} \| ξ - cξ \|$ is arbitrarily small, and therefore $cde ∊ aAa$ approximates $b$ on $F$ arbitrarily well. □

The following limiting examples for the property $(\ast)$, as well as for the conclusions of Theorem 3.4.2 and Theorem 3.4.5, will not be used elsewhere in this book.

Example 3.4.7. In the following examples $M ∊ ℬ(H)$ is a von Neumann algebra and $p ∊ ℬ(H)$ is a finite-rank projection. In (3) $A$ is a $C^*$-algebra WOT-dense in $M$.

1. An example of $M$ for which $(\ast)$ holds vacuously. Suppose $(X, μ)$ is a measure space, and let $H := L_2(X, μ)$. Then $L_∞(X, μ)$ is a maximal abelian $C^*$-subalgebra of $ℬ(H)$ and if some nonzero projection $p ∈ L_∞(X, μ)$ has finite rank then $μ$ has an atom. Therefore $(\ast)$ vacuously holds for $M = L_∞(X, μ)$ in the case when $μ$ is the Lebesgue measure (or any other atomless measure).

2. An example of $M, c ∊ M$, and $p_ε$ for $ε > 0$ such that no $b ∊ M$ satisfies $bp_ε = cp_ε$ and $∥b∥ ≤ (∥c∥ε)^{-1}$. Let $ν$ denote the Lebesgue measure, $H := L_2([0, 1], ν)$, and $M := L_∞([0, 1], ν)$. For every $g ∊ H$ and $h ∊ H$ such that $g(t) ≠ 0$ a.e. there exists at most one $f ∊ M$ such that $f = h$, satisfying $f(t) = h(t)/g(t)$ a.e. Let $g_ε ∊ H$ be defined by $g_ε(t) = ε$ if $0 ≤ t ≤ 1/2$ and $g_ε(t) = \sqrt{2 - t^2}$ if $1/2 < t ≤ 1$. Let $c ∊ M$ be $c := 1_{[0, 1/2]}$. Then $∥g_ε∥_2 = 1$ and $∥cg_ε∥_2 = 2^{-1/2}ε$. Hence (with $p_ε$ denoting the projection to the span of $g_ε$) $∥cp_ε∥ = 2^{-1/2}ε$. Since $∥c∥ = 1$ and $c$ is the unique $b ∊ M$ that satisfies $bp_ε = cp_ε$, the conclusion follows.

3. An example of a $C^*$-algebra $A ∊ ℬ(H)$, an $F ∊ H$, and an operator $c ∊ A''$ such that $a \upharpoonright F ≠ c \upharpoonright F$ for all $a ∊ A$. Let $ν$, $H$, $c$, $g := g_ε$, and $F := \{g\}$ for any $0 < ε < 1$ be as (2) and let $A := C([0, 1])$, as a concrete $C^*$-subalgebra of $M$. Since the unique $b ∊ M$ satisfying $bg = cg$ is not a.e. equal to a continuous function, there is no $a ∊ A$ satisfying $a \upharpoonright F = c \upharpoonright F$.

An important example of a situation in which Theorem 3.4.5 can be extended to show that for every $c ∊ A''$ and every $F ∊ H$ there exists $a ∊ A$ satisfying $a \upharpoonright F = c \upharpoonright F$ will be given in Theorem 3.5.4.

3.5 Transitivity Theorems, II: Direct Sums of Irreducible Representations

In this section we prove an extension of the Kadison Transitivity Theorem, due to Glimm and Kadison, to direct sums of inequivalent irreducible representations.

A subrepresentation of a representation $π : A → ℬ(H)$ is one of the form...
\[ \pi_p(a) := p \pi(a) p \]

for \( p \in \pi(A)' \). Exercise 1.11.56 implies that the map \( a \mapsto p \pi(a) p \) is a \(^*\)-homomorphism if and only if \( p \in \pi[A]' \). Thus subrepresentations of \( \pi \) correspond to projections in the von Neumann algebra \( \pi(A)' \).

Two representations \( \pi \) and \( \sigma \) of \( A \) are disjoint if they have no equivalent nonzero subrepresentations. An intertwiner of representations \( \pi \) and \( \sigma \) of \( A \) is a bounded linear operator \( T : \mathcal{B}(H_\pi) \to \mathcal{B}(H_\sigma) \) such that \( T \pi(a) = \sigma(a) T \) for all \( a \in A \). A representation \( \pi : A \to \mathcal{B}(H) \) is nondegenerate if \( \pi[A]' = \{0\} \). Every unital representation is clearly nondegenerate, and every representation is a direct sum of a nondegenerate representation and the zero representation.

**Lemma 3.5.1.** Suppose \( \pi : A \to \mathcal{B}(H) \) is a nondegenerate representation, \( p \) and \( q \) are orthogonal projections in \( \pi[A]' \). Then the following are equivalent.

1. \( \pi_p \) and \( \pi_q \) are disjoint.
2. \( \pi_p \) and \( \pi_q \) have no nontrivial intertwiners.
3. \( (p+q)\pi(A)'(p+q) \subseteq p\pi(A)'p \oplus q\pi(A)'q \).

**Proof.** Clearly (2) implies (1). To prove that (1) implies (2), suppose (2) fails and let \( T \) be a nontrivial intertwiner of \( \pi_p \) and \( \pi_q \). By the polar decomposition we have \( T = v|T| \) for a partial isometry \( v \). Then \( v, v^*v, v^*v \) all belong to \( \pi(A)' \), giving equivalent subrepresentations of \( \pi_p \) and \( \pi_q \). Thus (1) fails.

It remains to prove that (3) is equivalent to (2).

Since \( (p+q)\pi(A)'(p+q) \supseteq p\pi(A)'p \oplus q\pi(A)'q \), (3) is equivalent to the inclusion

\[ (p+q)\pi(A)'(p+q) \subseteq p\pi(A)'p \oplus q\pi(A)'q. \]

This is equivalent to all intertwiners being trivial, which is (2). \( \square \)

We will need Proposition 3.5.2 below only in the case when all representations \( \pi_i \) are irreducible.

**Proposition 3.5.2.** Suppose \( \pi_i : A \to \mathcal{B}(H_i) \), for \( i \in J \) are nondegenerate representations of a \( \ast \)-algebra \( A \), \( H := \bigoplus_i H_i \), \( \pi := \bigoplus_i \pi_i : A \to \mathcal{B}(H) \), and \( p_i \) is the projection of \( H \) onto \( H_i \). Then the following are equivalent.

1. All \( \pi_i \) are disjoint.
2. If \( i \neq j \) then \( \pi_i \) and \( \pi_j \) have no nontrivial intertwiners.
3. \( \pi(A)' = \bigoplus_{i \in J} p_i \pi(A)' p_i. \)
4. \( \pi(A)' = \bigoplus_{i \in J} p_i \pi(A)' p_i. \)

**Proof.** The case when \( J \) has two elements in Lemma 3.5.1. Since each of the assertions (1)–(4) is local, in the sense that it holds for \( J \) if and only if it holds for all of its two-element subsets, this concludes the proof. \( \square \)

A corollary of the equivalence of (1) and (4) in Proposition 3.5.2 is worth recording.
Corollary 3.5.3. Suppose \( \pi_i : A \to \mathcal{B}(H_i) \), for \( i \in J \), are irreducible representations of a C\(^*\)-algebra \( A \). Then these representations are inequivalent if and only if \( \bigoplus_{i < d} [\pi_i[A]]^{WOT} = \bigoplus_{i < d} \mathcal{B}(H_i) \).

We can now prove the following extension of Theorem 3.4.5.

Theorem 3.5.4 (Glimm–Kadison). Suppose \( \pi_i : A \to \mathcal{B}(H_i) \) for \( i < n \) are inequivalent irreducible representations, \( F_i \subset H_i \), and \( T_i \in \mathcal{B}(H_i) \) for \( i < n \). Let \( T := \bigoplus_i T_i \) and \( \varepsilon > 0 \).

1. Then there exists \( a \in A \) such that \( \pi(a) \upharpoonright F_i = T_i \upharpoonright F_i \) for all \( i < n \).

2. If each \( T_i \) is self-adjoint then \( a \) can be chosen to be self-adjoint and such that \( \|a\| \leq \|T\| + \varepsilon \).

3. If each \( T_i \) is a unitary in \( \mathcal{B}(H_i) \) then \( a \) can be chosen to be a unitary with \( \|a - 1\| \leq \|T - 1\| + \varepsilon \).

Proof. Since each \( \pi_i \) is irreducible, Corollary 3.5.3 implies \( \pi[A]'' = \bigoplus_i \mathcal{B}(H_i) \). By Lemma 3.4.4, \( \pi[A]'' \) satisfies (*), and Theorem 3.4.2 implies (1) and (2).

The proof of (3) from (1) and (2) is analogous to the proof of Theorem 3.4.5 (3).

The following consequence of Theorem 3.5.4 will be used in §5.1. Recall that \( U_0(A) \) is the subgroup of the unitary group \( U(A) \) consisting of all unitaries homotopic to 1.

Corollary 3.5.5. Suppose \( A \) is a unital C\(^*\)-algebra, \( \pi_i : A \to \mathcal{B}(H_i) \), for \( i < n \), are inequivalent irreducible representations, and \( E \) is a finite-rank projection onto a subspace of \( H = \bigoplus_{i < n} H_i \). Let \( U := \{ E \} \cap U_0(A) \). Then \( u \mapsto uE \) defines a surjective group homomorphism from \( U \) onto \( U(E[H]) \).

Proof. By Theorem 3.5.4 (3), for every \( u \in U(E[H]) \) there exists \( v \in U_0(A) \) such that \( vE = u \). In order to prove that \( v \) commutes with \( E \), note that

\[
vv^* = vEv^* + v(1 - E)v^* = uu^* + v(1 - E)v^* = E + v(1 - E)v^*.
\]

By multiplying with \( v \) on the right and cancelling, we obtain \( 0 = Ev - vE \).

3.6 Pure States

The purpose of this section is to study pure states (i.e., nonzero extreme points of the state space of a C\(^*\)-algebra) and irreducible representations of C\(^*\)-algebras. We prove a noncommutative version of the Radon–Nikodym theorem and give several equivalent characterizations of pure states. We also collect several characterizations of abelian C\(^*\)-algebras which will be used elsewhere in this book.
If $A$ is a unital C*-algebra then the state space of $A$, $S(A)$, is weak*-compact and convex. By the Krein–Milman theorem, it is equal to the closed convex hull of its extreme points.

**Definition 3.6.1.** An extreme point of $S(A)$ is called a pure state, and the space of pure states on $A$ is denoted $P(A)$.

**Proposition 3.6.2.** Suppose $A$ is an abelian C*-algebra.

1. Pure states of $A$ correspond to the point-evaluation functionals on $\hat{A}$.
2. A state on $A$ is pure if (and only if) it is a character (i.e., a unital $^*$-homomorphism into $\mathbb{C}$).
3. If $A$ is unital then its pure states comprise a weak*-closed subset of the dual unit ball.

**Proof.**

1. By Example 1.7.2, states correspond to probability measures on the spectrum $\hat{A}$ of $A$, and the nonzero extreme points correspond to the point-evaluation functionals.

2. A state that corresponds to a point mass measure is a point-evaluation state, and therefore a character. Conversely, if $\mu$ is not a point mass measure then the corresponding functional does not even preserve orthogonality and it therefore cannot be a character.

3. The space of point mass measures on $\hat{A}$ with respect to the weak*-topology is homeomorphic to $\hat{A}$, and therefore compact.

Needless to say, the structure of the pure states of a noncommutative C*-algebra is a bit more complex. An infinite-dimensional non-abelian C*-algebra may not have characters at all, and pure states typically do not form a weak*-closed subset of its state space. Pure states however form a Polish subspace (see Definition B.0.1) of the state space in any separable C*-algebra (Exercise 3.10.25). If $A$ is nonunital then the weak*-closure of $S(A)$ is equal to the convex closure of $S(A) \cup \{0\}$ and $S(A)$ is included in the closed convex hull of $P(A) \cup \{0\}$.

Recall from §1.10 that the left kernel of a state $\varphi$ is $L_\varphi := \{ a \in A : \varphi(a^*a) = 0 \}$. If $(\pi_\varphi, H_\varphi, \xi_\varphi)$ is the GNS triplet associated with $\varphi$, then since $\varphi(a) = (\pi_\varphi(a) \xi_\varphi | \xi_\varphi)$ for all $a \in A$, we have $L_\varphi = \{ a \in A : \pi_\varphi(a) \xi_\varphi = 0 \}$.

**Lemma 3.6.3.** For any state $\varphi$ of a C*-algebra $A$, $L_\varphi$ is a norm-closed left ideal of $A$ and $L_\varphi + L_\varphi^* \subseteq \ker(\varphi)$.

**Proof.** By Corollary 1.6.5 (4) we have $a^*b'ba \leq a^*a\|b^*b\|$ and therefore $a \in L_\varphi$ implies $ba \in L_\varphi$ for all $b$. Since $\varphi$ is continuous, $L_\varphi$ is norm-closed, and by Exercise 1.11.41 we have $(a + b)^*(a + b) \leq 2(a^*a + b^*b)$. Therefore $L_\varphi$ is closed under addition. This proves $L_\varphi$ is a left ideal.

The Cauchy-Schwarz inequality (applied in $\hat{A}$) implies $|\varphi(a)|^2 \leq \varphi(1)\varphi(a^*a)$, and in particular $L_\varphi \subseteq \ker(\varphi)$. Similarly $|\varphi(a)|^2 \leq \varphi(1)\varphi(aa^*)$ and $L_\varphi^* \subseteq \ker(\varphi)$. Hence $L_\varphi + L_\varphi^* \subseteq \ker(\varphi)$. 

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5 Quite the opposite is true: by Glimm’s Lemma (Lemma 5.2.6) $P(A)$ is weak*-dense in $S(A)$ if $A$ is simple.
3.6 Pure States

### 3.6.1 Irreducible Representations

As defined in §3.4, a representation $\pi: A \rightarrow \mathcal{B}(H)$ is irreducible if the only $\pi[A]$-invariant subspaces of $H$ are $\{0\}$ and $H$. Since $\pi[A]'$ is a von Neumann algebra (Theorem 3.1.6), $\pi$ is an irreducible representation if and only if $\pi[A]'' = \mathcal{B}(H)$.

Since the states on an abelian $C^*$-algebra $C_0(X)$ correspond to Borel probability measures on $X$, if $\varphi$ is a state on $A$ then $(A, \varphi)$ can be considered as a ‘noncommutative measure space.’ Lemma 3.6.4 below belongs to the ‘noncommutative measure theory.’ If $\psi \leq \varphi$ are states on $C_0(X)$ then there exists $h \in C_0(X)$, $0 \leq h \leq 1$, such that $\varphi(a) = \varphi(ha)$ for all $a \in C_0(X)$. Modulo the identification of states on $C_0(X)$ with Radon measures on $X$ this is the Radon–Nikodym Theorem. In the statement of the following non-commutative analog of the Radon–Nikodym Theorem and its proof we write $(\pi_\varphi, H_\varphi, \xi_\varphi)$ denotes the GNS triplet associated with $\varphi$.

**Lemma 3.6.4.** Suppose $\varphi$ is a state on a $C^*$-algebra $A$. There is an order-isomorphism between $\mathcal{X} := \{h \in \pi_\varphi[A]' : 0 \leq h \leq 1\}$ and $\mathcal{Y} := \{\psi \in S(A) : \psi \leq \varphi\}$ implemented by

$$\psi_h(a) := (\pi_\varphi(a) h \xi_\varphi | \xi_\varphi).$$

**Proof.** Fix $h \in \mathcal{X}$. Then $\psi_h = \omega_{h^{1/2} \xi_\varphi} \circ \pi_\varphi$. To prove $\psi_h \leq \varphi$, fix $a \in A_+$. The restrictions of $\varphi$ and $\psi_h$ to the abelian $C^*$-algebra $C^*(a, h)$ correspond to Radon measures $\mu$ and $v$ such that $v(f) = \int fh^{1/2} d\mu$ (Example 1.7.2). Since $h^{1/2} \leq 1$ we have $v \leq \mu$ and therefore $\psi_h(a) \leq \varphi(a)$. An analogous proof shows that $h \leq h'$ implies $\psi_h \leq \psi_{h'}$.

It remains to verify that every $\psi \in \mathcal{Y}$ is of the form $\psi_h$ for a unique $h \in \mathcal{X}$. Fix $\psi \in S(A)$. For $a, b, c$ in $A$, let $\alpha(a + L_\varphi, b + L_\varphi) := \varphi(b^* a)$.

Since $|\varphi(b^* a)|^2 \leq \varphi(b^* b) \varphi(a^* a)$, $\alpha$ is a well-defined sesquilinear form of norm $\leq 1$ on the GNS pre-Hilbert space $A/L_\varphi$. Let $h \in \mathcal{B}(H_\varphi)$ correspond to the continuous extension of $\alpha$ to $H_\varphi$. For $a, b, c$ in $A$ we have

$$(\pi_\varphi(a) h(b + L_\varphi))(c + L_\varphi)) = \varphi(c^* ab).$$

Since $(h(a + L_\varphi))(a + L_\varphi) = \psi(a^* a) \geq 0$, the operator $h$ is positive. Since

$$(\pi_\varphi(a) h(b + L_\varphi))(c + L_\varphi)) = \varphi(c^* ab) = (h\pi_\varphi(b + L_\varphi))(c + L_\varphi))$$

and $b, c$ were arbitrary, we have $h \in \pi_\varphi[A]'$.

To prove the uniqueness of $h$, observe that it is uniquely determined by the sesquilinear form $\alpha$ and that $\alpha$ is uniquely determined by $\psi$. \hfill \Box

**Proposition 3.6.5.** For every state $\varphi$ of $A$ the following are equivalent.

1. It is pure.
2. If $\psi \leq \varphi$ is a positive functional, then $\psi$ is a scalar multiple of $\varphi$.

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6 However, the theory of von Neumann algebras is generally considered to be the proper noncommutative analog of measure theory.
3. The GNS representation $\pi_\phi$ is irreducible.
4. We have $\ker(\phi) = L_\phi + L_\phi^*$.

Proof. (1) implies (2): If $\phi \in P(A)$ and $\psi \leq \phi$ then $\phi = \psi + (\phi - \psi)$ hence $\|\psi\|^{-1}\psi = \phi$.

(2) implies (1): If $\phi = t\psi_0 + (1-t)\psi_1$ with $0 < t < 1$ and $\psi_0 \neq \psi_1$ then at least one of the inequalities $t^{-1}\psi_0 \neq \phi$ and $(1-t)^{-1}\psi_1 \neq \phi$ holds.

Lemma 3.6.4 implies that (2) is equivalent to (3).

(3) implies (4): Lemma 3.6.3 implies $L_\phi + L_\phi^* \subseteq \ker(\phi)$ for any state $\phi$. Suppose that the GNS representation corresponding to $\phi$, $\pi: A \to \mathcal{B}(H)$, with cyclic vector $\xi$, is irreducible. If $a \in \ker(\phi)$ then $\langle \pi(a)\xi|\xi\rangle = 0$ and $\eta := \pi(a)\xi$ is orthogonal to $\xi$. By Theorem 3.4.5 there exists $b \in A_{sa}$ such that $\pi(b)\xi = 0$ and $\pi(b)\eta = \eta$. Then $b \in L_\phi^*$ since $\pi(b)\xi = \pi(b^*)\xi = 0$, and therefore $ba \in L_\phi^*$. Also $a - ba \in L_\phi$ because $\langle \pi(a - ba)\xi|\xi\rangle = 0$ and we can write $a = (a - ba) + ba \in L_\phi + L_\phi^*$.

Since $a \in \ker(\phi)$ was arbitrary, this proves (4).

(4) implies (2): Suppose that $\phi$ satisfies (4) and $\psi \leq \phi$ is positive and nonzero. Then $\psi(a^*a) \leq \phi(a^*a)$ and $L_\psi \supseteq L_\phi$. Lemma 3.6.3 implies

$$\ker(\psi) \supseteq L_\psi + L_\psi^* \supseteq L_\phi + L_\phi^* = \ker(\phi)$$

and $\ker(\psi) = \ker(\phi)$. Since every state is uniquely determined by its kernel (Exercise 1.11.51) we have $\psi = \|\psi\|^{-1}\phi$. $\square$

A modest collection of equivalent characterizations of abelian C*-algebras follows (see also Exercise 3.10.26).

**Proposition 3.6.6.** For a C*-algebra $A$ the following are equivalent.

1. $A$ is abelian.
2. If $\phi$ and $\psi$ are distinct pure states on $A$ then $\|\phi - \psi\| = 2$.
3. The space $P(A)$ with respect to the norm topology has no nontrivial connected components.
4. The space $P(A)$ with respect to the norm topology contains no homeomorphic copy of the unit circle $\mathbb{T}$.
5. For every irreducible representation $\pi: A \to \mathcal{B}(H)$ the space $H$ is one-dimensional.
6. No element of $A$ is nilpotent.

Proof. $(1) \Rightarrow (2)$: If $A$ is abelian then $A \cong C_0(X)$ where $X$ is the space of pure states on $A$ equipped with the weak*-topology (Example 1.7.2). For any two distinct points of $X$ the Tietze Extension theorem provides $a \in C_0(X)$ such that $\|a\| = 1$, $\phi(a) = 1$ and $\psi(a) = -1$. Therefore $\|\phi - \psi\| \geq 2$. Since $\|\phi - \psi\| \leq \|\phi\| + \|\psi\| = 2$ for all states $\phi$ and $\psi$, (2) follows.

$(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are trivial.

$(4) \Rightarrow (5)$: Suppose $(5)$ fails and let $\pi: A \to \mathcal{B}(H)$ be an irreducible representation with $\dim H \geq 2$. Fix orthogonal unit vectors $\xi$ and $\eta$ in $H$. Define $f: \mathbb{T} \to H$ by

---

7 Actually, both.
\[ f(\exp(i\theta)) := \cos \theta \xi + \sin \theta \eta. \] Then every vector in the range of \( f \) has norm 1, and
\( \varphi_0 := \omega_f(\exp(i\theta)) \circ \pi \) is a state on \( A \) with cyclic vector \( f(\theta) \) for \( 0 \leq \theta \leq 2\pi. \) Since \( \pi \) is irreducible, these states are distinct and pure, and they form a homeomorphic copy of \( \mathbb{T} \) in \( \mathcal{P}(A) \). Therefore (4) fails.

(5) \implies (1): Since the direct sum of all irreducible representations of \( A \) is faithful, (5) implies that \( A \) is isomorphic to a \( C^* \)-subalgebra of an abelian \( C^* \)-algebra and (1) follows.

(1) implies (6) because there are no nilpotent elements in \( C_0(X) \).

(6) implies (5): Suppose (6) fails and let \( \pi : A \to \mathcal{B}(H) \) be an irreducible representation of \( A \) with \( \dim(H) \geq 2 \). Let \( \xi \) and \( \eta \) be orthogonal unit vectors in \( H \). By Theorem 3.4.5 we can find \( a \in A_{sa} \) with \( \|a\| = 1 \) and \( y \in A \) such that \( a\xi = \xi, a\eta = -\eta, y\xi = \eta, \) and \( y\eta = 0 \). Then \( \omega_\xi \circ \pi \) and \( \omega_\eta \circ \pi \) are distinct pure states on the abelian \( C^* \)-algebra \( C^*(a,1) \). The elements \( b := \frac{1}{2}(|a| + a) \) and \( c := \frac{1}{2}(|a| - a) \) belong to \( A_{+1} \) and satisfy \( b\xi = \xi, c\eta = \eta, \) and \( bc = 0 \). Then \( x = byc \) satisfies \( x^2 = 0 \) and \( \|x\| \leq 1 \). Since \( x\xi = \xi \) it is nilpotent of degree two and norm 1. \( \square \)

This is a good moment to record one of the key steps in the proof of Glimm’s Theorem (Theorem 3.7.1).

**Corollary 3.6.7.** If \( A \) is a WOT-dense \( C^* \)-subalgebra of \( \mathcal{B}(H) \) and \( A \cap \mathcal{K}(H) = \{0\} \) then every hereditary \( C^* \)-subalgebra \( B \) of \( A \) contains a nilpotent of degree two.

**Proof.** Fix \( b \in B_{+1} \). Then Corollary 3.4.6 implies that the hereditary \( C^* \)-subalgebra \( \overline{bA'b} \) is WOT-dense in \( b[H] \). Since \( \overline{bA'b} \cap \mathcal{K}(H) = \{0\} \), \( H \) is infinite-dimensional and \( \overline{bA'b} \) is not abelian. It contains a nilpotent by Proposition 3.6.6. \( \square \)

An easy criterion for irreducibility of a representation will come handy in some concrete situations (Lemma 10.4.7).

**Lemma 3.6.8.** Suppose \( \pi : A \to \mathcal{B}(H) \) is a representation. The following are equivalent.

1. \( \pi \) is irreducible.
2. The set \( \{ \xi \in H_{\leq 1} : \pi[A_{\leq 1}]\xi \text{ is dense in } H_{\leq 1} \} \) is dense in \( H_{\leq 1} \).

**Proof.** If \( \pi \) is irreducible then for every nonzero vector \( \xi \in H \) we have \( \pi[A]\xi = H \) by the Kadison Transitivity Theorem. To prove the converse suppose that \( \pi \) is not irreducible and fix a nontrivial projection \( p \in \pi[A]' \). Let \( \xi_0 \in p[H] \) and \( \eta_0 \in p[H]^{\perp} \) be unit vectors. For unit vectors \( \xi \approx \xi_0 \) and \( \eta \approx \eta_0 \) and \( a \in A_{\leq 1} \) we have
\[
\|a\eta - \xi\| \geq \|p(a\eta - \xi)\| \geq \|\|a\eta\| - \|p\xi\|\| \approx \|a\eta\| - \|p\xi\| \approx 1,
\]
and therefore (2) fails. \( \square \)
3.7 Type I C*-algebras. Glimm’s Theorem

Notes are like handcuffs. Or brain-cuffs. I actually think one should forget everything about a piece and create a new piece in the moment. But that is probably illegal.

Patricia Kopatchinskaja

In this section we prove Glimm’s Theorem, asserting that every non-type I C*-algebra has a subalgebra with a quotient isomorphic to the CAR algebra. Our proof uses completely positive maps of order zero from §3.2 and the characterization of abelian C*-algebras given in §3.6.

The following theorem will not be used explicitly; we will be interested only in the C*-algebras that do not satisfy either of the equivalent conditions stated in it.

**Theorem 3.7.1 (Kaplansky).** For a C*-algebra $A$ the following are equivalent.

1. The image of $A$ under any irreducible representation $\pi: A \to \mathcal{B}(H)$ includes the compact operators $\mathcal{K}(H)$.
2. The weak closure of the image of $A$ under any representation $\pi: A \to \mathcal{B}(H)$ is a type I von Neumann algebra.

**Proof.** See e.g., [14, Theorem 2.4.1]. ⊓ ⊔

A C*-algebra $A$ that satisfies (1) in Theorem 3.7.1 is said to be GCR, and one that satisfies (2) in Theorem 3.7.1 is said to be of type I. Since the two definitions are equivalent, we will use only the latter. A C*-algebra is said to be of non-type I if it is not of type I.

**Theorem 3.7.2 (Glimm).** Every non-type-I C*-algebra has a C*-subalgebra isomorphic to an infinite tensor product of nonabelian C*-algebras. This C*-subalgebra has a quotient isomorphic to the CAR algebra.

A proof of Theorem 3.7.2 takes up the remainder of the present section. The following nonstandard bit of notation will be useful. For $a \in A_{+,1}$ let

$$A_{\preceq a} := \{ c \in A : ac = ca = aca = c \}.$$ 

This is a hereditary subalgebra of $\overline{aAa}$. The following lemma shows, among other things, that a nilpotent element $w$ acts like a partial isometry on $A_{\preceq w^*w}$.

**Lemma 3.7.3.** If $w$ is a nilpotent element of degree two in a C*-algebra $A$, then

1. The restriction of $\text{Ad}_w(x) := wxw^*$ to $A_{\preceq w^*w}$ is an isomorphism onto $A_{\preceq w^*w}$.
2. The restriction of $\text{Ad}_{w^*}(x) := w^*xw$ to $A_{\preceq w^*w}$ is an isomorphism onto $A_{\preceq w^*w}$ which is the inverse of the isomorphism in (1).
3. The algebra $\{ a + \text{Ad}_w(a) : a \in A_{\preceq w^*w} \}$ is isomorphic to $A_{\preceq w^*w}$ and it is included in $C^*(w)^\prime \cap A$.

**Proof.** A straightforward computation, left to the reader. ⊓ ⊔
Lemma 3.7.4. If $A$ is a WOT-dense $C^*$-subalgebra of $\mathcal{B}(H)$ and $A \cap \mathcal{K}(H) = \{0\}$ then for every non-scalar $b \in A_\omega$ there exist $a \in C^*(b)_{+1}$ and a nilpotent $w \in A_{\text{nil}}$ of degree two.

Proof. The assumptions imply that $H$ is infinite-dimensional. Being WOT-dense in $\mathcal{B}(H)$, $A$ is nonabelian. Corollary 3.6.7 implies that every nontrivial hereditary subalgebra of $A$ contains a nilpotent $w$ of order two. It therefore suffices to find $a \in C^*(b)$ such that $A_{\text{nil}}$ is nontrivial.

By the continuous functional calculus we may assume $\{0, 1\} \subseteq \text{sp}(b) \subseteq [0, 1]$. We will need $f$ and $g$ in $C_0((0, 1])_{+1}$ such that $f g = f$ and $\|f\| = \|g\| = 1.$ For example, let $f: [0, 1] \to [0, 1]$ be such that $f(t) = 0$ for $0 \leq t \leq 1/2$, $f(1) = 1$, and $f$ is linear on $[1/2, 1]$, and let $g: [0, 1] \to [0, 1]$ be such that $g(0) = 0$, $g(t) = 1$ for $1/2 \leq t \leq 1$, and $g$ is linear on $[0, 1/2]$ (see Figure 3.2). Let $a = g(b).$ Then $a \in A_{+1}$ and $a f(b) a = f(b)$, and therefore $f(b) \in A_{\text{nil}}.$ Since $f(b) \in A_{+1}$, $A_{\text{nil}}$ is a nontrivial hereditary $C^*$-subalgebra of $A$. As pointed out in the first paragraph, this completes the proof. \qed

Proposition 3.7.5. Suppose $A$ is a WOT-dense $C^*$-subalgebra of $\mathcal{B}(H)$ such that $A \cap \mathcal{K}(H) = \{0\}$.

1. There are nonzero c.p.c. order zero maps $\phi_n: M_2(\mathbb{C}) \to A$, for $n \in \mathbb{N}$, with commuting ranges.
2. The algebra $A$ has a $C^*$-subalgebra with a quotient isomorphic to the CAR algebra.

Proof. (1) By applying Lemma 3.7.4 recursively choose $w_n$, for $n \in \mathbb{N}$, satisfying the following.

1. Each $w_n$ is nilpotent of degree two.
2. $w_{n+1} \in A_{\text{nil}}^{w_n w_n}$.

Let $A_n := A_{\text{nil}}^{w_n w_n}$. Then (2) implies $A_{n+1} \subseteq A_n$ for all $n$, and Lemma 3.7.3 (3) implies that $\Phi_n(a) := a + w_n a w_n^*$ is an isomorphism from $A_{n+1}$ onto a $C^*$-subalgebra of $C^*(w_n)\cap A$. Define $C^*$-subalgebras $B_n$, for $n \in \mathbb{N}$, by $B_0 := C^*(w_0), B_1 := \Phi_0[C^*(w_1)]$, and...
Then $B_1 \subseteq C^*(w_0)' \cap A$ and in general $B_n \subseteq C^*(w_j : j < n)' \cap A$. Let

$$v_n := \Phi_0(\Phi_1(\ldots \Phi_{n-1}(w_n) \ldots)),\$$

for $n \in \mathbb{N}$. Then $v_n \in B_n$ and $v_n$, for $n \in \mathbb{N}$, are commuting nilpotents of degree two.

In addition, for all $m$ and all $s \in \{\omega, +\}^m$ with $v^s := v$ we have $\|\prod_{j<m} v^{s(j)}\| = 1$. A nilpotent $v$ of degree two satisfies $C^*(v) \cong C_0(\text{sp}(v), M_2(\mathbb{C}))$ by Lemma 2.3.8. This algebra is isomorphic to $C_0(\text{sp}(v), M_2(\mathbb{C}))$. For each $n$ fix an injective *-homomorphism $\psi_n : C_0(\text{sp}(v^n), M_2(\mathbb{C})) \to A$. Then $\phi_n : M_2(\mathbb{C}) \to A$ defined by

$$\phi_n(a) = \psi_n(\text{id}_{\text{sp}(v^n)}) \otimes a$$

is injective, c.p.c., and of order zero (Example 3.2.8 (1)).

The ranges of $\phi_n$ commute because the $v_n$'s commute.

(2) Proposition 3.2.11 implies that for every $n \in \mathbb{N}$ there exists a surjective *-homomorphism $\Psi_n : C^*(v_n) \to M_2(\mathbb{C})$. For an $m$-tuple $b_j \in C^*(v_j)$, for $j < m$, let

$$\Theta_m(\prod_{j<m} b_j) := \Psi_0(b_0) \otimes \Psi_1(b_1) \otimes \cdots \otimes \Psi_{m-1}(b_{m-1}) \in \otimes_{j<m} M_2(\mathbb{C})$$

The linear extension of $\Theta_m$ to $C^*(v_j : j < m)$ is a *-homomorphism onto $\otimes_{j<m} M_2(\mathbb{C})$. Since $\Theta_{m+1}$ extends $\Theta_m$, the limit $\Theta$ of these maps defines a *-isomorphism from $C^*(v_j : j \in \mathbb{N})$ onto a dense $C^*$-subalgebra of the CAR algebra. Since a *-homomorphic image of a $C^*$-algebra is a $C^*$-algebra, the range of $\Theta$ is the CAR algebra. □

**Proof (Theorem 3.7.2).** Suppose $A$ is non-type I. Fix an irreducible representation $\pi : A \to \mathcal{B}(H)$ whose range has trivial intersection with $\mathcal{K}(H)$. Then Proposition 3.7.5 implies that $\pi[A]$ (i.e., a quotient of $A$) has a $C^*$-subalgebra isomorphic to $\otimes_{i=1}^\infty C_0((0, 1], M_2(\mathbb{C}))$. Since $M_2(\mathbb{C})$ is a quotient of $C_0((0, 1], M_2(\mathbb{C}))$ (the quotient map sends $f$ to $f(1)$), the conclusion follows. □

### 3.8 Spatial Equivalence of States

In this section we use the Kadison Transitivity Theorem to study spatial equivalence of pure states. Spatial equivalence of $n$-tuples of inequivalent pure states is studied using the Glimm–Kadison extension of the former.

Two states $\phi$ and $\psi$ of a $C^*$-algebra $A$ are **unitarily equivalent** if $\phi \circ \text{Ad} w = \psi$ for some $w \in U(\hat{A})$. Recall from §1.10 that two representations $\pi_i : A \to \mathcal{B}(H_i)$ (for $i < 2$) are **spatially equivalent** (denoted $\pi_0 \sim \pi_1$) if there is a unitary $u : H_0 \to H_1$ such that $\pi_1 := \text{Ad} u \circ \pi_0$ (Fig. 3.3).

Since all automorphisms of $\mathcal{B}(H)$ are inner, any isomorphism between $\mathcal{B}(H_0)$ and $\mathcal{B}(H_1)$ is implemented by a unitary (Exercise 2.8.29). Therefore $\pi_0 \sim \pi_1$ if and only if $\pi_1 = \Phi \circ \pi_0$ for some isomorphism $\Phi : \mathcal{B}(H_0) \to \mathcal{B}(H_1)$.
3.8 Spatial Equivalence of States

Proposition 3.8.1. For pure states \( \varphi \) and \( \psi \) of a \( C^* \)-algebra \( A \) the following are equivalent.

1. The states \( \varphi \) and \( \psi \) are unitarily equivalent.
2. The representations \( \pi_\varphi \) and \( \pi_\psi \) are spatially equivalent.
3. There exists \( u \in U(A) \) such that \( \| \psi - \varphi \circ \text{Ad}u \| < 2 \).
4. There exists \( a \in A_+ \) with \( \| a \| \leq \pi \) such that \( w = \exp(ia) \) satisfies \( \varphi \circ \text{Ad}w = \psi \) and \( \psi \circ \text{Ad}w = \varphi \).
5. The states \( \varphi \) and \( \psi \) belong to the same path-connected component of \( P(A) \).

The proof of Proposition 3.8.1 will require two lemmas.

Lemma 3.8.2. Suppose that a \( C^* \)-algebra \( A \subseteq \mathcal{B}(H) \) is WOT-dense and \( \xi \) and \( \eta \) are unit vectors in \( H \). Then there exists \( a \in A_\text{sa} \) such that \( \| a \| \leq \pi \) and \( u = \exp(ia) \) satisfies \( u\xi = z\eta \) and \( u\eta = \bar{z}\xi \) for some \( z \in \mathbb{T} \).

Proof. Let \( z \in \mathbb{T} \) be such that \( \Re(\xi|\eta) \in \mathbb{R} \) and let \( \eta' := z\eta \). Then \( \xi - \eta' \) is orthogonal to \( \xi + \eta' \). By Theorem 3.5.4 we can find \( a \geq 0 \) with \( \| a \| \leq \pi \) which has \( \xi + \eta' \) as a 0-eigenvector and \( \xi - \eta' \) as a \( \pi \)-eigenvector. The unitary \( u = \exp(ia) \) has \( \lambda \)-eigenvectors of \( a \) as \( e^{\lambda} \)-eigenvectors. In particular, \( u \) has \( \xi + \eta' \) as a 1-eigenvector and \( \xi - \eta' \) as a \( -1 \)-eigenvector. Therefore

\[
u(\xi) = \frac{1}{2}(u(\xi + \eta') + u(\xi - \eta')) = \frac{1}{2}(\xi + \eta' + \bar{z}\xi + \bar{z}\eta') = \eta',
\]

and similarly \( u(\eta') = \xi' \). Since \( \eta' = z\eta \), \( u \) is as required. \( \square \)

Lemma 3.8.3. Suppose that each of \( \Phi \) and \( \Psi \) is an \( n \)-tuple of inequivalent pure states on a \( C^* \)-algebra \( A \) and \( \max_{j < n} \| \varphi_j - \psi_j \| < 2 \). Then there exists \( a \in A_+ \) such that \( u := \exp(ia2\pi) \) satisfies \( \varphi \circ \text{Ad}u = \psi \).

Proof. Suppose that \( \varphi_j \) and \( \psi_j \) are not equivalent for some \( j \). By Theorem 3.5.4 there exists \( c \in A_\text{sa} \) such that \( \| c \| = \varphi_j(c) = 1 \) and \( \psi_j(c) = -1 \), and \( \| \varphi_j - \psi_j \| = 2 \); contradiction.

Fig. 3.3 The equivalence of representations and the equivalence of states.
We may therefore assume that $\varphi_j \sim \psi_j$ for all $j < n$. By Lemma 3.8.2 for each $j < n$ there exists $0 \leq a_j \leq 2\pi$ such that $u_j := \exp(ia_j)$ satisfies $\varphi_j \circ \text{Ad} u_j = \psi_j$.

Since states $\varphi_j$, for $j < n$, are inequivalent, Theorem 3.5.4 implies the existence of $a$ such that $0 \leq a \leq 2\pi$ and $u := \exp(ia)$ moves the cyclic vector for $\psi_j$ to the cyclic vector for $\varphi_j$ for all $j < n$. Then $\varphi_j \circ \text{Ad} u = \psi_j$ for all $j < n$.\hfill $\square$

**Proof (Proposition 3.8.1).** Clearly (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Also (3) implies (4) by the case $n = 1$ of Lemma 3.8.3.

If (4) holds with $u = \exp(ia)$, then $u_t := \exp(ita)$, for $0 \leq t \leq 1$, is a continuous path of unitaries such that $u_0 = 1$, $\psi = \varphi \circ \text{Ad} u_1$, and $\varphi = \psi \circ \text{Ad} u_1$; thus (a strengthening of) (5) holds.

Finally assume that (5) holds. ‘Discretize’ the path between $\psi$ and $\varphi$ and find $m$ and pure states $\varphi_j$, for $j < m$, such that $\varphi_0 = \varphi$, $\varphi_{m-1} = \psi$, and $\|\varphi_j - \varphi_{j+1}\| < 2$ for all $j < m - 2$. Apply the case $n = 1$ of Lemma 3.8.3 $m - 2$ times and multiply thus obtained unitaries to find a unitary $u$ satisfying $\psi = \varphi \circ \text{Ad} u$. Hence (1) holds.\hfill $\square$

A corollary of the equivalence of (1) and (4) in Proposition 3.8.1 is worth stating.

**Corollary 3.8.4.** Two pure states on a unital $C^*$-algebra $A$ are unitarily equivalent if and only if they are unitarily equivalent via a unitary in $U_0(A)$.\hfill $\square$

We will also need a generalization of Proposition 3.8.1 to $m$-tuples of pure states.

**Proposition 3.8.5.** If both $\bar{\varphi}$ and $\bar{\psi}$ are $n$-tuples of inequivalent pure states on $A$ then the following are equivalent.

1. The states $\varphi_i$ and $\psi_i$ are unitarily equivalent for all $i < n$.
2. The representations $\pi_{\varphi_0}$ and $\pi_{\psi_0}$ are spatially equivalent for all $i < n$.
3. There exists $u \in U(\tilde{A})$ such that $\|\psi_i - \varphi_i \circ \text{Ad} u\| < 2$ for all $i < n$.
4. There exists $a \in A_+$ with $\|a\| \leq \pi$ such that $w = \exp(ia)$ satisfies $\varphi_i \circ \text{Ad} w = \psi_i$ and $\psi_i \circ \text{Ad} w = \varphi_i$.
5. The states $\varphi_i$ and $\psi_i$ belong to the same path-connected component of $P(A)$ for all $i < n$.

**Proof.** The equivalences of (1), (2), and (5) are a consequence of Proposition 3.8.1. By the same proposition, (3) implies (1). Also, (4) clearly implies (3). Finally, (3) implies (4) by Lemma 3.8.3.\hfill $\square$

### 3.9 The Universal Representation and the Second Dual

In this section we prove that the second Banach space dual $A^{**}$ of a $C^*$-algebra $A$ is isomorphic to the weak closure of the image of $A$ in its universal representation. We use $A^{**}$ to prove a transitivity-type proposition that will be used in the construction of an approximate diagonal in §5.1.

The *universal representation* of a $C^*$-algebra $A$ is the direct sum of all of its GNS representations,
Lemma 3.9.2. Suppose $X$ and $Y$ are Banach spaces and $T$ is an identity on $A$. Then $T$ can be extended to $S \in \mathcal{B}(X,Y^*)$ such that $S$ is weak$^*$-continuous and $\|S\| = \|T\|$.

Proof. Let $S := (T^* | Y)^*$. Then $S : X^* \to Y^*$. Since $X$ is weak$^*$-dense in $X^*$ and Proposition C.4.10 implies that both $S$ and $T^*$ are weak$^*$-weak$^*$-continuous, we have $S(x) = T^*(x)$ for all $x \in X$. As $T^*$ agrees with $T$ on $X$, $S$ extends $T$ and $\|T\| = \|S\|$. Since $\|T\| = \|T^* \circ Y\| = \|S\|$, the conclusion follows. □

Theorem 3.9.1. There is a linear isometry between the bicommutant of the image of $A$ under its universal representation and the second Banach space $A^{**}$ that is an identity on $A$.

The standard proof of Theorem 3.9.1 proceeds by representing every functional on $A$ as a linear combination of states. I have learned the more intuitive proof included below from Narutaka Ozawa (for the transpose of an operator $T$, denoted $T^*$) in the following proof, see Definition C.4.9 and Proposition C.4.10.

Lemma 3.9.2. Suppose $X$ and $Y$ are Banach spaces and $T \in \mathcal{B}(X,Y^*)$. Then $T$ can be extended to $S \in \mathcal{B}(X,Y^*)$ such that $S$ is weak$^*$-continuous and $\|S\| = \|T\|$.

Proof. Let $S := (T^* | Y)^*$. Then $S : X^* \to Y^*$. Since $X$ is weak$^*$-dense in $X^*$ and Proposition C.4.10 implies that both $S$ and $T^*$ are weak$^*$-weak$^*$-continuous, we have $S(x) = T^*(x)$ for all $x \in X$. As $T^*$ agrees with $T$ on $X$, $S$ extends $T$ and $\|T\| = \|S\|$. Since $\|T\| = \|T^* \circ Y\| = \|S\|$, the conclusion follows. □

Theorem 3.9.1. Let $\pi_u := \bigoplus_{\varphi \in S(A)} \pi_{\varphi}$ be the universal representation of $A$.

Theorem 3.1.16 implies that $\pi_u[A]''$ is the dual of a Banach space $Y$. Lemma 3.9.2 applied to $\pi_u : A \to \pi_u[A]''$ implies that a linear, weak$^*$-continuous, contraction $\pi : A^{**} \to \pi_u[A]''$ extends $\pi_u$.

We first prove that $\pi$ is a surjection. The Banach–Alaoglu theorem implies that the unit ball of $A^{**}$ is weak$^*$-compact. Since $\pi$ is weak$^*$-continuous on $A^{**}$, $\pi[A^{**}]$ is the weak$^*$-closure of $\pi[A_{\leq 1}]$. By the Kaplansky Density Theorem, $\pi[A_{\leq 1}]$ is weak$^*$-dense in the unit ball $\pi_u[A]''$, and therefore $\pi$ is surjective.

To prove that $\pi$ is an isometry, choose $b \in A^{**}$. Fix $\varepsilon > 0$ and $\varphi \in A^*$ such that $\|\varphi\| = 1$ and $|b(\varphi)| \geq \|b\| - \varepsilon$. By Theorem 1.10.8, there exists a cyclic representation $\pi_0 : A \to H$ and vectors $\xi$ and $\eta$ in $H$ such that $\|\varphi\| = \|\xi\| \|\eta\|$ and $\varphi(a) = (\pi_0(a)\xi|\eta)$ for all $a \in A$. We can identify $\pi_u$ with a subrepresentation of the universal representation $\pi_u$. Let $(a_\lambda)$ be a net in $A$ weakly converging to $b$. Then $\varphi(a_\lambda) = (\pi_u(a_\lambda)\xi|\eta)$ for all $\lambda$. Since $\pi$ is a weak$^*$-continuous extension of $\pi_u$, we have $\pi(b)\xi|\eta = b(\varphi)$ and $\|\pi(b)\| \geq \|b\| - \varepsilon$. Since $b \in A^{**}$ and $\varepsilon > 0$ were arbitrary this proves that $\pi$ is an isometry. □

Example 3.9.3. 1. The second dual of $\mathcal{K}(H)$ is $\mathcal{B}(H)$ by Theorem 3.1.14 and Exercise 1.11.50.

2. Theorem 3.9.1 and the Riesz Representation Theorem (Theorem C.3.8) together imply that the second dual of $C([0,1])$ is isomorphic to the dual of the space $M([0,1])$ of all complex Radon measures on $[0,1]$. This space is large. A satisfactory description of the second dual of $C([0,1])$ was given by Mauldin, but only using the Continuum Hypothesis (see [257, §2.2]). To the best of my knowledge, there is no simple description of $C([0,1])^{**}$ in ZFC.
The second dual of a C*-algebra is something of an analog of the completion of a Boolean algebra. The annihilator of a subset B of a C*-algebra A is

\[ B^\perp \cap A = \{ b \in A : ba = ab = 0 \text{ for all } a \in A \}. \]

**Proposition 3.9.4.** Suppose that B is a C*-subalgebra of a C*-algebra A. The following are equivalent.

1. B is a hereditary subalgebra of A.
2. There exists a projection \( p \in A^{**} \) such that \( B = pA^{**}p \cap A \).
3. \((B^\perp)^\perp = B\).

**Proof.** (3) \(\Rightarrow\) (1): Clearly both \( B^\perp \) and \((B^\perp)^\perp \) are hereditary C*-subalgebras of A.

(1) \(\Rightarrow\) (2): Let \( e_\lambda \), for \( \lambda \in A \), be an approximate unit of \( B \). Since \( A^{**} \) is a von Neumann algebra, \( p := \sup_\lambda e_\lambda \) belongs to \( A^{**} \) (Lemma 3.1.3). Then \( 0 \leq p \leq 1 \) and \( p \geq p^2 \geq e_\lambda \) for all \( \lambda \). Therefore \( p = p^2 \) and \( p \) is a projection.

Clearly \( B \subseteq pA^{**}p \), and we need to prove only \( pA^{**}p \cap A \subseteq B \). Fix \( c \in A \) such that \( pcp = c \). Since multiplication is WOT-continuous on bounded sets,

\[ \text{WOT-lim}_\lambda e_\lambda ce_\lambda = pc p = c. \]

By the Hahn–Banach separation theorem, \( c \in \overline{\text{conv}} \{ e_\lambda ce_\lambda : \lambda \in A \} \) and \( c \in B \). Therefore (2) follows.

(2) \(\Rightarrow\) (3): Given \( p \) as in (2) we have \( B^\perp = (1 - p)A^{**}(1 - p) \cap A \). Therefore \((B^\perp)^\perp = pA^{**}p \cap A = B \), and (3) follows. \(\square\)

An application of \( A^{**} \) given in §5.1 will require the following standard fact about second duals of Banach spaces.

**Lemma 3.9.5.** Suppose \( x_\lambda, \) for \( \lambda \in A \), is a net in a Banach space \( X \) that weak*-converges to \( y \in X^{**} \). Then there exists a net \( z_\lambda, \) for \( \lambda \in A' \), in \( \overline{\text{conv}} \{ x_\lambda : \lambda \in A \} \) which weak*-converges to \( y \) and satisfies \( \lim_\lambda \| z_\lambda \| = \| y \| \).

**Proof.** If \( z_\lambda \to y^* \) \( y \), then \( \| y \| \leq \lim \sup_\lambda \| z_\lambda \| \). Otherwise, there exists a unit functional \( \varphi \in X^* \) such that \( \Re\varphi(\varphi(y)) \leq \lim \sup_\lambda \| z_\lambda \| \) and therefore \( \lim_\lambda \varphi(z_\lambda) \neq \varphi(y) \); contradiction. It therefore suffices to find a net \( z_\lambda, \) for \( \lambda \in A' \), in \( \overline{\text{conv}} \{ x_\lambda : \lambda \in A \} \) such that \( \lim \sup_\lambda \| z_\lambda \| \leq \| y \| \).

We claim that for all \( \mu \in A \) and \( n \geq 1 \) there exists \( z_{\mu,n} \in \overline{\text{conv}} \{ x_\lambda : \lambda > \mu \} \) which satisfies

\[ \| z_{\mu,n} \| \leq \| y \| + \frac{1}{n}. \]

Suppose otherwise. By the Hahn–Banach separation theorem there exist \( r \in \mathbb{R} \) and \( \varphi \in X^* \) such that \( \Re\varphi(\varphi(y)) < r \leq \Re\varphi(\varphi(z_{\mu,n})) \) for all \( z_{\mu,n} \in \overline{\text{conv}} \{ x_\lambda : \lambda > \mu \} \); contradiction. Therefore \( z_{\mu,n} \), for \( (\mu,n) \in A \times \mathbb{N} \) (ordered coordinatewise), is a net as required. \(\square\)

**Proposition 3.9.6.** Suppose \( A \) is a C*-algebra and \( \pi : A \to \mathcal{B}(H) \) is a direct sum of finitely many inequivalent irreducible representations. If \( E \in \overline{\text{WOT}} \{ A \} \) is a projection of finite rank, \( n \geq 1, \varepsilon > 0 \), and \( a_j \), for \( j < n \) are elements of \( A \) such that \( \max_j \| \pi(a_j)E \| < \varepsilon \), then there exists \( e \in A_{n+1} \) satisfying \( E \leq e \) and \( \max_j \| a_j e \| < \varepsilon \).
Proof. Corollary 3.5.3 implies that there is a decomposition \( H = \bigoplus_{i \leq d} H_i \) for which \( \overline{\pi[A]}^{WOT} = \bigoplus_{i \leq d} \pi(H_i) \) holds. We can identify \( \pi \) with a direct summand of \( \pi_\nu \), so that \( A^{**} = \overline{\pi[A]}^{WOT} \oplus M \) for a von Neumann algebra \( M \). By the Kaplansky Density Theorem (Theorem 3.1.9) there is a net \( e_\lambda \in A_{+1} \), for \( \lambda \in \Lambda \), such that SOT-lim \( \pi_\nu(e_\lambda) = E \). Since multiplication is weakly continuous on bounded sets, WOT-lim \( \pi_\nu(a_j e_\lambda) = \pi_\nu(a_j)E \) for all \( j < n \). For every set \( \mathcal{X} \) and every \( a \) we have

\[
\overline{\text{conv}} \{ \pi(x) : x \in \mathcal{X} \} = \{ \pi(a) : a \in \overline{\text{conv}} \mathcal{X} \}.
\]

Applying Lemma 3.9.5 to \( \{ a_0 e_\lambda : \lambda \in \Lambda \} \), find a net \( e_\lambda^0 \), for \( \lambda \in \Lambda_0 \), such that WOT-lim \( \pi(e_\lambda^0) = E \) and \( \| \pi(a_0 e_\lambda^0) \| < \epsilon \) for all \( \lambda \). Applying the same lemma to \( \{ a_1 e_\lambda^0 : \lambda \in \Lambda_0 \} \), find a net \( e_\lambda^1 \), for \( \lambda \in \Lambda_1 \), such that WOT-lim \( \pi(e_\lambda^1) = E \) and \( \| \pi(a_1 e_\lambda^1) \| < \epsilon \) for all \( \lambda \). Then \( a_0 \) satisfies \( \| \pi(a_0 e_\lambda^1) \| < \epsilon \) for all \( \lambda \in \Lambda_1 \) because \( e_\lambda^1 \in \overline{\text{conv}} \{ e_\lambda^0 : \lambda \in \Lambda_0 \} \). By repeating this argument, we obtain a sequence of nets \( e_\lambda^{n-1} \), for \( \lambda \in \Lambda_{n-1} \), for all \( n \in \mathbb{N} \) such that WOT-lim \( \pi(e_\lambda^{n-1}) = E \) and max \( \| \pi(a_j e_\lambda^{n-1}) \| < \epsilon \).

We still need to assure that \( E \geq \pi(e) \). Let \( \delta := (\epsilon - \max_j \| \pi(a_j e_\lambda^{n-1}) \| )/2 \). Since the projection \( E \) has finite rank, \( \lim \| (1 - \pi(e_\lambda^{n-1}))E \| = 0 \). Let \( \lambda \) be large enough so that \( e_\lambda := e_\lambda^{n-1} \) satisfies \( \| (1 - \pi(e_\lambda))E \| < \delta \). Fix a unit vector \( \xi \in E[H] \). By the Cauchy–Schwarz inequality, \( \| (1 - \pi(e_\lambda))\xi \| \xi \| \leq \| (1 - \pi(e_\lambda))\xi \| \xi \| < \delta \). Therefore \( \epsilon := \min(\epsilon_0 + \delta, 1) \) satisfies \( \pi(e_\lambda)\xi \| \xi \| = 1 \). Since \( \xi \) was an arbitrary unit vector in \( E[H] \), we have \( \pi(e) \geq E \). Also, \( \max_j \| a_j e \| \leq \max_j \| a_j e_\lambda^{n-1} \| + \delta < \epsilon \). \qed

3.10 Exercises

Exercise 3.10.1. Prove that a convex subset of a Hilbert space \( H \) is norm-closed if and only if it is weakly closed. Prove that a convex subset of \( \mathcal{B}(H) \) is WOT-closed if and only if it is SOT-closed.

Exercise 3.10.2. Prove that a functional on \( \mathcal{B}(H) \) is normal if and only if it is of the form \( b \mapsto \text{tr}(ab) \) for a trace class operator \( a \).

Exercise 3.10.3. Suppose \( A \) is a \( C^* \)-subalgebra of \( \mathcal{B}(H) \) and \( e_\lambda \), for \( \lambda \in \Lambda \), is an approximate unit of \( A \). Prove that SOT-lim \( \lambda e_\lambda = 1 \mathcal{B}(H) \) if and only if \( A \) is nondegenerate.\(^8\)

Exercise 3.10.4. Suppose \( H \) is an infinite-dimensional complex Hilbert space.

1. Construct a faithful nondegenerate representation \( \pi: \mathcal{B}(H) \to \mathcal{B}(K) \) for some Hilbert space \( K \) such that \( \pi[\mathcal{B}(H)] \) is not a von Neumann \( C^* \)-subalgebra of \( \mathcal{B}(K) \).

\(^8\) Used in the proof of Lemma 3.2.10 and Proposition 13.2.1.
2. Conclude that a $^*$-homomorphism between von Neumann algebras is not necessarily WOT-continuous.

**Exercise 3.10.5.** Prove that the operator $v$ in the conclusion of Stinespring’s theorem (Theorem 3.2.5) can be taken to be a contraction, and that in the case of a u.c.p. map it can be taken to be a projection.

**Exercise 3.10.6.** Suppose $\phi$ is a c.p.c. map between $C^*$-algebras. Prove that $\phi$ is self-adjoint, i.e., $\phi(a^*) = \phi(a)^*$ for all $a$.

**Exercise 3.10.7.** Suppose $\phi : A \to B$ is a completely positive map between $C^*$-algebras. Prove that the following are equivalent.

1. $\phi$ is contractive.
2. (Kadison’s inequality) $\phi(a^*a) \geq \phi(a)^*\phi(a)$ for all $a \in A$.

**Exercise 3.10.8.** Suppose $\phi : A \to B$ is a u.c.p. map between $C^*$-algebras. The multiplicative domain of $\phi$ is $C := \{a \in A : \phi(a^*a) = \phi(a)^*\phi(a)\}$. Prove that $C$ is a subalgebra of $A$, and that the restriction of $\phi$ to $C$ is a $^*$-homomorphism.

**Hint:** To prove that $C$ is a subalgebra, consider $\phi \otimes \text{id}_2$ and apply the Cauchy–Schwartz inequality in $M_2(B)$.

**Exercise 3.10.9.** Suppose that a state $\phi$ of a $C^*$-algebra $A$ is pure. Prove that it is tracial if and only if it is a character.

Recall that $A_{+,1} = \{a \in A_+ : \|a\| = 1\}$.

**Exercise 3.10.10.** Suppose $A$ is a $C^*$-algebra, $D$ is an abelian $C^*$-subalgebra of $A$, and $h \in A_{+,1}$. Prove that the following are equivalent.

1. $h \in D' \cap A$.
2. For all $a$ and $b$ in $D_+$, $ab = 0$ implies $abh = 0$.
3. For all $a$ and $b$ in $W^*(D_+)$, $ab = 0$ implies $abh = 0$.

**Exercise 3.10.11.** Prove that a u.c.p. map $\phi$ between $C^*$-algebras has order zero if and only if it is a $^*$-homomorphism.

The following is a generalization of the first part of Proposition 3.2.11.

**Exercise 3.10.12.** Suppose $A$ is a unital $C^*$-algebra and $\phi : A \to \mathscr{B}(H)$ is a c.p.c. map of order zero. If $\phi[A] \neq \{0\}$ prove that there exists a $C^*$-algebra $C$ and a completely positive map $\psi : \phi[A] \to C$ such that $\psi \circ \phi : A \to C$ is a $^*$-homomorphism.

**Hint:** In the proof of Proposition 3.2.11 replace Lemma 2.4.7 by an argument using the second dual of $C^*(h, \pi[A])$.

**Exercise 3.10.13.** Suppose $A$ is a finite-dimensional $C^*$-algebra and $D$ is a masa in $A$. Prove that for every $a \in A_{\leq 1}$ one has $\text{dist}(a, D) \leq \sup_{b \in D_{\leq 1}} \|\{a, b\}\| \leq 2\text{dist}(a, D)$.

**Hint:** To prove the first inequality use averaging as in Example 3.3.1. The second inequality is straightforward.
Exercise 3.10.14. Suppose $A$ and $B$ are $C^*$-algebras, $B$ is unital, $a$ and $c$ belong to $A$, and $u$ and $v$ are unitaries in $B$. Prove that $\|a \otimes u - c \otimes v\| \geq \inf_{\lambda \in \mathbb{D}} \| \lambda a - c \|$. The following exercise uses the notation from §2.4.1.

Exercise 3.10.15. Suppose that $G$ is a subgroup of a discrete group $\Gamma$.

1. Prove that $C^*_r(G)$ is a isomorphic to a subalgebra of $C^*_r(\Gamma)$. 

2. Prove that there exists a conditional expectation of $C^*_r(\Gamma)$ onto $C^*_r(G)$.

Hint: After observing that (1) is not vacuous, observe that the left regular representation of $G$ on $\ell^2(\Gamma)$ is equivalent to a direct sum of its left regular representations on $\ell^2(G)$. This also gives a hint for how to construct a conditional expectation as in (2). It is worth noting that the latter coincides with the restriction of the orthogonal projection $\text{proj}_{\ell^2(G)}$ to $C^*_r(G)$.

Exercise 3.10.16. Give a short proof of the following poor man’s version of (*) with $M = \mathcal{B}(H)$.

\begin{equation}
(*) \quad \text{For every finite-rank projection } p \text{ in } Z(M)' \text{ and every self-adjoint } c \in M \text{ there exists a self-adjoint } b \in M \text{ such that }
\end{equation}

\[ (c - b)p = 0 \quad \text{and} \quad \|b\| \leq 2\|cp\|. \]

Then show that the conclusion of Theorem 3.4.2 holds for every von Neumann algebra $M$ which satisfies $(*)$ in place of $(*).$

Hint: For the first part use $b := pcp + (1 - p)cp + pc(1 - p)$.

Exercise 3.10.17. Show that Theorem 3.4.5 (2) can be improved as follows. If $A$ is a $C^*$-algebra which is WOT-dense in $\mathcal{B}(H)$, $p$ is a finite-rank projection in $\mathcal{B}(H)$ and $c \in \mathcal{B}(H)_{sa}$, then there exists $a \in A_{sa}$ such that $p(a - c)p = 0$ and $\|a\| = \|c\|$. Also show that this implies a similar improvement to Theorem 3.4.5 (3). Finally state and prove the analogous improvements to Theorem 3.5.4 (2) and (3).

Exercise 3.10.18. Suppose $\varphi$ is a pure state on a unital $C^*$-algebra $A$. Prove that there exists $a \in A_+$ such that $\|a\| = \varphi(a) = 1.9$

Exercise 3.10.19. Prove that for an irreducible representation $\pi : A \to \mathcal{B}(H)$ we have $\pi[A] \supseteq \mathcal{K}(H)$ if and only if $\pi[A] \cap \mathcal{K}(H) \neq \{0\}$.

Exercise 3.10.20. 1. Suppose $p$ is a projection in $\mathcal{B}(H)$ and $u$ is a unitary in $p\mathcal{B}(H)p$. If $a$ is such that $ap = u$ and $\|a\| = 1$, then $ap = pa$. 

2. Describe the set of all extreme points of $\mathcal{B}(H)_{\leq 1}$.

Exercise 3.10.21. Assume $\pi : A \to \mathcal{B}(H)$ is an irreducible representation and a projection $p \in \mathcal{B}(H)$ has finite rank. Prove that $B = \{a \in A : p\pi(a) = \pi(a)p\}$ is a $C^*$-subalgebra of $A$ such that $a \mapsto p\pi(a)p$ is a surjection of $B$ onto $p\mathcal{B}(H)p$.

Hint: You will need Theorem 3.4.5 and Exercise 3.10.20.

\footnote{9 Used in the proof of Theorem 5.2.1.}
Exercise 3.10.22. Suppose that $A$ is a unital $C^*$-algebra, $a \in A_{sa}$, and $b \in A$.\footnote{Used in the proofs of Lemma 5.6.6 and Lemma 5.6.7.}

1. Prove that $\| [b, \exp(ia)] \| < e^{\|a\|} \| [b, a] \|$ (this time, $e = \exp(1) \approx 2.71828$).
2. Prove that $\|a\| < \varepsilon$ implies $\| [b, \exp(ia)] \| < 2 \|b\| (e^\varepsilon - 1)$.

*Hint:* Estimate $\| [b, (ia)^n] \| \leq n \|a\|^{n-1} \| [b, a] \|$ and sum the series.

The following is high-school math but we will need it in the middle of a long proof in the fairly technical §5.6.

Exercise 3.10.23. Let $r \geq 0$ be a real. Prove the following.

1. If $h$ is self-adjoint and $-r \leq h \leq r$ then $\|1 - \exp(ih)\| \leq 2 \sin(r/2)$.
2. If $r < 2\pi$ and $u$ is a unitary such that $\|1 - u\| \leq 2 \sin(r/2)$ then $u = \exp(ih)$ for some self-adjoint $h$ such that $-r \leq h \leq r$.

Exercise 3.10.24. Suppose $\varphi$ is a state on a $C^*$-algebra $A$. We use the notation from the GNS construction (Proposition 1.10.3), in particular $(a \cdot b)_\varphi := \varphi(b^*a)$ and

$$L_\varphi := \{ a \in A : \varphi(a^*a) = 0 \}.$$

1. If $\varphi$ is a pure state, prove that $A/L_\varphi$ is complete with respect to the Hilbert space norm associated with $\langle \cdot, \cdot \rangle_\varphi$.
2. If $\varphi$ is the tracial state on the CAR algebra $A$, prove that $A/L_\varphi$ is not complete with respect to the Hilbert space norm associated with $\langle \cdot, \cdot \rangle_\varphi$.

Exercise 3.10.25. Let $A$ be a separable $C^*$-algebra. Prove that the space $P(A)$ of pure states is $G_\varphi$ in $S(A)$, and therefore Polish, in the weak*-topology.

Exercise 3.10.26. Prove that a $C^*$-algebra $A$ is abelian if and only if $\| [a, b] \| < 2$ for all contractions $a$ and $b$ in $A$.\footnote{This could have been used in the proof of Proposition 10.4.1.}

Exercise 3.10.27. Suppose $A$ is a $C^*$-algebra.

1. Prove that $A$ contains a nilpotent of degree $n$ if and only if $A$ has an irreducible representation on a Hilbert space of dimension $\geq n$.
2. Prove that $C^*(x) \cong C_0(\text{sp}(xx^*) \setminus \{0\}, \mathbb{M}_n(\mathbb{C}))$ if $x \in A$ is nilpotent of degree $n$.

Exercise 3.10.28. Prove that a $C^*$-algebra $A$ has distinct pure states if and only if it is distinct from $\mathbb{C}$.

Exercise 3.10.29. Prove that if $\varphi$ is a state on $A$ and $a \in A$ is normal with $\varphi(a) = \|a\|$, then the restriction of $\varphi$ to $C^*(a, 1)$ is pure.

Exercise 3.10.30. Suppose $A$ has a faithful irreducible representation $\pi : A \to \mathcal{B}(H)$ such that $\pi[A] \cap \mathcal{K}(H) = \{0\}$. Prove that every hereditary $C^*$-subalgebra $B$ of $A$ has a $C^*$-subalgebra isomorphic to $C_0((0, 1], \mathbb{M}_2(\mathbb{C}))$.

*Hint:* This is a consequence of Theorem 3.7.2. However the ‘honest’ way to solve this exercise is to combine the proof of Lemma 3.7.4 with Exercise 2.8.1.
Exercise 3.10.31. Suppose that $B$ is a UHF algebra and $A$ is a WOT-dense C*-subalgebra of $\mathcal{B}(H)$ such that $A \cap \mathcal{K}(H) = \{0\}$. Prove that $A$ has a C*-subalgebra with a quotient isomorphic to $B$.

Exercise 3.10.32. Suppose that $\varphi$ and $\psi$ are pure states on $A$. Prove that $\pi_\varphi \sim \pi_\psi$ if and only if there exists $\xi \in H_\varphi$ such that $\psi = \omega_\xi \circ \pi_\varphi$. Then prove that the connected component of a pure state $\varphi$ in $S(A)$ (considered with respect to the norm metric) is homeomorphic to the space of 1-dimensional projections in $\mathcal{B}(H_\varphi)$.\(^{12}\)

Recall that by $c$ we denote the cardinality of the continuum, $2^{\aleph_0}$.

Exercise 3.10.33. Suppose $A \neq \mathbb{C}$ and let $\pi$ be an irreducible representation of $A$. Prove that $\bigoplus_c \pi$ (the direct sum of $c$ copies of $\pi$) is a subrepresentation of the universal representation of $A$.

Hint: Since $A \neq \mathbb{C}$, it has more than one pure state and therefore $S(A)$ is uncountable. If $\varphi$ and $\psi$ are distinct pure states on $A$, then the GNS representation corresponding to $t\varphi + (1-t)\psi$ has $\pi_\varphi$ as a subrepresentation for all $t \in [0,1]$.

Suppose $\pi: A \to \mathcal{B}(H)$ is a nondegenerate representation of a C*-algebra $A$. It is a direct sum of cyclic representations (Proposition 1.10.10) and therefore there exists a normal *-homomorphism $\tilde{\pi}: A^{**} \to \mathcal{B}(H)$ extending $\pi$. The kernel of $\tilde{\pi}$ is equal to $(1-p)A^{**}(1-p)$ for a central projection $p$ in $A^{**}$. The projection $p$ is the central cover of $\pi$.

Exercise 3.10.34. Prove that two irreducible representations of a C*-algebra $A$ are spatially equivalent if and only if their central covers are equal.

Exercise 3.10.35. Prove that if $A$ is a separable non-type I C*-algebra, then there exist a C*-subalgebra $B$ of $A$ and a projection $q \in B' \cap A^{**}$ such that $qAq = Bq$ and $Bq$ is isomorphic to the CAR algebra.

Hint: Redo the proof of Proposition 3.7.5; see [194, Theorem 6.7.3].

Notes for Chapter 3

§3.1 More information on von Neumann algebras and W*-algebras can be found e.g., in [142], [215], [235], [59], or [27]. As the *-homomorphisms between C*-algebras are automatically norm-continuous, one could expect that the *-homomorphisms between von Neumann algebras are ultraweakly continuous. While not always the case (Exercise 3.10.4), this is true for the isomorphisms between von Neumann algebras (see e.g., [142, Theorem 7.2.1]).

The elegant proof of Lemma 3.1.13 is due to Larry Brown, and I learned it from Bruce Blackadar.

§3.2 Completely positive maps are at the heart of one of the equivalent definitions of nuclear C*-algebras via the completely positive approximation property, or CPAP

\(^{12}\) Used in the proof of Proposition 5.3.7.
An excellent source of information on completely positive maps is [193]. Theorem 3.2.9 was proved in [263] and [261]. The simple proofs of Lemma 3.2.10 and Theorem 3.2.9 were kindly provided by Narutaka Ozawa.

Example 3.2.4 (2) was provided by Aaron Tikuisis.

§3.5 The Kadison Transitivity Theorem was proved by Kadison, and its extension Theorem 3.5.4 was proved by Glimm and Kadison ([119, Corollary 7]). The present section was partly based on [58]. In particular Proposition 3.5.2 was adapted from [58, Theorem 5.2.1].

§3.7 Type I C*-algebras should not be confused with type I von Neumann algebras. For example, $\mathcal{B}(H)$ is a type I von Neumann algebra but not a type I C*-algebra. Theorem 3.7.2, together with Theorem 5.7.29, was proved by Glimm in [118]; see also [194, Theorem 6.8.7].

Theorem 3.7.2 (with its corollaries proved in §5.5) forms just a fragment of Glimm’s Theorem (“probably the deepest theorem in the subject of operator algebras when it was proved” according to [27, §IV.1.5]). Its full statement can be found in [27, IV.1.5.7] and in [194, §6.8]. See also Notes for §5.5.

The proof of Proposition 3.7.5 contains a germ of the result that became known as the Glimm–Effros Dichotomy ([127]). Together with results of Friedman and Stanley ([109]), this started the abstract classification theory, a subject that was to become a very prominent theme in the Descriptive Set Theory. See [130] and [112] for the general theory and [173], [84], and [174] for applications in operator algebras.

§3.9 The standard Lemma 3.9.5 was taken from [30, Lemma 1.1]. Proposition 3.9.6 is implicit in [161] and its proof was kindly provided by N. Ozawa.
Chapter 4
Tracial States and Representations of $C^*$-algebras

This is about $\text{II}_1$ factors in their ambient spaces. $\text{II}_1$ factors love to sit in their ambient spaces. $C^*$-algebras, you stick them anywhere.

Thomas Sinclair

The interplay between a $C^*$-algebra and the weak closure of its image under a representation $\pi$ is a powerful tool for analysing $C^*$-algebras. If $\pi : A \to \mathcal{B}(H)$ is irreducible, then Kadison’s Transitivity Theorem provides a mean for transferring the information between $A$ and the weak closure of its image. In the very different case when $\pi$ is the GNS representation associated to a tracial state $\tau$, the GNS Hilbert space $L^2(A, \tau)$ provides a valuable insight into $A$. This is in particular the case when $A$ is a reduced group $C^*$-algebra $C^*_r(\Gamma)$ for a discrete group $\Gamma$. This method has been borrowed from the study of $\text{II}_1$ factors, where the structure of an operator algebra is juxtaposed with the structure of the GNS pre-Hilbert space associated to a trace.

4.1 Finiteness and Tracial States

We review the bare minimum of information about finiteness and tracial states.

A projection $p$ in a $C^*$-algebra is infinite if there exists a partial isometry $v$ such that $v^*v = p$ and $p - vv^*$ is a nonzero projection. Otherwise, $p$ is finite. A unital $C^*$-algebra $A$ is finite if $1_A$ is finite. It is stably finite if $M_n(A)$ is finite for all $n$. Finally, a $C^*$-algebra is infinite if it contains an infinite projection.

Example 4.1.1. 1. Every finite-dimensional $C^*$-algebra is finite, and finiteness is preserved by inductive limits. Therefore all AF algebras are finite. Since the stabilization of an AF algebra is also AF, all AF algebras are stably finite.
2. All Cuntz algebras $\mathcal{O}_n$ (Example 2.3.10) are infinite.
3. If $H$ is an infinite-dimensional Hilbert space then both $\mathcal{B}(H)$ and $\mathcal{D}(H)$ are infinite, because each one of them contains a unital copy of $\mathcal{O}_2$. 

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4. An extension of a finite C*-algebra by a finite C*-algebra can be infinite. With \( s \) denoting the unilateral shift on a basis of \( \mathcal{B}(H) \) (Example 1.1.1), \( C^*(s) \) is infinite. However, \( C^*(s)/\mathcal{K}(H) \) is abelian (Proposition 12.4.3) and therefore finite.

**Definition 4.1.2.** A state \( \tau \) on a C*-algebra \( A \) is tracial if \( \tau(ab) = \tau(ba) \) for all \( a \) and \( b \). A state \( \phi \) on \( A \) is faithful if \( \phi(aa^*) = 0 \) implies \( a = 0 \) for all \( a \in A \). A trace is an additive and homogeneous function \( \tau: A \to [0, \infty] \) such that \( \tau(aa^*) = \tau(a^*a) \) for all \( a \in A \).

If a trace is bounded then its normalization \( \|\tau\|^{-1}\tau \) uniquely defines a tracial state since every element of \( A \) is a linear combination of positive elements.

**Lemma 4.1.3.** If \( \tau \) is a tracial state on a C*-algebra \( A \), then its left kernel \( L_\tau \) is a norm-closed, two-sided, self-adjoint ideal of \( A \).

**Proof.** Lemma 3.6.3 implies that \( L_\phi \) is a norm-closed ideal. If \( \tau \) is a tracial state then \( L_\tau = L_\tau^* \), and it is therefore a two-sided, norm-closed ideal. It is self-adjoint by Lemma 2.5.1. \( \square \)

**Corollary 4.1.4.** Every tracial state of every simple C*-algebra is faithful. \( \square \)

**Example 4.1.5.**
1. The proof of Lemma 4.1.3 shows that if \( \tau \) is a trace on \( A \), then \( \{ a : \tau(a^*a) < \infty \} \) is a two-sided, self-adjoint, (possibly not norm-closed) ideal of \( A \).
2. Suppose \( M \) is a II\(_\infty\) factor and let \( p \) be a projection in \( M \) such that \( pMp \) is a II\(_1\) factor. Let \( \tau \) be the unique tracial state on \( pMp \). The restriction of \( \tau \) to \( (pMp)_+ \) can be extended to an unbounded trace \( \sigma \) on \( M \). The norm-closed, two-sided, ideal on \( M \) generated by \( p \) is the Breuer ideal. Since \( M \) is a factor, this is an essential ideal (Definition 2.5.5).

**Example 4.1.6.** If \( \tau \) is a tracial state on a unital C*-algebra \( A \) then \( \tau(vv^*) = \tau(v^*v) \) for all \( v \) in \( A \), hence \( \tau \) is constant on Murray-von Neumann equivalence classes of projections. This implies that every unital C*-algebra with a faithful tracial state is finite. Even more is true: If \( \tau \) is a tracial state on \( A \), then on \( M_n(A) \) the formula

\[
\tau^a(a) := \frac{1}{n} \sum_i \tau(a_{ii})
\]

defines a tracial state. In particular, if \( A \) has a faithful tracial state, it is stably finite.

The space of all tracial states of \( A \) is denoted \( T(A) \). If \( \tau \) and \( \sigma \) are tracial states of \( A \) and \( 0 \leq t \leq 1 \) then \( t\tau + (1-t)\sigma \) is a tracial state. Therefore \( T(A) \) is a convex subset of the unit sphere of \( A^* \). Also, since being a tracial state is a closed condition in the weak*-topology, by the Birkhoff--Alaoglu theorem \( T(A) \) is compact in the weak*-topology. Being a compact and convex set, by the Krein--Milman theorem \( T(A) \) is the closure of the convex hull of its extreme points.

\footnote{It is not difficult to check that all compact operators belong to \( C^*(s) \): start with \( 1 - ss^* \).}
Lemma 4.1.8. Assume A is unital and abelian. By the Gelfand–Naimark theorem $A = C(X)$ for a compact Hausdorff space $X$. By the Riesz Representation Theorem (Theorem C.3.8), every continuous functional $\phi$ of $A$ is of the form $\phi(f) = \int f \, d\mu$ for a complex Radon measure $\mu$ on $X$. If $\phi$ is a state then $\mu_\phi$ is a probability measure. Since $A$ is abelian the condition $\phi(ab) = \phi(ba)$ is automatic and therefore all states are tracial.

Therefore $T(A)$ is affinely homeomorphic to $P(X)$, the space of Radon probability measures on $X$.

2. The tracial state on $M_n(\mathbb{C})$ is by the elementary linear algebra unique, and every UHF algebra carries a unique tracial state (Corollary 4.1.9).

3. If $A = B \oplus C$ then we clearly have

$$T(A) = \{ \lambda \tau + (1 - \lambda)\sigma : \tau \in T(B), \sigma \in T(C), 0 \leq \lambda \leq 1 \}. $$

Therefore if $A$ is a direct sum of $n$ matrix algebras then $T(A)$ is affinely homeomorphic to the $n$-simplex.

4. By (3) every tracial state $\tau$ of $\bigoplus_{i < n} M_{n(i)}(\mathbb{C})$ is a convex combination of traces on its direct summands. Therefore if $A = \bigoplus_{i < n} M_{n(i)}(\mathbb{C})$, $B = \bigoplus_{i < k} M_{k(i)}(\mathbb{C})$, and $\Phi : A \to B$ is a unital $^*$-homomorphism and $\tau$ is a tracial state on $B$, then $\sigma(a) := \tau(\Phi(a))$ is a tracial state on $A$. If $p_i$ is the identity of $M_{n(i)}(\mathbb{C})$ and $q_j$ is the identity of $M_{k(j)}(\mathbb{C})$ then $\sigma(p_i)$ is uniquely determined by $\tau(q_j)$ and the Bratteli diagram of $\Phi$.

Suppose $\Phi : B \to C$ is a unital $^*$-homomorphism. Its adjoint (denoting the Banach space duals of $B$ and $C$ by $B^*$ and $C^*$) $\Phi^* : C^* \to B^*$ is defined by $\Phi^*(\psi) := \psi \circ \Phi$. This linear map is both weak-weak continuous and norm-norm continuous (Proposition C.4.10). Since $\Phi$ is multiplicative, $\Phi^*$ sends $T(C)$ into $T(B)$.

Lemma 4.1.8. Assume $A = \lim_{n} A_n$, all $A_n$ are unital, and all connecting maps are unital. Then $T(A) = \varprojlim T(A_n)$, and it is nonempty.

Proof. The space $S(A)$ is the inverse limit of $S(A_n)$ with respect to the adjoints to the connecting maps. Since $T(B)$ is weak$^*$-closed in $S(B)$ for all $B$, $T(A)$ is the inverse limit of $T(A_n)$. Since an inverse limit of a sequence of compact Hausdorff spaces is nonempty, this completes the proof. \qed

Corollary 4.1.9. Every AM algebra, and in particular every UHF algebra, has a unique tracial state.

A $C^*$-algebra with a unique tracial state is said to be monotracial. The following proposition will be applied to reduced group $C^*$-algebras in \S4.3.

Proposition 4.1.10. Suppose $A$ is a unital $C^*$-algebra such that for every $a \in A$ the set $O_a := \text{conv}\{ uu^* : u \in U(A) \}$ intersects $\mathbb{C}$.

1. Then $A$ has a tracial state if and only if $O_a \cap \mathbb{C}$ is a singleton for every $a$.

2. If in addition $A$ has a faithful tracial state, then $A$ is simple.
Proof. (1) If \( \tau \) is a tracial state then \( \tau(uau^*) = \tau(a) \) for all \( a \), and \( \tau \) is constant on \( O_a \). Therefore \( O_a \subseteq \{ \tau(a) \} \) and \( O_a \cap \mathbb{C} \) has at most one element for every \( a \). Conversely, \( O_a \cap \mathbb{C} \) has exactly one element for every \( a \), then setting \( \tau(a) \) to be this unique element defines a tracial state on \( A \).

(2) If a tracial state \( \tau \) is faithful, then \( \tau(a^*a) = 0 \) if and only if \( a = 0 \). If \( a \in A \) is nonzero, then \( r := \tau(a^*a) > 0 \) and we have \( r \in O_a \). But this implies that \( 1_A \) can be approximated in \( A \) by a finite sum \( \sum x_i a^* a y_i \). Since the invertible elements form an open set (Lemma 1.2.6), the ideal generated by \( a \) is not proper. Since \( a \) was an arbitrary nonzero element, \( A \) has no nontrivial ideals.

The assumption of Proposition 4.1.10 is called the strong Dixmier property.

### 4.2 The \( L_2 \)-Space of a C*-Algebra Associated to a Tracial State

In this section we study C*-algebras with a distinguished tracial state \( \tau \) and the associated GNS Hilbert space. We introduce the \( \tau \)-orthogonality relation, prove that \( \tau \) has a unique extension to a tracial state on the weak closure of the image of the C*-algebra under the GNS representation associated with \( \pi \). We also prove that the strong operator topology on this weak closure is given by the 2-norm associated with \( \tau \).

We now take a closer look at the structure of GNS representations associated with tracial states. This is accomplished by juxtaposing the C*-algebra structure and the GNS-Hilbert space structure associated with a trace. This tool, borrowed from the theory of II_1 factors, will be used in the study of reduced group C*-algebras in §4.3.

Suppose \( \tau \) is a faithful state (not necessarily tracial, for now) on a C*-algebra \( A \), so that the left ideal \( L_{\tau} = \{ a : \tau(a^*a) = 0 \} \) is trivial. Then \( \langle a|b \rangle_{\tau} := \tau(b^*a) \) is an inner product on \( A \). The completion \( L_2(A, \tau) \) of \( A \) with respect to the associated norm

\[
\|a\|_{2,\tau} := \tau(a^*a)^{1/2}
\]

(denoted \( \|a\|_{2,\tau} \) if \( \tau \) is clear from the context) coincides with the Hilbert space \( H_\tau \). Since \( \tau \) is faithful, the GNS representation \( \pi_{\tau} : A \to B(H_\tau) \) is faithful as well.

**Example 4.2.1.** Consider the reduced group algebra \( C_r^*(\Gamma) \) associated with a discrete group \( \Gamma \) (§2.4.1). The standard trace on \( C_r^*(\Gamma) \) is faithful (Lemma 2.4.9 (5)) and each of \( \ell^2(\Gamma) \) and \( C_r^*(\Gamma) \) is isomorphic to the completion of the group algebra \( \mathbb{C}[\Gamma] \) with respect to the appropriate norm (\( \| \cdot \|_{2,\tau} \) and \( \| \cdot \| \), respectively). Since \( \tau(\lambda(g)) = 1 \) if \( g = e \) and \( \tau(\lambda(g)) = 0 \) otherwise, \( \lambda(g) \), for \( g \in \Gamma \), form an orthonormal basis for \( H_\tau \) (see Example 7.4.6 (3)). Also, \( \lambda(g) \mapsto \delta_g \) extends to an isometry between \( H_\tau \) and \( \ell^2(\Gamma) \) and the GNS representation associated with \( \tau \) is spatially equivalent to the left-regular representation.

Every C*-subalgebra \( B \) of a C*-algebra \( A \) (and every Banach subspace of \( A \)) corresponds to a closed subspace \( \overline{B^{\|L_2\|}} \) of \( H_\tau \). In this way the geometry of \( H_\tau \) is superimposed onto the C*-algebraic structure of \( A \).
4.2 The $L_2$-Space of a C$^*$-Algebra Associated to a Tracial State

If $A$ is unital and $B$ and $C$ are unital C$^*$-subalgebras of $A$, then $B \cap C$ includes the scalars and in particular the subspaces associated to $B$ and $C$ are not orthogonal. This, together with Lemma 4.2.4 below, justifies the following definition.

**Definition 4.2.2.** Suppose $A$ is a unital C$^*$-algebra and $\tau$ is a faithful state on $A$.

1. We say that $a \in A$ is $\tau$-orthogonal to a C$^*$-subalgebra $B$ of $A$, and write $a \perp \tau B$, if $\tau(a^*b) = 0$ for all $b \in B$ such that $\tau(b) = 0$.

2. C$^*$-subalgebras $B$ and $C$ of $A$ are $\tau$-orthogonal if $\tau(bc) = \tau(b)\tau(c)$ for all $b \in B$ and $c \in C$.\(^2\)

3. The $\tau$-orthogonal complement of $B$ is defined by

$$B^\perp : = \{ c : \tau(bc) = \tau(b)\tau(c) \text{ for all } b \in B \}. $$

**Lemma 4.2.3.** If $A$ is a unital C$^*$-algebra with a tracial state $\tau$, then $\tau$ is uniformly continuous with respect to $\| \cdot \|_{2, \tau}$. Its continuous extension to $L_2(A, \tau)$ is the $\| \cdot \|_{2, \tau}$-orthogonal projection onto $\mathbb{C}$. \(\square\)

If $H$ is a Hilbert space and $K$ is a subspace of $H$ then

$$H \ominus K := H \cap K^\perp. $$

If $\tau$ is a fixed tracial state on $A$ we drop the prefix $\tau$- and write `orthogonal’ instead of `$\tau$-orthogonal.'\(^3\)

**Lemma 4.2.4.** Suppose $\tau$ is a faithful state on a C$^*$-algebra $A$, and $B$ and $C$ are C$^*$-subalgebras of $A$. Then the following are equivalent.

1. $B$ and $C$ are $\tau$-orthogonal.
2. $B \ominus \mathbb{C}$ and $C \ominus \mathbb{C}$ are orthogonal subspaces of $H_\tau$.
3. $\tau(bc) = 0$ for all $b \in B$ and $c \in C$ such that $\tau(b) = \tau(c) = 0$.
4. $\tau(bc) = \tau(b)\tau(c)$ for all $b \in B$ and $c \in C$.

**Proof.** To see that (3) and (4) are equivalent, simplify $\tau((b - \tau(b))(c - \tau(c)))$. Lemma 4.2.3 implies that $B \ominus \mathbb{C} = \{ b \in B : \tau(b) = 0 \}$, and the analogous formula applies to $C$. The equivalence of (1), (2), and (3) is now straightforward. \(\square\)

We now step into the convenient world of von Neumann algebras.

**Lemma 4.2.5.** Suppose $\tau$ is a tracial state on a C$^*$-algebra $A$. Then $\tau$ can be extended to a normal tracial state on the WOT-closure of the image of $A$ under the GNS-representation associated with $\tau$.

---

\(^2\) This notation, going back to [204], should be taken with a grain of salt; see Lemma 4.2.4.

\(^3\) As a matter of fact, to the best of my knowledge nobody else ever bothers to emphasize the distinction. This practice is mostly harmless because this terminology is typically used in the context of monotracial algebras, especially II$_1$ factors.
Proof. Let $(\pi, L_2(A, \tau), \xi)$ be the GNS triplet associated with $\tau$. By identifying $A$ with $\pi[A]$, we have that $\tau$ is the restriction of the vector state $\omega_\xi$ to $A$. Its restriction to $M = \overline{A}^{\text{WOT}}$ is clearly a state. We denote this restriction by $\tau$ and prove that it is tracial. If $a$ and $b$ are such that $\omega_\xi(ab) \neq \omega_\xi(ba)$, then $(b_\xi^*a)^* \neq (a_\xi^*b)^*$. This gives a WOT-open neighbourhood of $(a, b, a^*b^*)$ disjoint from $A^4$. Since $\tau$ is the restriction of a vector state to $M$, it is normal. $\Box$

**Proposition 4.2.6.** Suppose $\tau$ is a tracial state on a $C^*$-algebra $A$. Then the strong operator topology on the unit ball of $\pi[A]^{\text{WOT}}$ is induced by $\|a\|_2 := \tau(a^*a)^{1/2}$.

Proof. Let $(\pi, H, \xi)$ be the GNS triplet for $\tau$. Kaplansky’s Density Theorem implies that the unit ball of $M := \overline{\pi[A]}^{\text{SOT}}$ is equal to the SOT-closure of $\pi[A_1]$.

A net $(a_\lambda)$ in $M_1 := \overline{\pi[A]}^{\text{SOT}}$ converges to 0 if and only if $\|a_\lambda\eta\| \to 0$ for a dense set of $\eta \in H$. Since $\xi$ is cyclic for $\pi[A]$, $\lim_\lambda a_\lambda = 0$ if and only if $\lim_\lambda \|a_\lambda b_\xi\| = 0$ for all $b \in A_1$. But

$$\|a_\lambda b_\xi\|^2 = (a_\lambda b_\xi | a_\lambda b_\xi) = (\pi((a_\lambda b)^* a_\lambda b) \xi | \xi) = \tau((a_\lambda b)^* a_\lambda b) = \tau(a_\lambda b^* a_\lambda) \leq \tau(a_\lambda^* a_\lambda) = \|a_\lambda\|^2$$

Therefore $a_\lambda \to 0$ if and only if $\|a_\lambda\|_2 \to 0$. By the linearity, the equivalence of the topologies follows. $\Box$

With a little bit of extra work, Proposition 4.2.6 can be proved with faithful states in place of tracial states (Exercise 4.5.13).

### 4.3 Reduced Group $C^*$-Algebras of Powers Groups

In this section we study reduced group $C^*$-algebras $C^*_r(\Gamma)$ associated with free products of countable groups. We give a norm estimate for a sum of unitary conjugates of a fixed element, sufficient conditions that every automorphism of $C^*_r(\Gamma)$ extends to an automorphism of $C^*_r(\Gamma * A)$, and also study Powers groups and the question of simplicity of reduced group $C^*$-algebras, and give sufficient conditions that every pure state on $C^*_r(\Gamma)$ extends to a pure state on $C^*_r(\Gamma * A)$.

Our analysis of the reduced group algebra of the free group requires the following norm estimate.

**Lemma 4.3.1.** Suppose $A$ is a unital $C^*$-subalgebra of $B(H)$ and $p$ is a projection in $B(H)$. Furthermore assume that $n \geq 1$, and $u_j$, for $j < n$, are unitaries in $A$ that satisfy $(1 - p)u_i^* u_j (1 - p) = 0$ whenever $i \neq j$. Then every contraction $b \in A$ satisfies the following.

1. If $pb = 0$ then $\left\| \frac{1}{n} \sum_{j=0}^{n-1} u_j b u_j^* \right\| \leq \frac{1}{\sqrt{n}}$.

---

1 The projection $p$ is not required to belong to $A$. 

2. If \( p b p = 0 \) then \( \| \frac{1}{n} \sum_{j=0}^{n-1} u_j b u_j^* \| \leq \frac{2}{\sqrt{n}} \).

**Proof.** Revealing the punchline of this proof at the beginning is a justified spoiler. If \((1 - p)a = 0\) and \(p b = 0\) then \(a \xi \) is orthogonal to \(b \bar{\xi}\) for every \(\xi \in H\) and therefore \(\|a + b\|^2 \leq \|a\|^2 + \|b\|^2\).

(1) The proof proceeds by induction on \(n\). For \(n = 1\) the assertion is trivially true. Suppose it is true for \(n\) and fix \(b\) and unitaries \(u_j\), for \(j < n + 1\) satisfying the assumptions. We now have

\[
\| \sum_{j=0}^{n} u_j b u_j^* \|^2 = \|u_0(b + \sum_{j=1}^{n} u_j^* b u_j^* u_0)\|^2 = \|b + \sum_{j=1}^{n} u_j^* b u_j^* u_0\|^2.
\]

Let \(a := \sum_{j=1}^{n} u_j^* b u_j^* u_0\). Since \((1 - p)b = b\), we have

\[
(1 - p)a = \sum_{j=1}^{n} (1 - p)u_j^* b u_j^* u_0 = 0.
\]

Therefore \(\| \sum_{j=0}^{n} u_j b u_j^* \|^2 \leq \|b\|^2 + \|a\|^2 \leq n + 1\), and the estimate follows.

(2) We work in \(C^*(A, p)\) and write

\[
F_n(b) := \frac{1}{n} \sum_{j=0}^{n-1} u_j b u_j^*.
\]

Let \(b_0 := b p\) and \(b_1 := (b - b_0)^* = (1 - p)b\). Then \(p b_0 = p b_1 = 0\), and therefore by the first part of this lemma \(\|F_n(b)\| \leq \sqrt{n}\) for \(i < 2\). Since \(b = b_0 + b_1\), by the linearity we have \(F_n(b) = F_n(b_0) + F_n(b_1)^*\) and \(\|F_n(b)\| \leq 2/\sqrt{n}\). \(\square\)

Every isomorphism between \(C^*\)-algebras is continuous (Corollary 1.2.11, Corollary 1.3.3), but this is in general not true for isomorphisms between dense *-subalgebras of \(C^*\)-algebras (e.g., by Example 1.2.12). In particular, an (algebraic) automorphism of a dense subalgebra of a \(C^*\)-algebra \(A\) need not have an extension to an automorphism of \(A\). The following lemma gives a sufficient condition for an automorphism of a dense *-subalgebra of a \(C^*\)-algebra \(A\) to extend to an automorphism of \(A\).

**Lemma 4.3.2.** Suppose \(B\) is a norm-dense (and not necessarily norm-closed) *-subalgebra of \(A\) and \(\Phi \in \text{Aut}(B)\). If there exists a GNS-faithful state \(\psi\) such that \(\psi = \psi \circ \Phi\), then \(\Phi\) has a unique extension to an automorphism of \(A\).

**Proof.** The formula \(\|c\|_2 := \psi(c^* c)^{1/2}\) defines an \(\ell_2\)-norm on the quotient \(A/L_\psi\) (see the GNS construction, Proposition 1.10.3) and \(H_\psi\) is the completion of \(A/L_\psi\) in this norm. Since \(\|c\|_2 \leq \|c\|\), \(B/(L_\psi \cap B)\) is a dense subspace of \(H_\psi\). Because of this and because the GNS representation \(\pi_\psi\) is faithful, every \(a \in A\) satisfies (see Exercise 1.11.53)

\[
\|a\| = \|\pi_\psi(a)\| = \sup_{\xi \in H_\psi, \|\xi\|_2 = 1} \|a \xi\|_2 = \sup_{b \in B, \|b\|_2 = 1} \|a b\|_2.
\]

In other words, \(\|\cdot\|\) is definable from \(\|\cdot\|_2\). Since \(\Phi\) is \(\psi\)-preserving, it is \(\|\cdot\|_2\)-preserving and therefore \(\|\cdot\|\)-preserving. By the continuity of algebraic operations, the unique continuous extension of \(\Phi\) to \(A\) is an automorphism of \(A\). \(\square\)
Lemma 4.3.2 is particularly applicable in a situation in which $A$ has a unique tracial state $\psi$; with this assumption the required $\Phi$-invariance of $\psi$ is almost, but not quite, automatic.

From this point on, our discussion is specialized to reduced group $\mathbb{C}^*$-algebras. Recall (§2.4.1) that if $\Gamma$ is a discrete group then $\mathbb{C}[\Gamma]$ denotes its group algebra, and $\mathbb{C}_r^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ in the norm obtained from the left regular representation of $\mathbb{C}[\Gamma]$ on $\ell_2(\Gamma)$. Lemma 2.4.9 (5) implies that the standard tracial state on $\mathbb{C}[\Gamma]$ and $\mathbb{C}_r^*(\Gamma)$ defined by $\tau(\sum_{g \in F} \alpha_g \lambda(g)) = \alpha_e$ is faithful for every $\Gamma$.

**Definition 4.3.3.** A discrete group $\Gamma$ has the Powers approximation property (or that it is a Powers group) if for every $a \in \mathbb{C}[\Gamma]$ and every $\varepsilon > 0$ there exists $F \in \mathcal{F}$ such that $\left\| \sum_{h \in F} \lambda(h)a\lambda(h^{-1}) - \tau(a) \right\| < \varepsilon$.

**Lemma 4.3.4.** If $\Gamma$ is a Powers group then $\mathbb{C}_r^*(\Gamma)$ has a unique tracial state and it is simple.

**Proof.** Every $a \in \mathbb{C}_r^*(\Gamma)$ can be approximated by elements of $\mathbb{C}[\Gamma]$, and therefore $0 \in \text{conv}\{u(a - \tau(a))u^* : u \in U(A)\}$ and $\tau(a) \in \text{conv}\{uau^* : u \in U(A)\}$. Since $\tau$ is a faithful tracial state on $\mathbb{C}_r^*(\Gamma)$ (Lemma 2.4.9), Proposition 4.1.10 implies that $\mathbb{C}_r^*(\Gamma)$ is simple and has a unique tracial state.

A few easy observations on free products of groups will be used to fix the notation and terminology. Suppose $\Gamma = *_{j \in J} \Gamma_j$ and each $\Gamma_j$ has more than one element. Then for every $g \in \Gamma \setminus \{e\}$ there are $n \in \mathbb{N}$ and a sequence $\bar{k}_g = \langle j : \bar{k}(j) : j < n \rangle$, such that $g = g_0g_1\cdots g_{n-1}$ for some $g_j \in \Gamma_{\bar{k}(j)} \setminus \{e\}$, for $j < n$. Since $\Gamma$ is the free product of $\Gamma_j$ for $j \in J$, the tuple

$$\langle n, \bar{k}, g_0, \ldots, g_{n-1} \rangle$$

is uniquely determined by $g$. We say that $g$ begins with an element $g_0$ of $\Gamma_{\bar{k}(0)}$ and ends with an element $g_{n-1}$ of $\Gamma_{\bar{k}(n-1)}$. If $h \in \Gamma_j$ is of infinite order, then $h^mgh^{-m}$ both begins and ends with an element of $\Gamma_j$ for all large enough $m$ (but these elements of $\Gamma_j$ are not necessarily powers of $g$). This observation will be used in the proof of Lemma 4.3.5, the assumption of which is far from optimal.

The presence of free nonabelian subgroups will play a role in the ongoing discussion. It may therefore be worth noting that there is only one nontrivial free product of groups with no free subgroup, although this fact will not be needed explicitly. In $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ the sets $F_n := \{g : |k_g| \leq n\}$ form a Følner sequence, and therefore this group is amenable. Every other nontrivial free product has a subgroup isomorphic to the free group $F_2$.

**Lemma 4.3.5.**

1. Suppose $\Gamma = G * H$ is a discrete group and elements $g \in G$ and $h \in H$ have infinite order. Then for every $a \in \mathbb{C}[\Gamma]$ and every $\varepsilon > 0$, for all large enough $m$ and $n$ we have $\left\| \frac{1}{m} \sum_{j=0}^{m-1} \lambda(g^jh^m)a\lambda(h^{-m}g^{-j}) - \tau(a) \right\| < \varepsilon$, hence $\Gamma$ satisfies the Powers approximation property.

2. If $\Gamma$ is a free nonabelian group or a free product of at least four nontrivial groups, then it satisfies the Powers approximation property.
3. In both (1) and (2), $C^*_r(\Gamma)$ has a unique trace and it is simple.

**Proof.** (1) Suppose $\Gamma = G \ast H$ and fix $g \in G$ and $h \in H$, each of an infinite order. Fix $a = \sum_{f \in F} \alpha_f \lambda(f)$ in $\mathbb{C}[\Gamma]$. Since $\tau(a) = \alpha_e$, by replacing $a$ with $a - \tau(a)\lambda(e)$ we may assume $\tau(a) = 0$. Since $F$ is finite, $e \notin F$, and $h$ has infinite order, using the discussion and terminology from the paragraph preceding this lemma, for every large enough $m$ we have that $\lambda(h^m)a\lambda(h^{-m})$ is a linear combination of unitaries of the form $\lambda(f)$ where $f$ both begins and ends with an element of $H$.

Let $p$ be the projection of $\ell_2(\Gamma)$ onto the closed linear span of the set 

$$\{ \delta_g : g \text{ is a reduced word that does not begin with an element of } H \}.$$ 

Then $p\lambda(h^m)a\lambda(h^{-m})p = 0$ for a large enough $m$. Since the order of $g$ is infinite, the unitaries $u_j := \lambda(g^j)$, for $j < n$, are distinct and $(1 - p)u_i^*u_j(1 - p) = 0$ if $i \neq j$. Lemma 4.3.1 now implies $$\left\| \frac{1}{n} \sum_{j=0}^{n-1} u_j \lambda(h^m)a\lambda(h^{-m})u_j^* \right\| \leq \frac{2}{\sqrt{n}}.$$ The proof is completed by choosing a large enough $m$ and $n > 4/e^2$.

(2) By the associativity of the free product, we may assume that $\Gamma$ is a free product of exactly four nontrivial groups. Suppose $G_0$ and $G_1$ are nontrivial groups and fix $g_0 \in G_0 \setminus \{e\}$ and $g_1 \in G_1 \setminus \{e\}$. Then $g_0g_1 \in G_0 \ast G_1$ is an element of infinite order. Therefore $\Gamma$ is equal to the free product of two groups each of which has an element of infinite order, and (1) implies that it has the Powers approximation property.

(3) By Lemma 4.3.4 the Powers approximation property implies that $C^*_r(\Gamma)$ has a unique tracial state and that it is simple. \qed

Recall that if $G$ is a subgroup of $\Gamma$ then $C^*_r(G)$ is a subalgebra of $C^*_r(\Gamma)$ (Exercise 3.10.15).

**Lemma 4.3.6.** Suppose that $G$ and $H$ are discrete groups and $C^*_r(G \ast H)$ has a unique trace. Then every automorphism $\Phi$ of $C^*_r(G)$ has an extension to an automorphism of $C^*_r(G \ast H)$, and there is a unique such extension $\tilde{\Phi}$ with the property that $\tilde{\Phi}(\lambda(h)) = \lambda(h)$ for all $h \in H$.

**Proof.** Let $B$ be the $^*$-subalgebra of $C^*_r(G \ast H)$ algebraically generated by $C^*_r(G)$ and $\{ \lambda(h) : h \in H \}$. By the freeness we can extend $\Phi$ to an automorphism $\Phi_B$ of $B$ that sends $\lambda(h)$ to itself for all $h \in H$. Since the unique tracial state vanishes on $\lambda(g)$ for all $g \neq e$, it is clearly preserved by $\Phi_B$. By Lemma 4.3.2 $\Phi_B$ has a unique extension to an automorphism of $C^*_r(G \ast H)$; this $\Phi$ is as required. \qed

**Corollary 4.3.7.** If $X \subseteq Y$ then every automorphism $\Phi$ of $C^*_r(F_X)$ has an extension to an automorphism of $C^*_r(F_Y)$, and there is a unique such extension $\tilde{\Phi}$ with the property that $\tilde{\Phi}(\lambda(h)) = \lambda(h)$ for all $h \in F_Y \setminus X$. \qed

**Lemma 4.3.8.** Suppose that groups $G$ and $H$ are such that $H$ has an element of infinite order and $G$ is countable. Then every pure state on $C^*_r(\mathbb{Z} \ast G)$ has a unique extension to a pure state on $C^*_r(\mathbb{Z} \ast G \ast H)$. 

- **Lemma 4.3.1**
- **Corollary 4.3.4**
- **Exercise 3.10.15**
- **Exercise 4.3.2**
- **Lemma 4.3.3**
- **Lemma 4.3.4**
- **Lemma 4.3.5**
- **Lemma 4.3.6**
- **Lemma 4.3.7**
- **Lemma 4.3.8**
Proof. A very special case is proved first: We assume that \( G \) is trivial and that \( \varphi \) is a specific pure state. Throughout this proof \( g \) denotes the generator of \( Z \) and \( u := \hat{\lambda}(g) \).

Claim. Suppose \( \varphi \) is a state on \( C^*_r(\mathbb{Z}) \) such that \( \varphi(u) = 1 \). Then every state extension \( \psi \) of \( \varphi \) to \( C^*_r(\mathbb{Z} \ast H) \) vanishes on \( Z := \{ \hat{\lambda}(h) : h \in \mathbb{Z} \ast H \setminus \mathbb{Z} \} \).

Proof. Since 1 is an extreme point of \( T = sp(\varphi) \), Proposition 1.7.8 and induction on \( n \) (and on \( -n \)) imply that \( \varphi(u^n) = 1 \) for all \( n \). Therefore \( \varphi \) is uniquely determined by the condition \( \varphi(u) = 1 \) and pure.

For every \( h \in \mathbb{Z} \ast H \) there exists a unique \( m \in \mathbb{Z} \) such that \( g^m h \) starts with an element of \( H \). Since \( \varphi(g^m) = 1 \) is an extreme point of \( sp(g^m) \), Proposition 1.7.8 implies that \( \psi(h) = \psi(g^m h) \). It therefore suffices to prove \( \psi(\hat{\lambda}(h)) = 0 \) for every \( h \in \mathbb{Z} \ast H \) that starts with an element of \( H \). Let

\[
K := \text{sp}\{ \delta_f : f \in \mathbb{Z} \ast H \text{ and } f \text{ does not start with an element of } H \}.
\]

Then \( \hat{\lambda}(g^n)[K] \perp K \) for all \( n \neq 0 \), and for all \( j \neq k \) the unitaries \( u_j := \hat{\lambda}(g^j) \) satisfy

\[
(1 - p)u^j u^{-k}(1 - p) = 0.
\]

Let \( p \in \mathcal{B}(\ell_2(\mathbb{Z} \ast H)) \) be the projection to the subspace \( K \). If \( h \in \mathbb{Z} \ast H \) starts with an element of \( H \), and \( b := \hat{\lambda}(h) \), then \( pb = 0 \) and by Lemma 4.3.1 we have \( \| \frac{1}{2} \sum_{j < n} u_j b u_j^{-1} \| \leq 2/\sqrt{n} \) for all \( n \). Since \( \varphi(U^j) = 1 \), Proposition 1.7.8 implies \( \varphi(u^j b u^{-j}) = \varphi(b) \) for all \( j \), and therefore \( \varphi(b) \leq 2/\sqrt{n} \). Since \( n \) was arbitrary, we have \( \varphi(b) = 0 \). Since \( h \) was arbitrary, any state extension of \( \varphi \) vanishes on \( Z \), as required.

Suppose \( \theta \) is a pure state on \( B := C^*_r(\mathbb{Z} \ast G) \). Since \( B \) is simple and separable, by Theorem 5.6.1 there exists an automorphism \( \Phi \) of \( B \) such that \( \theta \circ \Phi \) is a pure state extension of \( \varphi \) to \( C^*_r(\mathbb{Z} \ast G) \). By Lemma 4.3.6, \( \Phi \) can be extended to an automorphism of \( C^*_r(\mathbb{Z} \ast G \ast H) \). Therefore \( \theta \) has a unique extension to \( C^*_r(\mathbb{Z} \ast G \ast H) \) if and only if \( \theta \circ \Phi \) does. The claim implies that the restriction of the latter state to \( C^*_r(\mathbb{Z}) \) has a unique extension to \( C^*_r(\mathbb{Z} \ast G \ast H) \), and this concludes the proof.

\[\square\]

4.4 Diffuse Masas and Normalizers

In this section we use diffuse masas to compute normalizers of \( C^* \)-subalgebras of \( C^* \)-algebras equipped with a tracial state.

If \( \tau \) is a state on a unital abelian \( C^* \)-algebra \( D \), by the Riesz representation theorem (Example 1.7.2) there exists a unique Borel probability measure \( \mu = \mu_{\tau,D} \) on \( X = D \) such that (identifying \( a \in D \) with \( f_a \in C(X) \))

\[
\tau(a) = \int f_a \, d\mu
\]
for all $a \in D$.

A measure $\mu$ is said to be \textit{diffuse} if every $\mu$-measurable, $\mu$-positive, set can be partitioned into two $\mu$-measurable, $\mu$-positive, sets. A $\mu$-measurable, $\mu$-positive, set that cannot be partitioned into two $\mu$-measurable, $\mu$-positive, sets is an \textit{atom}.

\textbf{Definition 4.4.1.} Suppose $\tau$ is a state on a unital C*-algebra $A$ and $D$ is a unital abelian subalgebra of $A$. If $\mu_{\tau,D}$ is a diffuse measure it is then said that $\tau$ is \textit{diffuse on} $D$. If $a \in A$ is normal then $\tau$ is \textit{diffuse on} $a$ if it is diffuse on $C^*(a, 1)$. If $B$ is a unital C*-subalgebra of $A$, $\tau$ is \textit{diffuse on} $B$ if it is diffuse on some masa $D$ of $B$.

\textbf{Proposition 4.4.2.} Suppose that $\tau$ is a faithful tracial state on a von Neumann algebra $M$. The following are equivalent.

1. The restriction of $\tau$ to some masa in $M$ is not diffuse.
2. The restriction of $\tau$ to every masa in $M$ is not diffuse.
3. There exists a nonzero central projection $p$ in $M$ such that $pMp$ is finite-dimensional.

\textbf{Proof.} Before we start, note that a masa $D$ of $M$ is isomorphic to some $L_\infty$ space and that this space has an atom if and only if it contains a nonzero projection $q$ such that $qDq \cong \mathbb{C}$. The remaining part of the proof goes through with the assumption that $M$ is a C*-algebra.

(2) $\Rightarrow$ (1): The Axiom of Choice implies that masas exist.

(3) $\Rightarrow$ (2): If $D$ is a masa in $M$, then a central projection $p$ as in (3) belongs to $D$. Since $\tau$ is faithful, $\tau(p) > 0$ and since $pMp$ is finite-dimensional, any minimal nonzero projection $q$ in $pDp$ satisfies $\tau(q) > 0$ and $qDq \cong \mathbb{C}$.

(1) $\Rightarrow$ (3): Suppose that the restriction of $\tau$ to a masa $D$ of $M$ is not diffuse. Fix a projection $q$ in $D$ such that $qDq \cong \mathbb{C}$. Since every projection $r$ Murray–von Neumann equivalent to $q$ satisfies $\tau(r) = \tau(q) > 0$, there can be only finitely many orthogonal projections Murray-von Neumann equivalent to $q$ in $M$. Let $q_j$, for $j < m$, be a maximal family of orthogonal projections Murray–von Neumann equivalent to $q$.

\textbf{Claim.} The projection $p := \sum_{j < m} q_j$ is central in $M$

\textbf{Proof.} Assume not and fix $a \in M$ such that $pa \neq ap$. By subtracting $pap$ from both sides, we conclude that $pa(1 - p) \neq (1 - p)ap$, and therefore at least one of these products is nonzero. By replacing $a$ with $a^*$ if necessary, we may assume $pa(1 - p) \neq 0$. Fix $j < m$ such that $x := q_j a (1 - q)$ is nonzero. Then $xx^*$ is a non-zero element of $q_j M q_j$. This algebra is isomorphic to $pMp \cong \mathbb{C}$, hence $xx^*$ is a (necessarily positive) scalar multiple of $q_j$. Thus some scalar multiple $y$ of $x$ satisfies $yy^* = q_j$. Then $y^* y$ is a projection Murray–von Neumann equivalent to $q_j$ and orthogonal to $p$; contradiction. \qed

By the Claim, $p$ is a central projection and $q_j$, for $j < m$, generate a finite-dimensional masa in $pMp$. This implies that $pMp$ is finite-dimensional (Exercise 2.8.8, to which the proof of the Claim is a hint) and concludes the proof. \qed

The notion of $\tau$-orthogonality and the symbol $\perp_\tau$ associated with the GNS Hilbert space $H_\tau$, were introduced in Definition 4.2.2.
Lemma 4.4.3. Suppose $\tau$ is a faithful tracial state on a unital $C^*$-algebra $A$. If $B$ is a unital $C^*$-subalgebra of $A$ such that $\tau$ is diffuse on $B$ and $w$ is a unitary such that $wBw^*$ is orthogonal to $B$, then $\tau(w) = 0$.

Proof. Extend $\tau$ to a normal tracial state (denoted $\tau$) on $M := \overline{\pi[A]}^{\text{WOT}}$ using Lemma 4.2.5. Let $D$ be a masa in $B$ such that $\tau$ is diffuse on $D$. Then $D_0 := \overline{\pi[D]}^{\text{WOT}}$ is an abelian von Neumann subalgebra of $M$; extend it to a masa $D_1$ of $M$. Since the pullback of a diffuse measure is diffuse, $\tau$ is diffuse on $D_1$. By the continuity of $\tau$, $wD_1w^*$ is orthogonal to $D_1$. Since $D_1$ is a von Neumann algebra, for any $n \geq 1$ we can find orthogonal projections $e_j$, for $j < n$, in $D_1$ such that $\tau(e_j) = 1/n$ for all $j$ and $\sum_j e_j = 1$. We now have $\tau(w) = \sum_j \tau(w e_j) = \sum_j \tau(\sum_j w e_j) = \tau(\sum_j w e_j)$. The Cauchy–Schwarz inequality implies $|\tau(a)|^2 \leq \tau(a^*a)$ for all $a$, and

$$|\tau(\sum_j e_j w e_j)|^2 \leq \tau((\sum_i e_i w^* e_i)(\sum_j w e_j)) = \sum_j \tau(w e_j w e_j)$$

(using $e_i e_j = 0$ if $i \neq j$). Since $w^* D_1w$ is $\tau$-orthogonal to $D_1$, we have

$$\tau(w^* b w) = \tau(w^* b w) = \tau(b)$$

for all $b$ and $c$ in $D_1$. This implies $\sum_j \tau(w^* e_j w e_j) = \sum_j \tau(e_j)^2 = 1/n$. Therefore $|\tau(w)|^2 \leq 1/n$. Since $n \geq 1$ was arbitrary, we conclude that $\tau(w) = 0$. \qed

The normalizer of a subalgebra $B$ of a unital $C^*$-algebra $A$ is

$$\mathcal{N}(B) := \{ u \in U(A) : uB u^* = B \}.$$

Proposition 4.4.4. Suppose $\tau$ is a faithful tracial state on a unital $C^*$-algebra $A$, $B$ is a $C^*$-subalgebra of $A$, and $u$ is a unitary in $A$. If there is a normal $b \in B$ such that $\tau$ is diffuse on $\text{sp}(b)$ and $ubu^* \perp B$, then $u \perp \mathcal{N}(B)$.

Proof. Fix $v \in \mathcal{N}(B)$. Then $v^* ub u^* v \perp v^* B v = B$, hence $w := v^* u$ and $b$ satisfy the assumptions of Lemma 4.4.3. Therefore $\tau(v^* u) = 0$ and $v \perp_u u$. Since $v \in \mathcal{N}(B)$ was arbitrary, $u \perp \mathcal{N}(B)$. \qed

If $\Gamma$ is a discrete group and $G$ is a subgroup of $\Gamma$ then the normalizer of $G$ is

$$\mathcal{N}(G) := \{ h \in H : h G h^{-1} = G \}.$$

If $\mathcal{N}(G) = \Gamma$ then $G$ is a normal subgroup of $\Gamma$. By Exercise 3.10.15, if $G$ is a subgroup of $\Gamma$ we can identify $C_r^*(G)$ with a $C^*$-subalgebra of $C_r^*(\Gamma)$. In this case the standard trace of the larger algebra agrees with that of the smaller algebra; we slightly abuse the notation and use $\tau$ to denote both traces.

Definition 4.4.5. If $\Gamma$ is a discrete group, the group von Neumann algebra $L(\Gamma)$ is the von Neumann algebra in $\mathcal{B}(l_2(\Gamma))$ generated by the image of $\Gamma$ under its left regular representation $\lambda : \Gamma \to l_2(\Gamma)$.

By definition, $L(\Gamma)$ is the closure of the reduced group algebra $C_r^*(\Gamma)$ in the weak (or equivalently, strong) operator topology (see §2.4.1).
Lemma 4.4.6. Suppose $G < H < \Gamma$ are discrete groups and $G$ is an infinite Powers group.

1. If $fgf^{-1} \cap G = \{e\}$ for all $f \in \Gamma \setminus H$ and $\tau$ is diffuse on a von Neumann subalgebra $B$ of $L(G)$ then $\mathcal{N}(B) \subseteq L(H)$.

2. If $fgf^{-1} \cap G = \{e\}$ for all $f \in \Gamma \setminus H$ and $\tau$ is diffuse on a $C^*$-subalgebra $B$ of $C^*_r(G)$ then $\mathcal{N}(B) \subseteq C^*_r(H)$.

3. If for every $f \in \Gamma \setminus H$ there exists $g \in G$ of infinite order such that $fgf^{-1} \not\in G$ then $\mathcal{N}(C^*_r(G)) \subseteq C^*_r(H)$.

4. If, in addition to the assumption of (3), $G$ is a normal subgroup of $H$ then $\mathcal{N}(C^*_r(G)) = C^*_r(H)$.

Proof. (1) It is sufficient to prove $L(H) \subseteq \mathcal{N}(L(G))$ since $\tau$ is faithful. As $\{\lambda(f) : f \in \Gamma\}$ is an orthonormal basis for $L(\Gamma)$ with respect to $\|\cdot\|_2$ (Proposition 4.2.6), it suffices to prove that $\lambda(f) \in \mathcal{N}(L(G))$ for $f \in \Gamma \setminus H$. If $f \in \Gamma \setminus H$ then $fgf^{-1} \cap G = \{e\}$, hence $\lambda(f)B\lambda(f)^* \perp L(G)$. Since $\tau$ is diffuse, Proposition 4.4.4 implies that $\lambda(f)$ is orthogonal to $\mathcal{N}(L(G))$.

(2) Take the WOT-closures and apply (1). The normalizer of $B$ is included in $L(H)$, and it only remains to observe that $L(H) \cap C^*_r(\Gamma) = C^*_r(H)$. This is because $\lambda(g)$, for $g \in \Gamma$, is an orthonormal basis in both $C^*_r(\Gamma)$ and its completion $L(\Gamma)$.

(3) Fix $f \in \Gamma \setminus H$ and $g \in G$ of infinite order such that $fgf^{-1} \not\in G$. The group $G_0$ generated by $g$ is isomorphic to $\mathbb{Z}$, therefore $C^*_r(G_0) \cong C(\mathbb{T})$ (Example 2.4.8 (3)) and $\mu_\pi$ is the Lebesgue measure on the circle, hence diffuse on $sp(a)$. Therefore Proposition 4.4.4 applied to $w := \lambda(f)$ and $B := C^*(\lambda(g))$ implies $\lambda(f) \in \mathcal{N}(C^*_r(G))^\perp$. We have proved that $\mathcal{N}(C^*_r(G))^\perp \supseteq \bigcap \{\lambda(f) : f \in \Gamma \setminus H\}$ and (3) follows.

(4) The assumptions imply $H = \mathcal{N}(G)$. It therefore suffices to prove that $C^*_r(\mathcal{N}(G)) \subseteq \mathcal{N}(C^*_r(G))$ (with the normalizers computed in $\Gamma$ and $C^*_r(\Gamma)$, respectively). If $h \in \mathcal{N}(G)$ then $\lambda(h) \in \mathcal{N}(C^*_r(G))$. Since these elements form a basis for $C^*_r(\mathcal{N}(G))$, we have $C^*_r(\mathcal{N}(G)) \subseteq \mathcal{N}(C^*_r(G))$. □

4.5 Exercises

Exercise 4.5.1. Suppose $\tau$ is a tracial state on a $C^*$-algebra $A$. Prove that $\pi_\tau[A]^\prime\prime$ is a factor if and only if $\tau$ is an extreme point of the space $\mathcal{T}(A)$ of tracial states of $A$.

Exercise 4.5.2. Suppose $\tau$ is a faithful tracial state on a unital $C^*$-algebra $A$. Prove that if $a$ is a unitary such that $\tau(a) = 1$, then $a = 1$.

Exercise 4.5.3. 1. Prove that every separable $C^*$-algebra has a faithful state.

2. Find a $C^*$-algebra without a GNS-faithful state.

3. Find a separable, stably finite, $C^*$-algebra without a faithful tracial state.

Exercise 4.5.4. Find a separable $C^*$-algebra $A$ with no faithful tracial state. In addition, make sure that $A$ is unital and each one of its nontrivial quotients (including $A$ itself) has a tracial state.
Definition 4.5.5. The opposite algebra of a C*-algebra $A$ is the C*-algebra whose underlying Banach space structure and involution are the same as that of $A$, but the product of $x$ and $y$ is defined as $yx$ rather than $xy$. It is denoted by $A^{\text{op}}$.

Exercise 4.5.6. Suppose $\tau$ is a tracial state on a C*-algebra $A$ and $\pi_{\tau}: A \rightarrow \mathcal{B}(H_{\tau})$ is the associated GNS representation. Verify that the right multiplication by elements of $A$ on the GNS pre-Hilbert space $A/L_{\tau}$ defines a representation of $A^{\text{op}}$, whose range is included in the commutant $\pi_{\tau}[A]'$.

We will return to opposite algebras in Exercise 7.5.7 and Exercise 7.5.8.

Exercise 4.5.7. Let $X$ be a compact metric space and let $A$ be a C*-algebra with a unique tracial state. Prove that $T(C(X, A))$ is affinely homeomorphic to $P(X)$, the space of Borel probability measures on $X$.

Exercise 4.5.8. For every $n \geq 1$ construct a simple AF algebra $A$ such that $T(A)$ is affinely homeomorphic to the $n$-dimensional simplex. 

Hint: For $n = 2$, try $\bullet \quad \bullet \quad \bullet \quad \bullet \quad \ldots \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \ldots$

Exercise 4.5.9. Prove that the Breuer ideal of a II$_\infty$ factor (Example 4.1.5 (2)) is not $\sigma$-unital.

Exercise 4.5.10. Prove that Lemma 4.3.8 can be strengthened by dropping the assumption that the group $G$ be countable.

Exercise 4.5.11. Suppose that in a C*-algebra $A$ we have $aa^* \leq bb^* + cc^*$. Use Corollary 1.6.13 to prove that there are $x$ and $y$ in $A$ such that $x^*x \leq b^*b$, $y^*y \leq c^*c$, and $aa^* = xx^* + yy^*$.

Exercise 4.5.12. A discrete group $\Gamma$ is an ICC group (ICC stands for infinite conjugacy classes) if the conjugacy class of every non-identity element is infinite. Prove that $L(\Gamma)$ is a factor if and only if $\Gamma$ is an ICC group.

Exercise 4.5.13. Use Exercise 1.11.58 and the proof of Proposition 4.2.6 to prove the following. If $\varphi$ is a faithful state on a C*-algebra $A$, then the strong operator topology on the unit ball of $\overline{\pi_\varphi[A]}^{\text{WOT}}$ is induced by the norm $\|a\|_2 := \varphi(a^*a)^{1/2}$.

Exercise 4.5.14. Suppose that the equality $\sum_{i<m}a_i^*a_i = \sum_{j<n}b_j^*b_j$ holds in a C*-algebra $A$. Use Corollary 1.6.13 to prove that in $A$ there are $e_{ij}$, for $i < m$ and $j < n$, such that $\sum_{j<n}c_{ij}e_{ij} = a_i^*a_i$ and $\sum_{i<m}c_{ij}e_{ij} = b_j^*b_j$ for all $i$ and $j$.

Exercise 4.5.15. Suppose that $\tau$ is a tracial state on a unital C*-algebra $A$ and $u$ is a unitary in $A$ such that $\tau(u^n) = 0$ for all $n \geq 1$. Prove that $\tau$ is diffuse on $C^*(u)$. 


Notes for Chapter 4

§4.1 It is not known whether every stably finite C*-algebra has a tracial state (see Example 4.1.6). This is true for exact C*-algebras ([123]). Not every finite C*-algebra has a tracial state. An example of a finite and simple C*-algebra $A$ such that $M_2(A)$ is infinite was constructed in [209].

The space $T(A)$ is not just any old compact convex set. For every C*-algebra $A$, every trace $\tau \in T(A)$ is the barycenter of a unique measure that concentrates on the extreme boundary of $T(A)$. When $A$ is finite-dimensional, then $T(A)$ is a simplex and this is evident. A proof of the general statement can be found e.g., in [215]. A compact and convex set with this property is called a Choquet simplex.

§4.2, §4.4 These sections, and Proposition 4.4.4 in particular, are an adaptation of [204] where the free group factors were considered. The notion of orthogonality between subalgebras of a tracial C*-algebra prominent in this section is transferred from its natural habitat, the theory of II$_1$ factors. We have only scratched the surface, and the readers who enjoyed this section would love the theory of II$_1$ factors.

§4.3 Lemma 4.3.1 has been taken from [41]; see also [9]. Powers introduced his approximation property in order to prove that $C^*_r(F_2)$ is simple and has a unique tracial state. This property is equivalent to the simplicity of $C^*_r(\Gamma)$ and strictly weaker than the uniqueness of a trace on this algebra (see [154], [124], [31], Exercise 11.4.2).

Notably, a converse to Exercise 4.5.15 holds: If $\tau$ is a diffuse tracial state on an abelian unital C*-algebra $A$, then there exists a unitary $u \in A$ such that $\tau(u^n) = 0$ for all $n \geq 1$ ([67]).

A prominent open problem in operator algebras is whether the free group von Neumann algebras (Definition 4.4.5) $L(F_2)$ and $L(F_3)$ are isomorphic. The positive answer is equivalent to the simplicity of $C^*_r(\Gamma)$ and strictly weaker than the uniqueness of a trace on this algebra (see [154], [124], [31], Exercise 11.4.2).

Notably, a converse to Exercise 4.5.15 holds: If $\tau$ is a diffuse tracial state on an abelian unital C*-algebra $A$, then there exists a unitary $u \in A$ such that $\tau(u^n) = 0$ for all $n \geq 1$ ([67]).

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It is not too difficult to see that $L(F_2) \cong L(F_3)$ is a $\Sigma^1_2$ statement, and therefore absolute between transitive models of a large enough fragment of ZFC that contain all countable ordinals (Theorem B.2.12). It is therefore very unlikely (albeit not completely impossible) that the answer to this problem is independent from ZFC.
Chapter 5
Irreducible Representations of C*-algebras

This chapter is a continuation of Chapter 3. It starts with Kishimoto’s rather elementary construction of an approximate diagonal for an irreducible representation of a C*-algebra. Subsequently we prove the excision theorem for pure states. The first glimpse of the set-theoretic vantage point is provided in the section on quantum filters. These noncommutative analogs of ultrafilters are used to study pure states on C*-algebras. The chapter culminates with the proof of the Kishimoto–Ozawa–Sakai theorem on the homogeneity of the pure state space of a simple, separable, C*-algebra. This theorem is one of the components of the construction of a counterexample to Naimark’s problem.

5.1 Approximate Diagonals à la Kishimoto

This section contains Kishimoto’s elegant and elementary proof of a deep fact that every irreducible representation of a C*-algebra has an approximate diagonal. Along the way, we prove a stability result for unitaries and a combinatorial result about optimal transport of ε-nets of minimal cardinality in a compact metric space.

The proof uses the analysis of direct sums of inequivalent irreducible representations from §3.4. The use of approximate diagonals is crucial in the proof of Theorem 5.6.1. Without further ado, we define the approximate diagonals and state the theorem whose proof the present section is devoted to.

Definition 5.1.1. An approximate diagonal of a representation \( \pi : A \to \mathcal{B}(H) \) of a C*-algebra \( A \) is a net \( b_\lambda \), for \( \lambda \in \Lambda \), which satisfies the following.

1. \( b_\lambda := (b_{\lambda,0}, \ldots, b_{\lambda,n(\lambda)-1}) \) belongs to \( M_{n(\lambda)}(A) \) for some \( n(\lambda) \geq 1 \),
2. \( b_\lambda^* b_\lambda \leq 1 \),
3. For all \( a \in A \) the following holds

\[
\lim_{\lambda} \sup_{\|c\| \leq 1} \|a \sum_{i<n(\lambda)} (b_{\lambda,i} c b_{\lambda,i}^*) - \sum_{i<n(\lambda)} (b_{\lambda,i} c b_{\lambda,i}^*) a \| = 0,
\]
4. For every finite-dimensional projection $E$ in $\mathcal{B}(H)$ there exists $\lambda_0 \in \Lambda$ such that

$$E(1 - \pi(b_\lambda b_\lambda^*)) = 0 \text{ for all } \lambda \geq \lambda_0.$$ 

**Theorem 5.1.2.** Suppose $A$ is a $C^*$-algebra and $\pi : A \to \mathcal{B}(H)$ is a direct sum of inequivalent irreducible representations. Then $\pi$ has an approximate diagonal.

The first step towards our proof of Theorem 5.1.2 is a proof of a form of weak stability (Definition 2.3.12) for unitaries.

**Lemma 5.1.3.** For every $\varepsilon > 0$ there exists $\delta > 0$ such that the following holds. Suppose $A \subseteq \mathcal{B}(H)$ is a unital $C^*$-algebra, $E \in \widetilde{\mathcal{A}}^{\text{WOT}}$ is a finite-rank projection, and $u$ is a unitary in $A$ satisfying $||u, E|| < \delta$. Then there exists a unitary $v \in A$ such that $|v, E| = 0$, $||u - v|| < \varepsilon$, and $u$ and $v$ are homotopic.

**Proof.** Fix $A$, $u$, and $E$ as in the statement of the lemma. The smallness of $\delta$ will be discussed later. The projection $P := uEu^*$ satisfies $||P - E|| = ||u, E|| < \delta$. With $Q := \text{proj}_{\mathcal{B}(E[H], uE[H])}$ we have $E \leq Q$, $P \leq Q$, and

$$T := EP + (Q - E)(Q - P)$$

satisfies $T \approx 2\delta P^2 + (Q - P)^2 = Q$, that is $||T - Q|| < 2\delta$. Consider the polar decomposition $T = W|T|$. As $P, Q, \text{ and } T$ belong to $W^*(E, u) \subseteq \widetilde{\mathcal{A}}^{\text{WOT}}$, so does $W$.

Until further notice, we will work in the unital and finite-dimensional $C^*$-algebra $\mathcal{B}(Q[H])$; thus 'unitary' means 'unitary in $\mathcal{B}(Q[H])$', 'invertible' means 'invertible in $\mathcal{B}(Q[H])$', and so on. Suppose that $\delta < 1/2$. Then $||T - Q|| < 1$. By Lemma 1.2.6, $T$ is invertible and $W$ is a unitary.

Lemma 1.4.5 (2) implies $||W - Q|| < 3||T - Q|| = 6\delta$. Since $T = PT = T^*T = PEP + (Q - P)(Q - E)(Q - P),$ we have $T^*T = PEP + (Q - P)(Q - E)(Q - P),$ and $TPT^{-1} = E$. As $T$ commutes with $P$, we have $WPW^* = W|T|P|T|^{-1}W^* = TPT^{-1} = E$. Therefore $WP = EW$ and, since $P = uEu^*$, $WuE = EWu$.

The rank of $W$ is finite, and by the continuous functional calculus there exists $S \in C^*(W)_w$ such that $\exp(iS)Q = WQ = W$. Since $W \in \widetilde{\mathcal{A}}^{\text{WOT}}$, we have $S \in \widetilde{\mathcal{A}}^{\text{WOT}}$. Also, as $||W - Q||$ is small, we have $||S|| \approx ||W - Q||$; therefore $||S|| \to 0$ as $\delta \to 0$.

Back in $A$: Since the rank of $Q$ is finite, by Theorem 3.5.4 there exists $h \in A_{+,1}$ such that $||h|| = ||S||$ and $hQ = SQ$. Let $w_t := \exp(it)h$ for $0 \leq t \leq 1$. Then $v_t := w_tu$, for $0 \leq t \leq 1$, is a path of unitaries in $A$ connecting $v_0 = u$ and $v := v_1$. We have

$$vE = w_1uE = WuE = EWu = Ew_1u = Ev$$

and $||v - u|| = ||w_1 - 1|| \leq ||h||$ can be made arbitrarily small by choosing a sufficiently small $\delta > 0$, and this completes the proof. □

---

1 This does not exhaust the required degree of smallness of $\delta$. 

The proof of Theorem 5.1.2 requires a little gem that is Lemma 5.1.4 below. Given a metric space \((X,d)\) and \(\varepsilon > 0\), an \(\varepsilon\)-net is \(Y \subseteq X\) such that the \(\varepsilon\)-balls around elements of \(Y\) cover \(X\). If \(X\) is compact, then for every \(\varepsilon > 0\) there exists a finite \(\varepsilon\)-net in \(X\).

**Lemma 5.1.4.** Suppose \((X,d)\) is a compact metric space, \(\varepsilon > 0\), and \(Y\) and \(Z\) are \(\varepsilon\)-nets in \(X\) of the minimal possible cardinality. Then there exists a bijection \(f: Y \to Z\) such that \(\max_{y \in Y} d(y, f(y)) < 2\varepsilon\).

**Proof.** The proof uses Hall’s Matching Theorem (also known as Hall’s Marriage Lemma) found in any graph theory book. For \(F \subseteq Y\) let

\[G[F] := \{z \in Z : \min_{y \in F} d(z, y) < 2\varepsilon\}.\]

Since \(Z\) is an \(\varepsilon\)-net, we have \(\bigcup_{y \in F} B(y, \varepsilon) \subseteq \bigcup_{z \in G[F]} B(z, \varepsilon)\). Therefore \((Y \setminus F) \cup G[F]\) is an \(\varepsilon\)-net, and \(|G[F]| \geq |F|\). Since \(F\) was an arbitrary subset of \(Y\), Hall’s Matching Theorem implies that there exists an injection \(f: Y \to Z\) such that \(f(y) \in G[f(y)]\), and therefore \(d(y, f(y)) < 2\varepsilon\) for all \(y \in Y\). \(\square\)

Recall that \(U_0(A)\) is the connected component of the identity in the unitary group of a unital \(C^*\)-algebra \(A\).

**Proposition 5.1.5.** Suppose \(A\) is a unital \(C^*\)-algebra, \(\pi: A \to \mathcal{B}(H)\) is a direct sum of inequivalent, faithful, and irreducible representations of \(A\), \(\varepsilon > 0\), \(E\) is a finite-rank projection in \(\mathcal{B}(H)\), and \(U \subseteq U_0(A)\). Then there exist \(m \geq 1\) and \(W \subseteq M_1(A)\) such that \(\pi(w^*)E = E\) for all \(w \in W\) and for every \(u \in U\) there is a permutation \(f_u: W \to W\) satisfying \(\max_{w \in W} |uw - f_u(w)| < 3\varepsilon\).

**Proof.** Let \(\pi_i: A \to \mathcal{B}(H_i)\), for \(i < d\), be an enumeration of those irreducible summands of \(\pi\) such that \(E[H_i] \neq \{0\}\). We may assume that \(H = \bigoplus_{i < d} H_i\) and identify \(A\) with \(\pi[A] = \bigoplus_{i < d} \pi_i[A]\). Then Corollary 3.5.3 implies that \(\overline{A}_{\text{WOT}} = \bigoplus_{i < d} \mathcal{B}(H_i)\).

For every \(u \in U\) fix a continuous path of unitaries \(u_t\), for \(0 < t \leq 1\), such that \(u_0 = 1\) and \(u_1 = u\). With \(\delta > 0\) as guaranteed by Lemma 5.1.3, let \(m\) be large enough so that each one of these paths has a discretization

\[\tilde{u} := (u(j) : j < m),\]

with \(u(0) = 1\), \(u(m - 1) = u\), and \(\|u(i) - u(i + 1)\| < \delta\) for all \(i < m - 2\).

For every \(i < d\) fix a finite-rank projection \(E_i \in \mathcal{B}(H_i)\) such that \(E \leq \sum_{i < d} E_i\). Lemma 1.9.4 implies that there exists a finite-rank projection \(F_i \in \mathcal{B}(\mathbb{C}^m \otimes H_i)\) such that \(E_i \oplus 0 \oplus \cdots \oplus 0 \leq F_i\) and \(\|F_i, \tilde{u}\| < \delta\) for all \(u \in U\). Then \(F = \sum_{i < d} F_i\) is in \(\mathcal{B}(\mathbb{C}^m \otimes H)\) and it satisfies \(E \oplus 0 \oplus \cdots \oplus 0 \leq F\) and \(\|F, \tilde{u}\| < \delta\) for all \(u \in U\).

By the choice of \(\delta\), for each \(\delta\) there is \(\delta\)-net \(\mathcal{V}_u \subseteq U_0(M_m(A))\) such that \(\|\tilde{v}_u - \tilde{u}\| < \varepsilon\) and \(\tilde{v}_F = 0\). Let \(U' := \{\tilde{v}_u(0) : u \in U\}\).

Let \(\mathcal{W} := U(F[\mathbb{C}^m \otimes H])\) and \(\mathcal{Y}' := \{E\}' \cap U_0(M_m(A))\). Then \(M_m(A)\) is WOT-dense in \(\mathcal{B}(\mathbb{C}^m \otimes H)\) (Exercise 5.7.10). By Corollary 3.5.5, for every \(y \in \mathcal{W}\) there exists a unitary \(z \in \mathcal{Y}'\) such that \(zE = y\).
Fix $Y \subseteq \mathcal{Y}$ of the minimal possible size such that $\{yF : y \in Y\}$ is an $\varepsilon$-net in $\mathcal{Y}$. For $u \in U$ let $Y_u := \{\tilde{v}_u y : y \in Y\}$. Since $\tilde{v}_u \in \mathcal{Y}$ and the multiplication by $\tilde{v}_u$ is an isometry of $\mathcal{Y}$, we have $Y_u \subseteq \mathcal{Y}$ and $\{\tilde{v}_u yF : y \in Y_u\}$ is an $\varepsilon$-net in $\mathcal{Y}$ of the minimal size. By Lemma 5.1.4 there exists a bijection $f_u : Y \to Y_u$ such that

$$\max_{y \in Y} \| (f_u(y) - \tilde{v}_u y)F \| < 2\varepsilon.$$ 

By Proposition 3.9.6 there exists $e \in M_m(A)$ such that $F \leq e \leq 1$ and

$$\max_{y \in Y, a \in U} \| (f_u(y) - \tilde{v}_u y)e \| < 2\varepsilon.$$ 

Since $\|\tilde{v}_u - \tilde{u}\| < \varepsilon$, we have

$$\max_{y \in Y, a \in U} \| (f_u(y) - \tilde{u} y)e \| < 3\varepsilon.$$ 

For $y \in Y$ let $w_y \in M_{1m}(A)$ be the first row of $ye \in M_m(A)$. Since $y$ is a unitary and $\|e\| \leq 1$, we have $\|w_y\| \leq 1$.

Extend $f_u$ to $W := \{w_y : y \in Y\}$ by $f_u(w_y) := w_y f_u(y)$. Since $e \geq F \geq E$, we have $ww^* \geq E$ for all $w \in W$. Therefore $\max_{w \in W} \| f_u(w) - uw \| < 3\varepsilon$. This completes the proof.

Proof (Theorem 5.1.2). Fix a C*-algebra $A$ and a sum of finitely many inequivalent irreducible representations, $\pi : A \to \mathcal{B}(H)$. Fix $F \in A$, let $E$ be a finite-rank projection in $\pi[A]^{\text{WOT}}$, and let $\varepsilon > 0$. We need to find $n$ and $b \in M_{1n}(A)$ such that $bb^* \leq 1$, $E(1 - \pi(bb^*)) = 0$, and for all $a \in F$ the following holds:

$$\sup_{c \in A, \|c\| \leq 1} \| a \sum_{i<n} (b_i c b_i^*) - \sum_{i<n} (b_i c b_i^*)a \| = 0.$$ 

Since every element of $A$ is a linear combination of elements of $U_0(A)$, we may assume $F \subseteq U_0(A)$. By Proposition 5.1.5 there exists $m \geq 1$ and $W \subseteq M_{1m}(A)_{\leq 1}$ such that $ww^* E = E$ and for every $u \in F$ there is a permutation $f_u : W \to W$ such that $\max_{w \in W} \| uw - f_u(w) \| < 3\varepsilon$. Let $n := |W|m$ and let $b = (b_i : i < n) \in M_{1n}(A)$ be equal to the direct sum of all $w \in W$ divided by $|W|^{1/2}$.

Then $bb^* = \frac{1}{|W|} \sum_{w \in W} ww^* \leq 1$ and $bb^* E = E$. If $c \in A$ and $u \in F$ then for every $b \in W$ we have $u c b^* u^* \approx_{6\varepsilon} f(b) c f(b)^*$. By summing, we obtain

$$\| u \sum (b_i c b_i^*) u^* - \sum (b_i c b_i^*) \| < 6\varepsilon \|c\|.$$ 

Since $\varepsilon > 0$, $E$, and $F$ were arbitrary, this concludes the proof. \hfill $\square$

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2 Here $|W|$ denotes the cardinality of the set $W$. 
5.2 Excision of Pure States

In this section we use the analysis of pure states from §3.6 to prove the excision theorem for pure states. This theorem is used to study the weak$^*$-topology on the state space and prove that the pure states are weak$^*$-dense in the state space of a primitive C$^*$-algebra (Glimm’s Lemma). We also prove some refinements that will be used in the proof that the pure state space of a simple and separable C$^*$-algebra is homogeneous in §5.6. This section concludes with a proof of Kirchberg’s Slice Lemma and its applications to the ideal structure of $A \otimes B$.

Recall that given a C$^*$-algebra $A$ we write $A_{+1} = \{ a \in A_+ : \|a\| = 1 \}$. The following theorem, known as the excision of pure states, is (among other things) the key step in associating a ‘noncommutative ultrafilter’ to a pure state (§5.3).

Theorem 5.2.1. If $\varphi$ is a pure state on a C$^*$-algebra $A$ then there is a decreasing net $\{ a_\lambda : \lambda \in \Lambda \}$ in $A_{+1}$ such that $\varphi(a_\lambda) = 1$ for all $\lambda$ and for every $b \in A$ we have

$$\lim_\lambda \| a_\lambda b a_\lambda - \varphi(b)a_\lambda^2 \| = 0.$$  

A net $(a_\lambda)$ that satisfies the conclusion of Theorem 5.2.1 is said to excise $\varphi$.

Proof. We will first prove this under the additional assumption that $A$ is unital. Since $L_\varphi$ is a left ideal (Lemma 3.6.3), its adjoint $L_\varphi^*$ is a right ideal and

$$L_\varphi \cap L_\varphi^* = \{ a \in A : \varphi(a^* a) = \varphi(aa^*) = 0 \}$$

is a hereditary C$^*$-subalgebra of $A$. Let $e_\lambda$, for $\lambda \in A$, be an approximate unit in this algebra. Define

$$a_\lambda := 1 - e_\lambda.$$  

Then $0 \leq e_\lambda \leq 1$ implies $0 \leq a_\lambda \leq 1$. Since an approximate unit is upwards directed, the family $a_\lambda$, for $\lambda \in A$, is downwards directed. For $d \in L_\varphi \cap L_\varphi^*$ we have $\lim_\lambda \| d(1 - e_\lambda) \| = 0$. Therefore

$$\lim_\lambda \| a_\lambda d \| = \lim_\lambda \| da_\lambda \| = \lim_\lambda \| a_\lambda da_\lambda \| = 0.$$  

Fix $b \in A$. Since $\varphi$ is pure, by Proposition 3.6.5 we can write $\varphi(b) - b = c + d^*$ for $c$ and $d$ in $L_\varphi$. Since $L_\varphi \cap A_{sa} = L_\varphi^* \cap A_{sa}$, $c \in L_\varphi$ implies $c^* c \in L_\varphi \cap L_\varphi^*$, and

$$\lim_\lambda \| ca_\lambda \| = \lim_\lambda \| a_\lambda c^* c a_\lambda \| = 1/2 = 0.$$  

Similarly, $\lim_\lambda \| a_\lambda d^* \| = 0$. Therefore

$$\| a_\lambda b a_\lambda - \varphi(b)a_\lambda^2 \| = \| (1 - e_\lambda)(c + d^*)(1 - e_\lambda) \| \leq \| (1 - e_\lambda)d^* \| + \| c(1 - e_\lambda) \|$$

tends to 0 as $\lambda \to \Lambda$. Since $b \in A$ was arbitrary, we have proved that the net $(a_\lambda)$ excises $\varphi$. 

If $A$ is not unital, fix $a \in A_{+1}$ such that $\|a\| = \varphi(a) = 1$ (Exercise 3.10.18). Then the net defined by $a_\lambda := a^{1/2}(1 - e_\lambda)a^{1/2}$ excises $\varphi$ by a computation marginally more complex than in the unital case. \hfill $\square$

Theorem 5.2.1 provides one with a good grip on the the weak$^*$-topology on the state space of a C$^*$-algebra. The following may be the simplest example.

**Lemma 5.2.2.** Suppose $\varphi$ is a pure state on a C$^*$-algebra $A$. The sets

$$U_{a, \varepsilon} := \{ \psi \in S(A) : \psi(a) > 1 - \varepsilon \}$$

for $a \in A_{+1}$ such that $\varphi(a) = 1$ and $\varepsilon > 0$ form a local neighbourhood basis of $\varphi$ in the weak$^*$-topology on $S(A)$.

**Proof.** Since $U_{a, \varepsilon}$ is clearly a weak$^*$-open neighbourhood of $\varphi$, it suffices to prove that every weak$^*$-open neighbourhood $V$ of $\varphi$ contains one of this form. Fix $V$ and let $F \subset A_{<1}$ and $\delta > 0$ be such that $V \supseteq \{ \psi : |\psi(b) - \varphi(b)| < \delta \text{ for all } b \in F \}$.

Let $\varepsilon < \delta/4$ be such that $2\varepsilon^{1/2} < \delta/4$. Use Theorem 5.2.1 to find $a \in A_{+1}$ satisfying $|\psi(b) - a^{1/2}ba^{1/2}| < \delta/4$ for all $b \in F$. Fix $\psi \in U_{a, \varepsilon}$. For $b \in F$ the second part of Lemma 1.7.5 implies $|\psi(b) - \psi(a^{1/2}ba^{1/2})| < 2\varepsilon^{1/2}|b| < \delta/4$ and therefore

$$\frac{\delta}{4} > |\psi(b)a - a^{1/2}ba^{1/2}| = |\varphi(b)\psi(a) - \psi(a^{1/2}ba^{1/2})|$$

$$\approx_{\delta/4} |\varphi(b)\psi(a) - \psi(b)| \approx_{\varepsilon} |\varphi(b) - \psi(b)|.$$  

This shows $|\varphi(b) - \psi(b)| < \delta$. Since $b \in F$ was arbitrary we conclude that $\psi \in V$. Since $\psi \in U_{a, \varepsilon}$ was arbitrary, $U_{a, \varepsilon} \subseteq V$ and the assertion follows. \hfill $\square$

Recall that a state $\varphi$ is faithful if $\varphi(a^*a) = 0$ implies $a = 0$. We introduce a less common terminology and say that a state is GNS-faithful if the corresponding GNS representation is faithful (and therefore isometric by Lemma 1.2.10).

**Lemma 5.2.3.** Every faithful state on a C$^*$-algebra is GNS-faithful.

**Proof.** Suppose $\varphi$ is a state on a C$^*$-algebra $A$. If $a \in A$ then

$$\varphi(a^*a) = (\pi_\varphi(a^*a)\xi_{\varphi}) = \|\pi_\varphi(\xi_{\varphi})\|^2 \leq \|\pi_\varphi(a)\|^2.$$  

If $\varphi$ is faithful, then this implies $\|\pi_\varphi(a)\| > 0$ for all $a \in A$. \hfill $\square$

Not every GNS-faithful state is faithful (see also Exercise 5.7.1).

**Example 5.2.4.** 1. Every state on a simple C$^*$-algebra is GNS-faithful. Yet if $a$ is a positive contraction in a unital C$^*$-algebra $A$ such that $\|1 - a\| = 1$, then there exists a state $\varphi$ of $A$ such that $\varphi(a) = \varphi(a^2) = 0$.

2. On an abelian C$^*$-algebra a state is faithful if and only if it is GNS-faithful. The states on an abelian C$^*$-algebra $C(X)$ correspond to Radon probability measures on $X$ by Example 1.7.2. A measure on $X$ is strictly positive if the measure of
Lemma 5.2.5. Suppose $\varphi$ is a GNS-faithful pure state on a $C^*$-algebra $A$, and that the associated GNS representation $\pi : A \to \mathcal{B}(H)$ satisfies $\pi[A] \cap \mathcal{K}(H) = \{0\}$. Then for every finite-dimensional subspace $H_0$ of $H$ the set

$$\{ \omega_\zeta \mid A : \| \zeta \| = 1, \zeta \in H \cap H_0^\perp \}$$

is weak$^*$-dense in $P(A)$.

Proof. By Lemma 5.2.2, it suffices to prove that for every $a \in A_{+,1}$ and $\varepsilon > 0$ there exists $\zeta \in H_0^\perp$ such that $\omega_\zeta(a) > 1 - \varepsilon$. By the Spectral Theorem (Theorem C.6.11) there exists a unitary $u : H \to L_2(X, \mu)$ for some probability Radon measure space $(X, \mu)$ so that $uau^* = \mu$ is the multiplication operator $M_f$ for some $f \in L_\infty(X, \mu)$. Since $a \in A_{+,1}$, $b := (a - (1 - \varepsilon))_+$ is nonzero and $\pi(b)$ is not compact. Therefore, if $Y := \{ x \in X : f(x) > 1 - \varepsilon \}$ then $K := L_2(Y, \mu)$ is an infinite-dimensional subspace of $L_2(X, \mu)$ and we can find $\zeta \in H_0^\perp \cap K$. Then $\omega_\zeta(a) > 1 - \varepsilon$.

Lemma 5.2.6 (Glimm’s Lemma). Suppose $A \subseteq \mathcal{B}(H)$ and $A \cap \mathcal{K}(H) = \{0\}$. Then the set of restrictions of vector states $\omega_\zeta$, for $\zeta \in H$, to $A$ is weak$^*$-dense in $S(A)$.

Proof. Let $U$ be a nonempty weak$^*$-open set in $S(A)$. The Krein–Milman theorem implies that $U$ contains a convex combination $\varphi = \sum_{i<n} r_i \varphi_i$ of pure states.

Fix $F \subseteq A_{\leq 1}$ and $\varepsilon > 0$ such that $\max_{b \in F} |\varphi(b) - \psi(b)| < \varepsilon$ implies $\psi \in U$ for all $\psi \in S(A)$. By Theorem 5.2.1 we can find $a_i \in A_{+,1}$ for $i < n$ so that

$$\| \varphi_i(b) a_i - a_i^{1/2} b a_i^{1/2} \| < \varepsilon$$

for all $b \in F$.

Let $\delta$ be such that $2\delta^{1/2} < \varepsilon$. We will find vectors $\zeta_i$ such that $(a_i \zeta_i | \zeta_i) > 1 - \delta$. Likewise, by using Lemma 5.2.5 $n$ times, we can assure that

$$\zeta_k \in \{ b \zeta_j, b^* \zeta_j : b \in F, j < k \}$$

for all $k < n$. Let $\zeta := \sum_{k<n} r_k \zeta_k$. Fix $b \in F$. For $j \neq k$ we have $(b \zeta_j | \zeta_k) = 0$, and therefore (using the choice of $\delta$ and the second part of Lemma 1.7.5)

$$\omega_\zeta(b) = (b \sum_{k<n} r_k \zeta_k | \sum_{k<n} r_k \zeta_k) = \sum_{j<n} r_j \omega_\zeta(b) \approx_\varepsilon \sum_{j<n} r_j \varphi_j(b).$$

Therefore $|\omega_\zeta(b) - \varphi(b)| < \varepsilon$ for all $b \in F$, and $\omega_\zeta \mid A \subseteq U$.

Theorem 5.2.7. If a $C^*$-algebra $A$ is simple, unital, and infinite-dimensional then its pure state space $P(A)$ is weak$^*$-dense in $S(A)$.
Proposition 5.2.8. Suppose \( \varphi \) is a GNS-faithful pure state on a \( C^\ast \)-algebra \( A \) and that the associated GNS representation \( \pi : A \to \mathcal{B}(\mathcal{H}) \) satisfies \( \pi[A] \cap \mathcal{K}(\mathcal{H}) = \{0\} \). Then \( \Theta := \{ \varphi \circ \text{Ad} \exp(ia) : 0 \leq a \leq \pi \} \) is weak\(^*\)-dense in \( \mathcal{P}(A) \). If \( A \) is simple and infinite-dimensional, then \( \Theta \) is weak\(^*\)-dense in \( S(A) \).

Proof. Let \( \pi : A \to \mathcal{B}(\mathcal{H}) \) be the GNS representation associated with \( \varphi \). If \( U \) is a nonempty weak\(^*\)-open subset of \( \mathcal{P}(A) \) then by Lemma 5.2.5 we can find a unit vector \( \zeta \in \mathcal{H} \) such that \( \omega \zeta \circ \pi \in U \). Proposition 3.8.1 (4) implies that the restriction of \( \omega \zeta \circ \pi \) to \(\mathcal{H}_\gamma \) is unitarily equivalent to \( \varphi \) via a unitary \( \exp(ia) \) for \( 0 \leq a \leq \pi \). The second part is a consequence of the first part and Theorem 5.2.7.

Proposition 5.2.9. Suppose \( \varphi_\gamma \), for \( \gamma < \kappa \), are inequivalent GNS-faithful pure states on a unital \( C^\ast \)-algebra \( A \) and that the associated GNS representations satisfy \( \pi_\gamma[A] \cap \mathcal{K}(\mathcal{H}_\gamma) = \{0\} \). Then \( \Theta' := \{ (\varphi_\gamma \circ \text{Ad} \exp(ia)) : \gamma < \kappa \} \) is weak\(^*\)-dense in \( \mathcal{P}(A)^\kappa \).

Proof. By the definition of the product topology it suffices to prove the assertion for a finite \( \kappa \). For \( i < \kappa \) let \( \pi_i : A \to \mathcal{B}(\mathcal{H}_i) \) be the GNS representation associated with \( \varphi_i \). Fix nonempty weak\(^*\)-open subsets \( U_i \), for \( i < \kappa \), of \( \mathcal{P}(A) \). Lemma 5.2.5 implies the existence of unit vectors \( \zeta_i \in \mathcal{H}_i \) such that \( \omega \zeta_i \circ \pi_i \in U_i \) for \( i < \kappa \). Proposition 3.8.5 (4) gives a unitary of the form \( \exp(ia) \) for \( 0 \leq a \leq \pi \) such that \( \omega \zeta \circ \pi_i \) and \( \varphi_i \circ \text{Ad} \epsilon \) agree on \( A \) for all \( i < \kappa \). Therefore \( \varphi_i \circ \text{Ad} \epsilon \in \prod_{i < \kappa} U_i \) follows.

The remaining part of this section is devoted to a proof of Kirchberg’s Slice Lemma (Lemma 5.2.11), a fundamental result about the minimal tensor product, and its applications to the ideal structure of tensor products. These results are an end in themselves and will not be used later on in this text (Lemma 5.2.13 was used earlier on).

Lemma 5.2.10. Suppose \( A \) and \( B \) are \( C^\ast \)-algebras and \( c \in (A \otimes B)_+ \). Then for every \( \varepsilon > 0 \) there are \( a \in A_{+1} \) and \( b \in B_{+1} \) such that \( \| (a \otimes 1) c (a \otimes 1) - a \otimes b \| < \varepsilon \).

Proof. We may assume \( c \neq 0 \). Fix \( d \) such that \( c = d^* d \). By modifying \( \varepsilon > 0 \) and approximating \( d \) with a sum of elementary tensors, we may assume that \( c \) is equal to a sum of elementary tensors, \( c = \sum_{l \in n} x_l \otimes y_l \). Let \( \varphi \) and \( \psi \) be pure states of \( A \) and \( B \), respectively, such that \( (\varphi \otimes \psi)(c) > 0 \) and let \( b := (\varphi \otimes \text{id}_B)(c) \) using the conditional expectation from Example 3.3.5. Then \( \| b \| \geq \psi(b) = (\varphi \otimes \psi)(c) > 0 \). With \( \lambda_i := \varphi(x_i) \), we have \( b = \sum_{l \in n} \lambda_i y_i \). Since a conditional expectation is completely positive, \( b \geq 0 \).

By Theorem 5.2.1 there exists \( a \in A_{+1} \) such that \( \| a^{1/2} x_i a^{1/2} - \lambda_i a \| < \varepsilon / n \) for all \( i \). Then \( (a \otimes 1) c (a \otimes 1) \approx_\epsilon a \otimes \sum_{l \in n} \lambda_i y_i = a \otimes b \) as required.
Lemma 5.2.11 (Kirchberg’s Slice Lemma). Suppose \( D \) is a nonzero hereditary subalgebra of \( A \otimes B \). Then there exists a nonzero \( x \in A \otimes B \) such that \( x^*x \in D \) and \( xx^* \) is an elementary tensor.

Proof. Fix \( c \in D_{+1} \). By Lemma 5.2.10, there are \( a \in A_+ \) and \( b \in B_+ \) such that \( \|c - a \otimes b\| < 1/4 \). We may assume \( \|a\| = \|b\| = 1 \). Proposition 1.6.14 implies that there exists \( z \in C^*(c, a \otimes b) \) such that \( z \varepsilon^* = ((a \otimes b) - 1/4)_+ \). We claim that there exists \( y \in C^*(a, b) \) such that \( y((a \otimes b) - 1/4)_+ y^* = (a - 3/4)_+ \otimes (b - 3/4)_+ \). It will be convenient to write \( (s - \varepsilon)_+ := \max\{s, s - \varepsilon\} \) for \( s \in \mathbb{R}_+ \).

For \( (s, t) \in \text{sp}(a) \times \text{sp}(b) \) we have that \( \min(s, t) \geq 3/4 \) implies \( st > 1/4 \) and by the Tietze Extension Theorem there exists \( f \in C(\text{sp}(a) \otimes \text{sp}(b)) \) such that \( f(s, t) = 0 \) if \( st \leq 1/4 \) and \( f(s, t) = ((s - 3/4)_+ (t - 3/4)_+)^{1/2} (st - 1/4)_+^{1/2} \) if \( \min(s, t) \geq 3/4 \). Then \( f \) determines \( y \in C^*(a, b) \) as required. Let \( x := yz \varepsilon^{1/2} \). Then \( x^*x \leq c \) and it belongs to \( D \) and \( xx^* = (a - 3/4)_+ \otimes (b - 3/4)_+ \) is an elementary tensor. \( \Box \)

If \( A \otimes_\omega B \) is a non-spatial tensor product, then the kernel of the surjection of \( A \otimes_\omega B \) onto \( A \otimes B \) (see Takesaki’s Theorem, Exercise 5.7.10 (5)) is a nontrivial ideal disjoint from \( A \otimes B \). The ‘obvious’ ideals in \( A \otimes B \) are the ones of the form \( I \otimes J \) for ideals \( I \) and \( J \) of \( A \) and \( B \), respectively, and we can now prove that every nonzero ideal in \( A \otimes B \) contains one of the ‘obvious’ ideal.

Corollary 5.2.12. 1. For any two \( C^* \)-algebras \( A \) and \( B \), every nontrivial ideal \( J \) of \( A \otimes B \) contains a nonzero elementary tensor.
2. The ideal \( J \) of \( A \otimes B \) generated by \( a \otimes b \) includes \( J_a \otimes J_b \), where \( J_a \) is the ideal of \( A \) generated by \( a \) and \( J_b \) is the ideal of \( B \) generated by \( b \).
3. The minimal tensor product of simple \( C^* \)-algebras is simple.

Proof. (1) Suppose \( J \) is a nontrivial ideal in \( A \otimes B \). Lemma 5.2.11 implies that there exists \( x \in A \otimes B \) such that \( x^*x \in J \) and \( xx^* \) is an elementary tensor. By Corollary 1.6.10, there exists \( e \in A \otimes B \) such that \( x = e(x^*x)^{1/4} \) and therefore \( x \in J \). Also \( \|xx^*\| = \|x^*x\| > 0 \), and \( xx^* \) is a nonzero elementary tensor in \( J \).

(2) Since the ideal of a \( C^* \)-algebra \( D \) generated by an element \( d \) is equal to \( DdD \), this is a straightforward computation.

(3) is an immediate consequence of (1) and (2). \( \Box \)

We can now improve the conclusion of Lemma 2.4.7.

Lemma 5.2.13. Suppose \( A \) is a unital \( C^* \)-algebra, \( B \) and \( C \) are \( C^* \)-subalgebras of \( A \), \( B \) is simple and it contains an approximate unit of \( A \), \( C \) is nuclear, and \( bc = cb \) for all \( b \in B \) and all \( c \in C \). Then \( C^*(B, C) \) is isomorphic to \( B \otimes C \) via a map that sends \( b \otimes 1_c \) to \( b \) and \( 1_b \otimes c \) to \( c \) for all \( b \in B \) and all \( c \in C \).

Proof. Since \( C \) is nuclear, \( B \otimes C \) and \( B \otimes_{\text{max}} C \) are isomorphic via an isomorphism that is the identity on \( B \otimes C \). Let \( \Phi : B \otimes C \to C^*(B, C) \) be a surjective \( * \)-homomorphism as in Lemma 2.4.7.

Assume that \( \ker(\Phi) \neq \{0\} \). Corollary 5.2.12 (3) implies that \( \ker(\Phi) \) includes an ideal of the form \( B \otimes I \) (recall that \( B \) is simple) for some ideal \( I \) of \( C \). Since
5.3 Quantum Filters

The purpose of this section is to study quantum filters of positive contractions of norm 1 in a C*-algebra. We prove that a quantum filter determines a face of the state space of a C*-algebra and we also define a canonical bijection between maximal quantum filters and pure states. This section ends with a proof that our definition generalizes the original notion of maximal quantum filters of projections in C*-algebras of real rank zero.

The motivation for abstract quantum filters (Definition 5.3.3), comes from the need to analyze states on C*-algebras. States provide a blueprint for quantum filters.

Definition 5.3.1. Suppose \( \varphi \) is a state on a C*-algebra \( A \). The quantum filter associated with \( \varphi \) is \( \mathcal{F}_\varphi := \{ a \in A_{+,1} : \varphi(a) = 1 \} \).

Example 5.3.2. 1. Suppose \( A = C(X) \). By the Riesz Representation Theorem, every state on \( A \) is of the form \( \varphi(a) = \int a d\mu \) for a Radon probability measure \( \mu \) on \( X \). Then \( \mathcal{F}_\varphi = \{ a \in C(X)_{+,1} : a \mid \text{supp}(\mu) = 1 \} \). If \( \mu \) is the point mass measure concentrating at \( x \in X \), then \( \mathcal{F}_\varphi = \{ a \in C(X)_{+,1} : a(x) = 1 \} \) and \( \varphi(a) = a(x) \).

2. If \( \xi \) is a unit vector then on \( \mathcal{B}(H) \) we have \( \mathcal{F}_{\omega_\xi} = \{ a \in \mathcal{B}(H)_{+,1} : a\xi = \xi \} \), the set of all operators in \( \mathcal{B}(H)_{+,1} \) that have \( \xi \) as a 1-eigenvector.

3. If \( \varphi \) is a state on a C*-algebra \( A \) and \( \pi_\varphi \) is the corresponding GNS representation with cyclic vector \( \xi_\varphi \), then \( \mathcal{F}_\varphi = \pi_\varphi^{-1}[\mathcal{F}_{\omega_{\xi_\varphi}}] \).

We can now state the abstract definition.

Definition 5.3.3. A subset \( \mathcal{F} \) of \( A_{+,1} \) is a maximal quantum filter on a C*-algebra \( A \) if the following two conditions are satisfied.

1. For all \( n \geq 1 \) and \( a_j, \) for \( j < n, \) in \( \mathcal{F} \) we have \( \| \prod_{j=1}^n a_j \| = 1 \).

2. If \( \mathcal{F} \subseteq \mathcal{F}' \subseteq A_{+,1} \) and \( \mathcal{F}' \) satisfies (1) then \( \mathcal{F}' = \mathcal{F} \).

A subset \( \mathcal{F} \) of \( A_{+,1} \) is a quantum filter on \( A \) if it is an intersection of a (nonempty) family of maximal quantum filters.

Lemma 1.7.5 implies that \( \varphi(ab) = \varphi(b) \) for every \( a \in \mathcal{F}_\varphi \) and every \( b \in A \), and therefore \( \mathcal{F}_\varphi \) satisfies (1) of Definition 5.3.3 for every state \( \varphi \). We will talk about (maximal) quantum filters, omitting the reference to \( A \) if it is clear from the context.

An ordered set \( \mathbb{P} \) is downwards directed if every finite subset of \( \mathbb{P} \) has a lower bound. A quantum filter \( \mathcal{F}_\varphi \) in a C*-algebra \( A \), considered as a suborder of \( \langle A_{+,1}, \leq \rangle \)
is in general not downwards directed (Exercise 5.7.9). Proposition 5.3.4 gives a poor man’s version of being downwards directed. It is proved by recycling the idea used to construct an approximate unit in every C*-algebra. It will suffice for all practical purposes.

**Proposition 5.3.4.** Suppose \( \phi \) is a pure state on a C*-algebra \( A \). Then \( \mathcal{F}_\phi \) contains a norm-dense subset \( \mathcal{D} \) which excises \( \phi \) and is downwards directed.

**Proof.** Because \( a \in A_{+1} \) implies \( 0 \leq a \leq 1 \) and \( a \in L_\phi \cap L^*_\phi \) implies \( \phi(a) = 0 \) and therefore \( \|1 - a\| = 1 \), we have \( \mathcal{F}_\phi = \{a \in A_{+1} : 1 - a \in L_\phi \cap L^*_\phi \} \). Let

\[
\Lambda := \{a \in (L_\phi \cap L^*_\phi)_+ : a < 1\}.
\]

Clearly \( \mathcal{D} := \{a \in A_{+1} : 1 - a \in \Lambda \} \) is norm-dense in \( \mathcal{F}_\phi \), and it therefore excises \( \phi \).

As in the proof of Proposition 1.6.8, the function \( a \mapsto (1 - a)^{-1} - 1 \) is an order-isomorphism between \( \Lambda \) and \( (L_\phi \cap L^*_\phi)_+ \). Therefore \( (\Lambda, \leq) \) is upwards directed, and \( \mathcal{D} \) is downwards directed. \( \square \)

**Lemma 5.3.5.** If \( A \) is a C*-algebra and \( \mathcal{F} \) is a quantum filter on \( A \), then

\[
\mathcal{J}_\mathcal{F}(A) := \{\phi \in S(A) : \mathcal{F}_\phi \supseteq \mathcal{F}\}
\]

is a face of \( S(A) \).

**Proof.** For \( F \subseteq \mathcal{F} \) let

\[
\mathcal{J}_F := \{\phi \in S(A) : \phi(a) = 1 \text{ for all } a \in F\}.
\]

We will prove that \( \mathcal{J}_F \neq \emptyset \) for every \( F \subseteq \mathcal{F} \). Let \( n := |F| \) and let \( a_j \), for \( j < n \), enumerate \( F \). For \( 0 \leq k < n - 1 \) let \( b_k := \prod_{i \leq j < n} a_j \) and let \( b_{n-1} = 1 \). Then \( b_0 b^*_0 \) is a positive element of norm \( \|b_0\|^2 = 1 \), and therefore by Lemma 1.7.6 there exists \( \phi \in S(A) \) such that \( \phi(b_0 b^*_0) = 1 \).

We prove that \( \phi(a_j) = 1 \) and \( \phi(b_j b^*_j) = 1 \) for \( j < n \), by induction on \( j \). It is given that \( \phi(b_0 b^*_0) = 1 \). If \( j \) satisfies \( \phi(b_j b^*_j) = 1 \), then \( 0 \leq a_j \leq 1 \), Corollary 1.6.5 (3), and \( \|b_{j+1}\| = 1 \) together imply \( b_j b^*_j = a_j b_{j+1} b^*_j a_j \leq a_j^2 \leq a_j \). Since \( \phi \) is positive, \( \phi(a_j) = 1 \). Lemma 1.7.5 implies \( \phi(b_{j+1} b^*_j a_j) = \phi(a_j b_{j+1} b^*_j a_j) = 1 \) for all \( j < n \), and therefore \( \phi \in \mathcal{J}_F \) and \( \mathcal{J}_F \neq \emptyset \) as required.

Clearly every \( \mathcal{J}_F \) is weak*-compact, \( F \subseteq G \) implies \( \mathcal{J}_F \supseteq \mathcal{J}_G \), and therefore \( \mathcal{J}_\mathcal{F}(A) = \bigcap_{F \subseteq \mathcal{F}} \mathcal{J}_F \) is nonempty and weak*-compact. For every \( \phi \in S(A) \) and every \( a \in A_{+1} \) we have \( \phi(a) \in [0, 1] \) and therefore \( \mathcal{J}_\mathcal{F}(A) \) is convex. In order to prove that it is a face, suppose \( \phi \in \mathcal{J}_\mathcal{F}(A) \) is a nontrivial convex combination of \( \psi_0 \) and \( \psi_1 \) in \( S(A) \). We claim that both \( \psi_0 \) and \( \psi_1 \) belong to \( \mathcal{J}_\mathcal{F}(A) \). If \( a \in \mathcal{F} \), then \( \phi(a) = 1 \) and therefore \( \psi_0(a) = \psi_1(a) = 1 \). This implies \( \psi_j \in \mathcal{J}_\mathcal{F}(A) \) for \( i < 2 \) and concludes the proof. \( \square \)

**Lemma 5.3.6.** If \( \phi \) is a state on a C*-algebra \( A \) then the left kernel of \( \phi \) is uniquely determined by \( \mathcal{F}_\phi \) as \( L_\phi \setminus \{0\} = \{a \in A \setminus \{0\} : 1 - a^* a \|a\|^{-2} \in \mathcal{F}_\phi\} \). If \( \phi \) is pure and \( \mathcal{F}_\phi = \mathcal{F}_\psi \) for some state \( \psi \) then \( \phi = \psi \).
Proof. If \( \|a\| \leq 1 \) then \( \|1 - a^*a\| \leq 1 \), hence \( \varphi(a^*a) = 0 \) if and only if \( 1 - a^*a \in \mathcal{F}_\varphi \). Therefore \( L_\varphi \cap A_{\leq 1} = \{ a \in A \cup \{ 1 - a^*a \} \in \mathcal{F}_\varphi \} \). Since \( L_\varphi \) is a left ideal this uniquely determines \( L_\varphi \), and by renormalization it implies the required characterization of \( L_\varphi \).

If \( \varphi \) is pure, then \( \ker(\varphi) = L_\varphi + L_\varphi^* \) by Proposition 3.6.5 and therefore \( \ker(\varphi) \) is uniquely determined by \( \mathcal{F}_\varphi \). Finally, every state is uniquely determined by its kernel by Exercise 1.11.51.

**Proposition 5.3.7.** The following are equivalent for every quantum filter \( \mathcal{F} \) on a C*-algebra \( A \):

1. The face \( \mathcal{F}_\varphi(A) \) has more than one element.
2. The quantum filter \( \mathcal{F} \) is not maximal.
3. There exists \( a \in A_{+,1} \) such that each of \( \mathcal{F} \cup \{ a \} \) and \( \mathcal{F} \cup \{ 1 - a \} \) generates a quantum filter.

Proof. (3) clearly implies (2).

\((2) \Rightarrow (1)\): Suppose \( \mathcal{F} \) is not maximal. By Lemma 5.3.6, \( \mathcal{F} \) is not equal to \( \mathcal{F}_\varphi \) for any pure state \( \varphi \). Lemma 5.3.5 implies that \( \mathcal{F}_\varphi \) is a face of \( S(A) \). By the Krein–Milman theorem it has more than one extreme point and (1) follows.

\((1) \Rightarrow (3)\): Suppose that \( \mathcal{F}_\varphi \) has more than one element. Lemma 5.3.5 implies that \( \mathcal{F}_\varphi \) is a face of \( S(A) \) and therefore every extreme point of \( \mathcal{F}_\varphi \) is a pure state. The discussion now splits into two cases.

First, assume that \( \mathcal{F}_\varphi \cap \mathcal{P}(A) \) contains inequivalent pure states, \( \varphi \) and \( \psi \). By Proposition 3.5.2 the WOT-closure of \( (\pi_\varphi \oplus \pi_\psi)[A] = B(H_\varphi) \oplus B(H_\psi) \). By Theorem 3.5.4 applied to a self-adjoint operator in \( B(H_\varphi \oplus H_\psi) \) with eigenvectors \( \xi_\varphi \) and \( \xi_\psi \), we can find \( a \in A_{+,1} \) such that \( \varphi(a) = 1 \) and \( \psi(a) = 0 \), hence (3) follows.

Now consider the second possibility, that all pure states in \( \mathcal{X} := \mathcal{F}_\varphi \cap \mathcal{P}(A) \) are equivalent. Let \( \varphi \) be one of them. If \( \pi_\varphi : A \to B(H_\varphi) \) is the GNS-representation corresponding to \( \varphi \) then for every \( \psi \in \mathcal{X} \) there exists a unit vector \( \xi_\varphi \in H_\varphi \) such that \( \psi = \omega_\varphi \circ \pi_\varphi \) (Exercise 3.10.32). Fix \( a \in \mathcal{F} \). Then \( \pi_\varphi(a) \xi_\varphi = \xi_\varphi \) for all \( \psi \in \mathcal{X} \), and therefore

\[
H_0 := \operatorname{span}\{ \xi_\varphi : \psi \in \mathcal{X} \}
\]

is included in the 1-eigenspace of \( \pi_\varphi(a) \). Since \( \mathcal{X} \) has more than one element, and distinct states correspond to linearly independent vectors, \( \dim(H_0) \geq 2 \). Fix a unit vector \( \eta \) in \( H_0 \) orthogonal to \( \xi_\varphi \). By Theorem 3.4.5 there exists \( a \in A_{+,1} \) such that \( \varphi(a) = 1 \) and \( \pi_\varphi(a) \eta = 0 \). Therefore with \( \psi := \omega_\eta \circ \pi_\varphi \), we have that \( \mathcal{F} \cup \{ a \} \subseteq \mathcal{F}_\varphi \) and \( \mathcal{F} \cup \{ 1 - a \} \subseteq \mathcal{F}_\varphi \); hence (3) follows.

**Theorem 5.3.8.** Suppose \( A \) is a C*-algebra. Then \( \varphi \mapsto \mathcal{F}_\varphi \) is a bijective correspondence between pure states on \( A \) and maximal quantum filters of \( A \).

Proof. Lemma 5.3.6 and Proposition 5.3.7 together imply that for every maximal quantum filter \( \mathcal{F} \) there exists a unique pure state \( \varphi \) such that \( \mathcal{F} = \mathcal{F}_\varphi \). Since \( \mathcal{F}_\varphi \) is a face, it is the closed convex hull of pure states contained in it. Therefore \( \mathcal{F}_\varphi \) is a face of \( S(A) \). Conversely, every pure state \( \varphi \) determines \( \mathcal{F}_\varphi \) uniquely and by Lemma 5.3.6 this correspondence is an injection.
Lemma 5.3.9. Suppose $A$ is a C*-algebra and $\mathcal{F}$ is a quantum filter on $A$. Then

$$\mathcal{F} = \{ a \in A_{+1} : \varphi(a) = 1 \text{ for all } \varphi \in \mathcal{F}(A) \}.$$ 

Proof. The direct inclusion is trivial. If $a \in A_{+1}$ but $a \notin \mathcal{F}$, then there exists a maximal quantum filter $\mathcal{G} \supseteq \mathcal{F}$ such that $a \notin \mathcal{G}$. Then $\varphi_{\mathcal{G}}(a) \neq 1$ and $\varphi_{\mathcal{G}} \in \mathcal{F}(A)$, therefore $a \notin \mathcal{F}$. Since $a$ was arbitrary this proves the converse inclusion. \qed

Proposition 5.3.10. Suppose $\mathcal{F}$ is a quantum filter on a C*-algebra $A$. For every $b \in A_{+1}$ the following are equivalent.

1. $b \in \mathcal{F}$.
2. $b^n \in \mathcal{F}$ for some $n \geq 1$.
3. $b^n \in \mathcal{F}$ for all $n \geq 1$.
4. $(\forall \varepsilon > 0)(\exists n \in \mathbb{N})b + \varepsilon \geq a$

If in addition $\mathcal{F}$ is maximal then (1)–(4) are equivalent to each of the following.

5. $\|ba\| = 1$ for all $a \in \mathcal{F}$.
6. $\|aba\| = 1$ for all $a \in \mathcal{F}$.

Proof. We start by proving that the first three conditions are equivalent.

Trivially (3) implies (2).

(2) $\Rightarrow$ (1): Suppose that $b^n \in \mathcal{F}$ for some $n \geq 1$. Fix $\varphi \in \mathcal{F}(A)$. Then $\varphi(b^n) = 1$ for all $\varphi \in \mathcal{F}(A)$. Since $0 \leq b \leq 1$, we have $b^n \leq b$; therefore $1 = \varphi(b^n) \leq \varphi(b)$ and $\varphi(b) = 1$. Since $\varphi \in \mathcal{F}$ was arbitrary, $b \in \mathcal{F}$ and (1) follows from Lemma 5.3.9.

(1) $\Rightarrow$ (3): Assume (1) and fix $\varphi \in \mathcal{F}(A)$. By Proposition 1.7.8 we have $\varphi(aca') = \varphi(c)$ for all $a \in \mathcal{F}$, $a' \in \mathcal{G}$, and $c \in A$. By induction on $n$, $\varphi(b^n) = 1$ follows. Since $\varphi \in \mathcal{F}$ was arbitrary, (3) follows from Lemma 5.3.9.

(1) $\Rightarrow$ (4): take $a = b$.

(4) $\Rightarrow$ (1):

Suppose (4) holds, hence $b \in A_{+1}$ is such that for all $\varepsilon > 0$ we have $a_\varepsilon \in \mathcal{F}$ such that $b + \varepsilon \geq a_\varepsilon$. If $\varphi \in \mathcal{F}(A)$ then $\varphi(b) + \varepsilon \geq \varphi(a_\varepsilon) = 1$ for all $\varepsilon > 0$, and therefore $\varphi(b) = 1$ and (1) follows.

We have yet to prove that if $\mathcal{F}$ is a maximal quantum filter then (1) is equivalent to the last two conditions. In this case (6) implies (1) by the maximality.

By the definition (1) implies (6), and (6) clearly implies (5). Suppose (5) holds for $b$. By the C*-equality, $\|ab^2a\| = \|ba\|^2$ and therefore (6) holds for $b^2$. \qed

Example 5.3.11. In the case of non-maximal quantum filters, (6) is strictly weaker than (1). Take any unital C*-algebra $A \neq \mathbb{C}$ and let $\mathcal{F} = \{1_A\}$. Then every $b \in A_{+1}$ satisfies (6).

Definition 5.3.12. If $\mathcal{A} \subseteq A_{+1}$ satisfies condition (1) of Definition 5.3.3 then the quantum filter generated by $\mathcal{A}$ is

$$\mathcal{F}(\mathcal{A}) := \bigcap \{ \mathcal{F} : \mathcal{F} \supseteq \mathcal{A}, \mathcal{F} \text{ is a maximal quantum filter} \}.$$ 

A quantum filter $\mathcal{F}$ on a C*-algebra $A$ is generated by projections if $\mathcal{F}$ is the quantum filter generated by $\mathcal{F} \cap \text{Proj}(A)$. 

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Proposition 5.3.13. Suppose that a C*-algebra A has real rank zero. Then every quantum filter on A is generated by projections.

Proof. If a ∈ F, a ≠ 1, and 0 < ε < 1 then by Lemma 2.7.6 there exists a projection p = p(a, ε) ∈ A such that a ≤ p + (1 − ε)p and p ≤ a + ε. But a^n ≤ p + (1 − ε)^np, and Proposition 5.3.10 implies a^n ∈ F. Since (1 − ε)^n tends to 0 as n tends to ∞, Proposition 5.3.10 implies p ∈ F. If p(a, ε) ∈ F for all small enough ε > 0, then Proposition 5.3.10 implies a ∈ F. ⊓ ⊔

Example 5.3.14. Suppose A is a simple, unital, and infinite-dimensional C*-algebra. Theorem 5.2.7 and Theorem 5.2.1 together imply that for every state ϕ of A, every F ⊆ A, and every ε > 0 there exists a ∈ A_{+, 1} such that max_{b ∈ F} ∥aba − ϕ(b)a^2 ∥ < ε. If ϕ is a pure state then a can be chosen in F. However, if ϕ is not pure then this need not be the case. For example, if A is the CAR algebra and τ is its unique trace, then F = {1}: If a ∈ A_{+, 1} and a ≠ 1, then 1 − a > 0. Since τ is a faithful trace, we have τ(1 − a) > 0 and therefore a /∈ F.

5.4 Extensions of Pure States

In this section we study extensions of a given pure state on a C*-algebra to a larger C*-algebra. We use quantum filters to analyze uniqueness and equivalence of such extensions in the case of crossed products. These results will be used in the construction of a counterexample to Naimark's problem (Problem 5.5.2) in §11.2.

If A is a C*-subalgebra of B then by Lemma 1.7.6 every state on A can be extended to a state on B.

Lemma 5.4.1. Suppose A is a unital C*-algebra, B is a unital C*-subalgebra of A, and ϕ is a pure state on B.

1. The set of all extensions ψ ∈ S(A) of ϕ is a face of S(A).
2. The pure state ϕ has a pure state extension to A.
3. The pure state ϕ has a unique state extension to A if and only if ϕ has a unique pure state extension to A.

Proof. (1) Let F be the quantum filter of A generated by F_ϕ. Then ψ ∈ S(A) extends ϕ if and only if F_ψ ⊇ F; equivalently, if and only if F_ψ ⊇ F. The set of such ψ is a face by Lemma 5.3.5.

(2) Since every extreme point of a face is an extreme point of the original convex set, this is a consequence of (1).

(3) Since extreme points of a face are extreme points of the original convex set, (1) implies that all extreme points of the set of all extensions of ϕ to A are pure states on A. By the Krein–Milman theorem, a compact convex set is a singleton if and only if it has exactly one extreme point, and (3) follows. ⊓ ⊔

To analyze extensions of pure states we'll need a notion more precise than the one used in §5.2.
Lemma 5.4.2. If $\mathcal{X} \subseteq A_{+1}$ and $c \in A$, we say that $c$ is excited by $\mathcal{X}$ if there is $z \in \mathbb{C}$ such that for every $\varepsilon > 0$ there exists $a \in \mathcal{X}$ satisfying $\|aca - za^2\| < \varepsilon$. Let $\text{Exc}(\mathcal{X})$ denote the set of all $c \in A$ excited by $\mathcal{X}$.

A moment of reflection shows that $\text{Exc}(\mathcal{X})$ is a self-adjoint and norm-closed vector subspace of $A$. If a state $\varphi$ is pure then $\text{Exc}(\mathcal{F}_\varphi) = A$.

If $B$ is a $C^*$-subalgebra of $A$ and $\mathcal{F}$ is a maximal quantum filter on $B$, then the Axiom of Choice implies that $\mathcal{F}$ can be extended to a maximal quantum filter on $A$. Therefore all nontrivial implications in the following lemma are consequences of Theorem 5.3.8 and Lemma 5.4.1.

Lemma 5.4.3. Suppose $B$ is a $C^*$-subalgebra of $A$. For $\varphi \in \mathcal{P}(B)$ the following are equivalent.

1. The state $\varphi$ has a unique extension to a state on $A$.
2. The state $\varphi$ has a unique extension to a pure state on $A$.
3. The set $\mathcal{F}_\varphi$ generates a maximal quantum filter in $A$.
4. Every $c \in A$ is excited by $\mathcal{F}_\varphi$. □

Lemma 5.4.4. If $\varphi$ is a pure state on a $C^*$-algebra $A$, $\alpha \in \text{Aut}(A)$, and $u \in U(\hat{A})$ then $\mathcal{F}_{\varphi \circ \alpha} = \alpha^{-1}[\mathcal{F}_\varphi]$ and $\mathcal{F}_{\varphi \circ \text{Ad}_u} = u^\ast \mathcal{F}_\varphi u$.

Proof. For $a \in A_{+1}$ we have $\varphi(\alpha(a)) = 1$ if and only if $\alpha(a) \in \mathcal{F}$, or equivalently, if and only if $a \in \alpha^{-1}(\mathcal{F})$. This proves the first equality. The second is a special case of the first. □

Lemma 5.4.5. Suppose $A$ is a $C^*$-algebra and $\varphi_0$ and $\varphi_1$ are pure states, with the corresponding maximal quantum filters $\mathcal{F}_0$ and $\mathcal{F}_1$. The following are equivalent.

1. Pure states $\varphi_0$ and $\varphi_1$ are unitarily equivalent.
2. There is $u \in U_0(\hat{A})$ such that $\|a_0a_1\| = 1$ for all $a_0 \in \mathcal{F}_0$ and all $a_1 \in \mathcal{F}_1$.
3. There is $b \in A_{\leq 1}$ such that $\|a_0ba_1\| = 1$ for all $a_0 \in \mathcal{F}_0$ and all $a_1 \in \mathcal{F}_1$.
4. There are $b \in A_{\leq 1}$ and $\varepsilon > 0$ such that $\|a_0ba_1\| \geq \varepsilon$ for all $a_0 \in \mathcal{F}_0$ and all $a_1 \in \mathcal{F}_1$.

Proof. (1) $\Rightarrow$ (2): If (1) holds then Corollary 3.8.4 implies that there is $u \in U_0(\hat{A})$ such that $\varphi_0 = \varphi_1 \circ \text{Ad}_u$. Lemma 5.4.4 implies $\mathcal{F}_0 = u^\ast \mathcal{F}_1 u$. For all $a_j \in \mathcal{F}_j$, for $j < 2$, we have $a_0a_1u^\ast \in \mathcal{F}_0$ and therefore $\|a_0a_1\| = \|a_0a_1u^\ast\| = 1$, and this is (2).

Both (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1): We will suppose that (1) fails and prove that (4) fails. Fix a contraction $b \in A$. For $j < 2$ let $\pi_j : A \to \mathcal{B}(H_j)$ be the GNS representation corresponding to $\varphi_j$ with the cyclic vector $\xi_j$. Since $\varphi_0$ and $\varphi_1$ are pure and inequivalent, by Theorem 5.3.4 applied to $F := \{\xi_0, \pi_1(b^\ast)\xi_1\}$ we can find $a \in A_1$ such that $\pi_0(a)\xi_0 = \xi_0$ and $\pi_1(a)\pi_1(b^\ast)\xi_1 = 0$. By the continuous functional calculus, $a_0 := \min(a, 1)$ belongs to $A$. Since $\xi_0$ is a 1-eigenvector of $\pi_0(a)$, we have $a_0 \in \mathcal{F}_0$. Also, $\pi_1(b^\ast)\xi_1$ is a 0-eigenvector of $\pi_1(a_0)$ and therefore

$$
\varphi_1(ba_0^2b^\ast) = (\pi_1(ba_0^2b^\ast)\xi_1) = (\pi_1(ba_0b^\ast)\xi_1|\pi_1(a_0b^\ast)\xi_1) = 0.
$$
Let $c = ba_1^2b^*$. Since $0 \leq c \leq 1$, we have $1 - c \in A_{+,1}$ and since $\phi_1(1 - c) = 1$ we have $1 - c \in \mathcal{F}_1$. Proposition 5.3.10 implies $(1 - c)^n \in \mathcal{F}_1$ for all $n \geq 1$. We can now estimate a product of the form $a_1^2ba_0$ for $a_0$ in $\mathcal{F}_0$ and $a_1 := (1 - c)^n$ in $\mathcal{F}_1$ (the last equality is computed in an abelian $C^*$-algebra):

$$\|(1 - c)^n ba_0\|^2 = \|(1 - c)^n ba_0^2b^* (1 - c)^n\| = \|(1 - c)^{2n}c\|$$

As $\lim_{n \to \infty} \sup_{0 \leq t \leq 1} (1 - t)^{2n} = 0$, we have $\lim_n \|(1 - c)^{2n}c\| = 0$, and (4) fails. \[\Box\]

We now specialize to the study of extensions of pure states to crossed products.

**Lemma 5.4.6.** Suppose $A$ is a $C^*$-algebra, $\phi$ is a pure state on $A$, $\Gamma$ is a discrete group, and $\alpha : \Gamma \to \text{Aut}(A)$ is a group homomorphism. For every pure state $\phi$ on $A$ and every $g \in \Gamma$ the following are equivalent.

1. The pure states $\phi$ and $\phi \circ \alpha_{g^{-1}}$ are not unitarily equivalent.
2. For every state extension $\psi$ of $\phi$ to $A \rtimes \alpha$, $\Gamma$ we have $\psi(cu_g) = 0$ for all $c \in A$.

**Proof.** If $g = e$ then both assertions are false, so we may assume $g \neq e$. We will prove the equivalence directly. Lemma 5.4.5 implies that $\phi \not\sim \phi \circ \alpha_{g^{-1}}$ if and only if

$$\sup_{c \in A} \inf_{a \in \mathcal{F}_\phi} \|abc\| : a \in \mathcal{F}_{\phi}, b \in \mathcal{F}_{\phi \circ \alpha_{g^{-1}}} = 0. \quad (5.1)$$

Lemma 5.4.4 implies that $\mathcal{F}_{\phi \circ \alpha_{g^{-1}}} = u_g \mathcal{F}_\phi u_g^*$. Since $u_g^*$ is a unitary we have $\|acu_ga^*\| = \|acu_ga\|$ for all $a$, and since $\mathcal{F}_\phi$ has a norm-dense directed subset (Proposition 5.3.4), by Lemma 5.4.4 the formula (5.1) is equivalent to

$$\sup_{c \in A} \inf_{a \in \mathcal{F}_\phi} \|acu_ga\| = 0. \quad (5.2)$$

If $\psi$ is a state extension of $\phi$ to $A \rtimes \alpha, \Gamma$ and $a \in \mathcal{F}_\phi$ then $\psi(acu_ga) = \psi(cu_g)$ by Lemma 1.7.5. Therefore (5.2) is equivalent to asserting that $\psi(cu_g) = 0$ for every state extension $\psi$ of $\phi$. This completes the proof. \[\Box\]

**Proposition 5.4.7.** Suppose $A$ is a $C^*$-algebra, $\phi$ is a pure state on $A$, and $\Gamma$ is a discrete group. If $\alpha : \Gamma \to \text{Aut}(A)$ is a group homomorphism, then the following are equivalent.

1. The state $\phi$ has a unique pure state extension to $A \rtimes \alpha, \Gamma$.
2. The state $\phi$ is not unitarily equivalent to $\phi \circ \alpha_g$ for any $g \in \Gamma \setminus \{e\}$.

**Proof.** Let $\theta : B \to A$ be the conditional expectation defined in Proposition 3.3.9. Then $\psi := \phi \circ \theta$ is an extension of $\phi$ to a state on $B$. Since $\theta(a_2u_g) = 0$ if $g \neq e$ and $\theta(a_1u_e) = a_e$, Lemma 5.4.6 implies that (2) is equivalent to having every state extension of $\phi$ to $A \rtimes \alpha, \Gamma$ equal to $\psi$, and this is equivalent to (1). \[\Box\]

Theorem 5.4.8 certifies that we have acquired a complete control on when two pure states on $A$ have equivalent pure state extensions to a crossed product of $A$.

**Theorem 5.4.8.** Suppose $A$ is a $C^*$-algebra and $\alpha$ is an action of a discrete group $\Gamma$ on $A$. Then the following are equivalent for all pure states $\phi$ and $\psi$ of $A$. 

1. The state $\phi$ has a unique pure state extension to $A \rtimes \alpha, \Gamma$.
2. The state $\phi$ is not unitarily equivalent to $\phi \circ \alpha_g$ for any $g \in \Gamma \setminus \{e\}$.
5.5 Nonhomogeneity of the Pure State Space, I. Consequences of Glimm’s Theorem

1. There are pure state extensions ϕ of φ and ψ of ψ to A ⋉_α,Γ such that ϕ ∼ ψ.

2. There exists g ∈ Γ such that φ ∼ ψ ∘ α_{g^{-1}}.

3. There exists c ∈ A ⋉_α,Γ such that for every f(F)

Proof. Let B := A ⋉_α,Γ.

(1) ⇒ (2): Suppose (2) holds and let u ∈ U(Â) be such that φ = ψ ∘ α_{g^{-1}} ∘ Ad u. If ˜ψ is a pure state extension of ψ to B then ˜φ = ˜ψ ∘ Ad(u_{g^{-1}}u) is a pure state extension of φ equivalent to ˜ψ. This proves a strengthening of (1):

4. Every pure state extension ˜ψ of ψ to B is unitarily equivalent to some pure state extension φ of φ to B.

We will prove that (4) is equivalent to (1)–(3). Clearly (4) ⇒ (1).

(1) ⇒ (3): Use Lemma 5.4.5.

(3) ⇒ (2): Assume (2) fails and fix c ∈ B. For a moment fix g ∈ Γ and ε > 0. Find F ∈ Γ and d_f ∈ A, for f ∈ F, such that

\|c - \sum_{f ∈ F} d_f u_f\| < 1 - 2ε.

Let K := max_{f ∈ F} \|d_f\| and δ := ε/(2K|F|). Since the states φ and ψ ∘ α_{g^{-1}} are not unitarily equivalent and \(\mathfrak{F}_\psi \alpha_{g^{-1}} = \alpha_{\psi} \mathfrak{F}_\psi\) (Lemma 5.4.4), Lemma 5.4.5 implies that for every f ∈ F there exist a_f ∈ \mathfrak{F}_φ and b_f ∈ \mathfrak{F}_ζ such that \|a_f d_f \alpha_{g\gamma}(b_f)\| < δ.

Passing to B we have \|a_f d_f u_f b_f\| = \|a_f d_f u_f b_f u_f^\gamma\| < δ. Proposition 5.3.4 implies that \mathfrak{F}_φ has a norm-dense directed subset and we can find a ∈ \mathfrak{F}_φ and b ∈ \mathfrak{F}_ζ such that a ≤ a_f + δ and b ≤ b_f + δ for all f ∈ F. Therefore for all f ∈ F we have

\|a_f d_f u_f \| ≤ \|a_f d_f u_f b_f\| + 2δ\|d_f\|.

Putting all this together,

\|acb\| ≤ \|a(c - \sum_{f ∈ F} d_f u_f)\| + \sum_{f ∈ F} \|a_f d_f u_f b_f\| + 2|F|K\delta < (1 - 2ε) + 2ε < 1

and (3) fails. □

5.5 Nonhomogeneity of the Pure State Space, I. Consequences of Glimm’s Theorem

In this section we prove that all pure states on the algebra of compact operators are unitarily equivalent and discuss the converse, known as Naimark’s problem. We give a positive answer to Naimark’s problem in the case of C*-algebras faithfully represented on a Hilbert space of density character strictly smaller than the continuum. This is a consequence of the main result of the present section: Glimm’s Theorem.

Via the GNS construction, pure states on a C*-algebra A are in a bijective correspondence with irreducible representations of A (Proposition 3.6.5). The automorphism group Aut(A) naturally acts on the state space S(A) of A by the affine
homeomorphisms \( \varphi \mapsto \varphi \circ \alpha \). Since affine homeomorphisms send extreme points to extreme points, \( \text{Aut}(A) \) acts on \( P(A) \). Via the bijective GNS-correspondence between pure states and irreducible representations, \( \text{Aut}(A) \) also acts on the space of all irreducible representations of \( A \).

The automorphism group of \( A \) has several distinguished subgroups, such as the groups of inner automorphisms \( \text{Inn}(A) \), approximately inner automorphisms \( \overline{\text{Inn}}(A) \), asymptotically inner automorphisms \( A\text{Inn}(A) \), or inner automorphisms corresponding to unitaries in the connected component of the identity \( U_0(A) \) (§2.6). The unitary equivalence of pure states coincides with the (apparently smaller) orbit equivalence relation of the action of \( U_0(A) \) (Corollary 3.8.4). The equivalence classes of this relation are the connected components of \( P(A) \) taken with respect to the norm metric (Exercise 3.10.32). The GNS-correspondence sends unitarily equivalent pure states to spatially equivalent irreducible representations, and vice versa (Proposition 3.8.1).

Now consider the orbit equivalence relation \( \sim_{\text{Aut}} \) of \( \text{Aut}(A) \) on \( P(A) \). For pure states \( \varphi_0 \) and \( \varphi_1 \) we write \( \varphi_0 \sim_{\text{Aut}} \varphi_1 \) and say that \( \varphi_0 \) and \( \varphi_1 \) are conjugate if there exists an automorphism \( \Phi \) of \( A \) such that \( \varphi_0 = \varphi_1 \circ \Phi \). This relation is different from the unitary equivalence e.g., in the case of \( C(\{0,1\}^\mathbb{Z}) \) since all automorphisms are inner and the spectrum \( \{0,1\}^\mathbb{Z} \) is homogeneous. On the other hand, these two equivalence relations coincide when all automorphisms are inner, for example in the case of \( M_n(\mathbb{C}) \) for \( n \geq 1 \). This observation can be improved.

**Theorem 5.5.1 (Naimark).** Any two pure states on the algebra \( \mathcal{K}(H) \) of compact operators on any (not necessarily separable) Hilbert space \( H \) are equivalent.

**Proof.** If \( \xi \in H \) is a unit vector then the restriction of the vector state \( \omega_\xi \) (defined by \( \omega_\xi(a) := (a \xi \mid \xi) \)) to \( \mathcal{K}(H) \) is pure since the GNS representation corresponding to it is the standard irreducible representation of \( \mathcal{K}(H) \). Any two vector states, \( \omega_\xi \) and \( \omega_\eta \), are unitarily equivalent: take e.g., the unitary \( u \) that swaps \( \xi \) and \( \eta \) and is equal to the identity on \( \overline{\text{span}}(\{\xi, \eta\}) \). Then \( \omega_\xi = \omega_\eta \circ \text{Ad} u \) and \( u \in \mathcal{K}(H) \) since \( u \) is a finite-rank perturbation of the identity.

It therefore suffices to prove that every pure state on \( \mathcal{K}(H) \) is equal to the restriction of some vector state to \( \mathcal{K}(H) \). By Theorem 3.1.14 the Banach space dual of \( \mathcal{K}(H) \) is the space of trace class operators \( \mathcal{B}_1(H) \) and for every pure state \( \varphi \) of \( \mathcal{K}(H) \) there exists a trace class operator \( c \) such that \( ||c|| = ||\varphi|| \) and \( \varphi(a) = \text{tr}(ca) \) for all \( a \in \mathcal{K}(H) \) (where \( \text{tr}(ca) := \sum_{\xi \in \mathcal{E}} (ca \xi \mid \xi) \), for any orthonormal basis of \( H \)). An elementary argument shows that \( \varphi \) is positive if and only if \( c \) is positive, and that \( \varphi \) is a pure state if and only if \( c \) is a rank-1 projection. In this case, if \( \xi \) is a unit vector in the range of \( c \), then \( \text{tr}(ca) = \omega_\xi \). Since \( \varphi \) was an arbitrary pure state on \( \mathcal{K}(H) \), this completes the proof. \( \square \)

**Problem 5.5.2 (Naimark, 1951).** If all pure states on a \( C^* \)-algebra \( A \) are unitarily equivalent, is \( A \) isomorphic to \( \mathcal{K}(H) \) for some Hilbert space \( H \)?

If \( A \) has a nontrivial ideal, then it has irreducible representations with different kernels and therefore \( A \) cannot give a negative answer to this problem. Since every
simple type I algebra is isomorphic to \( \mathcal{X}(H) \) for some \( H \) ([14]), a negative answer to Naimark’s problem cannot be witnessed by an algebra of type I. Therefore Naimark’s Problem is about representations of simple, non-type-I, C*-algebras. We still haven’t seen an example of inequivalent pure states on a simple C*-algebra, or even of pure states that are not unitarily equivalent for a nontrivial reason. We’ll see that inequivalent pure states exist in any separable, simple, non-type I C*-algebra \( A \).

**Lemma 5.5.3.** Suppose \( A_\lambda \), for \( \lambda \in \Lambda \), is a inductive system of C*-algebras. Suppose furthermore that each \( A_\lambda \) has a distinguished state \( \phi_\lambda \) and that \( \lambda \leq \lambda' \) implies \( \phi_{\lambda'} \mid A_\lambda = \phi_\lambda \). Let \( A = \varprojlim A_\lambda \). Then there exists a uniquely defined state \( \phi \) of \( A \) extending all \( \phi_\lambda \). If each \( \phi_\lambda \) is pure, then so is \( \phi \).

**Proof.** By identifying a functional with its graph, let \( \phi := \bigcup_\lambda \phi_\lambda \). The assumptions imply that this is a well-defined function on \( \bigcup_\lambda A_\lambda \). It is a positive functional of norm 1, hence it has a continuous extension to \( A \) and this extension is a state.

Suppose \( \phi \) is not pure and fix distinct states \( \psi_0 \) and \( \psi_1 \) such that \( \phi = \frac{1}{2}(\psi_0 + \psi_1) \). Then \( \psi_0 \) and \( \psi_1 \) disagree on some \( A_\lambda \), hence \( \phi_\lambda \) is not pure.

The equivalence relation \( E_0 \) is defined on \( \{0,1\}^\mathbb{N} \) via

\[
x \in E_0 y \text{ if and only if } (\forall n) x(j) = y(j).
\]

This is an \( F_\sigma \) equivalence relation all of whose equivalence classes are countable. It is not difficult to find a perfect set of \( E_0 \)-inequivalent elements of \( \{0,1\}^\mathbb{N} \) (Exercise 5.7.20).

**Theorem 5.5.4 (Glimm).** There is a continuous map \( x \mapsto \psi_x \) from \( \{0,1\}^\mathbb{N} \) to \( \mathcal{P}(M_2\mathbb{C}) \) such that \( x \in E_0 y \) if and only if \( \psi_x \sim \psi_y \) and \( x \neq y \) implies \( \|\psi_x - \psi_y\| = 2 \).

**Proof.** On \( M_2(\mathbb{C}) \) let \( \varphi_0 \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) := a_{11} \) and \( \varphi_1 \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) := a_{22} \). Both \( \varphi_0 \) and \( \varphi_1 \) are pure states, being excised by the rank-1 projections \( p_0 := \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \) and \( p_1 := \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \), respectively. With \( c := p_0 - p_1 \) we have \( \varphi_0(c) = 1 \) and \( \varphi_1(c) = -1 \), and therefore \( \|\varphi_0 - \varphi_1\| = 2 \). For \( x \in \{0,1\}^\mathbb{N} \) define a product state \( \psi_x \) on \( S(M_2\mathbb{C}) \) as follows. If \( a = \bigotimes_{j \in \mathbb{N}} a_j \) is an elementary tensor in \( M_2\mathbb{C} \), then let \( \psi_x(a) := \prod_{j \in \mathbb{N}} \varphi_{x(j)}(a_j) \) and extend \( \psi_x \) linearly to all of \( M_2\mathbb{C} \). The restriction of \( \psi_x \) to \( \bigotimes_{j \in k} M_2(\mathbb{C}) \) is the pure state excised by the rank-1 projection \( \bigotimes_{j \in k} P_{x(j)} \). Therefore \( \psi_x \) is well-defined and pure by Lemma 5.5.3. Since the function \( x \mapsto \psi_x \) is clearly weak*-continuous, it will suffice to prove that the states \( \psi_x \) and \( \psi_y \) are unitarily equivalent if and only if \( x \in E_0 y \).

For \( n \in \mathbb{N} \) let \( A_n := \bigotimes_{j \leq n} M_2(\mathbb{C}) \), \( B_n := \bigotimes_{j > n} M_2(\mathbb{C}) \), and let \( C_n \) be the \( n \)th copy of \( M_n(\mathbb{C}) \) in \( M_2\mathbb{C} \). Therefore for all \( n \) we have
$$M_{2^n} = \bigotimes_{j=1}^n M_2(\mathbb{C}) = A_n \otimes C_n \otimes B_{n+1}$$

and each of the algebras $A_n, B_n, C_n$ is identified with a subalgebra of $M_{2^n}$.

Suppose that $x E_0 y$ and let $n$ be such that $x(j) = y(j)$ for all $j \geq n$. Since the restrictions of $\psi_x$ and $\psi_y$ to $A_n$ are pure and all pure states on a full matrix algebra are equivalent we can choose $u \in U(A_n)$ such that

$$\psi_x \upharpoonright A_n = \psi_y \circ \text{Ad} u \upharpoonright A_n.$$

The states $\psi_x$ and $\psi_y$ agree on $B_n$, and we have $\psi_x = \psi_y \circ \text{Ad} u$.

Now suppose $\psi_x \sim \psi_y$. Let $u \in U(M_{2^n})$ be such that $\psi_x = \psi_y \circ \text{Ad} u$. Since

$$U(M_{2^n}) = \lim_{n \to \infty} U(A_n),$$

for $n$ large enough there is a unitary $v \in A_n$ for which $\|u - v\| < 1/4$. Then for every $a \in (1 \otimes B_n)_{\leq 1}$ we have $\|ua - au\| < 1/2$. Therefore $\psi_x \circ \text{Ad} v \approx_{1/2} \psi_y \circ u = \psi_x$.

Also, $\text{Ad} v$ is the identity on $B_n$ and therefore $\psi_x \circ \text{Ad} v$ agrees with $\psi_x$ on $B_n$. The restriction of $\psi_x$ to $C_j$ is $\phi_0$ or $\phi_1$, and the same applies to $\psi_y$. Since $\|\phi_0 - \phi_1\| = 2$, for $j \geq n$ states $\psi_x$ and $\psi_y$ agree on $C_j$ and therefore $x(j) = y(j)$. Since $j \geq n$ was arbitrary, we conclude that $x E_0 y$. \qed

**Corollary 5.5.5.** Suppose $A$ is a non-type I $\mathbb{C}^*$-algebra. Then it has a family of pure states $\psi_f$, for $f \in \{0, 1\}^N$, such that $\|\psi_f - \psi_g\| = 2$ for all $f \neq g$.

**Proof.** Theorem 5.5.4 implies that the CAR algebra has a family of pure states $\psi_f$, for $f \in \{0, 1\}^N$, such that $\|\psi_f - \psi_g\| = 2$ if $f \neq g$. By Theorem 3.7.2 $A$ has a $\mathbb{C}^*$-subalgebra $B$ with quotient isomorphic to the CAR algebra. Each $\psi_f$ lifts to unique pure state on $B$ which by Lemma 5.4.1 extends to a pure state $\varphi_f$ of $A$. These extensions satisfy $\|\varphi_f - \varphi_g\| \geq \|\psi_f - \psi_g\| = 2$, and are therefore as required. \qed

**Corollary 5.5.6.** Every non-type-I $\mathbb{C}^*$-algebra with an irreducible representation on a Hilbert space of density character strictly smaller than $c$ has inequivalent pure states.

**Proof.** Suppose all pure states on a $\mathbb{C}^*$-algebra $A$ are equivalent. Fix a pure state $\psi$ of $A$ and let $(\pi, H, \xi)$ be the associated GNS triplet. Let $\varphi_f$, for $f \in \{0, 1\}^N$, be the family of pure states on $A$ guaranteed by Corollary 5.5.5. For $f \in \{0, 1\}^N$ fix a unitary $u_f$ of $A$ such that $\varphi_f = \psi \circ \text{Ad} u_f$. Then $\varphi_f(a) = (\pi(u_f a u_f^*) \xi(a) \xi)$ for all $a \in A$, and with $\xi(f) := \pi(u_f^*)^* \xi$ we have $\varphi_f = \omega_{\xi(f)} \circ \pi$.

If $f \neq g$ then $2 = \|\varphi_f - \varphi_g\| = \|\omega_{\xi(f)} - \omega_{\xi(g)}\|$, and $\xi(f) \perp \xi(g)$. Therefore $\xi(f), f \in \{0, 1\}^N$, are orthonormal, and the density character of $H$ is at least $c$. \qed

**Corollary 5.5.7.** A counterexample to Naimark’s problem cannot be separable, or even of density character strictly smaller than $c$.

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3 This follows from Exercise 2.8.11; in case you haven’t worked it out yet, try again using Lemma 1.4.4 in the relevant instance of Exercise 2.8.10
Proof. A C*-algebra of density character $\kappa$ can be faithfully represented on a Hilbert space of density character $\kappa$ (Corollary 1.10.4). \hfill $\Box$

Since a simple type I C*-algebra $A$ is isomorphic to $\mathcal{K}(\ell_2(\kappa))$ where $\kappa$ is the density character of $A$, Corollary 5.5.6 implies the following Glimm Dichotomy.

Corollary 5.5.8. Suppose $A$ is a simple C*-algebra whose density character is smaller than $c$. Then exactly one of the following applies.

1. All pure states on $A$ are unitarily equivalent and $A$ is isomorphic to $\mathcal{K}(\ell_2(\kappa))$ where $\kappa$ is the density character of $A$.
2. $A$ has at least $c$ unitarily inequivalent pure states and it is non-type I. \hfill $\Box$

Proof. Suppose that $A$ is not isomorphic to $\mathcal{K}(\ell_2(\kappa))$. Being simple, it is non-type I and by Corollary 5.5.5 it has a family $\phi_f$, for $f \in \{0, 1\}^N$, of pure states such that $\|\phi_f - \phi_g\| = 2$ for $f \neq g$. Let $\sim$ be the equivalence relation on $\{0, 1\}^N$ defined by $f \sim g$ if $\phi_f$ is unitarily equivalent to $\phi_g$. Let $\kappa$ denote the density character of $A$. Since every element of $A$ is a linear combination of four unitaries, $\kappa$ is equal to the density character of $U(A)$. Therefore every $\sim$-equivalence class has density character at most $\kappa$ in the norm topology. Since $\kappa < c$, there are $c$ equivalence classes. \hfill $\Box$

Therefore Naimark’s problem is about simple, non-type-I, C*-algebras of density character at least $c$. In order to say more, we’ll need tools from set theory.

5.6 Homogeneity of the Pure State Space

When used in a class or seminar, section 6 should be supplemented with coffee (not decaffeinated) and a light refreshment. We suggest Heatherton Rock Cookies. (Recipe:…)

Jon Barwise, Admissible Sets and Structures

In this, both technical and important, section we prove the strong homogeneity of the pure state space of every simple and separable C*-algebra $A$. The main result asserts that for any two sequences of unitarily inequivalent pure states on $A$ there exists an asymptotically inner automorphism that sends the $n$th state in the first sequence to a state unitarily equivalent to the $n$th state in the second sequence for every $n$. This is a key component in the construction of a counterexample to Naimark’s problem (§11.2). Its proof uses much of the material developed in chapters 3 and 5.

We continue the study of orbit equivalence relations $\sim_{\text{Aut}}$ and $\sim$ of $\text{Inn}(A)$ on $\mathcal{P}(A)$ started in §5.5. Recall that if $\phi_j$ are pure states on $A$ and $\pi_j: A \rightarrow \mathcal{B}(H_j)$ are the associated GNS representations for $j < 2$, then $\pi_0$ and $\pi_1$ are spatially equivalent if there exists a unitary operator $u: H_0 \rightarrow H_1$ such that $\pi_1 = \pi_0 \circ \text{Ad}u$. By Proposition 3.8.1, $\pi_0$ and $\pi_1$ are spatially equivalent if and only $\phi_0$ and $\phi_1$ are unitarily equivalent.

Two pure states $\phi_0$ and $\phi_1$ are conjugate, $\phi_0 \sim_{\text{Aut}} \phi_1$, if there exists an automorphism $\Phi$ of $A$ such that $\phi_1 = \phi_0 \circ \Phi$. 
Apart from the obvious fact that the unitary equivalence implies conjugacy, the relations \( \sim \) and \( \sim_{\text{Aut}} \) can be completely different from one another. E.g., in the case of \( C(\{0, 1\}^\mathbb{N}) \) (and any other abelian \( C^* \)-algebra) all inner automorphisms are trivial and therefore the unitary equivalence coincides with the equality on \( \mathcal{P}(A) \). On the other hand, \( \text{Aut}(C(\{0, 1\}^\mathbb{N})) \) acts transitively on \( C(\{0, 1\}^\mathbb{N}) \), and any two pure states on \( C(\{0, 1\}^\mathbb{N}) \) are conjugate by an automorphism.

The latter phenomenon occurs in all separable simple \( C^* \)-algebras: If \( A \) is separable and simple, then \( \text{Aut}(A) \) acts on \( \mathcal{P}(A) \) transitively. We will prove a strengthening of this result which is, together with \( \text{Aut}(A) \), the main ingredient in the Akemann–Weaver construction of a counterexample to Naimark’s problem.

**Theorem 5.6.1.** Suppose \( A \) is a simple and separable \( C^* \)-algebra. If \( n \leq \aleph_0 \) and \( \{ \phi_i : i < n \} \) and \( \{ \psi_i : i < n \} \), are two families of inequivalent pure states on \( A \) then there is an asymptotically inner automorphism \( \Theta \) of \( A \) such that \( \phi_i \circ \Theta \sim \psi_i \) for all \( i \).

See also Exercise 5.7.25, Exercise 5.7.26, and Exercise 5.7.27 for variations on the theme of Theorem 5.6.1. Its proof, divided into four more easily digestible chunks, will occupy the remainder of the present section.

### 5.6.1 Orthogonal States

The first lemma of this subsection belongs to finite-dimensional functional analysis, also known as linear algebra.

**Lemma 5.6.2.** Suppose \( \delta > 0 \), \( n \geq 1 \), \( H \) is a Hilbert space with \( \dim(H) \geq 4n \), and \( \xi_i, \eta_i, \) for \( i < n \), are vectors in \( H \) satisfying \( \max_{i,j} |(\xi_i|\xi_j) - (\eta_i|\eta_j)| < \delta \).

1. Then there is a unitary \( U \) such that \( \max_{i < n} \|U\xi_i - \eta_i\| \leq 2\sqrt{(n+1)\delta} \).
2. If in addition \( \xi_j \) is orthogonal to \( \eta_i \) for all \( i \) and \( j \) then there exists a self-adjoint unitary \( U \in \mathcal{B}(H) \) such that \( \max_i \|U\xi_i - \eta_i\| \leq 2\sqrt{(n+1)\delta} \) and \( \max_i \|U\eta_i - \xi_i\| \leq 2\sqrt{(n+1)\delta} \).

**Proof.** (1) Fix orthonormal \( e_j \), for \( j < n \), in \( \text{span}(\{\xi_i, \eta_i : i < n\})^\perp \cap H \). The vectors...
\[ \zeta_i := \xi_i + \sqrt{n} \delta e_i, \]

for \( i < n \), are linearly independent. On \( \text{span}\{\zeta_i : i < n\} \) define a linear operator \( S \) by \( S(\zeta_i) := \eta_i \) for \( i < n \).

**Claim.** We have \( \|S\| \leq 1 \).

**Proof.** Fix scalars \( \alpha_j \), for \( j < n \). Then \( \|\sum_j \alpha_j \zeta_j\|^2 = \sum_{i,j} \alpha_i \bar{\alpha}_j (\xi_i | \xi_j) + \sum_i |\alpha_i|^2 \delta \) and

\[
\|S(\sum_j \alpha_j \zeta_j)\|^2 = \sum_{i,j} \alpha_i \bar{\alpha}_j (\eta_i | \eta_j) \\
\leq \sum_{i,j} |\alpha_i|^2 (\xi_i | \xi_j) + \sum_i |\alpha_i|^2 \delta \\
= \sum_i |\alpha_i|^2 (\xi_i | \xi_i) + (\sum_j |\alpha_j|^2 \delta).
\]

Using the inequality between the quadratic mean and the arithmetic mean, we obtain

\[ \|S(\sum_j \alpha_j \zeta_j)\| \leq \sum_i |\alpha_i|^2 \delta. \]

Since \( \alpha_j \), for \( j < n \), were arbitrary, \( \|S\| \leq 1 \) follows. □

**Claim.** We have \( \max_{i<n} \|\zeta_i - S(\zeta_i)\| \leq \sqrt{(n+1)\delta} \).

**Proof.** By the previous claim, \( 0 \leq |S| \leq 1 \). Since \( (1-t)^2 \leq 1 - t^2 \) for \( 0 \leq t \leq 1 \), by the continuous functional calculus we obtain

\[ \|\zeta_i - S(\zeta_i)\|^2 \leq \|\zeta_i\|^2 - \|S(\zeta_i)\|^2 = (\xi_i | \xi_i) + n\delta - (\eta_i | \eta_i) \leq (n+1)\delta, \]

completing the proof. □

Let \( S = V|S| \) be the polar decomposition of \( S \). Extend \( V \) to an operator \( W \) whose restriction to \( \text{span}\{\zeta_i, e_i : i < n\} \) is an isometry.

We will prove that

\[ \max_{i<n} \|\eta_i - W(\zeta_i)\| \leq 2\sqrt{(n+1)\delta}. \]  

(5.3)

For \( i < n \) we have \( \|W(\zeta_i) - W(\zeta_i)\| = \|\zeta_i - \zeta_i\| = \sqrt{n} \delta \). On the other hand, Claim implies \( \|W(\zeta_i) - \eta_i\| = \|W(\xi_i - |S|\zeta_i)\| = \|\zeta_i - |S|\zeta_i\| \leq \sqrt{(n+1)\delta} \).

Therefore \( \|W(\zeta_i) - \eta_i\| \leq \sqrt{(n+1)\delta} + \sqrt{n} \delta < 2\sqrt{(n+1)\delta} \), as required. Let \( U \) be a unitary dilation of \( W \) (see Exercise 1.11.40). Then \( U(\zeta_i) = W(\zeta_i) \) for all \( i \), and (5.3) implies that \( U \) satisfies the requirements.

(2) Fix orthonormal vectors \( e_j \), for \( j < 2n \), in \( \text{span}\{\xi_i, \eta_i : i < n\} \} \cap H \). Then the spaces \( H_0 := \text{span}\{\xi_i, e_i : i < n\} \) and \( H_1 := \text{span}\{\eta_i, e_{n+i} : i < n\} \) are orthogonal. Define \( S \) and a partial isometry \( W \) extending \( V \) such that \( S = V|S| \) as in (1). Since the range of \( S \) is included in \( \text{span}\{\eta_i : i < n\} \) (see footnote 4) it is orthogonal to \( \text{span}\{\xi_i : i < n\} \) and a partial isometry \( W \) as in the proof of (1) can be defined so that \( W|H_0| \subseteq H_1 \). By applying the construction from (1) with the roles of \( \xi_i \) and \( \eta_i \) interchanged and each \( e_i \) replaced by \( e_{i+n} \), we obtain a partial isometry \( W_1 \) such that \( W_1|H_1| \subseteq H_0 \) and \( \max_i \|W_1(\eta_i) - \zeta_i\| < 2\sqrt{(n+1)\delta} \).

---

4 Pause for a moment to observe that the range of \( S \) is included in \( \text{span}\{\eta_i : i < n\} \)
Lemma 5.6.3. For every \( k < n \) satisfies \( p \) is a partial isometry. It is clearly self-adjoint. Finally, \( P := \text{proj}_{\text{span}(\xi, \eta, \varepsilon, \varepsilon_n, i \leq n)} \) satisfies \( P = U_0^* U_0 = U_0 U_0^* \), and \( U_0 + (1 - P) \) is a self-adjoint unitary as required. \( \square \)

Let \( \bar{\xi} \) be a unit vector in an infinite-dimensional Hilbert space \( H \) and \( b_k \), for \( k < n \), in \( \mathcal{B}(H) \) satisfy \( \sum_{k<n} b_k b_k^* \leq 1 \) and \( \sum_{k<n} b_k b_k^* \bar{\xi} = \bar{\xi} \). Suppose furthermore that \( \eta \) is a unit vector orthogonal to \( \bar{\xi} \), and all unit vectors \( \xi, \eta, \varepsilon, \varepsilon_n \) for \( k, n \) which satisfies

\[
\max_{j, k < n} |(b_k^* \bar{\xi} | b_j^*\bar{\xi}) - (b_k^* \eta | b_j^* \eta)| < \delta. \tag{5.4}
\]

Then there exists a projection \( q \) in \( \mathcal{B}(H) \) such that the unitary

\[ v := \exp(i\pi \sum_{k < n} b_k q b_k^*) \]

satisfies \( \| \eta - v \xi \| < 2\varepsilon \) and \( \| \xi - v \eta \| < 2\varepsilon \).

Proof. Let \( K \) be as in Lemma 1.2.8, so that with \( \varepsilon' := \varepsilon / K \) for all positive contractions \( a \), all unit vectors \( \bar{\xi} \), and all \( r \in [0, 1] \) we have that \( \| a \xi - r \bar{\xi} \| < \varepsilon' \) implies \( \| \exp(i\pi r) \xi - \exp(i\pi a) \bar{\xi} \| < \varepsilon \).

Fix \( \delta \) small enough to have \( 2\sqrt{(n+1)/\delta} < \varepsilon'/n \) (note that \( \delta < \varepsilon'/n \)). Fix \( \xi, b_k \), for \( k < n \), and \( \eta \) as in the statement of the lemma currently being proved. Since the assumption on \( \eta \) implies that \( \text{span} \{ b_j^* \xi : k < n \} \) and \( \text{span} \{ b_k^* \eta : k < n \} \) are orthogonal, by (5.4) and Lemma 5.6.2 we can find a self-adjoint unitary \( w \) such that

\[
\max_{k<n} \| w b_k^* \xi - b_k^* \eta \|, \max_{k<n} \| w b_k^* \eta - b_k^* \xi \| < \varepsilon'/n.
\]

Since \( \| b_k \| \leq 1 \) we have \( \| w b_k^* \xi - b_k^* \eta \| < \varepsilon'/n \) for all \( k \). After summing, we obtain

\[
\| \sum_{k<n} b_k w b_k^* \xi - \sum_{k<n} b_k^* b_k \eta \| < \varepsilon' . \tag{5.5}
\]

A similar argument gives

\[
\| \sum_{k<n} b_k w b_k^* \eta - \sum_{k<n} b_k^* b_k \xi \| < \varepsilon' . \tag{5.6}
\]

As each map \( a \mapsto b_k a b_k^* \) is completely positive, \( \lambda(a) := \sum_{k<n} b_k a b_k^* \) defines a completely positive map from \( A \) into \( A \). Since \( \lambda(1) = \sum_{k<n} b_k b_k^* \leq 1 \) and Stinespring’s theorem implies \( \| \lambda \| = \lambda(1) \), \( \lambda \) is contractive. The formulas (5.5) and (5.6) now read as follows:

\[
\max(\| \lambda(w) \xi - \lambda(1) \xi \|, \| \lambda(w) \xi - \lambda(1) \eta \|) \leq \varepsilon' . \tag{5.7}
\]

By our assumptions, \( \lambda(1) \xi = \bar{\xi} \), and \( (\lambda(1) \eta \xi = \sum_k \| b_k^* \eta \|^2 \approx_{n, \delta} \sum_k \| b_k^* \xi \|^2 = 1 \)

implies

\[
\| \eta - \lambda(1) \eta \| < \varepsilon'.
\]

Since \( w \) is a self-adjoint unitary, \( q := (1 - w)/2 \) is a projection. By (5.7) and linearity
\[ \lambda(q)(\eta + \xi) \approx_{\epsilon'} 0 \quad \text{and} \quad \lambda(q)(\eta - \xi) \approx_{\epsilon'} \eta - \xi. \]

Then \( v := \exp(i\pi aq) \) is a unitary and by the choice of \( \epsilon' = \epsilon/K \) with \( K \) as in the conclusion of Lemma 1.2.8 we have \( v(\eta + \xi) \approx_{\epsilon} \eta + \xi \) and \( v(\eta - \xi) \approx_{\epsilon} \xi - \eta \).

We therefore have \( \|\eta - v\xi\| < 2\epsilon \) and \( \|\xi - v\eta\| < 2\epsilon \), completing the proof. \( \square \)

### 5.6.2 Small Distance Adjustments

A compact notation will come handy.

**Definition 5.6.4.** Given a C*-algebra \( A \) and \( m \geq 1 \), we use the following notation.

1. \( P_m(A) \) is the set of \( m \)-tuples of inequivalent, GNS-faithful, pure states on \( A \). The elements of \( P_m(A) \) will be denoted \( \bar{\phi} = (\phi_j : j < m) \), and similarly for \( \bar{\psi}, \bar{\theta} \), and so on.

2. On \( P_m(A) \) consider the following relations.
   a. \( \bar{\phi} \sim \bar{\psi} \) abbreviates \( \phi_j \sim \psi_j \) for all \( j < m \).
   b. \( \|\bar{\phi} - \bar{\psi}\| := \max_{j < m} \|\phi_j - \psi_j\| \).
   c. \( \bar{\phi} \approx_{F, \epsilon} \bar{\psi} \) abbreviates \( \max_{a \in F, j < m} |\phi_j(a) - \psi_j(a)| < \epsilon \) for \( F \subseteq A \) and \( \epsilon > 0 \).

3. We’ll be using similar self-explanatory abbreviations, such as \( \bar{\phi} \circ \text{Ad} u = \bar{\psi} \).

The following lemma will be used tacitly.

**Lemma 5.6.5.** If \( A \) is a C*-algebra, \( m \geq 1 \), \( \bar{\phi} \in P_m(A) \), and \( \bar{\psi} \in P(A)^m \) satisfies \( \|\bar{\phi} - \bar{\psi}\| < 2 \), then \( \bar{\psi} \in P_m(A) \).

**Proof.** Proposition 3.8.1 implies \( \phi_i \sim \psi_i \) for \( i < m \). If \( i \neq j \) then \( \phi_i \not\sim \psi_j \) hence by transitivity \( \psi_i \not\sim \psi_j \). \( \square \)

The proof of Theorem 5.6.1 commences with a refinement of Proposition 3.8.1.

**Lemma 5.6.6.** For every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for all \( m \geq 1 \) the following holds. If \( A \) is a simple C*-algebra, \( \bar{\phi} \) and \( \bar{\psi} \) are in \( P_m(A) \), and \( \|\bar{\psi} - \bar{\phi}\| < \delta^2/2 \), then there exists a continuous path of unitaries \( u_t \), for \( 0 \leq t \leq 1 \) in \( A \) such that

1. \( u_0 = 1 \).
2. \( \bar{\phi} \circ \text{Ad} u_t = \bar{\psi} \) for all \( i < m \), and
3. \( \|b - \text{Ad} u_t(b)\| < \epsilon \|b\| \) for all \( b \in A \).

**Proof.** Let \( \delta < \pi \) be small enough so that \( |t| < \pi \) and \( |1 - \exp(it)| \) implies \( |t| < \epsilon/e \) (with \( e \) being the basis for the natural logarithm). For \( i < m \) let \( (\pi_i, H_i, \xi_i) \) be the GNS triplet associated to \( \phi_i \). Since \( \psi_i \) and \( \phi_i \) are pure and \( \|\psi_i - \phi_i\| < 2 \), Proposition 3.8.1 implies they are unitarily equivalent.

Fix \( i < m \) and let \( \zeta_i \) be the unit vector in \( H_i \) that satisfies \( \omega_{\zeta_i} \circ \pi_i = \psi_i \) and

\[ \|\zeta_i - \xi_i\| = \inf_{\xi \in \xi_i} \|\zeta_i - \zeta_i\|, \]

so that \( \|\zeta_i - \xi_i\| \) is the minimal possible.
Claim. We have \( \|\varphi_i - \psi_i\| = \|\omega_{\varphi_i} - \omega_{\psi_i}\| \).

Proof. Since \( A \) is simple, \( \pi_i \) is faithful and we have \( \|\varphi_i - \psi_i\| \leq \|\omega_{\varphi_i} - \omega_{\psi_i}\| \). To prove the converse equality, suppose \( r \geq 0 \) is such that some \( c \in \mathcal{B}(H) \) satisfies

\[
|\omega_{\varphi_i}(c) - \omega_{\psi_i}(c)| > r\|c\|.
\]

Theorem 3.4.2 applied to \( c \) and \( F := \{\xi, \zeta, c\xi, c\zeta\} \) implies that there is \( c_1 \in A \) such that \( |\omega_{\varphi_i}(\pi_i(c_1)) - \omega_{\psi_i}(\pi_i(c_1))| > r\|\pi_i(c_1)\| \), and therefore \( \|\varphi_i - \psi_i\| > r \). Since \( r \geq 0 \) was arbitrary, we conclude \( \|\varphi_i - \psi_i\| = \|\omega_{\varphi_i} - \omega_{\psi_i}\| \). \( \square \)

With \( p := \text{proj}_{C_{\xi}\zeta} \) we have \( \omega_{\varphi_i}(p) = 1 \), hence

\[
\|\varphi_i - \psi_i\| = \|\omega_{\varphi_i} - \omega_{\psi_i}\| = 1 - \|\omega_{\varphi_i}(p)\| = 1 - \|p\xi_i\|^2,
\]

and therefore \( \|p\xi_i\|^2 \geq 1 - \delta^2/2 \). Since \( \text{dist}(\xi_i, C\zeta) = \|\zeta_i - p\xi_i\| \) and (5.8), we have \( p\xi_i = r\xi_i \) for some \( r \in [(1 - \delta^2/2)^1/2, 1] \). From the right triangles whose vertices are \( 0 \) and the tips of \( \xi_i, \zeta_i \), and \( p\xi_i \), we have

\[
\|\xi_i - \zeta_i\|^2 = (1 - r)^2 + 1 - r^2 = 2 - 2r \leq \delta^2.
\]

Define a unitary \( w_0 \) in \( \mathcal{B}(H) \) that has each \( H_i \) as an invariant subspace as follows. For every \( i < m \) the restriction of \( w_0 \) to \( \overline{\text{span}}(\xi_i, \zeta_i) \) is the rotation sending \( \xi_i \) to \( \zeta_i \), for every \( i < m \), and its restriction to \( \overline{\text{span}}(\{\xi_i, \zeta_i : i < m\})^+ \cap H_i \) is the identity. Then \( \|1 - w_0\| = \max_{1 \leq i \leq m}\|\xi_i - \zeta_i\| \leq \delta < \pi \); hence \( -1 \notin \text{sp}(u_0) \) and by the continuous functional calculus and \( \delta < \varepsilon/e \) there exists \( a_0 \in \mathcal{B}(H)_{sa} \) such that \( \|a_0\| < \varepsilon/e \) and \( w_0 = \exp(ia_0) \).

Since all \( \pi_i \) are inequivalent, Theorem 3.5.4 implies that there exists \( a \in A_{sa} \) such that \( |a| < \varepsilon/e \) and \( w := \exp(ia) \) satisfies \( \pi_i(w)\xi_i = \zeta_i \) for all \( i \).

The unitaries \( u_t := \exp(it\alpha) \), for \( 0 \leq t \leq 1 \), satisfy \( \varphi_i \circ \text{Ad}\, u_t = \psi_t \) for all \( i \) and \( u_0 = 1 \). Also, \( \|b - (\text{Ad}\, u_t)b\| < \|b\| (\text{exp}(\delta) - 1) \| < \varepsilon \) for all \( t \) and \( b \in A \) (Exercise 3.10.22), as required. \( \square \)

### 5.6.3 The Fundamental Lemma for the Proof of Theorem 5.6.1

We will now state and prove the main technical result used in the proof of Theorem 5.6.1. It is the improvement to Proposition 5.2.9 in which the path of unitaries connecting tuples of inequivalent pure states is chosen in the ‘approximate commutant’ of elements of a prescribed finite subset \( F \) of \( A \).

In the statement of the following Lemma we use the notation and conventions introduced in Definition 5.6.4.

**Lemma 5.6.7.** Assume \( A \) is an infinite-dimensional and simple \( C^* \)-algebra. If \( m \geq 1 \), \( \Phi \in P_m(A) \), \( F \subseteq A \), and \( \varepsilon > 0 \), then there exist \( G \subseteq A \) and \( \delta' > 0 \) such that whenever
\( \psi \in P_m(A), \psi \sim \phi, \) and \( \psi \approx_{G, \delta'} \phi, \) then there exists a continuous path of unitaries \( u_t, \) for \( 0 \leq t \leq 1, \) in \( \tilde{A} \) satisfying the following.

1. \( u_0 = 1, \)
2. \( \bar{\phi} \circ \text{Ad} u_1 = \psi, \)
3. \( \max_{b \in F, 0 \leq t \leq 1} \|b - \text{Ad} u_t(b)\| < 24\varepsilon. \)

**Proof.** For the reader's convenience we explicit the quantifiers in the statement of Lemma 5.6.7. Given \( A, m, \) and \( \bar{\phi} \in P_m(A) \) it asserts that

\[
(\forall F \in A)(\forall \varepsilon > 0)(\exists G \in A)(\exists \delta' > 0)
(\exists \bar{\psi} \in P_m(A))(\psi \sim \bar{\phi} \text{ and } \psi \approx_{G, \delta'} \bar{\phi}) \text{ implies that }
\]

some continuous path of unitaries \( u_t, \) for \( 0 \leq t \leq 1, \) in \( \tilde{A} \) satisfies conditions (1)–(3).

Fix \( F \) and \( \varepsilon. \) Lemma 5.6.6 implies that there is \( \delta > 0 \) such that if \( \bar{\phi} \) and \( \psi \) are in \( P_m(A) \) and \( \| \bar{\phi} - \psi \| < \delta \) then there exists a continuous path of unitaries \( u_t, \) for \( 0 \leq t \leq 1 \) such that \( u_0 = 1, \bar{\phi} = \psi \circ \text{Ad} u_1, \) and \( \sup_{a \in A, 0 \leq t \leq 1} \| [a, u_t] \| < \varepsilon \|a\| \). Since \( \varepsilon > 0 \) can be arbitrarily small, it suffices to prove the existence of \( G \) and \( \delta' > 0 \) such that for all \( \bar{\psi}, \) if \( \psi \sim \bar{\phi} \) and \( \psi \approx_{G, \delta'} \bar{\phi} \) then there is a continuous path \( (u_t) \) of unitaries satisfying (1), (3), and the weakening of (2), requiring \( \| \bar{\phi} \circ \text{Ad} u_1 - \psi \| < \varepsilon \).

Let \( (\pi_i, H_i, \xi_i) \) be the GNS triplet associated with \( \varphi_i, \) for \( i < m. \)

With \( H := \bigoplus_{i < m} H_i \) and \( \pi := \bigoplus_{i < m} \pi_i, \) in \( \pi(H) \) let \( p_i \) be the projection to \( H_i \) and let \( p \) be the projection to \( \bigoplus_{i < m} \xi_i : i < m. \) Since the states \( \varphi_i \) are pure and inequivalent, Theorem 5.1.1 implies that \( \pi \) has an approximate diagonal. Fix \( n \geq 1 \) and \( b_i, \) for \( i < n, \) in \( A \) satisfying

4. \( \| \sum_{i=1}^{n} b_i b_i^* \| \leq 1, \)
5. \( p(1 - \sum_{i=1}^{n} \pi(b_i b_i^*)) = 0, \) and
6. \( \max_{a \in F} \sup_{c \in A, |c| \leq 1} \| a \sum_{i=1}^{n} b_i c b_i^* - \sum_{i=1}^{n} b_i c b_i^* a \| < \varepsilon. \)

We will prove that \( \delta' := \delta/2, \) with \( \delta \) as provided by Lemma 5.6.3, and

\[
G := \{ b_j b_k^*: j, k < n \}
\]

are as required.

Suppose \( \psi \in P_m(A) \) is such that \( \psi \sim \phi \) and \( \psi \approx_{G, \delta/2} \phi. \) Fix \( i < m. \) Since \( \psi_i \sim \varphi_i, \) there is a unitary \( w_i \) such that \( \psi_i = \varphi_i \circ \text{Ad} w_i. \) The vector \( \eta_i := w_i \xi_i \) satisfies

\[
(\pi(a) \eta_i | \eta_i) = \psi_i(a)
\]

for all \( a \in A. \) By Lemma 5.2.5 there is a unit vector

\[
\zeta_i \in \text{span} \{ \pi(b_j^* \xi_i, \pi(b_k^*) \eta_i : k < n, i < m \} \cap H_i
\]

such that \( \vartheta_i := \omega_{\zeta_i} \circ \pi_i \) satisfies \( \vartheta_i \approx_{G, \delta/2} \psi_i. \)

Therefore \( \max_{j, k < n} \| \varphi_i (b_j b_k^*) - \vartheta_i (b_j b_k^*) \| < \delta, \) and this condition reduces to

\[
\max_{j, k < n} |(\pi(b_j)^* \zeta_i | \pi(b_k)^* \zeta_i) - (\pi(b_j)^* \zeta_i | \pi(b_k)^* \zeta_i) | < \delta.
\]
By Lemma 5.6.3 with $b_k' := \pi(b_k)$, for $k < n$, obtain a projection $q_i \in \mathcal{B}(H_i)$ such that the unitary
\[
u_i := \exp(i\pi\sum_{k<n} \pi(b_k)q_i\pi(b_k'))
\]
satisfies $\|\zeta_i - \nu_i\zeta_i\| < \varepsilon$ and $\|\zeta_i - u_i\zeta_i\| < \varepsilon$. As we are slogging through this proof, now that $q_i$ and $u_i$ were chosen for all $i$, let $p'$ denote the projection to
\[
\text{span}\{\pi(b_k')\zeta_i, \pi(b_k)\zeta_i : i < m, k < n\}.
\]
Since $p'$ has finite rank and $\pi_i$ are pairwise inequivalent, by Theorem 3.5.4 there exists $h \in A$, such that $p'\pi(b_k) = p'\sum_i q_i p'$ and $\|h\| = 1$. Then (6) implies that $h' := \sum_k b_k hb_k^*$ satisfies $\max_{a \in F} \|\pi(a)\| < 24\varepsilon$ by Exercise 3.10.22. Also, $v_i = \exp(i\pi h')$ satisfies $\|\zeta_i - \pi(v_i)\zeta_i\| < \varepsilon$ for all $i$.

By an analogous application of Lemma 5.6.3 we can find a path of unitaries $w_i$, for $0 \leq t \leq 1$, so that $\|\eta_i - \pi(w_i)\zeta_i\| < \varepsilon$ for all $i$ and $\|a, w_i\| < 24\varepsilon$ for all $a \in F$ and all $i \in [0, 1]$. By merging the two paths of unitaries we obtain $u_i$, for $0 \leq t \leq 1$, such that $\|\eta_i - \pi(u_i)\zeta_i\| < \varepsilon$ for all $i$ and $\|a, u_i\| < 24\varepsilon$ for all $a \in F$ and all $t$. This is equivalent to the weakening of the statement of the proposition being proved in which (2) is replaced by $\|\Psi - \Phi \circ \Ad u_i\| < \varepsilon$. As pointed out at the beginning of this proof, by applying Lemma 5.6.6 twice one obtains a continuous path of unitaries connecting $\Phi$ with $\bar{\theta}$ and another connecting $\bar{\theta}$ and $\bar{\Psi}$. These two paths can be merged without affecting the commutation requirements; this completes the proof. \qed

### 5.6.4 The Proof of Theorem 5.6.1

We will prove only the case when $n = k_0$. This suffices since it subsumes the case of finite tuples of pure states. Suppose $\varphi_i$, for $i \in \mathbb{N}$, are inequivalent pure states on a simple and separable $\mathcal{C}^*$-algebra $A$ and $\psi_i$, for $i \in \mathbb{N}$, are inequivalent pure states on $A$. We will define the following objects.

1. A continuous path of unitaries $v_i$, for $0 \leq t < \infty$, such that $\Phi := \lim_{t \to \infty} \Ad v_t$
   (in the point-norm topology) is an asymptotically inner automorphism of $A$.
2. A continuous path of unitaries $w_i$, for $0 \leq t < \infty$, such that $\Psi := \lim_{t \to \infty} \Ad w_t$
   (in the point-norm topology) is an asymptotically inner automorphism of $A$.
3. Pure states $\psi'_i \sim \psi_i$ such that $\varphi_i \circ \Phi = \psi'_i \circ \Psi$ for all $i \in \mathbb{N}$.

Granted that such $\Phi$ and $\Psi$ exist, $\Psi \circ \Phi^{-1}$ is an asymptotically inner automorphism $\lim_{t \to \infty} \Ad(v_t w_t^*)$ of $A$ satisfying $\bar{\phi} = \bar{\psi}' \circ \Phi^{-1}$.

Let $a_n$, for $n \in \mathbb{N}$, be an enumeration of a dense subset of $A$. We will assure the following conditions for all $i \leq n$, all $j \leq n$ in $\mathbb{N}$, and all $t \geq 0$: 
4. \[ \|v_n v_n^* a_i\| < 2^{-n} \text{ and } \|v_n v_n^* v_{n+i} a_i\| < 2^{-n} \]
5. \[ \|w_n w_n^* a_i\| < 2^{-n} \text{ and } \|w_n w_n^* v_{n+i} a_i\| < 2^{-n} \]
6. \[ \phi(\text{Ad}v_{n+i}(a_i)) \approx_{\{a_i: i < n\}, 2} \psi(\text{Ad}v_{n+i}(a_i)) \]

Once proved, conditions (4) and (5) will imply that both \( \Phi := \lim_{t \to \infty} \text{Ad}v_t \) and \( \Psi := \lim_{t \to \infty} \text{Ad}w_t \) are automorphisms of A (see Lemma 2.6.4). At the same time, (6) will imply that \( \varphi \circ \Phi = \psi' \circ \Psi \), as required.

The recursive construction is propelled by Lemma 5.6.7. By Proposition 5.2.9, for any sequence \( \theta_i \), for \( i \in \mathbb{N} \), of inequivalent pure states the orbit

\[ \{ (\theta_i \circ \text{Ad}u : i \in \mathbb{N}) : u \in U_0(\tilde{A}) \} \]

is weak*-dense in \( \mathcal{P}(A)^N \). Together with Lemma 5.6.7, this implies the following.

7. For all \( m \) and every \( \tilde{\theta} \in \mathcal{P}_m(A) \),

\[ (\forall F \in A)(\forall \epsilon > 0)(\exists G \in A)(\exists \delta' > 0)(\forall \tilde{u} \in \mathcal{P}_m(A))(\tilde{u} \sim \tilde{\theta} \text{ and } \tilde{u} \approx_{G, \delta'} \tilde{\theta}) \]

implies that for all \( K \in A \) and all \( \epsilon' > 0 \) there exists a continuous path of unitaries \( u_t \), for \( 0 \leq t \leq 1 \), such that

a. \( u_0 = 1 \)

b. \( \tilde{\theta} \circ \text{Ad}u_1 \approx_{K, \epsilon'} \tilde{u} \), and

c. \( \|b - (\text{Ad}u_t)b\| < \epsilon \) for all \( b \in F \).

All is in place for our construction. Apply (7) to \( F_0 := \{ a_0 \} \) and \( \epsilon_0 = 1 \) to find \( G_0 \in A \) and \( \delta_0 > 0 \) so that for all \( \psi_0 \in \mathcal{P}(A) \) satisfying \( \psi_0 \approx \varphi_0 \) and \( \varphi_0 \approx_{G_0, \delta_0} \psi_0 \) and for all \( K \in A \) and \( \epsilon' > 0 \) there exists a continuous path of unitaries \( u_t \), for \( 0 \leq t \leq 1 \), satisfying \( u_0 = 1 \), \( \psi_0 \circ \text{Ad}u_1 \approx_{K, \epsilon'} \psi_0 \), and \( \|b - (\text{Ad}u_t)b\| < 1/2 \) for all \( b \in F_1 \). Choose a continuous path of unitaries \( v_t \), for \( 0 \leq t \leq 1 \), such that \( v_0 = 1 \), \( \|b - (\text{Ad}v_t)b\| < 1 \) for all \( b \in F_0 \), and

\[ \varphi_0 \circ \text{Ad}v_1 \approx_{G_0, \epsilon_0} \psi_0 \]

Apply (7) again to \( F_1 := \{ a_0, a_1 \} \) and \( \epsilon_1 = 1/2 \) to find \( G_1 \in A \) and \( \delta_1 > 0 \) so that for all \( \psi_0 \in \mathcal{P}(A) \) satisfying \( \psi_0 \approx \varphi_0 \) and \( \varphi_0 \approx_{G_1, \delta_1} \psi_0 \) and for all \( K \in A \) and \( \epsilon' > 0 \) there exists a continuous path of unitaries \( u_t \), for \( 0 \leq t \leq 1 \), satisfying \( u_0 = 1 \), \( \psi_0 \circ \text{Ad}u_1 \approx_{K, \epsilon'} \psi_0 \), and \( \|b - (\text{Ad}u_t)b\| < 1/2 \) for all \( b \in F_1 \). Choose a continuous path of unitaries \( v_t \), for \( 0 \leq t \leq 1 \), such that \( v_0 = 1 \), \( \|b - (\text{Ad}v_t)b\| < 1 \) for all \( b \in F_0 \), and

\[ \varphi_0 \circ \text{Ad}v_1 \approx_{G_1, \epsilon_1} \psi_0 \]

Apply (7) again to \( F_2 := F_0 \cup \{ (\text{Ad}v_t)b : b \in F_0 \} \) and \( \epsilon = 1/4 \) to find \( G_2 \in A \) and \( \delta_2 > 0 \) so that for all \( \psi_0 \in \mathcal{P}(A) \) satisfying \( \psi_0 \approx \varphi_0 \) and \( \varphi_0 \approx_{G_2, \delta_2} \psi_0 \) and for all \( K \in A \) and \( \epsilon' > 0 \) there exists a continuous path of unitaries \( u_t \), for \( 0 \leq t \leq 1 \), satisfying \( u_0 = 1 \), \( (\psi_0 \circ \text{Ad}u_1, \varphi_1 \circ \text{Ad}v_1) \approx_{K, \epsilon'} \tilde{u} \), and \( \|b - (\text{Ad}u_t)b\| < 1/4 \)
for all \( b \in F_2 \). Choose a continuous path of unitaries \( w_t \), for \( 0 \leq t \leq 1 \), such that 
\[ w_0 = 1, \| b - (\text{Ad} w_t) b \| < 1/2 \text{ for all } b \in F_1, \]
and
\[ \varphi_0 \circ \text{Ad} v_1 \approx_{G_2, [a_0, a_1, a_2], \delta_2} \psi'_0 \circ \text{Ad} w_1. \]
Finally, fix \( \psi'_1 \sim \psi_1 \) such that
\[ \varphi_1 \circ \text{Ad} v_1 \approx_{G_2, [a_0, a_1, a_2], \delta_2} \psi'_1 \circ \text{Ad} w_1. \]
We will start mending paths of unitaries in the forthcoming step of the construction.

Apply (7) again to \( F_3 := F_1 \cup \{ \text{Ad} w_1(b) : b \in F_1 \} \) and \( \varepsilon = 2^{-3} \) to find \( G_3 \subseteq A \) and \( \delta_3 > 0 \) so that for all \( \hat{u} \in P_2(A) \) satisfying \( \hat{u} \sim (\psi_0, \psi'_1) \) and
\[ \hat{u} \approx_{G_3, \delta_3} (\psi_0 \circ \text{Ad} w_1, \psi'_1 \circ \text{Ad} w_1) \]
and for all \( K \subseteq A \) and \( \varepsilon' > 0 \) there exists a continuous path of unitaries satisfying the relevant instance of (7a)—(7c). Choose a continuous path of unitaries \( u_{t, 2} \), for \( 0 \leq t \leq 1 \), such that \( u_{0, 2} = 1, \| b - \text{Ad} u_{t, 2} b \| < 1/4 \) for all \( b \in F_2 \),
\[ \varphi_0 \circ \text{Ad} v_1 \circ \text{Ad} u_{t, 2} \approx_{G_3, [a_0, a_1, a_2, a_3], \delta_3} \psi_0 \circ \text{Ad} w_1, \]
and
\[ \varphi_1 \circ \text{Ad} v_1 \circ \text{Ad} u_{t, 2} \approx_{G_3, [a_0, a_1, a_2, a_3], \delta_3} \psi'_1 \circ \text{Ad} w_1. \]
Let \( v_{t+1} := v_1 u_{t, 2} \) for \( 0 \leq t \leq 1 \).

The remaining \( k_0 \) steps in the construction contain no new ideas. In the \( n \)th step we find continuous paths \( u_{t, n} = v_{n} v_{n+t} \) and \( u'_{t, n} = w^* nw_{n+t} \) such that \( \| [u_{t,n}, a_i] \| < 2^{-n} \) and \( \| [u'_{t,n}, a_i] \| < 2^{-n} \) for all \( 0 \leq t \leq 1 \) and \( i \leq n \). In the \( n \)th step we also use Proposition 5.2.8 to choose \( \psi'_n \sim \psi_n \) in a sufficiently small weak*-open neighbourhood of \( \varphi_n \circ \text{Ad} v_n \circ \text{Ad} w_n^* \). Lemma 2.6.4 now implies that \( \Phi \) and \( \Psi \) are asymptotically inner automorphisms. These continuous paths were chosen so that
\[ \varphi_i \circ \text{Ad} v_{t} \approx_{[a_i, i \leq n], 2^{-n}} \psi'_t \circ \text{Ad} w_t \]
for all \( 0 < i < n \leq t \). Therefore \( \varphi_i \circ \Phi = \psi'_i \circ \Psi \) for all \( i \). This completes the proof of Theorem 5.6.1.

### 5.7 Exercises

**Exercise 5.7.1.** Prove that \( C \) is the only \( C^* \)-algebra with a faithful pure state.

**Definition 5.7.2.** A \( C^* \)-algebra is **primitive** if it has a faithful irreducible representation.

Equivalently, \( A \) is primitive if and only if it has a GNS-faithful pure state. Every simple \( C^* \)-algebra is primitive, and \( B(H) \) is an example of a primitive, but not simple, \( C^* \)-algebra.
Exercise 5.7.3. Prove that the following are equivalent for a C*-algebra $A$.

1. $A$ is primitive.
2. For all $a$ and $b$ in $A \setminus \{0\}$ there exists $c \in A$ such that $acb \neq 0$.

Exercise 5.7.4. Suppose $A$ is a C*-algebra and $\varphi$ is a state on $A$.

1. Prove that the following are equivalent.
   a. There exists a family $\mathcal{F} \subseteq A_{+1}$ satisfying the following. For every $\mathcal{F} \subseteq A$ and every $\varepsilon > 0$ there exists $a \in \mathcal{F}$ such that $\max_{b \in \mathcal{F}} \|aba - \varphi(b)a^2\| < \varepsilon$.
   b. The state $\varphi$ is a weak*-limit of pure states.
2. Conclude that every state on a primitive, infinite-dimensional C*-algebra can be excised.
3. Prove that $\mathcal{F}$ as in (1a) can be chosen as a subset of $\{a \in A_{+1} : \varphi(a) = 1\}$ if and only if $\varphi$ is pure.

Exercise 5.7.5. Suppose $A$ is a separable C*-algebra and $\varphi$ is a pure state on $A$. Prove that there exists $a \in A_{+1}$ such that $\varphi(a) = 1$ and $\psi(a) = 1$ implies $\psi = \varphi$ for any state $\psi$ of $A$.

Exercise 5.7.6. Prove that every quantum filter on $\mathcal{K}(H)$ is of the form

$$\mathcal{F}^p := \{a \in A_{+1} : p \leq a\}$$

for some projection $p$. Such quantum filter is said to be principal.

Exercise 5.7.7. A state $\varphi$ of $\mathcal{B}(H)$ is singular if it vanishes on $\mathcal{K}(H)$. Prove that a nonsingular state on $\mathcal{B}(H)$ is pure if and only if it is a vector state.

Exercise 5.7.8. Suppose $\mathcal{F}$ is a maximal quantum filter on C*-algebra $A$ and $\varphi$ is a pure state such that $\mathcal{F} = \mathcal{F}_\varphi$. Prove that every $b \in A_+$ satisfies

$$\varphi(b) = \inf \{\|a_1 \cdots a_n b a_n \cdots a_1\| : n \in \mathbb{N}, a_1, \ldots, a_n \in \mathcal{F}\}.$$ 

The following exercise is a companion to Proposition 5.3.4.

Exercise 5.7.9. Let $A := C([0,1], M_2(\mathbb{C}))$.

1. Prove that $\varphi(f) := a$ for $f \in A$ such that if $f(1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines a pure state on $A$.
2. Prove that $\mathcal{F}_\varphi$ is not downwards directed.

Hint: For the second part you need Exercise 1.11.43.

Exercise 5.7.10 culminates in a proof that the spatial tensor product of C*-algebras does not depend on the choice of representations and a proof of Takesaki's theorem of the minimality of the spatial tensor product.
Exercise 5.7.10. Suppose that $A$ and $B$ are $C^*$-algebras, $\varphi \in S(A)$, $\psi \in S(B)$, and $A \otimes_{\alpha} B$ is a tensor product of $A$ and $B$.

1. Use excision to prove that if $\varphi$ and $\psi$ are pure then there is a unique state $\varphi \otimes \psi$ on $A \otimes_{\alpha} B$ such that $(\varphi \otimes \psi)(a \otimes b) = \varphi(a)\psi(b)$ for all $a \in A$ and $b \in B$.\(^5\)
2. Extend the conclusion of (1) to all states $\varphi$ and $\psi$.
3. Suppose $\varphi$ and $\psi$ are faithful and let $A \otimes_{\varphi, \psi} B$ be the spatial tensor product computed using the GNS representations associated with $\varphi$ and $\psi$. Prove that the identity map on $A \otimes B$ extends to a continuous surjection $A \otimes_{\alpha} B \to A \otimes_{\varphi, \psi} B$.
4. Conclude that if both $A$ and $B$ have GNS-faithful states then $A \otimes B$ is the minimal tensor product of $A$ and $B$.\(^6\)
5. Prove that $A \otimes B$ is the minimal tensor product of $A$ and $B$.\(^7\)

Exercise 5.7.11. Suppose that $A$, $B$, $\varphi$, and $\psi$ as as in Exercise 5.7.10.

1. Prove that the state $\varphi \otimes \psi$ is pure if and only if both $\varphi$ and $\psi$ are pure.
2. Prove that (1) implies that if a representation $\pi: A \to \mathcal{B}(H)$ is irreducible, then for every $n \geq 2$ the map $\pi \otimes \id_n: M_n(A) \to \mathcal{B}(\mathbb{C}^n \otimes H)$ is irreducible.

Exercise 5.7.12. Suppose $A$ and $B$ are $C^*$-algebras.

1. Prove that for every nonzero $a \in A \otimes B$ there are pure states $\varphi$ on $A$ and $\psi$ on $B$ such that $(\varphi \otimes \psi)(a) \neq 0$.
2. Prove that if for every nonzero $a \in A \otimes_{\alpha} B$ there are pure states $\varphi$ on $A$ and $\psi$ on $B$ such that $(\varphi \otimes \psi)(a) \neq 0$ then $A \otimes_{\alpha} B$ is the spatial tensor product.

Exercise 5.7.13. Find a $C^*$-algebra $A$ and a face of $S(A)$ that is not of the form $\mathcal{F}_{\varphi}(A)$ for any quantum filter $\mathcal{F}$.

Exercise 5.7.14. Prove that $\mathcal{F} \subseteq A$ is a quantum filter if and only if there exists a projection $p_{\mathcal{F}} \in A^{**}$ such that (after identifying $A$ with a $C^*$-subalgebra of $A^{**}$) we have $\mathcal{F} = \{a \in A_{+1} : p_{\mathcal{F}} \leq a\}$. Prove that $\mathcal{F}$ is maximal if and only if $p_{\mathcal{F}}$ is a scalar projection in $A^{**}$.

Exercise 5.7.15. Suppose that $A$ is a $C^*$-algebra, $B$ is a $C^*$-subalgebra of $A$, a state $\varphi$ of $B$ has a unique extension to a state $\psi$ of $A$, and $\psi$ is pure. Prove that $\varphi$ is pure.

Exercise 5.7.16. 1. Give an example of a $C^*$-algebra $A$ and its proper subalgebra $B$ such that every state on $B$ has a unique extension to a state on $A$.

   Hint: This is a trick question. Read (3) below.

2. Give an example of a $C^*$-algebra $A$ and a proper unital subalgebra $B$ of $A$ such that every pure state on $B$ has a unique extension to a pure state on $A$.

3. Suppose that $A$ is a $C^*$-algebra, $B$ is a $C^*$-subalgebra of $A$, and every element of $\text{conv}(S(B) \cup \{0\})^8$ has a unique extension to an element of $\text{conv}(S(A) \cup \{0\})$. Prove that $A = B$.

---

\(^5\) Used in the proof of Theorem 15.4.5.

\(^6\) Take a look at Exercise 4.5.3.

\(^7\) This can be done by using a rudimentary variant of the methods of Part II.

\(^8\) The elements of $\text{conv}(S(A) \cup \{0\})$ are called quasi-states and the set of quasi-states on $A$ was denoted $Q$ in [194].
Exercise 5.7.17. Suppose that $A$ is a $C^*$-algebra, $B$ is a $C^*$-subalgebra of $A$, and $\varphi$ and $\psi$ are inequivalent pure states on $B$ with equivalent pure state extensions to $A$. Prove that every convex combination of $\varphi$ and $\psi$ has a pure state extension to $A$.

Exercise 5.7.18. Assuming $B$ is a $C^*$-subalgebra of an abelian $C^*$-algebra $A$, prove the following.

1. Every pure state on $A$ restricts to a pure state on $B$ if $A$ is abelian.
2. If every pure state on $B$ has a unique pure state extension to $A$, then $A = B$.
   *Hint:* In the unital case use Exercise 1.11.8.
3. Give an example showing that the conclusion of (1) can fail if $A$ is not assumed to be abelian.

Exercise 5.7.19. Consider the following properties of a state $\varphi$ on a $C^*$-algebra $A$.

1. There exists a net $(a_\lambda)$ in $A_{+1}$ such that for every $b \in A$ we have
   \[ \lim_{\lambda} \| a_\lambda ba_\lambda - \varphi(b)a_\lambda^2 \| = 0. \]

2. There exists a net $(a_\lambda)$ as in (1) that in addition generates a quantum filter.

Prove that, for every state $\varphi$, (1) is equivalent to $\varphi$ being a weak $^*$-limit of pure states, and (2) is equivalent to $\varphi$ being pure.

Exercise 5.7.20. With the equivalence relation $E_0$ as defined in the paragraph preceding Theorem 5.5.4, find a continuous injection $f: \{0, 1\}^N \to \{0, 1\}^N$ such that $f(x) E_0 f(y)$ if and only if $x = y$.

Exercise 5.7.21. Suppose all pure states on a $C^*$-algebra $A$ are equivalent and let $\pi: A \to \mathcal{B}(H)$ be an irreducible representation. Prove that for every normal $a \in A$ and every $\lambda \in \text{sp}(a)$ there exists a $\lambda$-eigenvector for $\pi(a)$ in $H$. (That is, the spectrum of every normal $a \in A$ is equal to the pure point spectrum of $\pi(a)$.)

Exercise 5.7.22. Suppose that all pure states on a $C^*$-algebra $A$ are spatially equivalent. Let $\pi: A \to \mathcal{B}(H)$ be an irreducible representation and let $a \in A$ be a normal element. Prove that $H$ has an orthonormal basis consisting of eigenvectors for $\pi(a)$.

Exercise 5.7.23. Use the idea of the proof of Theorem 5.5.4 to find a continuous map $\psi: \{0, 1\}^N \to P(M_2^-)$ such that $\psi(f)$ is unitarily equivalent to $\psi(g)$ if and only if $\lim_n \| f(n) - g(n) \| = 0$.

If familiar with Hjorth’s theory of turbulence ([130]) conclude that the unitary equivalence of pure states on $M_2^-$ is not classifiable by countable structures.

The following exercise requires acquaintance with models of ZFC (§A).

Exercise 5.7.24. Suppose $V \subseteq W$ are transitive models of ZFC such that $W$ contains a subset of $\mathbb{N}$ that does not belong to $V$.\footnote{The standard phrase describing this situation is ‘$W$ contains a new real’; see Remark B.1.6.} If $A$ is a non-type I $C^*$-algebra in $V$, prove that its completion in $W$ is a $C^*$-algebra that has a pure state not equivalent to any pure state on $A$ in $V$.\footnote{If familiar with Hjorth’s theory of turbulence ([130]) conclude that the unitary equivalence of pure states on $M_2^-$ is not classifiable by countable structures.}
Exercise 5.7.25. Suppose $A$ is a separable and simple C*-algebra, $m \geq 1$, and $\phi$ and $\psi$ are $m$-tuples of inequivalent pure states on $A$. Prove that there exists an asymptotically inner automorphism $\Phi$ of $A$ such that $\phi_i = \psi_i \circ \Phi$ for all $i < m$.

Exercise 5.7.26. Prove that Exercise 5.7.25 cannot be extended to the case when $m$ is infinite. More precisely, find a separable and simple C*-algebra $A$ and two sequences of inequivalent pure states $\phi_i$, for $i \in \mathbb{N}$, and $\psi_i$, for $i \in \mathbb{N}$, such that for no automorphism $\Phi$ of $A$ one has $\phi_i = \psi_i \circ \Phi$ for all $i$.

Exercise 5.7.27. Suppose $A$ is a separable C*-algebra.

1. Suppose that $m \geq 1$ and $\phi$ and $\psi$ are $m$-tuples of inequivalent pure states on $A$ such that $\ker(\pi_{\phi_i}) = \ker(\pi_{\psi_i})$ for all $i < m$. Prove that there exists an asymptotically inner automorphism $\Phi$ of $A$ such that $\phi_i = \psi_i \circ \Phi$ for all $i < m$.

2. Suppose that $\phi_i$, for $i \in \mathbb{N}$, and $\psi_i$, for $i \in \mathbb{N}$, are sequences of inequivalent pure states on $A$ and that $\ker(\pi_{\phi_i}) = \ker(\pi_{\psi_i})$ for all $i \in \mathbb{N}$. Prove that there exists an asymptotically inner automorphism $\Phi$ of $A$ such that $\phi_i$ is equivalent to $\psi_i \circ \Phi$ for all $i \in \mathbb{N}$.

Exercise 5.7.28. Suppose that $M$ is a von Neumann factor with separable predual. Prove that the action of $\text{Aut}(M)$ on $\mathcal{P}(M)$ is not transitive.

Hint: You need the fact that every automorphism of $M$ is ultraweakly continuous, [142, Theorem 7.2.1].

Notes for Chapter 5

§5.1 Approximate diagonals of C*-algebras (a notion stronger than being an approximate diagonal of a representation) were defined, constructed, and applied in the seminal work of Haagerup, [122] to prove that all nuclear C*-algebras are amenable Banach algebras (see Notes to §15.5 for more on amenable Banach algebras). Theorem 5.1.2, implicit in [122], was first explicitly stated in [162]. The elementary proof of Theorem 5.1.2 is taken from [161, Proposition 4.1].

For another significant use of Hall’s Matching Theorem in the theory of C*-algebras see [209].

§5.2 Theorem 5.2.1 was proved in [7, Proposition 2.2]. Lemma 5.2.6 is known as Glimm’s Lemma. Lemma 5.2.11 is due to Kirchberg and its proof is taken from [208, Lemma 4.19].

§5.3 Quantum filters of projections in $\mathcal{B}(H)$ and $\mathcal{Q}(H)$ were introduced by Weaver and the author as noncommutative analogs of ultrafilters associated with pure states ([99]). Proposition 5.3.7 was proved in [23]. The authors of [23] and [25] consider variants of the notion of a quantum filter and use terminology slightly different from ours. In [99] quantum filters on $\mathcal{B}(H)$ and $\mathcal{Q}(H)$ were defined as quantum filters of projections. Proposition 5.3.13 comes from [36]. It justifies the change in terminology: on any other C*-algebra with real rank zero, every quantum
filter is generated by the quantum filters of its projections. Maximal quantum filters associated with measurable cardinals were studied in [29].

§5.4 Most of this section, and Proposition 5.4.7 in particular, has been adapted from [5]. Theorem 5.4.8 was taken from [92].

§5.5 As promised, we continue the discussion of Glimm’s Theorem started in the Notes for §3.7. While Sakai generalized many of the implications in Glimm’s theorem to nonseparable C*-algebras, the truth—or even consistency with ZFC—of some of them remains open. One of these implications will be discussed §11.2. To state another we need a definition.

A representation \( \pi : A \to \mathcal{B}(H) \) of a C*-algebra is factorial if the WOT-closure of the image of \( A \) is a factor von Neumann algebra, and the type of a factorial representation is the type of the corresponding factor (see §3.1.5). For example, Exercise 4.5.12 characterizes when the GNS representation associated with a trace on \( C^*_r(\Gamma) \) is factorial. Hence factorial representations of type I are exactly the irreducible representations and a C*-algebra (separable or not) is of type I if and only if all of its factorial representations are of type I. The following is closely related to Theorem 3.7.2.

**Theorem 5.7.29 (Glimm).** If \( A \) is a simple and separable C*-algebra then the following are equivalent.

1. The algebra \( A \) is of non-type I.
2. The algebra \( A \) has a factorial representation of type II.
3. The algebra \( A \) has a factorial representation of type III.

Sakai proved that (1) implies (3) for every C*-algebra \( A \). The following question is open, and it is not unreasonable to believe that its solution will involve set theory.

**Question 5.7.30.** Does every non-type I C*-algebra have a factorial representation of type II?

Corollary 5.5.6, Corollary 5.5.7, and Corollary 5.5.8 were all taken from [5].

§5.6 Theorem 5.6.1 was first stated in its present form in [5], but most of the ideas in its proof are contained in [162] and [111].

The key application of Theorem 5.6.1 to set theory (and back to the theory of C*-algebras) is given in Proposition 11.2.1. Further refinements of these two results can be proved and used (together with \( \diamondsuit_{\mathcal{R}_1} \)) to construct interesting examples of C*-algebras; see [244].

Lemma 5.6.2 and its elegant proof, replacing [111, Lemma 3.2 and 3.3], were kindly provided by Narutaka Ozawa.

Exercise 5.7.23 was taken from [155] and [83]. Exercise 5.7.28 and the case when \( m = 1 \) of Exercise 5.7.27 (2) are taken from [162].
Part II

Set Theory and Nonseparable $C^*$-algebras
Chapter 6
Infinitary Combinatorics, I.

By the look of it, cerebral property in humans once dedicated to numeric memory has, in the six million years since we diverged from chimpanzees, been co-opted for grander purposes, like the ability to judge whether a sentence like this is true: “There is no non-vanishing continuous tangent vector field on even dimensional spheres.”

Natalie Angier, ‘Many Animals Can Count, Some Better Than You’

Brain areas that tens of thousands of years ago probably dealt with odours were put to work on more urgent tasks such as mathematics. The system prefers that our neurons solve differential equations rather than smell our neighbours.

Yuval Noah Harari, Homo Deus: A Brief History of Tomorrow

In this chapter we introduce elementary combinatorics of uncountable structures. Only a small fragment of the sophisticated machinery of contemporary set theory has been put to use in the study of operator algebras, and the basic set-theoretic principles used in this book are remarkably simple and few. One of them is the pigeonhole principle, asserting that if a set of regular cardinality $\kappa$ is covered by fewer than $\kappa$ of its subsets, then at least one of these subsets has cardinality $\kappa$. Another is a reflection principle that roughly asserts that any function from an uncountable set into itself will have many closure points. The depth of this statement lies in the choice of the definition of ‘many,’ leading us to the notion of (sadly, misnamed) closed unbounded sets, also known as clubs. Some far-reaching instances and consequences of this principle are the Downwards Löwenheim–Skolem Theorem (known to C*-algebraists as Blackadar’s method), and the analysis of the club filter and the ideal of nonstationary sets, in particular the Pressing Down Lemma. The $\Delta$-System Lemma provides structure theory for uncountable families of finite sets, or more generally, the structure theory for sufficiently large families of small sets.
6.1 Prologue: Elliott intertwining and AM algebras

In this, mostly motivational, section we discuss how the back-and-forth method is used to construct isomorphisms between separable AF $C^*$-algebras.

A Banach space is separable if and only if it has an increasing family of finite-dimensional subspaces with a dense union. Therefore an isomorphism between separable Banach space-based structures, such as $C^*$-algebras, can be recursively constructed from finite pieces in a process in which at each stage we are committed only to an, appropriately coded, finite approximation of the isomorphism. An early example is the proof of Glimm’s Theorem 6.1.3 (see also the discussion of the general back-and-forth method, §8.2). A simple instance of this theorem states that a unital and separable $C^*$-algebra is AM (an inductive limit of full matrix algebras) if and only if it is UHF (a tensor product of full matrix algebras). In order to put the current chapter into the proper perspective, we describe a general back-and-forth tool suitable for handling separable $C^*$-algebras and other complete metric structures.

Suppose $A = \lim_{\infty} A_n$ and $B = \lim_{\infty} B_n$ are $C^*$-algebras and $\alpha_m : A_m \to B_m$ and $\beta_m : B_m \to A_{m+1}$ are $^*$-homomorphisms for $m \in \mathbb{N}$ such that all triangles in Fig. 6.1 commute. Then $\alpha_m$ and $\alpha_n$ agree on $A_m$ whenever $m \leq n$, and a $^*$-homomorphism $\alpha : A \to B$ extends all $\alpha_m$. Similarly, a $^*$-homomorphism $\beta : B \to A$ extends all $\beta_m$.

Moreover, $\beta \circ \alpha = \text{id}_A$ and $\alpha \circ \beta = \text{id}_B$, hence $A$ and $B$ are isomorphic.

There is a much deeper ‘approximate’ version of this intertwining argument. In the following theorem, Fig. 6.1 still applies but the $2k$-th triangle is only assumed to $\varepsilon_k$-commute on a finite set $F_k$ and the $2k + 1$-st triangle is only assumed to $\delta_k$-commute on a finite set $G_k$. The following theorem is a special case of [208, Proposition 2.3.2] in this book it is used only as an example of what can be proved for separable $C^*$-algebras.

**Theorem 6.1.1.** Suppose $A$ and $B$ are separable $C^*$-algebras, $A_m$, for $m \in \mathbb{N}$, is an increasing sequence of sub-$C^*$-algebras of $A$ with dense union, $B_m$, for $m \in \mathbb{N}$, is an increasing sequence of sub-$C^*$-algebras of $B$ with dense union, there are $F_m \subseteq A_m$, $G_m \subseteq B_m$, and $^*$-homomorphisms $\alpha_m : A_m \to B_m$ and $\beta_m : B_m \to A_{m+1}$, for $m \in \mathbb{N}$ such that the following conditions hold.

---

1 The assumption that the connecting maps $f_m$ and $g_m$ be injective is, given additional assumptions, unnecessary.
6.1 Prologue: Elliott intertwining and AM algebras

1. With \( \epsilon_m := \max_{a \in F_m} \| \beta_{m+1} \alpha_m(a) - a \| \) we have \( \sum_m \epsilon_m < \infty \).

2. With \( \delta_m := \max_{b \in B_m} \| \alpha_m \beta_m(b) - b \| \) we have \( \sum_m \delta_m < \infty \).

3. \( F_m \subseteq F_{m+1}, \alpha_m[F_m] \subseteq G_m, G_m \subseteq G_{m+1} \), and \( \beta_m[G_m] \subseteq F_{m+1} \) for all \( m \).

4. \( A_n \cap \bigcup_{m \geq n} F_m \) is dense in \( A_n \) and \( B_n \cap \bigcup_{m \geq n} G_m \) is dense in \( B_n \) for all \( n \).

Then \( A \) and \( B \) are isomorphic.

Theorem 6.1.1 is a simple form of the Elliott intertwining. Its variant was used in the proof of Theorem 5.6.1. Another example is provided by the classification of separable AF algebras (see e.g., [208], [52]). More sophisticated instances in which one assumes that the morphisms in Fig. 6.1 are only approximately commuting are a basic tool in the Elliott classification programme (see e.g., [208, §2.3]).

Let us return to the simplest and earliest application of the back-and-forth method to the classification of \( C^* \)-algebras. This is Glimm’s characterization of separable UHF algebras (§2.2).

**Definition 6.1.2.** A \( C^* \)-algebra \( A \) is said to be **locally matricial** (or LM) if for every \( \epsilon > 0 \) and \( F \in A \) there exist \( n \) and a \( * \)-homomorphism \( \Phi : M_n(\mathbb{C}) \to A \) such that \( F \subseteq \epsilon \Phi[M_n(\mathbb{C})] \). In other words, for every finite subset of \( A \) there exists a full matrix \( C^* \)-subalgebra \( B \) of \( A \) such that each element of \( F \) is within \( \epsilon \) of \( B \).

Dixmier introduced LM algebras in [57] under the name of matroid algebras.

**Theorem 6.1.3 (Glimm).** For a separable, unital \( C^* \)-algebra the following are equivalent.

1. \( A \) is Uniformly HyperFinite (UHF).
2. \( A \) is Approximately Matricial (AM).
3. \( A \) is Locally Matricial (LM).

An algebra as in (1) is necessarily unital, and (2) and (3) remain equivalent in the case when \( A \) is separable, but not necessarily unital.

A complete proof of this theorem can be found for example in [52] or [208]. We give a rough sketch in order to indicate the role of separability.

**Proof.** Clearly, UHF implies AM and AM implies LM. To prove (2) implies (1), fix a separable AM algebra \( A \). Since \( A \) is separable, it is an inductive limit of a countable inductive system of full matrix algebras. By passing to a cofinal subset of this system, we obtain \( A \) as a unital inductive limit of the form

\[
M_{n(0)}(\mathbb{C}) \to M_{n(1)}(\mathbb{C}) \to M_{n(2)}(\mathbb{C}) \to \ldots
\]

Lemma 2.2.4 implies that \( k(j) := n(j + 1)/n(j) \) is a natural number for every \( j \) and that \( A \) is the inductive limit of the unital inductive system

\[
M_{n(0)}(\mathbb{C}) \to M_{n(0)}(\mathbb{C}) \otimes M_{k(1)}(\mathbb{C}) \to M_{n(0)}(\mathbb{C}) \otimes M_{k(1)}(\mathbb{C}) \otimes M_{k(2)}(\mathbb{C}) \to \ldots
\]

where the \( j \)-th connecting map is of the form \( a \mapsto a \otimes 1_{k(j)} \). Therefore \( A \) is UHF.
To prove (3) implies (2), assume that $A$ is LM and fix a countable dense subset $a_n$, for $n \in \mathbb{N}$. We build an increasing chain $A_j \cong M_{n(j)}(\mathbb{C})$, for $j \in \mathbb{N}$, of unital full matrix subalgebras of $A$ such that $\text{dist}(a_i, A_j) < 2^{-j}$ for all $i \leq j$. Since $A$ is LM, we can find $A_0$ as required. Suppose that $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_j$ have been chosen. Fix a small enough $\delta_j$ (to be specified later) and a set $E$ of matrix units for $A_j$. Let $B$ be a unital subalgebra of $A$ isomorphic to a full matrix algebra such that $\text{dist}(x, B) < \delta_j$ when $x = a_{j+1}$ or $x \in E$.

By the weak stability of the defining relations for the matrix units of full matrix algebras (Exercise 2.8.12), if $\delta_j$ is small enough then there is a unital isomorphic copy $A'_j$ of $M_{n(j)}(\mathbb{C})$ in $B$. Moreover, there is a unitary $u \in A$ such that $uA'_j u^* = A_j$ and $\|1 - u\|$ can be made arbitrarily small by choosing $\delta_j$ small. Then $A_{j+1} := u^*Bu$ is as required.

Since $a_n$, for $n \in \mathbb{N}$, was an enumeration of a dense subset of $A$, the union of the sequence $A_j$, for $j \in \mathbb{N}$, is dense in $A$. $\Box$

Can Theorem 6.1.3 be proved without the assumption of separability? The following question appears in [57, 8.1].

**Question 6.1.4 (Dixmier).**

1. Is every unital LM algebra AM?
2. Is every unital AM algebra UHF?

One may attempt to apply the back-and-forth method to construct an isomorphism between nonseparable $C^*$-algebras $A$ and $B$. After countably many steps, one is committed to an isomorphism $\Phi_\omega$ between separable subalgebras $A_\omega$ and $B_\omega$ of $A$ and $B$, respectively. Without an access to a very strong ‘prediction’ device, it is not possible to control the way these separable subalgebras ‘sit’ inside $A$ and $B$. This may result in $\Phi_\omega$ not being extendable to larger subalgebras of $A$ and $B$.

Obstructions to back-and-forth constructions in the nonseparable realm, as well as some workarounds, will be studied in the later sections of §6. The full transfinite back-and-forth method will be recovered in §8.1 and §15.1, but only under an additional model-theoretic assumption (saturation) satisfied only by certain massive $C^*$-algebras, such as ultraproducts and asymptotic sequence algebras, and only under suitable set-theoretic assumptions.

The second part of Question 6.1.4 will be answered in Theorem 10.3.1; for the first part see Notes to this Chapter.

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2 Here $\omega$ stands for the smallest infinite ordinal.
3 A sophisticated set-theoretic device known as morass can sometimes be used to construct an uncountable set from finite pieces (see [54]). However, construction along a morass requires one to amalgamate previously defined objects, often presenting formidable technical difficulties. In addition, a morass cannot be constructed in ZFC.
The purpose of this section is to study club filters in directed, $\sigma$-complete posets. This section is an introduction to §6.3 and all of its material is a part of the folklore of set theory.

The basic notions from naive set theory such as transitive sets, ordinals, and cardinals are reviewed in §A.3. A partially ordered set, or poset, is a set equipped with a partial (i.e., not necessarily linear) ordering. Given a poset $(P, \leq)$, a subset $X$ of $P$ is cofinal in $P$ if $$(\forall p \in P)(\exists x \in X)p \leq x.$$ An example justifying a particularly pedantic notation $f[X]$ for the pointwise $f$-image of a set follows.

**Example 6.2.1.** If $X$ is a transitive set and $\alpha \in X$ is an ordinal, then by von Neumann’s convention that $\alpha = \{\beta : \beta < \alpha\}$ we have $\alpha \subseteq X$ and if $\text{dom}(f) \supseteq \alpha + 1$ then the sets $f(\alpha)$ and $f[\alpha] = \{f(\beta) : \beta < \alpha\}$ may be different.

Most of the time this does not lead to a confusion, but we will take the due precautions and if $f : X \to Y$ and $Z \subseteq X$ we will write $f[Z] := \{f(z) : z \in Z\}$.

(Some authors write $f"Z$ instead; this is the standard notation in the theory of large cardinals, see [149].)

The following lemma is provided as a motivation; its key strengthening is given in Example 6.2.8 (1) below.

**Lemma 6.2.2.** Suppose $\kappa$ is a regular uncountable cardinal. If $f : \kappa \to \kappa$ then the set $C_f := \{\alpha < \kappa : f[\alpha] \subseteq \alpha\}$ is unbounded in $\kappa$. If $X \subseteq C_f$ and $\sup(X) < \kappa$, then $\sup(X) \in C_f$.

**Proof.** For $\beta < \kappa$ let $f^+(\beta) := \sup_{\gamma < \beta} f(\gamma)$. The assumption that $\kappa$ be regular assures that $f^+(\beta) < \kappa$ for all $\beta < \kappa$, and therefore $f^+ : \kappa \to \kappa$ is nondecreasing and $f \leq f^+$. Starting with any $\alpha_0 < \kappa$ recursively define $\alpha_{n+1} := f^+(\alpha_n) + 1$ for $n \geq 0$. Then $\alpha := \sup_\alpha \alpha_n$ is, by the regularity and the uncountability of $\kappa$, strictly smaller than $\kappa$ and an element of $C_f \setminus \alpha_0$. Since $\alpha_0$ was arbitrary, $C_f$ is unbounded.

Suppose $X$ is a bounded subset of $C_f$ and $\alpha := \sup X$ does not belong to $X$. Then $f[\alpha] = \bigcup_{\beta \in X \cap \alpha} f[\beta] \subseteq \bigcup_{\beta \in X \cap \alpha} \beta = \alpha$, and therefore $\alpha \in C_f$.

**Definition 6.2.3.** A directed set is $\sigma$-complete if every countable increasing sequence\(^4\) of its elements has the supremum. If $\kappa$ is a cardinal we say that a directed set is $\kappa$-complete if every increasing chain of its elements of cofinality less than $\kappa$ has the supremum.

---

\(^4\) In this book a ‘sequence’ need not be countable.
Thus $\sigma$-complete is synonymous with $\aleph_1$-complete. While the latter notion offers a straightforward generalization to limit cardinals, some authors—including the author of this book—prefer using the former terminology whenever this does not lead to a confusion.

**Example 6.2.4.** Each of the following posets is directed and $\sigma$-complete.

1. An ordinal $\alpha$ of uncountable cofinality ordered by $\leq$.
2. The ideal of Lebesgue null subsets of $\mathbb{R}$ ordered by inclusion.
3. The ideal of first category subsets of a Polish space (also known as *meager* subsets) ordered by inclusion.
4. The family of all countable subsets of an uncountable set $X$,
   
   $|X|^\mathfrak{c}_0 := \{ Y \subseteq X : |Y| = \aleph_0 \}$

ordered by inclusion.
5. Suppose $\lambda$ is a cardinal of uncountable cofinality and $X$ is a set of cardinality at least $\lambda$. Consider the poset of all subsets of $X$ of cardinality smaller than $\lambda$,
   
   $|X|^<\lambda := \{ Y \subseteq X : |Y| < \lambda \}$

ordered by inclusion. Some authors denote this set by $\mathcal{P}_\lambda(X)$.

The assumption that every directed countable subset has a supremum should not be confused with the assumption that every countable directed subset has an upper bound. The latter requirement is much weaker (see Example 9.5.1).

**Example 6.2.5.**

1. The lattice of projections with separable range in $\mathcal{B}(\ell_2(\kappa))$ for an uncountable $\kappa$, ordered by $p \leq q$ if $pq = p$ is directed and $\sigma$-complete. The supremum of a directed family of projections is the projection to the subspace generated by their ranges.
2. Suppose $M$ is a nonseparable $C^*$-algebra. Let $\text{Sep}(M)$ denote the set of all separable $C^*$-subalgebras of $M$ ordered by inclusion. This poset is directed and $\sigma$-complete. The supremum of a directed subset of $\text{Sep}(M)$ is equal the closure of its union, and it is typically strictly larger than its union.

The following definition is interesting if every ‘small’ (with ‘small’ taken to be countable, of cardinality less than $\kappa$, or of density character less than $\kappa$ for a given regular cardinal $\kappa$ that depends on the context and the target) directed subset of $\mathbb{P}$ has a supremum.

**Definition 6.2.6.**Suppose $\mathbb{P}$ is a directed set. We say that $C \subseteq \mathbb{P}$ is a **club** if the following two conditions are satisfied.

1. $C$ is *closed*: for every chain $X \subseteq C$, if $\sup(X)$ exists in $\mathbb{P}$ then $\sup(X) \in C$.
2. $C$ is *cofinal*: For every $p \in \mathbb{P}$ there exists $q \in C$ such that $p \leq q$.

---

5 We are not assuming that every directed and bounded subset of $\mathbb{P}$ has a supremum, and therefore this definition is in some (uninteresting) cases trivialized.
For a self-strengthening of this definition, see Exercise 6.7.5. This terminology merits an explanation.

**Example 6.2.7.** Suppose $\mathbb{P}$ is a directed set. The *order topology* on $\mathbb{P}$ is defined via the closure operator as follows. For $X \subseteq \mathbb{P}$ we let

$$\bar{X} := X \cup \{\sup(Y) : Y \subseteq X, Y \text{ is directed and } \sup(Y) \text{ exists}\}.$$ 

Thus a subset of $\mathbb{P}$ is closed if and only if it is of the form $X$ for some $X$. The reader can verify that this defines a topology on $\mathbb{P}$, and that $Z \subseteq \mathbb{P}$ is a neighbourhood of $p \in \mathbb{P}$ if and only if $\sup(Y) \neq p$ for every directed $Y \subseteq \mathbb{P} \setminus Z$ that has a supremum. The condition (1) in Definition 6.2.6 is equivalent to asserting that $C$ is closed in the order topology on $\mathbb{P}$.

‘Club’ stands for ‘closed and unbounded’ and Example 6.2.7 justifies the first half of the word. The ‘ub’ in ‘club’ is a historical accident much harder to justify, but easy to explain. This terminology dates back to the times when ‘club’ referred to closed and unbounded subsets of an ordinal. A subset of a linearly ordered set with no maximal element is cofinal if and only if it is unbounded. This is false for general posets (Example 9.5.4).

**Example 6.2.8.** 1. Suppose $\kappa$ is a regular cardinal and $f : \kappa \to \kappa$. By the proof of Lemma 6.2.2, the set

$$C_f := \{\alpha < \kappa : f[\alpha] \subseteq \alpha\}$$

is a club. Conversely, every club $C \subseteq \kappa$ includes a club of this form. To see this define $f_C := f : \kappa \to \kappa$ by $f_C(\xi) := \min C \setminus (\xi + 1)$. It is straightforward to check that $C_f \subseteq C$.

2. Suppose $X$ is an uncountable set. As before let

$$|X|^{|\mathbb{R}|} := \{s \subseteq X : s \text{ is finite}\}$$

and fix $f : |X|^{|\mathbb{R}|} \to X$. If $\lambda < |X|$ is an infinite cardinal then the set

$$C_f := \{Y \in |X|^\lambda : f[|Y|]\subseteq Y\}$$

is a club in $|X|^\lambda$. The sets in $C_f$ are said to be *closed under* $f$. If $Y_a$, for $\alpha < \lambda$, is an increasing sequence of elements of $C_f$ and $Y := \bigcup_{\alpha<\lambda} Y_a$, then $|Y|^{|\mathbb{R}|} = \bigcup_{\alpha<\lambda} [Y_a]^{|\mathbb{R}|}$ and therefore $Y$ is closed under $f$. Since $|Y|^{|\mathbb{R}|}$ has the same cardinality as $Y$, a closing-off argument shows that every subset of $X$ of cardinality $\lambda$ is included in a subset of $X$ of cardinality $\lambda$ closed under $f$.

We will see that every club in $|X|^{|\mathbb{R}|}$ includes a club of the form $C_f$ for some $f : |X|^{|\mathbb{R}|} \to X$ (Theorem 6.4.1).

**Proposition 6.2.9.** Suppose $\mathbb{P}$ is a directed $\kappa$-complete partially ordered set. Then the intersection of fewer than $\kappa$ clubs in $\mathbb{P}$ is a club.
Proof. Fix a cardinal $\lambda < \kappa$ and let $C_\alpha$, for $\alpha < \lambda$ be a family of clubs in $\mathcal{P}$. Then $\bigcap_\alpha C_\alpha$ is closed, being an intersection of closed sets. In order to see that it is cofinal fix $p_0 \in \mathcal{P}$. Since $|\lambda^2| = \lambda$ we can re-enumerate $C_\alpha$’s so that each member of the family occurs cofinally often. Recursively choose $p_\beta \in \mathcal{P}$, for $\beta \leq \lambda$ so that for all $\beta$ they satisfy the following requirements:

1. $p_{\beta+1} \in C_\beta$.
2. $p_{\beta+1} \geq p_\beta$, and
3. $p_\beta = \sup_{\gamma < \beta} p_\gamma$ if $\beta$ is a limit ordinal.

For $\alpha < \lambda$ we have $p_\lambda = \sup\{p_\gamma : C_\gamma = C_\alpha\}$ and therefore $p_\lambda \in C_\alpha$. We have proved that $p_\lambda \in \bigcap_\alpha C_\alpha$; since $p_0 \in \mathcal{P}$ was arbitrary, the intersection $\bigcap_\alpha C_\alpha$ is cofinal. $\square$

6.3 Concretely Represented, Directed, $\sigma$-Complete, Posets

In this section we study a nonstandard generalization of posets of the form $(\kappa, <)$ or $(|X|^\aleph_0, \subseteq)$ and prove that the diagonal intersection of clubs is a club.

Definition 6.3.1. A directed, $\kappa$-complete, poset $\mathcal{P}$ is concretely represented if there are a set $X$ and an order-embedding $\Phi : \mathcal{P} \to (|X|^{|<\kappa|}, \subseteq)$ such that every linearly ordered $Z \subseteq \mathcal{P}$ of cardinality smaller than $\kappa$ satisfies $\Phi(\sup Z) = \bigcup \Phi[Z]$.

A directed set $\mathcal{P}$ is isomorphic to a suborder of $(|\mathcal{P}|^{|<\kappa|}, \subseteq)$ if $|\{q : q \leq p\}| < \kappa$ for every $p \in \mathcal{P}$, via $\Phi(p) := \{q \in \mathcal{P} : q \leq p\}$. However, it is not always possible to choose an isomorphism so that the sup=union requirement of Definition 6.3.1 is satisfied (Example 6.3.2 (4)).

Example 6.3.2. 1. Suppose $\kappa$ is an uncountable regular cardinal. To see that the directed and $\kappa$-complete poset $(\kappa, \leq)$ is concretely represented, recall that an ordinal is identified with the set of all smaller ordinals.

2. If $X$ is an uncountable set then the directed and $\sigma$-complete poset $(|X|^\aleph_0, \subseteq)$ is concretely represented.

3. More generally, if $\kappa$ is an uncountable regular cardinal and the cardinality of $X$ is at least $\kappa$, the directed and $\kappa$-complete poset $|X|^{|<\kappa|}$ is concretely represented.

4. The poset Sep($A$) of separable C$^*$-subalgebras of a C$^*$-algebra (Example 6.2.4) ordered by the inclusion is directed and $\sigma$-complete. Later on, we will see that it is not concretely represented. This inconvenience will force us to jump through a hoop or two in Chapter 7.

Definition 6.3.3. Suppose $\mathcal{P}$ is a concretely represented directed and $\kappa$-complete set. The diagonal intersection of an indexed family of clubs $C_x$, for $x \in X$, is

$$\bigtriangleup_{x \in X} C_x := \{p \in \mathcal{P} : (\forall x \in p)p \in C_x\}.$$

While the definition of $\bigtriangleup_\alpha C_\alpha$ apparently depends on both the concrete representation of $\mathcal{P}$ and the indexing of clubs, this notion is rather robust.
Proposition 6.3.4. Suppose \( \kappa \) is an uncountable regular cardinal and \( \mathbb{P} \) is a concretely represented, directed, and \( \kappa \)-complete poset. Then the diagonal intersection of a family of clubs in \( \mathbb{P} \) is a club in \( \mathbb{P} \).

Proof. Let \( X \) be such that \( \mathbb{P} \subseteq [X]^\kappa \), and \( \mathbb{P} \) is ordered by \( \subseteq \) and closed under taking suprema of its subsets of cardinality less than \( \kappa \). To prove that \( \bigtriangleup_{x \in X} C_x \) is closed, choose an increasing chain \( p_\alpha \) for \( \alpha < \lambda \), in \( \mathbb{P} \) for some \( \lambda < \kappa \). Let \( p_\lambda := \bigcup_{\alpha < \lambda} p_\alpha \). Fix \( x \in p_\lambda \). The set \( Z_x := \{ \alpha < \lambda : x \in p_\alpha \} \) is co-bounded in \( \lambda \). Also, \( p_\alpha \in C_x \) for all \( \alpha \in Z_x \). Since \( C_x \) is a club and \( \mathbb{P} \) is \( \kappa \)-complete, \( p_\lambda \in C_x \). Since this is true for all \( x \in p_\lambda \), we have \( p_\lambda \in \bigtriangleup_{x \in X} C_x \).

To see that \( \bigtriangleup_{x \in X} C_x \) is cofinal, pick \( p(0) \in \mathbb{P} \). Recursively define \( p(m) \in \mathbb{P} \) and clubs \( D_m \), for \( m \in \mathbb{N} \), so that for all \( m \in \mathbb{N} \) the following conditions hold.

1. \( D_m := \bigcap_{x \in p(m)} C_x \).
2. \( p(m+1) \in D_m \), and
3. \( p(0) \leq p(m) \leq p(m+1) \).

Suppose \( m \in \mathbb{N} \) is such that \( p(m) \) and \( D_j \), for \( j < m \), were chosen and satisfy the requirements. Since \( |p(m)| < \kappa \) and \( \mathbb{P} \) is directed and \( \kappa \)-complete, \( D_m \) as defined above is a club and we can choose \( p(m+1) \geq p(m) \) in \( D_m \). Since \( \mathbb{P} \) is concretely represented and \( \sigma \)-complete, \( p(\omega) := \sup_n p(n) = \bigcup_{n < \omega} p(n) \) belongs to \( \mathbb{P} \). By the construction \( p(\omega) \) is an element of \( \bigcap_{n < \omega} D_n = \bigcap_{x \in p(\omega)} C_x \subseteq \bigtriangleup_{x \in X} C_x \). Since \( p(0) \) was arbitrary, this shows that \( \bigtriangleup_{x \in X} C_x \) is cofinal in \( \mathbb{P} \) and completes the proof. \( \square \)

We explicitly state the two most important instances of Proposition 6.3.4.

Corollary 6.3.5. Suppose \( \mathbb{P} \) is (\( \kappa, \subseteq \)) for a regular cardinal \( \kappa \) or (\( [X]^\kappa, \subseteq \)) for an uncountable set \( X \). Then the diagonal intersection of clubs in \( \mathbb{P} \) is a club in \( \mathbb{P} \). \( \square \)

6.4 Kueker’s theorem

In this section we prove that for an uncountable set \( X \) every club in \( [X]^\lambda \) includes the family of all sets closed under a single function \( f: [X]^\lambda \to X \).

For \( C \subseteq [X]^\kappa \) consider a two-player perfect information game \( G(C) \) in which two players, called I and II, collaborate to produce an infinite subset of \( X \). They alternately choose elements of \( X \) as follows:

<table>
<thead>
<tr>
<th>I</th>
<th>( x_0 )</th>
<th>( x_2 )</th>
<th>( x_4 )</th>
<th>\ldots</th>
<th>( x_{2n} )</th>
<th>\ldots</th>
</tr>
</thead>
<tbody>
<tr>
<td>II</td>
<td>( x_1 )</td>
<td>( x_3 )</td>
<td>( x_5 )</td>
<td>\ldots</td>
<td>( x_{2n+1} )</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

I wins if \( Y := \{ x_j : j \in \mathbb{N} \} \notin C \), and II wins if \( Y \in C \). A winning strategy for II is a function \( f: \bigcup_{n \in \mathbb{N}} [X]^{2n+1} \to X \) such that for every choice of \( x_{2n+1} \in X \), for \( n \in \mathbb{N} \), with \( x_{2n} := f(\{ y_j : j < 2n \}) \) the set \( \{ x_n : n \in \mathbb{N} \} \) belongs to \( C \). A winning strategy
for I is \( f : \bigcup_n [X]^{2n} \to X \) such that for every choice of \( x_{2n} \in X \), for \( n \in \mathbb{N} \), with \( x_{2n+1} := f(\{x_j : j < 2n + 1\}) \) the set \( \{x_n : n \in \mathbb{N}\} \) belongs to \( C \).

In other words, II has a winning strategy for \( G(C) \) if the following infinitary statement holds (all quantifiers range over \( X \)):

\[
(\forall x_0)(\exists x_1)(\forall x_2)(\exists x_3)\ldots(\forall x_{2n})(\exists x_{2n+1})\ldots\{x_j : j \in \mathbb{N}\} \in C.
\]

Analogously, I has a winning strategy for \( G(C) \) if

\[
(\exists x_0)(\forall x_1)(\exists x_2)(\forall x_3)\ldots(\exists x_{2n})(\forall x_{2n+1})\ldots\{x_j : j \in \mathbb{N}\} \notin C.
\]

At most one of the players can have a winning strategy.\(^6\) If \( f : [X]^{<\kappa} \to X \) and \( Y \subseteq X \), we say that \( Y \) is closed under \( f \) if \( f([Y]^{<\kappa}) \subseteq Y \). As in Example 6.2.8, let

\[
C_f := \{Y \in [X]^{\kappa} : Y \text{ is closed under } f\}.
\]

**Theorem 6.4.1.** Suppose \( X \) is a set and \( C \subseteq [X]^{\kappa} \). The following are equivalent.

1. \( C \) includes a club.
2. II has a winning strategy in \( G(C) \).
3. There exists \( f : [X]^{<\kappa} \to X \) such that \( C_f \subseteq C \).

**Proof.** (2) \(\Rightarrow\) (3): Suppose II has a winning strategy in \( G(C) \). Then any function \( f : [X]^{<\kappa} \to X \) that extends \( f \) has the property that \( C_f \subseteq C \).

(3) \(\Rightarrow\) (1): Suppose (3) holds and fix \( f : [X]^{<\kappa} \to X \) such that \( C_f \subseteq C \). We need to prove that \( C_f \) is a club. Clearly \( C_f \) is unbounded. If \( Y_n \), for \( n \in \mathbb{N} \), is an increasing family of subsets of \( X \) then \( \bigcup_n Y_n^{<\kappa} = \bigcup_n Y_n^{<\kappa} \), and therefore \( C_f \) is closed under unions of increasing chains. Therefore \( C_f \) is a club and (1) follows.

(1) \(\Rightarrow\) (2): Suppose (1) holds. By the Axiom of Choice we may assume \( X \) is a cardinal, \( \kappa \).

**Claim.** There is a family \( Y(s) \in C \), for \( s \in [\kappa]^{<\kappa} \), such that for all \( s \subseteq t \) we have \( s \subseteq Y(s) \) and \( Y(s) \subseteq Y(t) \).

**Proof.** This is proved by induction on \( n \in \mathbb{N} \), in which all \( Y(s) \) for \( |s| = n \) are chosen simultaneously in the \( n \)th step.

If \( n = 0 \), choose any \( Y(\emptyset) \in C \). Suppose \( n \geq 1 \) and \( Y(s) \) as required were chosen for all \( s \in [\kappa]^{<\kappa} \) such that \( |s| \leq n \). Fix \( t \in [\kappa]^{<\kappa} \) such that \( |t| = n + 1 \). Since \( P \) is directed and \( C \) is cofinal we can choose \( Y(t) \in C \) such that \( \bigcup_{s \subseteq t} Y(s) \subseteq Y(t) \); we then automatically have \( t \subseteq Y(t) \). The sets \( Y(s) \) chosen in this manner clearly satisfy the requirements. \( \square \)

For \( s \in \kappa \) fix an enumeration

\[
Y(s) = \{y^j_s : j \in \mathbb{N}\}.
\]

---

\(^6\) A less obvious fact that we will not need is that the infinitary analogs of the De Morgan laws do not hold in general, and in certain situations neither of the players has a winning strategy (Exercise 6.7.3).
Also fix a surjection \( g: \mathbb{N} \to \mathbb{N}^2 \) such that \( g(k) = (m,n) \) implies \( m \leq k \); for example, take \( g(2^m(2n+1)) := (m,n) \).

The definition of \( f(s) \) depends on whether \( |s| \) is even or odd. Suppose \( k \in \mathbb{N} \) and \( s \in [\kappa]^{2k} \). If \( g(k) = (m,n) \) and \( t \) consists of the first \( m \) elements of \( s \) in the ordering inherited from \( \kappa \) then let \( f(s) := y^s_n \). If \( |s| \) is odd let \( f(s) := \max(s) + 1 \). This defines the function \( f \). In order to prove \( C_f \subseteq C, \) fix \( Z \in C_f \). Since for every \( s \in Z \) of odd cardinality we have \( \max(s) + 1 \in Z \), \( \sup(Z) \) is a limit ordinal that does not belong to \( Z \). We claim that

\[
Z = \bigcup \{Y(s): s \in Z\}.
\]

Fix \( s \in Z \) and let \( m := |s| \). Since \( \sup(Z) \) is a limit ordinal that does not belong to \( Z \), by recursion we can choose an infinite increasing sequence \( z_j \), for \( j \in \mathbb{N} \), in \( Z \) such that \( \max(s) < z_0 \). Fix \( n \in \mathbb{N} \) and \( k \) such that \( g(k) = (m,n) \) and let

\[
t := s \cup \{z_j : j < 2k - m\}.
\]

Since \( k \geq m \), we have \( |t| = 2k \), \( t \in Z \), \( s \) is an initial segment of \( t \), and \( g(k) = (m,n) \). Therefore \( f(t) = y^s_n \). Since \( n \) was arbitrary, \( Y(s) \subseteq Z \). Since \( s \in Z \) was arbitrary,

\[
\bigcup \{Y(s): s \in Z\} \subseteq Z.
\]

Since the reverse inclusion is trivial, \( Z = \bigcup \{Y(s): s \in Z\} \) holds. If \( x_j \), for \( j \in \mathbb{N} \), is an enumeration of \( Z \), then the sets \( Y\{\{x_j : j < n\}\} \), for \( n \in \mathbb{N} \), form an increasing family of elements of \( C \) and their union— which is equal to \( Z \)— belongs to \( C \). This proves (2).

If \( X \subseteq Y \) and \( C \) is a club in \( [X]^{\aleph_0} \) then \( \{p \in [Y]^{\aleph_0} : p \cap X \in C\} \) is a club in \( [Y]^{\aleph_0} \), but a natural-sounding converse to this fact is false (Exercise 6.7.6). A weak converse is provided by the following corollary to Theorem 6.4.1.

**Corollary 6.4.2.** Suppose \( X \subseteq Y \) and \( C \) is a club in \( [Y]^{\aleph_0} \). Then \( \{p \cap X : p \in C\} \) includes a club in \( [X]^{\aleph_0} \).

**Proof.** By Theorem 6.4.1 fix \( f: [Y]^{\aleph_0} \to Y \) such that \( C_f \subseteq C \). For \( p \in [Y]^{\aleph_0} \) define the closure of \( p \) under \( f \) as follows. Recursively define \( p_k \), for \( k \in \mathbb{N} \), by \( p_0 := p \) and

\[
p_{k+1} := \bigcup_{n \leq k} f([p_k]^{\aleph_0})
\]

for \( k \geq 1 \). Let \( p^+ := \bigcup_k p_k \). Then \( p^+ \) is the minimal element of \( C_f \) that includes \( p \).

Let \( D := \{p^+ \cap X : p \in [X]^{\aleph_0}\} \). Then \( D \subseteq \{p \cap X : p \in C\} \) and it will suffice to prove that \( D \) is a club in \( [X]^{\aleph_0} \). The set \( D \) is cofinal in \( [X]^{\aleph_0} \) since \( p \subseteq p^+ \) for all \( p \).

If \( q_n \), for \( n \in \mathbb{N} \), is an increasing sequence in \( D \) then \( p := \bigcup_n q_n \) satisfies \( p^+ = \bigcup_n q_n^+ \) and hence \( p \cap X = p \). This proves that \( D \) is closed and completes the proof. \( \Box \)
6.5 Stationarity and Pressing Down

In this section we study stationary sets. We prove the Pressing Down Lemma and show that every stationary subset of $\kappa^+$ can be partitioned into $\kappa^+$ stationary sets, for every regular uncountable cardinal $\kappa$.

**Definition 6.5.1.** Suppose $\mathbb{P}$ is a directed set. A subset $S$ of $\mathbb{P}$ is *stationary* if it intersects every club in $\mathbb{P}$ nontrivially. It is *nonstationary* if it is disjoint from some club in $\mathbb{P}$.

Since the intersection of two clubs is a club, every club is stationary. The set

$$\text{Club}(\mathbb{P}) := \{ C \subseteq \mathbb{P} : C \text{ is nonstationary in } \mathbb{P} \}$$

is called the *club filter* in $\mathbb{P}$. Equivalently, a subset of $\mathbb{P}$ belongs to the club filter if it includes a club. Stationary subsets of $\mathbb{P}$ are the subsets of $\mathbb{P}$ positive with respect to the club filter. The set

$$I_{NS}(\mathbb{P}) := \{ Z \subseteq \mathbb{P} : Z \text{ is nonstationary in } \mathbb{P} \}$$

is the *ideal of nonstationary subsets of $\mathbb{P}$*. If $\mathbb{P} = \kappa$ is a cardinal, some authors write $NS_\kappa$ for $I_{NS}(\kappa)$.

We will follow a common practice and say e.g., ‘Let $S \subseteq \omega_1$ be stationary,’ with the understanding that $S$ is assumed to be stationary in $\omega_1$. Clearly, a subset of $\omega_1$ is never stationary in any larger ordinal.

**Remark 6.5.2.** A family of subsets of a set $X$ is a *filter* if it is closed under taking finite intersections and supersets of its elements. It is an *ideal* if it is closed under taking finite unions and subsets of its elements (see Definition 9.1.1). The club filter is a filter, and the ideal of nonstationary sets is an ideal, but these facts will be used only implicitly.

If a poset $\mathbb{P}$ is directed and $\kappa$-complete then the filter Club($\mathbb{P}$) is $\kappa$-closed by Lemma 6.2.9. The dual form of this fact is worth stating.

**Proposition 6.5.3.** Suppose $\mathbb{P}$ is a directed and $\kappa$-complete partially ordered set. Then $I_{NS}(\mathbb{P})$ is a directed and $\kappa$-complete poset under inclusion.

**Proof.** Suppose otherwise. Then there are $\lambda < \kappa$ a stationary subset $S$ of $\mathbb{P}$, and nonstationary sets $T_\alpha$, for $\alpha < \lambda$ such that $S = \bigcup_{\alpha < \lambda} T_\alpha$. Fix a club $C_\alpha$ disjoint from $T_\alpha$ for each $\alpha$. Proposition 6.2.9 implies that $\bigcap_\alpha C_\alpha$ is a club and it therefore intersects $S$. Therefore $C_\alpha \cap T_\alpha$ is nonempty for some $\alpha$; contradiction. $\Box$

If $\mathbb{P}$ is a concretely represented directed and $\kappa$-complete poset for some regular cardinal $\kappa$ then Proposition 6.3.4 implies that the club filter Club($\mathbb{P}$) is closed under diagonal intersections. As in the case of Proposition 6.5.3, this fact is worth stating in its dual form. This is the *Pressing Down Lemma* (also known as *Fodor’s Lemma*),
6.5 Stationarity and Pressing Down

**Definition 6.5.4.** Suppose $P$ is a concretely represented, directed, and $\kappa$-complete poset for some regular cardinal $\kappa$ and identify $P$ with a subset of $([\kappa]^{<\kappa}, \subseteq)$ as in Definition 6.3.1. If $S \subseteq P$, then $f : S \to X$ is regressive if $f(p) \in p$ for all $p \in S$.

**Example 6.5.5.**
1. If $S \subseteq \kappa$ then $f : S \to \kappa$ is regressive if $f(\alpha) < \alpha$ for all $\alpha$.
2. If $S \subseteq [\kappa]^{<\kappa}$ then $f : S \to X$ is regressive if $f(p) \in p$ for all $p$.

**Proposition 6.5.6 (Pressing Down Lemma).** Suppose $P$ is a concretely represented, directed, and $\kappa$-complete poset. Then every regressive function $f$ with stationary domain is constant on a stationary set.

**Proof.** We will prove the contrapositive. Suppose $S \subseteq P$ is stationary and $f$ is a regressive function on $S$ such that $Q_x := \{ p \in S : f(p) = x \}$ is nonstationary for all $x \in X$. Then $C := \triangle_i(P \setminus Q_i)$ includes a club by Proposition 6.3.4. However $C$ is disjoint from $S$, and therefore $S$ is nonstationary.

**Example 6.5.5** and **Proposition 6.5.6** combine to give a proof of the following.

**Corollary 6.5.7.** If $\kappa$ is a regular cardinal, $S \subseteq \kappa$ is stationary, and $f : S \to \kappa$ is regressive, then $f$ is constant on a stationary subset of $S$.

If $X$ is an uncountable set, $\lambda < |X|$ is a regular cardinal, $S \subseteq [X]^{<\lambda}$ is stationary, and $f : S \to X$ is regressive, then $f$ is constant on a stationary subset of $S$.

**Example 6.5.8.** Suppose $\Phi$ is an automorphism of an uncountable group $\Gamma$. If $Z_\Phi := \{ G \in [\Gamma]^\mathfrak{c} : G \text{ is a subgroup and } \Phi \restriction G \text{ is an inner automorphism of } G \}$ includes a stationary set, then $\Phi$ is inner. This is a consequence of Corollary 6.5.7. The function $f : Z_\Phi \to \Gamma$ defined by $f(G) = g$ if $g$ implements the restriction of $\Phi$ to $G$ is regressive, and therefore constant on a stationary subset $Z$ of $Z_\Phi$. If $h = f(G)$ for all $G \in Z$, then $h$ implements $\Phi$ on $\Gamma$.

A metric variant of the Pressing Down Lemma will be discussed in §7.2. We move on to study stationary subsets of cardinals.

Nothing that we have said so far implies that $\text{I}_{NS}(\kappa)$ is not a maximal ideal. Exercise 6.7.7 shows that a regular cardinal can be partitioned into at least as many stationary subsets as there are regular cardinals below it. We can do better. An ideal $\mathcal{I}$ is $\lambda$-complete if the union of any family of fewer than $\lambda$ sets in $\mathcal{I}$ belongs to $\mathcal{I}$. In other words, $\mathcal{I}$ is $\lambda$-complete if the poset $(\mathcal{I}, \subseteq)$ is $\lambda$-complete.

**Theorem 6.5.9 (Ulam).** Suppose that $\mathcal{I}$ is a $\lambda$-complete ideal on a successor cardinal $\lambda$. Then $\lambda$ can be partitioned into $\lambda$ sets neither of which belongs to $\mathcal{I}$.

**Proof.** We may assume that $\mathcal{I}$ includes all singletons, and therefore $\beta \in \mathcal{I}$ for all $\beta < \lambda$. Let $\kappa$ be the cardinal such that $\lambda = \kappa^+$. For every ordinal $\alpha$ such that $\kappa \leq \alpha < \kappa^+$ fix a bijection $f_\alpha : \alpha \to \kappa$. For $\xi < \kappa$ and $\beta < \kappa^+$ let

$$A_{\xi, \beta} := \{ \alpha : f_\alpha(\beta) = \xi \}.$$
Then $\bigcup_{\xi < \kappa} A_{\xi, \beta} = \kappa^+ \setminus \beta$, and since $\mathbf{J}$ is $\kappa$-complete, for every $\beta$ there exists $\xi(\beta) < \kappa$ such that $A_{\xi(\beta), \beta}$ does not belong to $\mathbf{J}$. By the pigeonhole principle there exists $\xi < \kappa$ such that $X := \{\beta < \kappa^+ : \xi(\beta) = \xi\}$ has cardinality $\kappa^+$. Since each $f_\alpha$ is a function, the sets $A_{\xi, \beta}$, for $\beta \in X$, are pairwise disjoint and neither one of them belongs to $\mathbf{I}$. By adding $\lambda \setminus \bigcup_{\beta \in X} A_{\xi, \beta}$ to one of these sets, we obtain the required partition.

\[\square\]

**Corollary 6.5.10.** If $\kappa$ is an infinite cardinal then every stationary subset of its successor $\kappa^+$ can be partitioned into $\kappa^+$ stationary sets.

**Proof.** Let $\mathbf{J}$ denote the ideal of nonstationary subsets of a fixed stationary subset of $\kappa^+$. Proposition 6.5.3 implies that $\mathbf{J}$ is a $\kappa^+$-complete ideal on a set of cardinality $\kappa^+$ that contains all singletons. Therefore the conclusion follows from Theorem 6.5.9. \[\square\]

The following result will be used to construct $2^\lambda$ nonisomorphic AM algebras in every density character $\lambda \leq \aleph_1$.

**Corollary 6.5.11.** If $\lambda$ is a regular cardinal then there exists a family $F$ of $2^\lambda$ subsets of $\lambda$ such that the symmetric difference of any two distinct sets in $F$ is stationary.

**Proof.** There exists a partition of $\lambda$ into stationary sets, $S_\xi$, for $\xi < \lambda$. If $\lambda$ is a successor, this is Corollary 6.5.10. If it is a limit, this is [134, Theorem 8.10] discussed in Notes for the present chapter. For $Y \subseteq \lambda$ let $T(Y) := \bigcup_{\eta \in Y} S_\eta$. Then $Y \neq Z$ implies $T(Y) \Delta T(Z) \supseteq S_\xi$ for $\xi \in Y \Delta Z$, and $F := \{T(Y) : Y \subseteq \lambda\}$ is as required. \[\square\]

### 6.6 The $\Delta$-System Lemma

This section is devoted to standard variations of the $\Delta$-System Lemma. We also segue into the territory of Chapter 7 by proving a metric pigeonhole principle.

**Definition 6.6.1.** A $\Delta$-system with root $R$ is a family $\mathcal{F}$ of sets such that $F \cap G = R$ for all distinct $F$ and $G$ in $\mathcal{F}$.

Here is the ‘vanilla’ flavour of the $\Delta$-System Lemma.

**Proposition 6.6.2.** Every family of finite sets of cardinality $\aleph_1$ includes a $\Delta$-system of cardinality $\aleph_1$.

**Proof.** Consider a family $F_\alpha$, for $\alpha < \aleph_1$, of finite sets. As $\bigcup_\alpha F_\alpha$ has cardinality at most $\aleph_1$, we identify it with a subset of $\aleph_1$. Let $g : \aleph_1 \to \aleph_1$ be defined by $g(\beta) := \sup(\bigcup_{\alpha < \beta} F_\alpha)$ and let $C \subseteq \aleph_1$ be the club of all limit ordinals such that $g(\beta) \subseteq \beta$ for all $\beta \in C$ (Example 6.2.8 (1)). Define $f : C \to \aleph_1$ by $f(\alpha) := \sup(F_\alpha \cap \alpha)$
The Pressing Down Lemma (Proposition 6.5.6) implies that there are a stationary set \( S_0 \subseteq C \) and \( \gamma \in \mathbb{R}_1 \) such that \( F_\alpha \cap \gamma \subsetneq \gamma \) for all \( \alpha \in S_0 \). For every \( R \in \gamma \) let \( S_R := \{ \alpha \in S_0 : F_\alpha \cap \gamma = R \} \). Since \( \gamma \) is a countable ordinal, the set of all finite subsets of \( \gamma \) is countable. By Proposition 6.5.3, the union of countably many non-stationary sets is nonstationary, and therefore \( S_R \) is countable, \( \alpha \in S_R \). For every \( \alpha < \beta \in S_R \) we have \( F_\alpha \subseteq F_\beta \). Therefore \( F_\alpha \cap F_\beta = R \) and \( F_\alpha \), for \( \alpha \in S_R \cap C \), is a \( \Delta \)-system with root \( R \). \( \square \)

An equivalent (modulo the Axiom of Choice) reformulation of Proposition 6.6.2 is obtained by replacing some (or all) occurrences of ‘cardinality \( \mathbb{R}_1 \)’ with ‘uncountable.’ We proceed to discuss some other variations. The proof of Proposition 6.6.2 gives the following, occasionally useful, statement.

**Proposition 6.6.3.** For every stationary set \( T \subseteq \mathbb{R}_1 \) and a family \( F_\alpha \), for \( \alpha \in T \), of finite sets there exists \( n \in \mathbb{N} \) and a stationary set \( S \subseteq T \) such that \( \{ F_\alpha : \alpha \in S \} \) is a \( \Delta \)-system and \( |F_\alpha| = n \) for all \( \alpha \in S \). \( \square \)

Two functions \( f_j \), for \( j < 2 \), with finite domains \( F_j \), for \( j < 2 \), are isomorphic if there is a bijection \( t : F_0 \to F_1 \) such that \( f_0 = f_1 \circ t \).

**Proposition 6.6.4.** Suppose \( S \subseteq \mathbb{R}_1 \) is stationary, \( Z \) is countable, \( F_\alpha \) is a finite set, and \( g_\alpha : F_\alpha \to Z \) for \( \alpha \in S \). Then there is a \( \Delta \)-system \( \{ F_\alpha : \alpha \in T \} \) with root \( R \) such that \( T \subseteq \mathbb{R}_1 \) is stationary, and the functions \( g_\alpha \) and \( g_\beta \) are isomorphic via an isomorphism that fixes \( R \) pointwise for all \( \alpha \) and \( \beta \) in \( T \).

**Proof.** By Proposition 6.6.3 we may assume that \( F_\alpha \), for \( \alpha \in S \), form a \( \Delta \)-system with root \( R \) and that for some \( n \in \mathbb{N} \) and all \( \alpha \) we have \( |F_\alpha| = n \). For each \( \alpha > 0 \) fix a bijection \( t_\alpha : F_0 \to F_\alpha \) that is equal to the identity on the root \( R \). Since there are only countably many distinct functions among \( g_\alpha \circ t_\alpha \) for \( \alpha < \mathbb{R}_1 \), this gives a partition of \( S \) into countably many sets such that all functions in each one of them are isomorphic via an isomorphism that fixes the root. By Proposition 6.5.3 at least one of these sets is stationary. \( \square \)

We now consider a metric pigeonhole principle. A subset of a topological space is perfect if it is closed and it has no isolated points. A complete accumulation point of a subset \( X \) of a topological space is a point such that for each of its open neighbourhoods \( U \) we have \( |U \cap X| = |X| \). A space is Polish if it is separable and completely metrizable (Definition B.0.1).

**Lemma 6.6.5.** Suppose \( \mathcal{P} \) is a separable metrizable space and \( X \subseteq \mathcal{P} \) is such that \( |X| \) has uncountable cofinality. Then the set \( Z \) of complete accumulation points of \( X \) is an uncountable perfect subset of \( \mathcal{P} \).

**Proof.** Let \( U_n \), for \( n \in \mathbb{N} \), be an enumeration of a basis for \( \mathcal{P} \) and

\[
V := \bigcup \{ U_n : |X \cap U_n| < |X| \}.
\]
Lemma 6.6.6. If $M$ is a metric space of density character one in the proof of Proposition 10.4.1 (see also Exercise 6.7.12). Let $z$ be an accumulation point of $X$. By the regularity of $|x|$ we may assume that $d$ is a complete metric on $X\setminus V$ is a closed uncountable set. If $z \in Z$ and $U_m \ni z$ then $U_m \setminus X$ is uncountable.

We have yet to prove that $Z$ has no isolated points. Otherwise, let $z \in Z$ be an isolated point. Fix a compatible metric $d$ on $\mathcal{P}$ and let $V_n$ denote the $1/n$-ball centered at $z$. If $m$ is large enough so that $V_m \cap Z = \{z\}$, then $X \cap (V_n \setminus V_{n+1})$ is countable for all $n > m$. Therefore $X \cap V_m$ is countable, contradicting $z \in Z$. □

The set of known variations of Lemma 6.6.5 is uncountable. Many of them have the same proof and quite a few of them are fairly useful. We will need a particular one in the proof of Proposition 10.4.1 (see also Exercise 6.7.12).

Lemma 6.6.6. If $M$ is a metric space of density character $\kappa$, $n \geq 1$, and $X \subseteq M^n$ satisfies cof$(\{|x|\}) > \kappa$, then $X$ has a complete accumulation point.

Proof. Since $M^n$ has the same density character as $M$, it suffices to prove the case when $n = 1$. Let $\mathcal{P}$ be a basis of $M$ of cardinality $\kappa$ and let

$$V := \bigcup \{U \in \mathcal{P} : |X \cap U| < |X|\}.$$Then $|X \cap V|$ is a union of at most $\kappa$ sets, each of them of cardinality smaller than $|X|$. By the regularity of $|X|$, $X \setminus V$ is nonempty and therefore $M \setminus V$ is nonempty. If $z \in M \setminus V$ then no open neighbourhood of $z$ is included in $V$ and $z$ is a complete accumulation point of $X$. □

Two functions $f_j$ with finite domains $F_j$, for $j < 2$, and ranges in a metric space $(M,d)$ are $\varepsilon$-isomorphic for $\varepsilon > 0$ if there is a bijection $t : F_0 \to F_1$ such that $d(f_0(x), (f_1 \circ t)(x)) < \varepsilon$ for all $x \in F_0$. The following is a metric variation of Proposition 6.6.4.

Proposition 6.6.7. Suppose $\mathcal{P}$ is a separable metric space, $\varepsilon > 0$, $F_\alpha$ is a finite set, and $g_\alpha : F_\alpha \to \mathcal{P}$ is a function for $\alpha < \mathbb{R}_1$. There is an uncountable $S \subseteq \mathbb{R}_1$ such that $\{F_\alpha : \alpha \in S\}$ is a $\Delta$-system with root $R$ and functions $g_\alpha$ and $g_\beta$ are $\varepsilon$-isomorphic via an isomorphism that fixes $R$ for all $\alpha$ and $\beta$ in $S$.

Proof. By Proposition 6.6.2 we may assume that $F_\alpha$, for $\alpha < \mathbb{R}_1$, form a $\Delta$-system with root $R$, and that for some $n \in \mathbb{N}$ all $\alpha$ we have $|F_\alpha| = n$. For each $\alpha > 0$ fix a bijection $t_\alpha : F_0 \to F_\alpha$ that is equal to the identity on $R$. With $F_0$ enumerated as $\xi(j)$, for $j < n$, let $\hat{x}_\alpha := ((g_\alpha \circ t_\alpha)(\xi(j)) : j < n)$. If there exists $\hat{x} \in \mathcal{P}^n$ such that $Z := \{\alpha : \hat{x}_\alpha = \hat{x}\}$ is uncountable then $\{g_\alpha : \alpha \in Z\}$ is an uncountable family of isomorphic functions.

Otherwise, the set of all $\hat{x}_\alpha$ is uncountable and by Lemma 6.6.5 it has a complete accumulation point $\hat{x} \in \mathcal{P}^n$. The set $\{\alpha : \max_{j < n} d(\hat{x}_\alpha(j), \hat{x}(j)) < \varepsilon\}$ is uncountable and as required. □

Additional versions of the $\Delta$-system Lemma can be found in Exercise 6.7.14, Exercise 6.7.15, and Exercise 6.7.16.
6.7 Exercises

**Exercise 6.7.1.** Identifying every ordinal with the set of all smaller ordinals, prove that \( S \subseteq \aleph_1 \) is a club in \( \aleph_1 \) if and only if it a club in \( [\aleph_1]^\omega \).

**Exercise 6.7.2.** Suppose that \( C \) is a club in \( [\aleph_2]^\omega \). Prove that \(|C| \geq c\), regardless of whether the Continuum Hypothesis holds or not.

**Exercise 6.7.3.** Use the Axiom of Choice to prove that there exists \( S \subseteq [\aleph_1]^\omega \) such that neither I nor II has a winning strategy in the game \( G(S) \) (Definition 6.7.3).

**Exercise 6.7.4.** Suppose \( \kappa \) is an uncountable regular cardinal. Prove that each of the following sets is a club in \( \kappa \) (see §A.3 for the ordinal arithmetic),

1. \( \{ \omega \cdot \alpha : \alpha < \kappa \} \).
2. \( \{ \alpha : \alpha = \omega \cdot \alpha, \alpha < \kappa \} \).
3. \( \{ \alpha : \alpha = \alpha^2, \alpha < \kappa \} \).

The following is a self-strengthening of Definition 6.2.6.

**Exercise 6.7.5.** Suppose that \( C \) is a club in a directed set \( P \) Prove that for every directed \( X \subseteq C \) such that \( \sup(X) \) exists in \( P \) we have \( \sup(X) \in C \).

The following gives a limiting example for Corollary 6.4.2.

**Exercise 6.7.6.** Suppose \( X \subseteq Y \) are uncountable sets. Prove the following.

1. If \( X \subseteq Y \) and \( C \) is a club in \( [X]^\omega \) then \( \{ p \in [Y]^\omega : p \cap X \in C \} \) is a club in \( [Y]^\omega \).
2. If \( Y \setminus X \) is infinite, there exists a club \( C \) in \( [Y]^\omega \) such that \( \{ p \cap X : p \in C \} \) is not a club in \( [X]^\omega \).

**Exercise 6.7.7.** Suppose \( \lambda < \kappa \) are regular cardinals. Use a closing-off argument as in the proof of Lemma 6.2.2 to prove that every club \( C \subseteq \kappa \) contains many ordinals of cofinality \( \lambda \). Conclude that \( S_\kappa^\lambda := \{ \alpha < \kappa : \text{cof}(\alpha) = \lambda \} \) is stationary in \( \kappa \).

**Exercise 6.7.8.** Suppose \( \kappa \) is an uncountable cardinal. Consider the Hilbert space \( H := \ell_2(\kappa) \) with a distinguished basis \( e_\gamma \), for \( \gamma < \kappa \). For a partition \( P \) of \( \kappa \) into countable sets let \( \mathcal{D}[P] \) denote the set of all \( a \in \mathcal{B}(H) \) such that \( \langle ae_\xi|e_\eta \rangle \neq 0 \) implies \( \xi \) and \( \eta \) belong to the same piece of partition.

1. Prove that \( \mathcal{D}[P] \) is a von Neumann algebra.
2. Prove that \( \mathcal{B}(H) = \bigcup_P \mathcal{D}[P] \), where \( P \) ranges over partitions of \( \kappa \) into countable sets.
3. Prove that the von Neumann algebras \( \mathcal{D}[P] \) form a directed and \( \sigma \)-complete family of subalgebras of \( \mathcal{B}(H) \), and that \( \mathcal{B}(H) \) is equal to the union of this family.

We will see that the analog of \( \mathcal{D}[P] \) in the case when \( H \) is separable (Definition 9.7.5) is somewhat more difficult to analyze.
Exercise 6.7.9. Suppose $A$ is an LM algebra. Prove that all of its hereditary C*-subalgebras are LM.

Exercise 6.7.10. Suppose that $\kappa$ is an uncountable cardinal and that $A_\xi$ and $B_\xi$, for $\xi < \kappa$, are unital separable C*-algebras distinct from $C$. Prove that (using the minimal tensor product) $\bigotimes_{\xi < \kappa} A_\xi \cong \bigotimes_{\xi < \kappa} B_\xi$ if and only if there exists a partition $\kappa = \bigcup_{\eta < \lambda} X(\eta)$ such that each $X(\eta)$ is countable and $\bigotimes_{\xi \in X(\eta)} A_\xi \cong \bigotimes_{\xi \in X(\eta)} B_\xi$ for all $\eta < \lambda$.

Exercise 6.7.11. 1. Use Exercise 6.7.10 to prove that for every UHF algebra $A$ (separable or not) there exists a sequence of cardinals $\kappa(A) = (\kappa_p : p$ prime) such that $A \cong B$ if and only if $\kappa(A) = \kappa(B)$.

2. How many isomorphism types of UHF algebras of density character $\aleph_\alpha$ are there for an ordinal $\alpha$?

Exercise 6.7.12. Prove that for every infinite $\kappa \leq \mathfrak{c}$ the following are equivalent.

1. If $P$ is a Polish space and $X \subseteq P$ has cardinality $\kappa$ then the set $Z$ of complete accumulation points of $X$ is an uncountable perfect subset of $P$.

2. The cofinality of $\kappa$ is uncountable.

Exercise 6.7.13. Prove the following variant of Proposition 6.6.7.

Suppose $\mathcal{P}$ is a Polish space, $\varepsilon > 0$, $S \subseteq \mathcal{R}$ is stationary and for all $\alpha \in S$ a finite set $F_\alpha$ and $g_\alpha : F_\alpha \to \mathcal{P}$ are given. There is a stationary $S' \subseteq S$ such that $\{F_\alpha : \alpha \in S'\}$ is a $\Delta$-system with root $R$ and functions $g_\alpha$ and $g_\beta$ are $\varepsilon$-isomorphic via an $\varepsilon$-isomorphism that fixes $R$ for all $\alpha$ and $\beta$ in $S$.

Exercise 6.7.14. 1. Find an infinite family of finite sets that does not include an infinite $\Delta$-system.

2. Prove that for every $n \in \mathbb{N}$ every infinite family of $n$-element sets includes an infinite $\Delta$-system.

Hint: For the second part: If $F_j = \{f(j,i) : i < n\}$ consider the colouring $c : \mathcal{P}(n)$ by $c(j,k) := \{i < n : f(j,i) \in F_k\}$. Apply Ramsey’s theorem (see e.g., [120]) and analyze the resulting homogeneous set.

Exercise 6.7.15. Assuming the Continuum Hypothesis, prove that every family of countable sets of cardinality $\aleph_2$ includes a $\Delta$-system of cardinality $\aleph_2$.

More generally, suppose $\kappa$ is an infinite cardinal such that $2^\kappa = \kappa^+$. Prove that every family of $\kappa^{++}$ sets each of which has cardinality at most $\kappa$ includes a $\Delta$-system of cardinality $\kappa^{++}$.

A weak $\Delta$-system with root $R$ is a collection $\mathcal{F}$ of sets such that $F \cap G \subseteq R$ for all distinct $F$ and $G$ in $\mathcal{F}$. Every $\Delta$-system is a weak $\Delta$-system with the same root. Every collection $\mathcal{F}$ is a weak $\Delta$-system with root $\bigcup \mathcal{F}$ and the utility of a weak $\Delta$-system is inversely proportional to the cardinality of its root.

Exercise 6.7.16. Prove that every collection of cardinality $\aleph_2$ of countable sets includes a weak $\Delta$-system of cardinality $\aleph_2$ and a countable root.
Exercise 6.7.17. Suppose $X$ is a Polish space. Prove that for every cardinal $\kappa$ there are no uncountable families of disjoint open sets in the product space $X^\kappa$.

For an infinite cardinal $\kappa$, $H_\kappa$ is the set of all sets of hereditary cardinality less than $\kappa$ (Definition A.7.1).

Exercise 6.7.18. Suppose $\kappa$ is an uncountable cardinal and $M$ is a countable elementary submodel of $H_\kappa$.

1. Prove that $\delta_M := M \cap \mathbb{R}_1$ is a countable ordinal.
2. Prove that $\delta_M = \min(\bigcap\{C : C \in M, C \text{ is a club in } \mathbb{R}_1\})$.

Exercise 6.7.19. Suppose $M$ is an elementary submodel of $H_\theta$ for some uncountable cardinal $\theta$. Prove that every countable set that belongs to $M$ is also a subset of $M$.

Exercise 6.7.20. Prove the $\Delta$-System Lemma by using elementary submodels.

Hint: Given an uncountable family $\mathcal{F}$ of finite sets, find a large enough cardinal $\theta$ and a countable elementary submodel $M \subseteq H_\theta$ (Definition A.7.1) that has $\mathcal{F}$ as an element. Pick $F \in \mathcal{F} \setminus M$. Use elementarity in $M$ to find an uncountable $\Delta$-system with the root $R := F \cap M$.

Exercise 6.7.21. Let $\kappa$ be an uncountable cardinal.

1. Suppose $\kappa$ is regular and prove that $(H_\kappa, \in)$ satisfies all axioms of ZFC except possibly the power set axiom.
2. Prove that $(H_\kappa, \in)$ satisfies the power set axiom if and only if $\lambda < \kappa$ implies $2^\lambda < \kappa$ for all cardinals $\lambda$. (A cardinal $\kappa$ with this property is said to be a strong limit cardinal.)
3. Prove that strong limit cardinals exist (in ZFC).
4. Prove that $(H_\kappa, \in)$ is a model of ZFC if and only if $\kappa$ is both regular and strong limit. (Such $\kappa$ is said to be a strongly inaccessible cardinal.)
5. Suppose that ZFC is consistent. Prove that the theory ZFC+‘there are no strongly inaccessible cardinals’ is consistent.

The following exercises show that from a certain point of view the only clubs in posets of the form $(|X|^{<\theta}, \subseteq)$ that matter are the clubs in posets of the form $(|H_\theta|^{<\theta}, \subseteq)$ for a large enough $\theta$.

Exercise 6.7.22. Suppose $X$ is uncountable and $C$ is a club in $|X|^{<\theta}$. Prove that for every large enough cardinal $\theta$ some club $D$ in $|H_\theta|^{<\theta}$ satisfies $\{p \cap X : p \in D\} \subseteq C$.

Exercise 6.7.23. Suppose that $X$ is an uncountable set in $H_\kappa$ and that $C \subseteq |X|^{<\theta}$ is a club defined by a first-order formula with a parameter $\vec{a}$ in $H_\kappa$. Prove that every countable $M \subseteq H_\kappa$ such that $\{\vec{a}, X\} \in M$ satisfies $M \cap X \in C$.7

7 Used in the proof of Lemma 8.2.9.
Notes for Chapter 6

§6.1 AM and LM algebras were named and studied in [93]. The following is [93, Theorem 1.3], and it answers the first part of Question 6.1.4.

Theorem 6.7.24. 1. Every locally matricial $C^*$-algebra of density character not greater than $\aleph_1$ is approximately matricial.
2. For every cardinal $\kappa \geq \aleph_2$ there exists a locally matricial $C^*$-algebra of density character $\kappa$ that is not approximately matricial. \qed

The first part is a consequence of a standard fact about separable AF algebras ([52, Corollary III.3.3]). The second part uses a cocycle crossed product construction.\(^8\)

§6.3 The results of this section are a part of the folklore of set theory with the exception of the notion of concretely represented, directed and $\kappa$-complete sets. To be specific, it is our study of clubs in directed $\kappa$-complete sets that are not concretely represented (such as Sep$(A)$ for a nonseparable metric structure $A$) that is somewhat novel.

§6.4 Theorem 6.4.1 is due to Kueker. Exercise 6.7.11 is taken from [94].

The importance of infinite games in modern set theory cannot be overestimated. See e.g., [149].

§6.5 The results of this section are a part of the folklore of set theory. Theorem 6.5.9 is due to Ulam and the family of sets $A_\xi, \beta$, for $\xi < \kappa$ and $\kappa \leq \beta < \kappa^+$, used in its proof is known as an Ulam matrix. The conclusion of Theorem 6.5.9 may fail for a $\kappa$-complete ideal on a regular cardinal\(^10\) but every stationary subset of an arbitrary regular cardinal $\lambda$ can be partitioned into $\lambda$ stationary sets, by a result of Solovay ([134, Theorem 8.10]). Another remark is worth making. The proof of Theorem 6.5.9 requires the Axiom of Choice for the simultaneous choice of functions $f_\alpha$ for all $\kappa \leq \alpha < \kappa^+$. Large cardinals (or the Axiom of Determinacy in $L(\mathbb{R})$) imply that the club filter on $\mathfrak{C}_{\kappa}$ is an ultrafilter in $L(\mathbb{R})$, probably the most important model of ZF in which the Axiom of Choice fails, (see [149]).

§6.6 Set-theorists use the $\Delta$-system Lemma all the time, and calling it the ‘Kaplansky Density Theorem of Set Theory’ (cf. [194, §2.3.4]) would not be an exaggeration. A form of Exercise 6.7.17 is quite important in Paul Cohen’s proof that the negation of the Continuum Hypothesis is relatively consistent with ZFC.

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\(^8\) The way in which the crossed product is twisted in this proof has a similar flavour to the way that the group $C^*$-algebra is twisted in §10.1.

\(^9\) This property is admittedly underwhelming. The depth of this proof comes from operator algebras, and not set theory.

\(^10\) Where the qualifier ‘may fail’ is interpreted appropriately. For example, a measurable cardinal carries a nonprincipal $\kappa$-complete ideal whose complement is an ultrafilter. For more on this fascinating subject see [149, §2, §16, §17] and [104].
Chapter 7
Infinitary Combinatorics, II: The Metric Case

The basics of infinitary combinatorics as introduced in Chapter 6 require a bit of tweaking before they can be applied to metric structures. The correct metric analogs of these concepts, suitable for analyzing C*-algebras and other metric structures, are rarely obvious and their direct translations are typically wrong (or ‘not even wrong,’ to paraphrase the famous Wolfgang Pauli’s quote). In this chapter we generalize clubs and stationary subsets of $\mathcal{X}_{\aleph_0}$ to clubs and stationary subsets of $\text{Sep}(A)$, the space of all separable substructures of a nonseparable metric structure $A$. We study Löwenheim–Skolem-type reflection results in some detail and give a metric version of the Pressing Down Lemma.

7.1 Spaces of Models

Natasha and Pierre, left alone, also began to talk as only a husband and wife can talk, that is, with extraordinary clearness and rapidity, understanding and expressing each other’s thoughts in ways contrary to all rules of logic, without premises, deductions, or conclusions, and in a quite peculiar way. Natasha was so used to this kind of talk with her husband that for her it was the surest sign of something being wrong between them if Pierre followed a line of logical reasoning. When he began proving anything, or talking argumentatively and calmly and she, led on by his example, began to do the same, she knew that they were on the verge of a quarrel.

L.N. Tolstoy, War and Peace

In this section we use the results of Chapter 6 to study countable submodels of a given uncountable first-order structure and prove the Downwards Löwenheim–Skolem theorem. Metric versions of these results will be proved in §7.1.2. The two subsections of this section require familiarity with the syntax and semantics of first order logic (§D.1 and §D.2), while following the lines of logical reasoning as closely as possible.
7.1.1 The Space of Models of a First-Order Theory

An onslaught of apparently mindless (and yet largely necessary, or at least useful) notation that awaits the unsuspecting reader will hopefully be eased by a motivating example.

Example 7.1.1. 1. If $\Gamma$ is an uncountable group then
\[
\{G \in [\Gamma]_{\aleph_0} : G \text{ is a subgroup of } \Gamma\}
\]
is a club in $[\Gamma]_{\aleph_0}$. Let $f : \Gamma \times \Gamma \to \Gamma$ and $g : \Gamma \to \Gamma$ be defined by $f(a, b) = ab$ and $g(a) = a^{-1}$. The set of all $G \in [\Gamma]_{\aleph_0}$ closed under both $f$ and $g$ is a club.

The elements of this club are the subgroups of $\Gamma$.

2. Now suppose that $\Gamma$ is a divisible uncountable group. Then
\[
\{G \in [\Gamma]_{\aleph_0} : G \text{ is a divisible subgroup of } \Gamma\}
\]
is a club in $\Gamma$. To prove this, for $n \geq 2$ choose $h_n : \Gamma \to \Gamma$ such that $nh_n(a) = a$ for all $a \in \Gamma$. The set of all $G \in [\Gamma]_{\aleph_0}$ closed under $f$, $g$ as in (1) and all $h_n$ for $n \geq 2$ is a club, and every $G$ in this set is a divisible subgroup of $\Gamma$.

3. Suppose $R$ is a countable ring and $M$ is an uncountable left $R$-module. Then
\[
\{N \in [M]_{\aleph_0} : N \text{ is a left } R\text{-module}\}
\]
is a club. By (1) the set $C := \{N \in [M]_{\aleph_0} : N \text{ is an abelian group}\}$ includes a club.

For $r \in R$ let $f_r : M \to M$ be the left multiplication by $r$. Since $R$ is countable, the set $\{N \in C : f_r[N] \subseteq N\}$ includes a club, and each of its elements is a left $R$-module.

Example 7.1.1 shows that each of the properties of being a group, a divisible group, or an $R$-module 'reflects' to countable substructures in the sense that the set of all substructures with this property includes a club. One could use analogous closing-off arguments to prove that the membership in many other algebraic categories reflects to countable structures: fields, algebraically closed fields, division rings, groups with a trivial center, divisible abelian groups, to name but a few.

Instead, we take a rather general route. As reviewed in §D.1, a (single-sorted, first-order) language $L$ consists of function, relation, and constant symbols. An $L$-structure $\mathfrak{A}$ is a set $A$ (the domain of $\mathfrak{A}$) equipped with the interpretations of all function, relation, and constant symbols of $L$. A constant symbol $c$ is interpreted as an element $c^\mathfrak{A}$ of $A$, an $n$-ary relation symbol $R$ is interpreted as a subset $R^\mathfrak{A}$ of $A^n$, and an $n$-ary function symbol $F$ is interpreted as a function $F^\mathfrak{A}$ from $A^n$ to $A$. If a relation or function symbol is $n$-ary then $n$ is known as its arity. The terms and formulas in the language $L$ are defined recursively (see §D.1).

---

1 Set-theorists will notice that it is unrelated to the reflection of stationary sets. See, however, Theorem A.6.1.
The \textit{cardinality} of a language $\mathcal{L}$, $|\mathcal{L}|$, is the cardinality of the set of all function, relation, and constant symbols in $\mathcal{L}$. The cardinality of the set of all $\mathcal{L}$-formulas is equal to $\max(|\mathcal{L}|, \Upsilon_0)$ (Lemma D.1.7).

In this section we will consider only languages $\mathcal{L}$ with no function or constant symbols. This convention will not prevent us from considering languages with function and constant symbols whenever convenient; this common practice is justified by Remark D.1.4.

\textbf{Definition 7.1.2.} For a set $X$ and language $\mathcal{L}$, by $\text{Struct}(\mathcal{L}, X)$ we denote the space of all $\mathcal{L}$-structures with domain $X$.

Suppose $\mathfrak{A} \in \text{Struct}(\mathcal{L}, X)$ and $Y \subseteq X$. The $\mathcal{L}$-substructure of $\mathfrak{A}$ with domain $Y$ is $(n(R))$ denotes the arity of $R$)

$$\mathfrak{A} \upharpoonright Y := \bigcup_{R \subseteq \mathcal{L}} \mathfrak{A}(R^\mathfrak{A} \cap \Gamma_n(R)),$$

Such structure is a \textit{submodel} of $\mathfrak{A}$. For an $\mathcal{L}$-formula $\varphi(\bar{x})$ and tuple $\bar{a}$ of the appropriate sort, the interpretation of $\varphi(\bar{x})$ at $\bar{a}$ in $\mathfrak{A}$ (see §D.1 and §D.2) is denoted by $\varphi^\mathfrak{A}(\bar{a})$. A submodel $\mathfrak{B}$ of $\mathfrak{A}$ is said to be \textit{elementary}, or $\mathfrak{B} \preceq \mathfrak{A}$ in symbols, if for every $\mathcal{L}$-formula $\varphi(\bar{x})$ and every $\bar{a}$ in the domain of $\mathfrak{B}$ we have $\varphi^\mathfrak{B}(\bar{a}) = \varphi^\mathfrak{A}(\bar{a})$.

\textbf{Theorem 7.1.3 (Downwards Löwenheim–Skolem Theorem).} Suppose $\mathcal{L}$ is a language of cardinality $\lambda$ and $\mathfrak{A}$ is an $\mathcal{L}$-structure with domain $A$ of cardinality greater than $\lambda$. Then the set $\{Y \subseteq |A|^\lambda : \mathfrak{A} \upharpoonright Y \preceq \mathfrak{A}\}$ is a club in $|A|^\lambda$.

\textbf{Proof.} By the Tarski–Vaught test (Theorem D.1.3) $\mathfrak{A} \upharpoonright Y \preceq \mathfrak{A}$ if and only if for every $n \geq 0$, every $\mathcal{L}$-formula $\varphi(\bar{x}, y)$ with $n + 1$ free variables, and every $\bar{a} \in Y^n$ we have

$$(\exists y \in A) \varphi^\mathfrak{A}(\bar{a}, y) \rightarrow (\exists y \in Y) \varphi^\mathfrak{A}(\bar{a}, y).$$

Let $d_0$ be a distinguished element of $A$. Define $f_\varphi : |A|^n \rightarrow A$ by $f_\varphi(\bar{a}) = d_0$ if there is no $b \in A$ such that $\varphi^\mathfrak{A}(\bar{a}, b)$ holds, and $f_\varphi(\bar{a}) = b$ if $b \in A$ is such that $\varphi^\mathfrak{A}(\bar{a}, b)$ holds. (This is a \textit{Skolem function} for $\varphi$.)

By Example 6.2.8, the set $D_\varphi := \{B \subseteq |A|^\lambda : f[B^n] \subseteq B\}$ is a club. Since the cardinality of $\mathcal{L}$ is $\lambda$, by Proposition 6.2.9 the intersection of all $D_\varphi$ is a club. Every element of this club satisfies the Tarski–Vaught test and is therefore an elementary submodel of $\mathfrak{A}$. \hfill \Box

The following variant of the Löwenheim–Skolem theorem will be useful in §8.3.

\textbf{Theorem 7.1.4.} Suppose $\kappa$ is a regular cardinal, $\mathcal{L}$ is a language of cardinality smaller than $\kappa$ and $\mathfrak{A}$ is an $\mathcal{L}$-structure with domain $\kappa$. Then $\{\alpha < \kappa : \mathfrak{A} \upharpoonright \alpha \preceq \mathfrak{A}\}$ is a club in $\kappa$.

\textbf{Proof.} For every existential formula $\varphi$ let $f_\varphi$ be the Skolem function as in the proof of Theorem 7.1.3. By Lemma D.1.7, the set of all $\mathcal{L}$-formulas has cardinality smaller than $\kappa$. For $\varphi$ of arity $n$ let $C_\varphi := \{\alpha < \kappa : f_\varphi(\alpha^n) \subseteq \alpha\}$. The required club is equal to $\bigcap_{\varphi} C_\varphi$. \hfill \Box
7.1.2 The Space of Models of a Metric Theory

As in §7.1.1, we start with a friendly warmup example meant to appease a reader suspicious of syntax-heavy mathematics.

**Example 7.1.5.** Suppose \( A \) is a nonseparable \( C^* \)-algebra and let \( X \) be a dense subset of \( A \). Then the set

\[
\{ Y \in [X]^{\aleph_0} : Y \text{ is a } C^*\text{-subalgebra} \}
\]

includes a club in \([X]^{\aleph_0}\). This is because as in Example 7.1.1, the set of countable \( \mathbb{Q} + i\mathbb{Q}^* \)-subalgebras of \( X \) includes a club, and the closure of any element of this club is a \( C^* \)-subalgebra of \( A \).

**Example 7.1.6.** Suppose that a nonseparable \( C^* \)-algebra \( A \) is in addition simple. Then the set

\[
\{ Y \in [X]^{\aleph_0} : Y \text{ is a simple } C^*\text{-subalgebra} \}
\]

includes a club. We provide a proof in the case when \( A \) is unital, as the general case is only notationally different. For a nonzero \( a \in A \) there are \( n \geq 1, \bar{b} \in A^n \), and \( \bar{c} \in A^n \) such that \( 1 = \sum_{i<n} b_i a c_i \). Since \( X \) is dense in \( A \), there are functions \( f_i : X \to X \) and \( g_i : X \to X \), for \( i \in \mathbb{N} \), such that for every \( a \in X \setminus \{0\} \) there is \( n \geq 1 \) satisfying

\[
\|1 - \sum_{i<n} f_i(a) a g_i(a)\| < 1.
\]

Lemma 1.2.6 implies that \( \sum_{i<n} f_i(a) a g_i(a) \) is invertible. The set of all \( Y \in [X]^{\aleph_0} \) such that \( Y \) is a \( C^* \)-subalgebra of \( A \) that are closed under all \( f_i \) and all \( g_i \) includes a club. If \( Y \) belongs to this club then \( Y \) is a simple \( C^* \)-subalgebra of \( A \).

The logic of metric structures and the terminology used in the following example are reviewed in §D.2.

**Example 7.1.7.** In the language of \( C^* \)-algebras the sorts correspond to the \( n \)-balls for \( n \geq 1 \), the metric is the uniform metric \( d(x,y) := \|x - y\| \), and the function symbols are \(+, \cdot, \text{ and } ^*\), with the standard interpretations. A \( C^* \)-algebra \( A \) uniquely defines the metric structure \( \mathcal{M}(A) \), defined as follows. Its sorts are the \( n \)-balls of \( A \), denoted \( A_n \), for \( n \geq 1 \). For every \( n \), the restriction of functions \(+, \cdot, \text{ and } ^*\) to the \( n \)-ball is easily checked to be uniformly continuous.

The **domain** of a metric structure \( \mathfrak{M} \) is the union of all of its sorts. Given a language \( \mathcal{L} \), terms and formulas in \( \mathcal{L} \) are defined recursively (see §D.2). Analogously to §7.1.1, by replacing a function with the distance to its graph (taking \( A^n \) as a metric space with respect to \( d(\bar{a}, \bar{b}) := \max_{i<n} d(a_i, b_i) \)) we may assume that \( \mathcal{L} \) has only relation symbols.

Recall that the **density character** of a metric structure \( M \) is the minimal cardinality of a dense subset. In metric structures, it coincides with the weight\(^3\) and (if the language of \( M \) is countable) with the minimal cardinality of a dense substructure of \( M \).

\( ^2 \) Note however that passing to the unitization would not work here, as a unitization is never simple.

\( ^3 \) In an arbitrary topological space the density may be strictly smaller than the weight; consider for example \( \beta \mathbb{N} \).
Definition 7.1.8. If $M$ is a metric structure in a separable language then $\text{Sep}(M)$ denotes the poset of its (closed, by definition) separable substructures ordered by inclusion. As in Example 6.2.5, this poset is $\sigma$-complete. The supremum of a directed family is equal the closure of its union.

Suppose that the language $\mathcal{L}$ contains only relation symbols and $A$ is an $\mathcal{L}$-structure with metric $d$ and a distinguished dense subset $X$. The restriction of $d$ to $X^2$ is uniquely determined by

$$\text{Code}_d(A) := \{(\bar{\alpha}, q) \in X^2 \times \mathbb{Q} : d(\alpha_0, \alpha_1) > q\},$$

and the restriction of an $n$-ary relation $R$ of $A$ to $X^n$ is uniquely determined by

$$\text{Code}_R(A) := \{(\bar{\alpha}, q) \in X^n \times \mathbb{Q} : R(\bar{\alpha}) > q\}.$$

Therefore, $A$ is uniquely determined by the appropriately coded disjoint union,

$$\text{Code}(A) := \text{Code}_d(A) \sqcup \bigsqcup R \text{Code}_R(A)$$

and every metric structure with $X$ as a distinguished subset has a code included in

$$\mathcal{X}(X, \mathcal{L}) := (X^2 \sqcup \bigsqcup R X^n(R)) \times \mathbb{Q}.$$

Let us see how one 'unravels' $A$, given $\text{Code}(A)$. The latter set uniquely determines the metric $d^A$ on $X$. The completion of this space is the domain of $A$, and the interpretation $R^A$ of an $n$-ary relation $R$ in $A$ is uniquely determined as the continuous extension of its restriction to $X^n$.

The set $\text{Code}(A)$ is a code for $A$. The space $\text{Struct}(\mathcal{L}, X)$ of all codes for $\mathcal{L}$-structures with a distinguished dense set $X$ is a subset of $\mathcal{P}(X^2 \sqcup \bigsqcup R X^n(R)) \times \mathbb{Q}$.

By fixing a bijection between $X$ and $\sqcup R X^n(R) \times \mathbb{Q}$, $\text{Struct}(\mathcal{L}, X)$ is identified with a subset of the power set of $X$. Not every subset of $\mathcal{X}(X, \mathcal{L})$ codes an $\mathcal{L}$-structure, but see Exercise 7.5.2.

Suppose $A$ is a metric structure and $B$ is a substructure of $A$. As in the discrete case, we say $B$ is an elementary submodel of $A$ is and write $B \preceq A$ if for every $\mathcal{L}$-formula $\phi(\bar{x})$ and every $\bar{a}$ in the domain of $B$ we have $\phi(\bar{a})^B = \phi(\bar{a})^A$. Proofs virtually identical to those of Theorem 7.1.3 and Theorem 7.1.10 using the metric Tarski–Vaught test (Theorem D.2.9) give metric variants of the Löwenheim–Skolem Theorem (for the density character of a theory see Definition D.2.4).

Theorem 7.1.9 (Downwards Metric Löwenheim–Skolem Theorem). Suppose $T$ is a separable metric theory and $\mathfrak{A}$ is a model of $T$ of density character greater than $\lambda$ with domain $A$. Then $\{ Y \in \text{Sep}(A) : \mathfrak{A} \models T \mid Y \preceq A \}$ is a club in $\text{Sep}(A)$. □

Theorem 7.1.10. Suppose $\kappa$ is a regular cardinal, $\mathcal{L}$ is a metric language of density character smaller than $\kappa$, and $\mathfrak{A}$ is an $\mathcal{L}$-structure of density character $\kappa$. Identifying $\kappa$ with a dense subset of $\mathfrak{A}$, the set $\{ \alpha < \kappa : \mathfrak{A} \models T \mid A \preceq A \}$ is a club in $\kappa$. □
7.2 A Metric Pressing Down Lemma

In this section we prove the analog of the Pressing Down Lemma in the context of metric structures. We also prove that every nonseparable metric structure \( M \) has a dense subset \( D \) such that the function \( p \mapsto p \) is an injection on a club in \([D]^{\aleph_0}\).

The Pressing Down Lemma (Proposition 6.5.6) asserts that every regressive function whose domain is a stationary subset of a concretely represented directed and \( \kappa \)-complete poset is constant on a stationary set. The metric analog of \([X]^{\aleph_0}\) is the space \( \text{Sep}(M) \) of all separable substructures of a metric structure \( M \) (Example 7.1.8).

**Definition 7.2.1.** A function \( f \) is regressive if \( f(A) \in A \) for all \( A \in \text{dom}(f) \).

The fact that the supremum of an increasing sequence in \( \text{Sep}(M) \) is strictly larger than its union (i.e., that \( \text{Sep}(M) \) is not concretely represented in the sense of Definition 6.3.1) presents some amusing technical difficulties when handling clubs and stationary sets. For example, the Pressing Down Lemma is false in metric context.

**Example 7.2.2.** Take your favourite nonseparable C*-algebra (if any), call it \( M \). Let us assume that it has density character \( \aleph_1 \) and fix an injection \( f: \aleph_1 \to M \) with a dense range. Then \( \{f[\alpha] : \alpha < \aleph_1\} \) is a club in the poset of closed separable subsets of \( M \). By the Downwards Löwenheim–Skolem Theorem, it includes a club \( C \) of separable C*-subalgebras of cardinality \( \aleph_1 \). Let \( A_0 \) be the minimal element of \( C \). Any injection from \( C \) into \( A_0 \) is a regressive function which is not constant on any stationary subset of \( C \).

We will state and prove the correct metric version of the Pressing Down Lemma.

**Definition 7.2.3.** Let \( M \) be a metric structure in a countable language. A countable substructure of \( M \) is an element of \([M]^{\aleph_0}\) closed under all functional symbols in the language of \( M \). Let

\[
\text{Ctble}(M) := \{p \in [M]^{\aleph_0} : p \text{ is a substructure of } M\}.
\]

If the language of a metric structure \( M \) contains no functional symbols then \([M]^{\aleph_0} = \text{Ctble}(M)\). Since countable substructures of \( M \) are rarely complete, they are usually not metric structures in the sense of logic of metric structures (§7.1.2). If the language of \( M \) is countable, then \( \text{Ctble}(M) \) is a club in \([M]^{\aleph_0}\). It is therefore a concretely represented directed and \( \sigma \)-complete set.

The following is really more of a motivational slogan than a lemma that requires a proof.

**Lemma 7.2.4.** Suppose \( \mathbb{P}_0 \) and \( \mathbb{P}_1 \) are directed and \( \sigma \)-complete posets and \( \Phi \) is an order-isomorphism between \( \mathbb{P}_0 \) and a club \( C \) in \( \mathbb{P}_1 \). Then

1. A set \( X \subseteq \mathbb{P}_1 \) is stationary (nonstationary, includes a club) in \( \mathbb{P}_1 \) if and only if \( \Phi^{-1}[X \cap C] \) is stationary (nonstationary, includes a club, respectively) in \( \mathbb{P}_0 \).
2. A set \( X \subseteq \mathbb{P}_0 \) is stationary (nonstationary, includes a club) in \( \mathbb{P}_0 \) if and only if \( \Phi[X] \) is stationary (nonstationary, includes a club, respectively) in \( \mathbb{P}_1 \).
Proof. If $X \subseteq \mathcal{P}_j$ and $C \subseteq \mathcal{P}_j$, then $X$ contains a club if and only if $X \cap C$ contains a club. All six assertions follow immediately.

By Theorem 6.4.1, the clubs in $[D]^{|\aleph_0|}$ are well understood. It is therefore desirable for a nonseparable metric structure $M$ to have a club in $\text{Sep}(M)$ isomorphic to a club in $[D]^{|\aleph_0|}$ for a well-chosen discrete set $D$. Sometimes the choice of $D$ is obvious.

Example 7.2.5. 1. Suppose $\Gamma$ is an uncountable group and $M := C^*_\alpha(\Gamma)$. Let $C := \{ G \in [\Gamma]^{|\aleph_0|} : G$ is a subgroup $\}$

and define $\Psi : C \to \text{Sep}(C^*_\alpha(\Gamma))$ by $\Psi(G) := C^*_\alpha(G)$ (this is well-defined by Exercise 3.10.15). Then $\Psi$ is an isomorphism between $C$ and a club in $\text{Sep}(C^*_\alpha(\Gamma))$.

2. Suppose that $A_j$, for $j \in \mathbb{J}$, is a family of unital and separable $C^*$-algebras and let $A := \bigotimes_{j \in \mathbb{J}} A_j$. For a nonempty $X \subseteq \mathbb{J}$ the algebra $A_X := \bigotimes_{j \in X} A_j$ can be naturally identified with a subalgebra of $A$. Then $\Psi : [\mathbb{J}]^{|\aleph_0|} \to \text{Sep}(A)$ defined by $\Psi(X):= A_X$ is an isomorphism between $[\mathbb{J}]^{|\aleph_0|}$ and a club in $\text{Sep}(A)$.

Not every metric structure $M$ permits an obvious isomorphism between a club in $\text{Sep}(M)$ and a club in $[D]^{|\aleph_0|}$ for some discrete set $D$ as in Example 7.2.5, but we will prove that such $D$ always exists. If $M$ is a nonseparable metric structure and $D \subseteq M$ is dense, define $\Phi_D : [D]^{|\aleph_0|} \to \text{Sep}(M)$ by sending a set to its closure,

$$\Phi_D(p) := \overline{p}.$$ 

The range of $\Phi_D$ is easily seen to be cofinal in $\text{Sep}(M)$. However, in order to assure that $\Phi_D$ is an isomorphism between clubs in $[D]^{|\aleph_0|}$ and $\text{Sep}(M)$, some care needs to be taken when choosing $D$. In general, if $p$ and $q$ are elements of $[D]^{|\aleph_0|}$ then $\Phi_D(p) \subseteq \Phi_D(q)$ does not necessarily imply $p \subseteq q$, and $\Phi_D(p) \cap D$ may not be equal to $p$. Some other technical annoyances are summarized in the following example and in Exercise 7.5.3.

Example 7.2.6. If $A$ is a $C^*$-algebra of density character $\aleph_1$, then there are a dense $D \subseteq A$ and a nonstationary $X \subseteq [D]^{|\aleph_0|}$ such that $\Phi_D[X]$ is a club in $\text{Sep}(A)$.

Write $A$ as an inductive limit of a continuous, strictly increasing, chain of separable $C^*$-subalgebras $A_\alpha$, for $\alpha < \aleph_1$. Every limit $\alpha$ satisfies $A_\alpha \neq \bigcup_{\beta < \alpha} A_\beta$. Choose a countable dense subset $p_\alpha$ of $A_\alpha \backslash \bigcup_{\beta < \alpha} A_\beta$. Let $D := \bigcup_{\alpha \in \text{limit}} p_\alpha$. For a limit $\alpha$ let $q_\alpha := \bigcup_{\beta < \alpha} p_\beta$ and $r_\alpha := D \cap A_\alpha = \bigcup_{\beta \leq \alpha} p_\beta$.

Then $C := \{ q_\alpha : \alpha \text{ limit} \}$ is a club in $[D]^{|\aleph_0|}$, $X := \{ r_\alpha : \alpha \text{ limit} \}$ is disjoint from $C$, but $\Phi_D(r_\alpha) = r_\alpha = A_\alpha$ for all $\alpha$ and $\Phi_D[X] = \Phi_D[C]$ is a club in $\text{Sep}(A)$.

Proposition 7.2.7. If $M$ is a nonseparable metric space, then there exists a dense $D \subseteq M$ such that $\{ p \in [D]^{|\aleph_0|} : D \cap \overline{p} \neq p \}$ includes a club $\mathcal{C}_0$. Therefore the function $\Phi_D(p) := \overline{p}$ is an isomorphism between $\mathcal{C}_0$ and a club in $\text{Sep}(M)$.

Proof. For $n \geq 1$ fix $D_n \subseteq M$ such that (i) $d(x, y) \geq 2^{-n}$ for all distinct $x$ and $y$ in $D_n$, and (ii) $D_n$ is a maximal subset of $M$ with this property. Hence for every $x \in M$ there
is at least one \( y \in D_n \) such that \( d(x, y) < 2^{-n} \), and there is at most one \( z \in D_n \) such that \( d(x, z) < 2^{-n-1} \). If such \( z \) exists we write \( f_n(x) := z \); otherwise let \( f_n(x) := y_n \) for a fixed \( y_n \in D_n \). The set \( D := \bigcup_{n \geq 1} D_n \) is clearly dense in \( M \). Let

\[
C_0 := \{ p \in [D]^{\aleph_0} : (\forall j \geq 1) f_j[p] \subseteq p \}.
\]

By Example 6.2.8 (2) and Proposition 6.2.9, \( C_0 \) is a club.

Suppose \( p \in C_0 \). In order to verify \( p \cap D \subseteq p \), fix \( x \in p \cap D \) and \( j \) such that \( x \in D_j \). Since for every \( i < j \) there exists at most one \( y \in D_i \) satisfying \( d(x, y) < 2^{-i-1} \), we can fix an \( m \) such that \( d(x, y) \geq 2^{-m} \) for all \( y \in \bigcup_{i < j} D_i \setminus \{ x \} \).

Suppose \( z \in p \setminus \{ x \} \) and \( d(x, z) < \min(2^{-m}, 2^{-j-1}) \). Then \( z \in D_n \) for some \( n > j \), and therefore \( x = f_m(z) \) belongs to \( p \). Since \( x \in p \cap D \) was arbitrary, \( p \cap D \subseteq p \), and therefore \( p \cap D = p \), as required.

It remains to prove that the function \( \Phi_D(p) := p \) has the required properties. It is clearly order-preserving and order-continuous. Since \( \Phi_D[C_0] \) is cofinal in Sep(\( M \)) and \( \subseteq \)-preserving by the choice of \( D \), \( C_1 := \Phi_D[C_0] \) is a club in Sep(\( M \)). The function \( \Psi : C_1 \to C_0 \) defined by \( \Psi(A) = A \cap D \) is an order-preserving and order-continuous inverse of \( \Phi_D \), and \( \Phi_D \) is an isomorphism between \( C_0 \) and \( C_1 \).

Proposition 7.2.7 and ‘motivational Lemma 7.2.4’ together yield the following.

**Corollary 7.2.8.** Every nonseparable metric space \( M \) has a dense subset \( D \) such that the following holds.

1. Some \( X \subseteq [D]^{\aleph_0} \) is stationary (nonstationary, includes a club) in \([D]^{\aleph_0}\) if and only if \( \{ p : p \in [D]^{\aleph_0} \} \) is stationary (nonstationary, includes a club, respectively) in Sep(\( M \)).
2. Some \( Y \subseteq \text{Sep}(M) \) is stationary (nonstationary, includes a club) in Sep(\( M \)) if and only if \( \{ p \in [D]^{\aleph_0} : p \in Y \} \) is stationary (nonstationary, includes a club, respectively) in \([D]^{\aleph_0}\).

It is important that \( x \) in the statement of Proposition 7.2.9 does not depend on the choice of \( \varepsilon > 0 \).

**Proposition 7.2.9 (Metric Pressing Down Lemma).** If \( M \) is a nonseparable metric structure in a separable language then for every regressive function \( g \) whose domain is a stationary subset of Sep(\( M \)) there exists \( x \in M \) such that for every \( \varepsilon > 0 \) the set \( \{ A \in \text{dom}(g) : d(g(A), x) < \varepsilon \} \) is stationary.

**Proof.** Suppose otherwise. For every \( x \in M \) fix \( \varepsilon(x) > 0 \) such that

\[
S_x := \{ A \in \text{dom}(g) : d(g(A), x) < \varepsilon(x) \}
\]

is nonstationary. The open balls \( B_{\varepsilon(x)}(x) \), for \( x \in M \), form an open cover of \( M \) and since \( M \) is (being metrizable) paracompact (e.g., [73, Theorem 5.1.3]), this cover has a locally finite open refinement \( \mathcal{U} \). By Proposition 7.2.7 there exists a dense \( D \subseteq M \) such that \( \{ p \in [D]^{\aleph_0} : p \cap D = p \} \) includes a club \( C_0 \) in \([D]^{\aleph_0}\) and \( \Phi_D(p) := p \) defines an isomorphism between \( C_0 \) and a club in Sep(\( M \)). Therefore
is a stationary subset of \([D]^{\aleph_0}\). Define \(h : R \to D\) so that
\[ (*) \ h(p) \in p \cap U \text{ for some } U \in \mathcal{U} \text{ satisfying } g(p) \in U. \]
Since \(h\) is regressive, by the Pressing Down Lemma (Proposition 6.5.6) there exists \(x_0 \in D\) such that
\[ Q := \{ p \in R : h(p) = x_0 \} \]
is stationary in \([D]^{\aleph_0}\). Since \(\mathcal{U}\) is locally finite there is \(\varepsilon < \varepsilon(x_0)\) such that \(B_\varepsilon(x_0)\) intersects only finitely many \(W \in \mathcal{U}\). Enumerate these sets as \(W_j\) for \(j < n\). For each \(j < n\) the set \(P_j := \{ A \in \dom(g) : g(A) \in W_j \}\) is included in a nonstationary set \(S_z\) for some \(z\), and is therefore nonstationary itself. Let \(C_j\) be a club in \(\text{Sep}(M)\) disjoint from \(P_j\). Then \(C := \bigcap_{j<n} C_j\) is a club in \(\text{Sep}(M)\). Fix \(A \in C\). Then \(g(A) \notin \bigcup_{j<n} W_j\), and therefore \(g(A) \notin B_\varepsilon(x_0)\), and \((*)\) implies \(h(A \cap D) \notin B_\varepsilon(x_0)\).

Since \(\{ A \cap D : A \in C \}\) is a club in \([D]^{\aleph_0}\) we can choose \(p \in Q\) such that \(p \cap D \in C\). Then \(h(p) \notin B_\varepsilon(x_0)\), but this contradicts \(p \in Q\) and completes the proof. \(\square\)

By an \(n\)-fold application of Proposition 7.2.9 we obtain the following.

**Corollary 7.2.10.** Suppose \(M\) is a nonseparable metric structure in a separable language. Then for every \(n \geq 1\) and regressive functions \(g_j\), for \(j < n\), whose domain is the same stationary subset \(S\) of \(\text{Sep}(M)\) there exist \(\bar{\varepsilon} \in M^n\) such that for every \(\varepsilon > 0\) the set \(\{ A \in S : d(g_j(A), x_j) < \varepsilon \text{ for all } j < n \}\) is stationary. \(\square\)

### 7.3 Reflection to Separable Substructures, I

In this section we consider a given nonseparable \(C^*\)-algebra \(C\) as an inductive limit of a closed unbounded set of its separable \(C^*\)-subalgebras. Examples of properties that reflect to separable subalgebras (as well as those that don’t) are given.

Suppose \(A\) is a nonseparable \(C^*\)-algebra. The family \(\text{Sep}(A)\) of all separable \(C^*\)-subalgebras of \(A\) is a directed and \(\sigma\)-complete partially ordered set with respect to the inclusion in which the supremum of an increasing sequence is the closure of its union (Example 6.2.4, also §7.1.2).

In the following discussion the terms ‘structure’ and ‘metric structure’ are used in their model-theoretic sense, as in §7.1.2 and. In particular, all (metric) structures are assumed to be complete.

**Definition 7.3.1.** A property \(P\) of metric structures reflects to separable substructures if for every nonseparable metric structure \(A\) that satisfies \(P\) the set
\[ \{ B \in \text{Sep}(A) : B \text{ satisfies } P \} \]
includes a club. The property \(P\) may involve parameters or predicates such as automorphisms or (if \(A\) is a \(C^*\)-algebra) states on \(A\).
The intersection of countably many clubs is a club (Proposition 6.2.9), and an immediate consequence is worth recording.

**Lemma 7.3.2.** If $P_i$, for $i \in \mathbb{N}$, are properties of metric structures each of which reflects to separable substructures, then their conjunction also reflects to separable substructures. \qed

The Löwenheim–Skolem theorem (Theorem 7.1.9) implies the following.

**Lemma 7.3.3.** If $P$ is an axiomatizable property of metric structures in a countable language then both $P$ and its negation reflect to separable substructures. \qed

**Example 7.3.4.** Each of the following properties of metric structures reflects to separable substructures. The negation of each of these properties also reflects to separable substructures.

1. Being a Banach algebra, or being a C$^*$-algebra. These properties are axiomatizable ([90]).
2. Being a C$^*$-algebra with any of the following additional properties: Abelian, finite, stably finite, tracial, having a character, infinite, real rank zero, purely infinite and simple... All of these (as well as many other) properties of C$^*$-algebras are axiomatizable by [87, Theorem 2.5.1].

**Example 7.3.5.** Some non-axiomatizable, or at least not obviously axiomatizable, properties of C$^*$-algebras also reflect to separable substructures.

1. Being a C$^*$-algebra with any of the following properties: Simple, nuclear, AF, and AM, reflects to separable substructures. This follows from [87, Theorem 5.7.3] (see §D.2.4).
2. Being a reduced group C$^*$-algebra (Example 7.2.5).
3. If $A$ is a nonseparable C$^*$-algebra and $\Phi$ is an automorphism of $A$, consider the expanded structure $\mathcal{A} := (A, \Phi)$. In the case of C$^*$-algebras, the assertion that $\Phi$ is an automorphism is axiomatizable by the conditions

$$\sup_{\|x\| \leq 1} \inf_{\|y\| \leq 1} \|\Phi(x) - y\| = 0,$$

$$\sup_{\|x\| \leq 1, \|y\| \leq 1} \|\Phi(xy) - \Phi(x)\Phi(y)\| = 0,$$

and the analogous conditions for the addition, multiplication by scalars, and the adjoint operation. Therefore the Löwenheim–Skolem Theorem implies that the property ‘$\Phi$ is an automorphism of $A$’ reflects to separable C$^*$-subalgebras.
4. Item (3) is a special case of a general principle. In it, $\Phi$’s being an automorphism is axiomatizable in the language obtained by adding a new function symbol for $\Phi$. Lemma 7.3.3 implies that any property axiomatizable in a countable expanded language reflects to separable substructures. One example is non-simplicity of a C$^*$-algebra. A C$^*$-algebra $A$ is not simple if and only if the associated metric structure $\mathbb{M}(A)$ has an expansion (Definition D.2.10) to a language with an additional predicate for the distance to a proper ideal of $A$ that satisfies the appropriate first-order theory.
5. The property ‘$\Phi$ is an approximately inner automorphism of a C*-algebra’ (§2.6) is an axiomatizable property of a structure of the form $(A, \Phi)$ as in (3). It is axiomatizable by the following sequence of sentences indexed by $n \in \mathbb{N}$:

$$\sup_{x_0, \ldots, x_{n-1}} \inf_{y} \max_{j < n} (\|\Phi(x_j) - y x_j y^*\| + \|yy^* - 1\| + \|y^* y - 1\|).$$

Another consequence of the Downward Löwenheim–Skolem theorem and the fact that the intersection of a countable family of clubs is a club is worth stating (for the theory of a metric structure $A$, $\text{Th}(A)$, see Definition D.2.7).

**Proposition 7.3.6.** For a theory $T$ in a countable language of metric structures, the property $\text{Th}(A) = T$ reflects to separable substructures. $\square$

**Proposition 7.3.7.** The property of being a monotracial C*-algebra reflects to separable C*-subalgebras.

**Proof.** If $A$ is a C*-algebra then the Cuntz–Pedersen nullset $A_0$ of $A$ is the norm-closure of the linear span of self-adjoint commutators $[a, a^*]$. It is a closed subspace of the real Banach space $A_{sa}$. By [50], the space of tracial states on $A$ is affinely homeomorphic to the dual unit sphere of $A_{sa}/A_0$. Therefore $A$ has a unique trace if and only if the quotient $A_{sa}/A_0$ is one-dimensional.

If $A$ is a (nonseparable) C*-algebra then $\{B \in \text{Sep}(A) : B \cap A_0 = B_0\}$ includes a club. This implies that being monotracial reflects to separable C*-subalgebras. $\square$

**Example 7.3.8.** The following properties of metric structures do not reflect to separable substructures.

1. Not being a UHF C*-algebra. Any of the AM algebras constructed in §10.3 serves as an example.
2. Being a C*-algebra not isomorphic to its opposite algebra. Jensen’s $\diamondsuit_{\mathfrak{p}}$ implies that there exists a C*-algebra not isomorphic to its opposite algebra such that club many of its separable subalgebras are isomorphic to the CAR algebra, and therefore to their opposite algebra ([92]).$^5$
3. The property of being a Banach algebra not isomorphic to a C*-algebra (Corollary 15.6.36).

In each of the properties given in Example 7.3.8 reflection fails in a very strong way. In order to show that a property $P$ does not reflect, it suffices to have a non-separable structure with property $P$ such that the set of separable substructures with property $P$ does not include a club. In Example 7.3.8 (1)–(3) this set is even nonstationary.

The discussion started in Example 7.3.4 (3) continues.

**Proposition 7.3.9.** Both properties ‘$\Phi$ is an inner automorphism of $A$’ and ‘$\Phi$ is an outer automorphism of $A$’ of a metric structure $(A, \Phi)$ reflect to separable substructures.

---

$^4$ It shows a bit more; see Exercise 7.5.5.

$^5$ This leaves open the possibility that in some models of ZFC this property reflects to separable subalgebras.
Proof. Suppose \( A \) is a nonseparable \( C^* \)-algebra and \( \Phi \in \text{Aut}(A) \). Let

\[
S := \{ B \in \text{Sep}(A) : \text{the restriction of } \Phi \text{ to } B \text{ is in } \text{Inn}(B) \}.
\]

We will prove that if \( S \) is stationary then \( \Phi \) is inner.

Suppose \( S \) is stationary. The function that sends \( B \in S \) to some \( u_B \in U(B) \) such that \( \text{Ad } u_B \) and \( \Phi \) agree on \( B \) is regressive (Definition 7.2.1). By the Metric Pressing Down Lemma (Proposition 7.2.9) there exists \( u \in U(A) \) such that the set

\[
S_\varepsilon := \{ B \in S : \| u_B - u \| < \varepsilon \}
\]

is stationary for all \( \varepsilon > 0 \). Then \( \| \text{Ad } u(b) \upharpoonright B - \text{Ad } u_B(b) \upharpoonright B \| < 2\varepsilon \) for all \( B \in S_\varepsilon \).

Since \( S_\varepsilon \) is cofinal we have \( \| \text{Ad } u - \Phi \| \leq 2\varepsilon \). Since \( \varepsilon > 0 \) was arbitrary, \( \Phi = \text{Ad } u \) is inner. We have proved that if \( S \) is stationary then \( \Phi \) is inner.

If \( \Phi \) is inner, say \( \Phi = \text{Ad } u \), then the set \( \{ B \in \text{Sep}(A) : u \in B \} \) is a club included in \( S \). We have therefore proved the following.

1. \( S \) includes a club if and only if \( \Phi \) is inner.
2. \( S \) is nonstationary if and only if \( \Phi \) is outer.

This completes the proof that for a given automorphism \( \Phi \) both properties of being inner and being outer reflect to separable \( C^* \)-subalgebras. \( \square \)

The second part of Proposition 7.3.9 cannot be improved by weakening its assumption to require only that the restriction of \( \Phi \) to every separable \( C^* \)-subalgebra of \( A \) is implemented by a unitary in \( A \) (see Example 17.1.14).

**Proposition 7.3.10.** Suppose \( A \) is a nonseparable \( C^* \)-algebra and \( \varphi \) is a state on \( A \). The following are equivalent:

1. The state \( \varphi \) is pure.
2. The set \( S := \{ B \in \text{Sep}(A) : \text{the restriction of } \varphi \text{ to } B \text{ is pure} \} \) includes a club in \( \text{Sep}(A) \).
3. The set \( S \) as in (2) is cofinal in \( \text{Sep}(A) \).

**Proof.** The structure of this proof is similar to that of the proof of Proposition 7.3.9. (3) implies (1) by Lemma 5.5.3 and (2) trivially implies (3).

Suppose (2) fails. For each \( B \in \text{Sep}(A) \setminus S \) such that the restriction of \( \varphi \) to \( B \) is not pure choose \( a_B \in B_{<1} \), \( m = m_B \geq 1 \), and states \( \psi_{B,0} \) and \( \psi_{B,1} \) of \( B \) such that \( \frac{1}{2}(\psi_{B,0} + \psi_{B,1}) = \varphi \upharpoonright B \) and \( |\psi_{B,0}(a_B) - \psi_{B,1}(a_B)| > 1/m \). By Proposition 7.2.9 there exist \( m \) and \( a \in A_{<1} \) such that

\[
T := \{ B \in \text{Sep}(A) \setminus S : m_B = m \text{ and } \| a_B - a \| < 1/(3m) \}
\]

is stationary. Choose an ultrafilter \( \mathcal{U} \) on \( T \) that contains all clubs. Therefore every element of \( \mathcal{J} \) is stationary. Then

\[
\psi_j(c) := \lim_{B \to \mathcal{U}} \psi_{B,j}(c)
\]
for \( c \in A \) defines a state on \( A \) for \( j < 2 \). These states satisfy \( \frac{1}{2}(\psi_0 + \psi_1) = \varphi \) and \( |\psi_0(a) - \psi_1(a)| > 1/(3m) \). Therefore \( \varphi \) is not pure.

The equivalence of the assertions (1) and (2) implies that \( \varphi \) is a pure state on a \( C^* \)-algebra \( A' \) reflects to separable \( C^* \)-subalgebras. The equivalence of the assertions (1) and (3) implies that \( \varphi \) is a state on a \( C^* \)-algebra \( A \) which is not pure also reflects to separable \( C^* \)-subalgebras. \( \square \)

**Corollary 7.3.11.** Both properties \( \varphi \) is a pure state on a \( C^* \)-algebra \( A \)’ and \( \varphi \) is a state on a \( C^* \)-algebra \( A \) which is not pure’ reflect to separable \( C^* \)-subalgebras. \( \square \)

### 7.4 Reflection to Separable Substructures, II. Relativized Reflection

In this section we consider properties that involve not only a subalgebra but also its position inside the original algebra. The section concludes with a brief discussion of Schauder bases in nonseparable Banach spaces.

By using the tools introduced in §7.2 we prepare the grounds for applying the results of §6.2 to study isomorphisms between nonseparable \( C^* \)-algebras.

Recall that the relative commutant of a \( C^* \)-subalgebra \( B \) of \( A \) is (Definition 3.1.19)

\[
B' \cap A := \{ a \in A : ab = ba \text{ for all } b \in B \}.
\]

**Definition 7.4.1.** A \( C^* \)-subalgebra \( B \) of a \( C^* \)-algebra \( A \) is **complemented** in \( A \) if \( C^*(B, B' \cap A) = A \).\(^6\)

**Example 7.4.2.** In a large tensor product of separable \( C^* \)-algebras club many \( C^* \)-subalgebras are complemented. Suppose \( A_j \), for \( j \in J \), is an uncountable family of separable and unital \( C^* \)-algebras. If \( X \subseteq J \), then \( A_X := \bigotimes_{j \in X} A_j \) is naturally identified with a \( C^* \)-subalgebra of \( A = \bigotimes_{j \in J} A_j \). Since the relative commutant of \( A_X \) includes \( A_{J \setminus X} \),\(^8\) \( A_X \) is complemented. As \( \{ A_X : X \in [J]^{\aleph_0} \} \) is a club (Example 7.2.5 (2)), this completes the proof.

**Definition 7.4.3.** If \( P \) is a property of \( C^* \)-subalgebras of a nonseparable \( C^* \)-algebra \( A \), we say that **club many separable \( C^* \)-subalgebras of \( A \) satisfy property \( P \)** if

\[
\{ C \in \text{Sep}(A) : C \text{ satisfies property } P \}
\]

includes a club. A property \( P \) is **preserved under isomorphisms** if for every isomorphism \( \Phi : A \to B \) between \( C^* \)-algebras a \( C^* \)-subalgebra \( C \) of \( A \) satisfies \( P \) if and only if the \( C^* \)-subalgebra \( \Phi[C] \) of \( B \) satisfies \( P \).

\(^6\) In [93] \( B' \cap A \) was denoted \( Z_\delta(B) \).

\(^7\) This (nonstandard) \( C^* \)-algebraic terminology is unrelated to the eponymous (well-established) terminology in the theory of Banach spaces briefly mentioned in Lemma 7.4.7 below.

\(^8\) If some of \( A_j \) have a nontrivial center, it can be larger.
A straightforward proof of the following lemma is omitted.

**Lemma 7.4.4.** Suppose \( \Phi: A \to B \) is an isomorphism between nonseparable C*-algebras.

1. If \( C \subseteq \text{Sep}(A) \) is a club, then \( \{ \Phi[C] : C \in C \} \) is a club in \( \text{Sep}(B) \).
2. If \( D \subseteq \text{Sep}(B) \) is a club, then \( \{ \Phi^{-1}[D] : D \in D \} \) is a club in \( \text{Sep}(A) \).
3. If a property \( P \) is preserved under isomorphisms, then club many separable C*-subalgebras of \( A \) satisfy \( P \) if and only if club many separable C*-subalgebras of \( B \) satisfy \( P \).

We conclude this section with a discussion of a standard notion in Banach space theory, to be used in \( \S \) 10.2.

**Definition 7.4.5.** A Schauder basis of a Banach space \( X \) is an indexed family \( x_j \), for \( j \in J \), of vectors in \( X \) such that for every \( y \in X \) there exist unique scalars \( \lambda_j(y) \), for \( j \in J \), for which the partial sums \( \sum_{j \in F} \lambda_j(y)x_j \), for \( F \subseteq J \), converge to \( y \) in norm.

**Example 7.4.6.** 1. An orthonormal basis in a Hilbert space is a Schauder basis by Parseval’s identity.
2. Both the standard basis of any classical sequence space \( \ell_p(J) \), for \( 1 \leq p < \infty \), and the standard basis of \( c_0(J) \) are Schauder bases ([169]).
3. In the reduced group algebra \( C_r^*(\Gamma) \) (\( \S \) 2.4.1) the unitaries \( u_g \), for \( g \in \Gamma \), form a Schauder basis. The finite linear combinations of the \( u_g \)'s are dense in \( C_r^*(\Gamma) \) by definition. The contractive surjection from \( C_r^*(\Gamma) \) onto \( \ell_2(\Gamma) \) sends \( u_g \) to \( \delta_g \) for all \( g \in \Gamma \). Since \( \delta_g \), for \( g \in \Gamma \), is a Schauder basis for \( \ell_2(\Gamma) \), its preimage \( u_g \), for \( g \in \Gamma \), is a Schauder basis of \( C_r^*(\Gamma) \).

Suppose \( x_j \), for \( j \in J \), is a Schauder basis of a Banach space \( X \). The support of \( y \) with respect to this basis is

\[ \text{supp}(y) := \{ j \in J : \lambda_j(y) \neq 0 \}. \]

Clearly, the support of a vector depends on the choice of the basis and it is countable. We do not assume that the Schauder basis \( J \) is countable. Quite the contrary: much of the ongoing discussion is vacuous unless \( J \) is uncountable.

A closed subspace \( Y \) of a Banach space \( X \) is **complemented** if there exists a closed subspace \( Z \) of \( X \) such that every \( x \in X \) can be uniquely written as \( x = y + z \) for \( y \in Y \) and \( z \in Z \). Every closed subspace of a Hilbert space is complemented: take \( Z := Y^\perp \).

**Lemma 7.4.7.** Suppose \( x_j \), for \( j \in J \), is a Schauder basis of a Banach space \( X \). Then \( \Phi: [J]^{\mathbb{R}_0} \to \text{Sep}(X) \) defined by \( \Phi(p) := \text{span}(p) \) is an isomorphism between \( [J]^{\mathbb{R}_0} \) and a club in \( \text{Sep}(X) \), and every \( Y \) in the range of \( \Phi \) is complemented in \( X \).

**Proof.** The fact that \( \Phi \) is an isomorphism is straightforward (and easier than the proof of Proposition 7.2.7).

If \( p \subseteq J \) is such that \( Y = \text{span}(p) \) then \( Z := \text{span}(J \setminus p) \) is a closed subspace of \( X \) such that \( X \cong Y \oplus Z \). Therefore every subspace \( Y \) of \( X \) in the range of \( \Phi \) is complemented. \( \square \)
7.5 Exercises

Exercise 7.5.1. Suppose $A$ is a metric space and $A_\xi$, for $\xi < \aleph_1$, is a strictly increasing family of closed subspaces of $A$. Prove that $A$ is nonseparable.

Exercise 7.5.2. Suppose $\mathcal{L}$ is a countable language. Prove that the set of codes for separable $\mathcal{L}$-structures is a Borel subset of $\mathcal{P}(X(\mathbb{N}, \mathcal{L}))$ when $\mathcal{P}(X(\mathbb{N}, \mathcal{L}))$ is identified with $\mathcal{P}(\mathbb{N})$.

Exercise 7.5.3. Suppose that $M$ is a nonseparable metric space. Prove that there are a dense $D \subseteq M$ and a club $C$ in $\big[\mathbb{D}\big]^{\omega_0}$ such that $\{x : x \in C\}$ is not a club in $\text{Sep}(M)$.

Exercise 7.5.4. Prove that an ideal $J$ of a $C^*$-algebra $A$ is essential (Definition 2.5.5) if and only if $\sup_{\|x\| \leq 1} \inf_{\|y\| \leq 1} \|xy - \|x\|| = 0$ holds in $A$. Conclude that $J$ being an essential ideal of a $C^*$-algebra reflects to separable subalgebras.

Exercise 7.5.5. Suppose $A$ is a nonseparable $C^*$-algebra. Prove that club many separable $C^*$-subalgebras of $A$ satisfy each of the following properties:

1. $\{B \in \text{Sep}(A) : \text{every tracial state on } B \text{ extends to a tracial state on } A\}$.
2. $\{B \in \text{Sep}(A) : \text{every ideal of } B \text{ is of the form } J \cap B \text{ for some ideal of } A\}$.

Exercise 7.5.6. Suppose $M$ is a $\Pi_1$ factor with separable predual. Prove that there exists a separable $C^*$-subalgebra $A$ of $M$ such that every automorphism of $A$ extends to an automorphism of $M$.

Exercise 7.5.7. There exists a $\Pi_1$ factor $M$ not isomorphic to its opposite algebra ([45]). Consider $M$ as a $C^*$-algebra, and prove that club many of its separable $C^*$-subalgebras are simple, have a unique trace, and are not isomorphic to their opposite.

Exercise 7.5.8. There exists a $\Pi_1$ factor $M$ isomorphic to its opposite algebra, but such that there is no isomorphism $\Phi : M \rightarrow M^{\text{op}}$ of order two ([141]). Consider $M$ as a $C^*$-algebra, and prove that club many of its separable $C^*$-subalgebras $A$ are simple, have a unique trace, are isomorphic to their opposite, but there is no isomorphism $\Phi : A \rightarrow A^{\text{op}}$ of order two.

Exercise 7.5.9. Suppose $M$ is a metric structure of density character at least $\aleph_2$ and $C$ is a club in $\text{Sep}(M)$. Consider the directed and $\aleph_2$-complete set

$\mathcal{P} := \{N : N \text{ is a substructure of } M \text{ of density character } \aleph_1\}$.

Prove that $\{N \in \mathcal{P} : C \cap \text{Sep}(N) \text{ is a club in } \text{Sep}(N)\}$ is a club in $\mathcal{P}$.

Exercise 7.5.9 and its generalizations show that if $M$ has density character $\aleph_2$ or greater, then every club in $\text{Sep}(M)$ reflects to a club in $\text{Sep}(N)$ for some substructure $N$ of $M$ of density character $\aleph_1$. The analogous (much more profound) question of the reflection of stationary sets is, regrettably, well beyond the scope of this book (see e.g., [134]).
Exercise 7.5.10. Suppose $X$ is a (nonseparable) Banach space and $B \subseteq X$ has $X$ as its closed linear span. Prove that the following are equivalent.

1. $B$ is a Schauder basis for $X$.
2. The set $T := \{ Y \in [B]^{\aleph_0} : Y$ is a Schauder basis for $\text{span}(Y)\}$ includes a club.
3. The set $T := \{ Y \in [B]^{\aleph_0} : Y$ is a Schauder basis for $\text{span}(Y)\}$ is cofinal in $[B]^{\aleph_0}$.

Exercise 7.5.11. Suppose that $\kappa$ is an infinite cardinal and $\mathcal{L}$ is a metric language of cardinality not greater than $\kappa$. Prove that there are at most $2^\kappa$ nonisomorphic metric structures of density character $\kappa$.

Conclude that there are at most $2^\kappa$ nonisomorphic $C^*$-algebras of density character $2^\kappa$.

Exercise 7.5.12. Suppose $A$ is a simple $C^*$-algebra and $\varphi$ and $\psi$ are pure states on $A$. Prove that there exists a $C^*$-algebra $C$ that has $A$ as a $C^*$-subalgebra and equivalent pure states $\tilde{\varphi}$ and $\tilde{\psi}$ extending $\varphi$ and $\psi$, respectively.

The followings exercises can be solved by appealing to Lemma 7.3.3, [87, Theorem 2.5.1], and [87, Theorem 5.7.3] (see also Example 7.3.5 (1)), but some readers may prefer solving them by using their ‘bare hands.’

Exercise 7.5.13. Prove that each of the following properties reflects to separable subalgebras: having real rank zero, being finite, being infinite.

Exercise 7.5.14. Prove that each of the following properties reflects to separable subalgebras: being AF, being AM, being simple, reflects to separable subalgebras.

Notes for Chapter 7

Most of the material of this chapter is original.

§7.1 Logic of metric structures was introduced in [22] and adapted to operator algebras in [90] and [87].

§7.3 In the context of $C^*$-algebras the Downwards Löwenheim–Skolem theorem (Theorem 7.1.3) was rediscovered by Blackadar. Blackadar’s method involves the following notion.

Definition 7.5.15. A property $P$ of $C^*$-algebras is separably inheritable, or (SI), ([27, §II.8.5]) if it has the following properties.

1. For every $C^*$-algebra $A$ with property $P$ and every separable $C^*$-subalgebra $B$ of $A$ there exists a separable $C^*$-algebra $C$ such that $B \subseteq C \subseteq A$ and $C$ has property $P$.

9 This estimate is optimal, see Theorem 10.3.4.
10 Compare with Proposition 10.5.5.
11 It is unrelated to the property (SI) introduced in [179].
2. If $A$ is an inductive limit of separable $C^*$-algebras $A_n$, for $n \in \mathbb{N}$, with injective connecting maps, and each $A_n$ has property $P$, then $A$ has property $P$.

Therefore a property is separably inheritable if it is inherited by club many separable $C^*$-subalgebras. Every (SI) property of $C^*$-algebras reflects to separable $C^*$-subalgebras and if $P_n$, for $n \in \mathbb{N}$, are separably inheritable properties, then so is their conjunction. Some separably inheritable properties are not elementary; see also Example 7.3.4.

§7.4 Exercise 7.5.7 and Exercise 7.5.5 were adapted from [198] and [191, Lemma 9], respectively.
You raise up your head
And you ask, “Is this where it is?”
And somebody points to you and says
“It’s his”
And you say, “What’s mine?”
And somebody else says, “Where what is?”
And you say, “Oh my God
Am I here all alone?”

Because something is happening here
But you don’t know what it is...

Bob Dylan, Ballad of a Thin Man

You’ve just read the most cryptic paragraph of this chapter. Trust me. This chapter contains every additional set-theoretic axioms used in the proof of any of the consistency results given in this book. We discuss Cantor’s Continuum Hypothesis and its strengthening, Jensen’s $\Diamond_{\aleph_1}$. Attached to the sections introducing these axioms are sections on $\sigma$-complete back-and-forth systems of partial isomorphisms and Suslin trees, respectively. While the Continuum Hypothesis has many consequences, its negation is a rather weak axiom. Its most common (and, arguably, most plausible) strengthenings are the so-called forcing axioms. The weakest one of them, asserting that the real line cannot be covered by fewer than $\mathfrak{c}$ nowhere dense sets, will be discussed in this chapter. The Chapter concludes with the study of OCA, a Ramseyan consequence of strong forcing axioms. It will be used to prove that all automorphisms of the Calkin algebra are inner (Theorem 17.8.5).

8.1 The Continuum Hypothesis

This section is a warmup for §8.2. In it we prove Cantor’s Theorem, that the power set of any set has strictly higher cardinality than the original set. We also provide a
list of sets of cardinality equal to the continuum, and a working reformulation of the Continuum Hypothesis.

For sets $X$ and $Y$ we write

$$X^Y := \{ f : f : X \to Y \}.$$  

The power set of $X$ is the set of all of its subsets, $\mathcal{P}(X) := \{ Y : Y \subseteq X \}$. The existence of the power set of every set is guaranteed by the axioms of ZFC (§A.1). Sets $X$ and $Y$ are equinumerous if there exists a bijection between them.

The cardinal exponentiation is monotonic, as $|X| \leq |Y|$ implies $2^{|X|} \leq 2^{|Y|}$. However, $|X| < |Y|$ does not necessarily imply $2^{|X|} < 2^{|Y|}$; see e.g., [166].

**Lemma 8.1.1.** For an infinite set $X$ the sets $\mathcal{P}(X)$, $\{0, 1\}^X$, $\mathbb{N}^X$, and $X^X$ are equinumerous.

**Proof.** A bijection between $\mathcal{P}(X)$ and $\{0, 1\}^X$ is defined by sending a subset of $X$ to its characteristic function. Therefore $|\mathcal{P}(X)| = |\{0, 1\}^X| = 2^{|X|}$. The other equalities follow by the monotonicity of the exponentiation. \(\square\)

The following theorem was among Cantor’s late 19th century results that marked the birth of set theory.

**Theorem 8.1.2.** For every set $X$, the cardinality of $\mathcal{P}(X)$ is strictly larger than the cardinality of $X$.

**Proof.** An injection of $X$ into $\mathcal{P}(X)$ is obtained by sending each $x \in X$ to $\{x\}$, and proves that $|X| \leq |\mathcal{P}(X)|$. If the converse inequality holds, let $f : \mathcal{P}(X) \to X$ be an injection. Extend its inverse to a surjection $g : X \to \mathcal{P}(X)$ by letting $g(x) = \emptyset$ if $x$ is not in the range of $f$. Let $S := \{ x \in X : x \notin g(x) \}$. If $S = g(y)$ then $y \notin S$, and therefore $y \in S$. This implies $y \notin S$; contradiction. \(\square\)

Theorem 8.1.2 has a far-reaching consequence. After accepting the existence of an infinite set $\mathbb{N}$, the power set axiom produces an infinite sequence of cardinals, $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$.

The list given in the following example is by no means exhaustive.

**Example 8.1.3.** The cardinality of each of the following sets is equal to $\mathfrak{c} := 2^{\aleph_0}$.

1. Any uncountable Polish space. In particular, $\mathbb{R}$, $\mathcal{P}(\mathbb{N})$, and any separable Banach space.
2. Any uncountable Borel subset of a Polish space. This is because there exists a bijection between any two uncountable Borel subsets of Polish spaces (Theorem B.2.4).
3. The spaces $\mathbb{N}^\mathbb{N}$, $\mathbb{R}^\mathbb{N}$, and $\prod_n A_n$ for any sequence $A_n$, $n \in \mathbb{N}$, of nontrivial separable metric spaces. This is because $2^\kappa \leq \kappa^\kappa \leq (2^\kappa)^\kappa = 2^{\kappa}$ for all cardinals $\kappa$.
   In particular, if $A$ is a separable $C^*$-algebra, then $\ell_\infty(A)$ and $c_0(A)$ have cardinality $\mathfrak{c}$.

---

1 This is just the beginning. The axioms of replacement and union (see §A.1) together imply that the supremum of any set of cardinals is a cardinal.
4. If \( X \) and \( Y \) are topological spaces, \( X \) is separable, and \( Y \) is second countable, then the set \( C(X,Y) \) of all continuous functions from \( X \) to \( Y \) has cardinality at most \( c \). Let \( D \subseteq X \) be a dense subset and let \( U_n \), for \( n \in \mathbb{N} \), enumerate a countable basis of \( Y \). Then every continuous \( f : X \to Y \) is uniquely determined by sets \( f^{-1}(U_n) \cap D \), for \( n \in \mathbb{N} \). This defines an injection from \( C(X,Y) \) into \( \mathcal{P}(D)^\mathbb{N} \).

5. If \( X \) is an infinite Polish space, then it has \( c \) Borel sets. Since \( X \) is infinite and all of its countable subsets are Borel, we have the lower bound. The upper bound is proved by induction on the Borel rank of a Borel set (the Borel rank of every Borel subset of \( X \) is a countable ordinal; see e.g., [152]).

6. If \( X \) and \( Y \) are Polish spaces and \( X \) is infinite, the set

\[ \{ f : X \to Y : f \text{ is Borel-measurable} \}. \]

Fix a basis \( U_n \), for \( n \in \mathbb{N} \), of \( Y \). Then every \( f : X \to Y \) is uniquely determined by the sequence \( f^{-1}(U_n) \), for \( n \in \mathbb{N} \). Since all of these sets are Borel and there are \( c \) Borel subsets of \( X \) by (5), the conclusion follows.

7. The dual space \( X^* \) of any separable Banach space \( X \). Every continuous linear functional \( \varphi \in X^* \) is a continuous function from \( X \) into the scalars. Since \( X \) is Polish, (4) implies \( |X^*| \leq c \). For the converse inequality we need the Hahn–Banach theorem; it implies that \( X^* \neq \emptyset \). Since the cardinality of \( X^* \) is not smaller than that of the field of scalars and \( |\mathbb{R}| = |\mathbb{C}| = c \), the conclusion follows.\(^2\)

8. By (7), every von Neumann algebra with a separable predual has cardinality \( c \).

By Theorem 3.1.16, \( \mathcal{B}(\ell_2(\mathbb{N})) \) has cardinality \( c \).

All of the above examples have the same definable cardinality as \( \mathbb{R} \), in the sense that they carry an intrinsic Borel space structure and a Borel-measurable bijection with \( \mathbb{R} \). This is not necessarily the case with the following examples.

9. The set of all isomorphism classes of countable structures in a fixed countable language \( \mathcal{L} \) has cardinality at most \( c \). This is because there is an injection from the space of \( \text{Struct}(\mathcal{L}, \mathbb{R}_0) \) (Definition 7.1.2) into \( \mathcal{P}(\mathbb{N}) \).

10. The set of all isomorphism classes of separable C*-algebras has cardinality \( c \).

   More generally, the set of all isomorphism classes of separable metric structures in a fixed countable language has cardinality \( c \), because there is an injection from \( \text{Struct}(\mathcal{L}, \mathbb{R}_0) \) into \( \mathcal{P}(\mathbb{N}) \).

11. If \( A_n \), for \( n \in \mathbb{N} \), are nontrivial separable C*-algebras and \( \mathcal{U} \) is an ultrafilter on \( \mathbb{N} \), then the cardinality of the ultraproduct \( \prod_{\mathcal{U}} A_n \) is \( c \). Also, the cardinality of the asymptotic sequence algebra \( \prod_{\mathbb{N}} A_n / \bigoplus_{\mathbb{N}} A_n \) (see §16.7) is \( c \).

Cantor’s Continuum Hypothesis (CH) asserts that every infinite \( X \subseteq \mathbb{R} \) satisfies either \( |X| = |\mathbb{N}| \) or \( |X| = |\mathbb{R}| \). The following gives a more useful equivalent reformulation of the Continuum Hypothesis.

**Proposition 8.1.4.** The Continuum Hypothesis is equivalent to the following:

\(^2\)The fact that \( |X^*| = c \) for every separable Banach space \( X \) is misleading since the density character of \( X^* \) can be \( \mathbb{R}_0 \) (if \( X = c_0 \)) or \( 2^{\mathbb{R}_0} \) (if \( X = \ell_1 \)), and the density character is the right analog of the cardinality for metric structures.
Every set of cardinality $c$ has a well-ordering such that each of its proper initial segments is countable.

**Proof.** Fix a set $X$ of cardinality $c$. The Axiom of Choice implies that $X$ can be well-ordered. If a well-ordering of $X$ has the minimal possible length, then every proper initial segment of $X$ has cardinality smaller than that of $X$ and is therefore countable by the Continuum Hypothesis.

Conversely, suppose that $\mathbb{R}$ has a well-ordering such that all of its proper initial segments are countable. Therefore the order-type of this ordering is the least uncountable cardinal, $\aleph_1$. Since $\aleph_1$ is a regular cardinal, the cardinality of $X \subseteq \aleph_1$ is countable if it is bounded and $\aleph_1$ if it is unbounded. □

The characteristic function of $\mathbb{Q}$ is everywhere discontinuous, but its restriction to $\mathbb{R} \setminus \mathbb{Q}$ is continuous (even constant). We can do better (or worse?).

**Example 8.1.5.** The Continuum Hypothesis implies that there exists $g: \mathbb{R} \to \mathbb{R}$ such that the restriction of $g$ to $X$ is discontinuous for every uncountable $X \subseteq \mathbb{R}$. The salient point is that if $Y \subseteq \mathbb{R}$ and $f: Y \to \mathbb{R}$ is continuous, then the set

$$Z := \{z \in \mathbb{R}: (\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in Y)(\forall y' \in Y)(\max(|y - z|, |y' - z|) < \delta \Rightarrow |f(y) - f(y')| < \varepsilon)\}$$

is $G_\delta$ and $f$ has a continuous extension to $Z$. Let $(Z_\alpha, f_\alpha)$, for $\alpha < \aleph_1$, be an enumeration of all pairs such that $Z_\alpha$ is a $G_\delta$ set of reals and $f_\alpha: Z_\alpha \to \mathbb{R}$ is continuous. Also let $x_\alpha$, for $\alpha < \aleph_1$, be an enumeration of $\mathbb{R}$. Define $g$ by recursion, so that $g(x_\beta) \notin \{f_\alpha(x_\beta) : \alpha \leq \beta\}$. Suppose $X \subseteq \mathbb{R}$ is uncountable and the restriction of $g$ to $X$ is continuous. Fix $\alpha$ such that $f_\alpha$ agrees with $g$ on $X$. Then $g(x_\beta) \neq f_\alpha(x_\beta)$ for all $\beta \geq \alpha$, contradicting the assumption that $X$ is uncountable.

This example, together with Proposition 8.6.3, implies that OCA_T and CH are incompatible. See also Exercise 8.7.10 and Exercise 8.7.11.

### 8.2 The Back-and-Forth Method. I

... The first was never to accept anything as true that I did not incontrovertibly know to be so; that is to say, carefully to avoid both prejudice and premature conclusions; and to include nothing in my judgments other than that which presented itself to my mind clearly and distinctly, that I would have no occasion to doubt it.

René Descartes, A Discourse on the Method, p. 17.

... Then, too, we know from this that the true function of breathing is to bring enough fresh air into the lungs to cause the blood entering them from the right cavity into the heart, where it has been rarefied and, as it were, changed into vapour, to thicken and convert itself once more into blood, before falling back into the left cavity; if it did not do this, it would not be fit to nourish the fire that is there.

René Descartes, A Discourse on the Method, p. 44.
8.2 The Back-and-Forth Method

In this section we prove that all countable dense linear orderings with no endpoints are isomorphic. This is followed by a study of Hausdorff’s $\eta_1$ sets. Afterwards we move on to $\sigma$-complete back-and-forth systems of separable substructures of a given nonseparable metric structure. These systems will be used in §16.6 and §16.7 to apply the Continuum Hypothesis to massive $C^*$-algebras such as ultraproducts or asymptotic sequence algebras.

We use a classical result of Cantor as a warmup. A linear ordering $(A, <_A)$ is dense if for any two elements $x <_A y$ there exists $z$ such that $x <_A z$ and $z <_A y$. The endpoints in a linear ordering are its maximal element and its minimal element, if they exist.

**Proposition 8.2.1.**
1. Any two countable dense linear orderings without endpoints are order-isomorphic.
2. Every dense linear ordering with at least two elements contains an order-isomorphic copy of the rationals.

**Proof.** We prove (1) by a ‘back-and-forth’ argument. Let $\mathfrak{A}$ and $\mathfrak{B}$ be countable linear orderings without endpoints. Fix enumerations $\mathfrak{A} = \{a_n : n \in \mathbb{N}\}$ and $\mathfrak{B} = \{b_n : n \in \mathbb{N}\}$. Define increasing sequences $F_n \subset A$, $G_n \subset B$, and $f_n : F_n \to G_n$ such that $f_n$ is an order-isomorphism, $a_n \in F_n$, and $b_n \in G_n$ for all $n$. Start by letting $F_0 := \{a_0\}$ and $G_0 := \{b_0\}$. The assumptions on $\mathfrak{A}$ and $\mathfrak{B}$ imply that $f_n$ can be extended to an order-isomorphism that includes any given $b \in B \setminus G_n$ in its range and any given $a \in A \setminus F_n$ in its domain.

A proof of (2) is similar to (1), but going only forth. $\square$

This was as straightforward as the back-and-forth arguments get to be. At any given point of the construction we were committed only to a finite piece of the final order-isomorphism, so a first-order property of $\mathfrak{A}$ and $\mathfrak{B}$ (not having endpoints and being dense) enabled us to continue the construction at any given stage. $\square$ The idea that extends Cantor’s back-and-forth machinery to get us past the limit stages of countable cofinality goes back all the way to Hausdorff.

**Definition 8.2.2.** A linear ordering $(L, <)$ is an $\eta_1$ set if it has no endpoints and it has the following property. If $A$ and $B$ are at most countable (possibly empty) subsets of $L$ such that $a < b$ for all $a \in A$ and all $b \in B$, then there exists $c \in L$ such that $a < c$ and $c < b$ for all $a \in A$ and all $b \in B$.

A proof of the following is left as Exercise 8.7.6.

**Proposition 8.2.3.** Any two $\eta_1$ sets of cardinality $\aleph_1$ are order-isomorphic. $\square$

This definition is our first (implicit) encounter with gaps (see §9.3 and §14.1) and the earliest example of a (model-theoretically) saturated structure (Definition 16.1.5).

---

3 Without going back. Or multiplying, for that matter.

4 The fact that the theory of dense linear orderings admits elimination of quantifiers also helped. See Notes to this section.
Hausdorff defined η₁ sets for every ordinal α. Suffice it to say that Proposition 8.2.1 asserts that all countable η₀ sets are order-isomorphic, and the reader can extrapolate the definition of an ηα set for an ordinal α > 1. In the terminology introduced in §9.3, η₁ sets have no (ℵ₀, ℵ₀) gaps, ℵ₀ limits, or endpoints, and no element has an immediate successor (see Exercise 9.10.4).

**Lemma 8.2.4.** Every η₁ set L contains an order-isomorphic copy of ℜ and its cardinality is therefore at least c.

*Proof.* By Proposition 8.2.1, there is an order-embedding of ℚ into L. Since ℚ is countable, the image of every Dedekind cut in ℚ can be filled by an element of L and the order-embedding of ℚ can be extended to an order-embedding of ℜ into L. □

An η₁ set cannot have a countable subset that intersects every nonempty interval. In particular, ℜ is not an η₁ set.

**Example 8.2.5.** An η₁ set. On the set \{0, 1\} of all functions from ℵ₁ into \{0, 1\} consider the lexicographical ordering defined as follows. Given a ≠ b in \{0, 1\} let

\[ \Delta(a, b) = \min\{\alpha < \aleph_1 : a(\alpha) ≠ b(\alpha)\} \]

and let a < _lex b if a(Δ(x, y)) < b(Δ(a, b)). Then \{(0, 1)^{< \aleph_1}, <_{lex}\} has the property that every subset has the supremum but it is not an η₁ set for two reasons: it has endpoints, and it is not dense. To remedy this, remove the two constant functions from \{0, 1\}^{< \aleph_1}. Also remove all a ∈ \{0, 1\}^{< \aleph_1} that are eventually equal to 0. A proof that the obtained subordering L is an η₁ set is left to the reader (Exercise 8.7.5).

**Proposition 8.2.6.** There exists an η₁ set of cardinality ℵ₁ if and only if the Continuum Hypothesis holds.

*Proof.* Lemma 8.2.4 implies that every η₁ set has cardinality at least c. Since \{0, 1\}^{α} has cardinality c for every countably infinite α, the η₁ ordering constructed in Example 8.2.5 has cardinality ℵ₁ · c = c. □

We are ready to transfer these ideas to the context of metric structures, and C*-algebras in particular. As in §7.3, Sep(M) is the poset of all separable substructures of a nonseparable metric structure M, ordered by inclusion. This poset is directed and σ-complete (Example 6.2.5). The full power of σ-complete back-and-forth systems is unleashed in connection with basic model theory. Therefore this section requires familiarity with §D.

Once Definition 8.2.7 and Definition 8.2.8 are internalized, the subsequent proofs and applications are obtained by assembling the results from the earlier sections.

**Definition 8.2.7.** A partial isomorphism between metric structures C and D is a triple \((A, B, \Phi)\) such that \(A ∈ \text{Sep}(C), B ∈ \text{Sep}(D),\) and \(\Phi : A → B\) is an isomorphism.

**Definition 8.2.8.** Suppose C and D are nonseparable metric structures in the same language. A σ-complete back-and-forth system between C and D is a poset \(F\) with the following properties.
1. The elements of $\mathcal{F}$ are partial isomorphisms $p = (A^p, B^p, \Phi^p)$.
2. The ordering is defined by $p \leq q$ if $A^p \subseteq A^q$, $B^p \subseteq B^q$, and $\Phi^q | A^p = \Phi^p$.
3. For every $p \in \mathcal{F}$ and all $a \in A$ and $b \in B$ there exists $q \geq p$ in $\mathcal{F}$ such that $a \in A^q$ and $b \in B^q$.
4. $\mathcal{F}$ is $\sigma$-complete: For every increasing sequence $p(n)$, for $n \in \mathbb{N}$, in $\mathcal{F}$ we require that (identifying a function with its graph) $p := \{ \bigcup_n A_n^{p(n)} \cup_n B_n^{p(n)} \cup_n \Phi_n^{p(n)} \}$ belongs to $\mathcal{F}$. We write $p = \sup_n p_n$.

Recall that $A \preceq C$ stands for ‘$A$ is an elementary submodel of $C$’.

**Lemma 8.2.9.** Suppose that there exists a $\sigma$-complete back-and-forth system $\mathcal{F}$ between nonseparable metric structures $C$ and $D$. Then the following hold.

1. The set $\{(A,B) \in \text{Sep}(C) \times \text{Sep}(D) : A \equiv B\}$ includes a club in $\text{Sep}(C) \times \text{Sep}(D)$.
2. The set $\{p \in \mathcal{F} : A^p \preceq C$ and $B^p \preceq D\}$ is, with respect to the coordinatewise inclusion, a $\sigma$-complete back-and-forth system.

**Proof.** (1) By Proposition 7.2.7 we can find a dense set $X \subseteq C$ such that

$$
C_0 := \{p \in [X]^\emptyset : X \cap p = p\}
$$

includes a club and a dense $Y \subseteq D$ such that $D_0 := \{p \in [Y]^\emptyset : Y \cap p = p\}$ includes a club. Let $\theta$ be a regular cardinal large enough to assure that $C, D, X, Y,$ and $\mathcal{F}$ all belong to $H_\theta$ (see Definition A.7.1). Let $Z := X \cup Y$ (the disjoint union of $X$ and $Y$). Let $C := \{M \subseteq H_\theta : M$ is countable and $C(D, X, Y, \mathcal{F}) \subseteq M\}$. By Exercise 6.7.23, the set $E := \{M \cap Z : M \in C\}$ includes a club in $|Z|^\emptyset$. The set

$$
\mathcal{F} := \{(A,B) \in \text{Sep}(C) \times \text{Sep}(D) : (\exists M \in C) A = M \cap X, B = M \cap Y\}
$$

is, by the choice of $X$, $Y$, and $E$, a club in $\text{Sep}(C) \times \text{Sep}(D)$. It therefore suffices to prove the following.

**Claim.** If $(A,B) \in \mathcal{F}$ then there exists $\Phi$ such that $(A,B,\Phi) \in \mathcal{F}$, and $A \equiv B$.

**Proof.** Fix $M \in E$ such that $A = M \cap X$ and $B = M \cap Y$. In addition, fix enumerations $M \cap X = \{a_n : n \in \mathbb{N}\}$ and $M \cap Y = \{b_n : n \in \mathbb{N}\}$. Choose $p_n = (A_n, B_n, \Phi_n) \in \mathcal{F}$ with the following properties.

1. $p_n \leq p_{n+1}$ for all $n$.
2. $a_j \in A_n$ for all $j \leq n$.
3. $b_j \in B_n$ for all $j \leq n$.
4. $p_n \in M$ for all $n$.

---

5 More precisely, we require $\theta \geq (2^\lambda)^+$, where $\lambda$ is a cardinal at least equal to the density characters of $C$ and $D$. This assures that isomorphic copies of $C$, $D$, and $\mathcal{F}$ belong to $H_\theta$.

6 The enumerations $a_n$, $b_n$, for $n \in \mathbb{N}$ cannot belong to $M$—$M$ ‘knows’ $C$ is nonseparable! However all we need is that the finite initial segments of these enumerations belong to $M$.
If $p_n$ has been chosen, then some $p'_{n+1} \in F$ satisfies the first three properties because $F$ is a $\sigma$-complete back-and-forth system. Since $M \preceq H_\theta$ and all relevant parameters $(p, a_j, b_j, \text{for } j \leq n+1)$ belong to $M$, we can find $p_{n+1} \in M$ as required.

Let $(A_\infty, B_\infty, \Phi_\infty) = \sup_n p_n$. We claim that $A_\infty = M \cap X$ and $B_\infty = M \cap Y$. By construction, $M \cap X \subseteq A_n$. Since $A_n$ is separable and an element of $M$, we have $A_n \subseteq \overline{M \cap X}$ for all $n$, and $A_\infty \subseteq \overline{M \cap X}$ follows. Having proved both inclusions, we conclude that $A_\infty = M \cap X$. The proof of $B_\infty = M \cap Y$ is analogous.

By the elementarity of $M$, $M \cap X$ is dense in $M \cap A$ and $M \cap Y$ is dense in $M \cap B$.

We conclude that $\Phi_\infty$ is an isomorphism between $M \cap A$ and $M \cap B$. Since $M$ was arbitrary, this proves the claim.

(2) By Proposition 7.3.6, club many separable substructures of $C$ are elementary submodels, and the same applies to $D$.

In Corollary 10.3.2 we will see that a simple-minded converse to Lemma 8.2.9 does not hold in general.

The $\sigma$-complete back-and-forth systems will be used again in §16.6.

8.3 Diamonds

Diamonds are a set theorist’s best friend.

Menachem Magidor

In this section we introduce Jensen’s diamond, $\diamondsuit_{\aleph_1}$. After proving that it implies the Continuum Hypothesis, we consider some variants of diamond suited for capturing subsets of first-order structures (discrete and metric) in a fixed language.

Definition 8.3.1. For an uncountable regular cardinal $\kappa$, Jensen’s $\diamondsuit_\kappa$ is the following statement. There exists a family $S_\alpha$, for $\alpha < \kappa$, such that

1. $S_\alpha \subseteq \alpha$ for all $\alpha$, and
2. for every $X \subseteq \kappa$ the set \( \{ \alpha : X \cap \alpha = S_\alpha \} \) is stationary.

A family $S_\alpha$, for $\alpha < \kappa$, that satisfies Definition 8.3.1 is commonly called a $\diamondsuit_\kappa$ sequence or, if $\kappa = \aleph_1$, a $\diamondsuit$ sequence or a diamond sequence (readers used to indexing their sequences by $\mathbb{N}$ will notice that a diamond sequence is rather long).

Example 8.3.2. Suppose $\kappa$ is a regular cardinal.

1. Suppose that $S_\alpha \subseteq \alpha$ and $T_\alpha \subseteq \alpha$ for $\alpha < \kappa$, and that $\{ \alpha < \kappa : S_\alpha = T_\alpha \}$ includes a club in $\kappa$. Then $S_\alpha$, for $\alpha \in \kappa$ is a diamond sequence if and only if $T_\alpha$, for $\alpha \in \kappa$ is a diamond sequence (see Exercise 6.7.14). It therefore suffices to define $S_\alpha$ for $\alpha$ in a closed unbounded set.
2. If $S_\alpha$, for $\alpha < \kappa$, is a diamond sequence, then $S_\alpha$, for $\alpha = \omega \cdot \alpha, \alpha < \kappa$, is a diamond sequence because Exercise 6.7.4 implies that $\{ \alpha < \kappa : \alpha = \omega \cdot \alpha \}$ is a club in $\kappa$. 
Theorem 8.3.3. If ZFC is consistent, then the assertion that \( \diamond \kappa \) holds for every uncountable regular cardinal \( \kappa \) is consistent with ZFC.

Proof. In Gödel’s constructible universe \( L \), \( \diamond \kappa \) holds for all regular uncountable \( \kappa \) (see e.g., [166, §III.7.13]). □

Proposition 8.3.4. \( \diamond \kappa \) implies \( 2^{\kappa} = \kappa \). In particular \( \diamond \aleph_1 \) implies the Continuum Hypothesis.

Proof. Suppose \( S_\alpha \), for \( \alpha < \kappa \), witnesses \( \diamond \kappa \). Fix \( \alpha < \kappa \). Since stationary subsets are unbounded, for every \( X \subseteq \alpha \) there exists \( \beta > \alpha \) such that \( S_\beta = X \cap \beta = X \).

Since \( X \subseteq \alpha \) was arbitrary, we have \( \mathcal{P}(\alpha) = \{ S_\beta \cap \alpha : \beta \leq \alpha < \kappa \} \) and therefore \( |\mathcal{P}(\alpha)| \leq \kappa \). Since \( \alpha \) was an arbitrary ordinal smaller than \( \kappa \), and there are only \( \kappa \) ordinals smaller than \( \kappa \), this completes the proof. □

Although \( \diamond \kappa \) captures subsets of \( \kappa \), it is well-known among logicians that \( \diamond \kappa \) implies its self-strengthening which captures small elementary submodels of any structure of cardinality \( \kappa \) in a first-order language of cardinality smaller than \( \kappa \). This easily extends to metric structures, by the following argument.

Fix an uncountable regular cardinal \( \kappa \) and a first-order language \( \mathcal{L} \) of cardinality smaller than \( \kappa \). Recall (Definition 7.1.2) that \( \text{Struct}(\mathcal{L}, \theta) \) denotes the space of \( \mathcal{L} \)-structures with domain \( \theta \), and that it is identified with a subset of \( \mathcal{P}(\bigcup_{\theta \in \mathcal{L}} \theta^n(\mathcal{R})) \), where \( \mathcal{R} \) ranges over the relation symbols in \( \mathcal{L} \). (As explained in §7.1.1, assuming that \( \mathcal{L} \) has only relation symbols is not a loss of generality since constants and functions can be coded by relations, and \( \mathcal{B} \preceq \mathcal{A} \) implies that \( \mathcal{B} \) is closed under all functions in the original language.)

For \( \mathcal{A} \in \text{Struct}(\mathcal{L}, \kappa) \) and \( \alpha < \kappa \) let \( \mathcal{A} \upharpoonright \alpha := \mathcal{A} \cap \bigcup_{\theta \in \mathcal{L}} \alpha^n(\mathcal{R}) \). By Theorem 7.1.4), the set \( \{ \alpha < \kappa : \mathcal{A} \upharpoonright \alpha \preceq \mathcal{A} \} \) includes a club.

Definition 8.3.5. For an uncountable regular cardinal \( \kappa \), by \( \diamond \kappa(\mathcal{L}) \) we denote the following statement. There exists a family \( R_\alpha \), for \( \alpha < \kappa \), such that

1. \( R_\alpha \in \text{Struct}(\mathcal{L}, \alpha) \) for all \( \alpha < \kappa \), and
2. for every \( \mathcal{A} \in \text{Struct}(\mathcal{L}, \kappa) \) the set \( \{ \alpha : \mathcal{A} \upharpoonright \alpha \preceq \mathcal{A} \text{ and } \mathcal{A} \upharpoonright \alpha = R_\alpha \} \) is stationary.

If \( \mathcal{L} \) has a single unary predicate symbol, then \( \text{Struct}(\mathcal{L}, \kappa) \) is naturally identified with \( \mathcal{P}(\kappa) \) and \( \diamond \kappa \) is equivalent to \( \diamond \kappa(\mathcal{L}) \).

Proposition 8.3.6. Suppose \( \kappa \) is a regular cardinal and \( \mathcal{L} \) is a language of cardinality smaller than \( \kappa \). Then \( \diamond \kappa \) implies \( \diamond \kappa(\mathcal{L}) \).

Proof. Fix a bijection \( \chi : \kappa \to \bigcup_{\theta \in \mathcal{L}} \kappa^n(\mathcal{R}) \) and write \( \chi(\alpha) := \bigcup_{\theta \in \mathcal{L}} \alpha^n(\mathcal{R}) \). Let \( g : \kappa \to \kappa \) be the function defined by

\[
g(\alpha) := \min\{ \beta > \alpha : \chi[\alpha] \subseteq \chi(\beta) \text{ and } \chi^{-1}[\chi(\alpha)] \subseteq \beta \}.
\]

Since \( \kappa \) is a regular cardinal, \( g \) is well-defined. The set \( C := \{ \alpha : g(\alpha) = \alpha \} \) is a club in \( \kappa \) (Example 6.2.8). Since \( g(\alpha) > \alpha \) for all \( \alpha \), every element of \( C \) is a limit
The choice of \(X\). There exists a family
\[\gamma \in \mathcal{P}(\mathcal{X}(\gamma)) \subseteq \gamma.\] Since \(g\) is a bijection, we have \(C = \{\alpha < \kappa : \chi[\alpha] = \mathcal{X}(\alpha)\}\). Let \(S_\alpha\), for \(\alpha < \kappa\), be a \(\diamond\) sequence. For \(\alpha \in C\) let \(R_\alpha := \chi[S_\alpha]\). Since \(\mathcal{L}\) has no function symbols, this defines an element of \(\text{Struct}(\mathcal{L}, \alpha)\). We claim that \(R_\alpha\), for \(\alpha < \mathfrak{R}_1\), is a \(\diamond\) family.\(^7\)

Fix \(\mathfrak{A} \in \text{Struct}(\mathcal{L}, \kappa)\). The set \(C_\mathfrak{A} := \{\alpha \in C : \mathfrak{A} \upharpoonright \alpha \leq \mathfrak{A}\}\) is a club, and for every \(\alpha \in C_\mathfrak{A}\) we have \(\chi^{-1}[\mathfrak{A} \upharpoonright \alpha] = \chi^{-1}[\mathfrak{A}] \cap \alpha\). By comparing the \(\diamond\) sequence with \(\chi^{-1}[\mathfrak{A}]\), we conclude that the set \(R := \{\alpha \in C_\mathfrak{A} : \chi^{-1}[\mathfrak{A} \upharpoonright \alpha] = S_\alpha\}\) is stationary. For every \(\alpha \in R\) we have \(T_\alpha = \mathfrak{A} \upharpoonright \alpha \leq \mathfrak{A}\), as required.

The metric version of Proposition 8.3.6 is slightly trickier to state and prove. Fix a regular cardinal \(\kappa\) and a language \(\mathcal{L}\) in the logic of metric structures of cardinality smaller than \(\kappa\). Recall from §7.1.2 that \(\text{Struct}(\mathcal{L}, \theta)\) denotes the space of codes for \(\mathcal{L}\)-structures with domain \(\theta\) and that this space is identified with a subset of \(\mathcal{P}((\theta^2 \uplus \bigcup \theta^{\eta(R)}) \times \mathbb{Q})\), where \(R\) ranges over the relation symbols in \(\mathcal{L}\). (As explained in §7.1.1, assuming that \(\mathcal{L}\) consists only relation symbols is not a loss of generality since functions can be coded by their graphs.)

For \(\mathfrak{A} \in \text{Struct}(\mathcal{L}, \kappa)\) and \(\alpha < \kappa\) let
\[\mathfrak{A} \upharpoonright \alpha := \mathfrak{A} \cap (\{\alpha^2 \uplus \bigcup \theta^{\eta(R)}\} \times \mathbb{Q}).\]

Writing \(\overline{\mathfrak{A}}\) the metric completion of a code \(\mathfrak{A} \in \text{Struct}(\mathcal{L}, \alpha)\), Theorem 7.1.10 implies that \(\{\alpha < \kappa : \overline{\mathfrak{A}} \upharpoonright \alpha \leq \overline{\mathfrak{A}}\}\) is a club.

\[\text{Definition 8.3.7.}\] For a metric language \(\mathcal{L}\), by \(\diamond(\mathcal{L})\) we denote the following statement. There exists a family \(R_\alpha\), for \(\alpha < \kappa\), such that

1. \(R_\alpha \in \text{Struct}(\mathcal{L}, \alpha)\) for all \(\alpha < \kappa\), and
2. for every \(\mathfrak{A} \in \text{Struct}(\mathcal{L}, \kappa)\) the set \(\{\alpha : \overline{\mathfrak{A}} \upharpoonright \alpha \leq \overline{\mathfrak{A}}\}\) is stationary.

\[\text{Proposition 8.3.8.}\] Suppose \(\kappa\) is a regular cardinal and \(\mathcal{L}\) is a metric language of cardinality smaller than \(\kappa\). Then \(\diamond\) implies \(\diamond(\mathcal{L})\).

\[\text{Proof.}\] The proof is similar to that of Proposition 8.3.6. Let \(\chi : \kappa \to \bigcup \kappa^{\eta(R)}\) be a bijection. Since \(\kappa\) is a regular cardinal, the set \(C := \{\alpha < \kappa : \chi[\alpha] = \mathcal{X}(\alpha)\}\) is a club in \(\kappa\). Let \(S_\alpha\), for \(\alpha < \kappa\), be a \(\diamond\) family. Fix \(\alpha \in C\) for a moment, and let \(R_\alpha := \chi[S_\alpha]\) if \(\chi[S_\alpha]\) belongs to \(\text{Struct}(\mathcal{L}, \kappa)\). Otherwise, let \(R_\alpha\) be an arbitrary element of \(\text{Struct}(\mathcal{L}, \kappa)\) (the choice of \(R_\alpha\) will make no difference, see Example 8.3.2). We claim that \(R_\alpha\), for \(\alpha \in C\), is a \(\diamond(\mathcal{L})\) family.

Fix \(\mathfrak{A} \in \text{Struct}(\mathcal{L}, \kappa)\). The L"owenheim–Skolem Theorem implies that
\[C_\mathfrak{A} := \{\alpha \in C : \mathfrak{A} \upharpoonright \alpha \leq \mathfrak{A}\}\]
is a club. For \(\alpha \in C_\mathfrak{A}\) we have \(\chi^{-1}[\mathfrak{A} \upharpoonright \alpha] = \chi^{-1}[\mathfrak{A}] \cap \alpha\). By comparing the \(\diamond\) sequence with \(\chi^{-1}[\mathfrak{A}]\), we conclude that the set

\(^7\) The choice of \(R_\alpha\) for \(\alpha\) in the nonstationary set \(\mathfrak{R}_1 \setminus \mathcal{C}\) is inconsequential; see Example 8.3.2 (1).
The oldest application of \( \diamond_{\aleph_1} \) can be found in the next section. It is older than \( \diamond_{\aleph_1} \) itself, because the diamond principle was extracted from the construction of a Suslin tree.

### 8.4 Trees

In this section we introduce trees. The complete binary tree of \( \{0, 1\}^{<\aleph_1} \) will later be used as a bookkeeping device for constructing large families of automorphisms. The section ends with an obligatory Suslin tree construction from \( \diamond_{\aleph_1} \).

**Definition 8.4.1.** A poset \((T, \leq_T)\) is a tree if the set \( (\cdot, t] := \{ s \in T : s \leq_T t \} \) of predecessors of \( t \in T \) is well-ordered by \( \leq_T \). It is therefore order-isomorphic to a unique ordinal, height(\( t \)). For an ordinal \( \alpha \) the \( \alpha \)th level of \( T \) is the set

\[ T_\alpha := \{ t \in T : \text{height}(t) = \alpha \}. \]

The height of a tree \( T \) is the minimal \( \alpha \) such that the \( \alpha \)th level of \( T \) is empty. Some \( B \subseteq T \) is a chain if it is linearly ordered by \( \leq_T \). Some \( A \subseteq T \) is an antichain if for all distinct \( s \) and \( t \) in \( A \) neither \( s \leq_T t \) nor \( t \leq_T s \) holds. Such elements of \( T \) are said to be incomparable.

**Example 8.4.2.** The complete binary tree of height \( \aleph_1 \). Let \( \{0, 1\}^{<\aleph_1} \) denote the set of all \( s : \alpha \to \{0, 1\} \) for \( \alpha < \aleph_1 \). This set is ordered by the end-extension:

\[ s \sqsubseteq t \text{ if } t \upharpoonright \alpha = s \text{ for some } \alpha. \]

Then \( \{0, 1\}^{<\aleph_1}, \sqsubseteq \) is a tree of height \( \aleph_1 \), and for \( \alpha < \aleph_1 \) its \( \alpha \)th level is \( \{0, 1\}^\alpha \).

Given \( s \in \{0, 1\}^{<\aleph_1} \), by \( s0 \) and \( s1 \) we denote the sequences whose domain is \( \text{dom}(s) + 1 \) obtained by adding 0 (1, respectively) at the end of the sequence \( s \). These are the immediate successors of \( s \).

Any discussion of \( \diamond_{\aleph_1} \) would be incomplete without constructing a Suslin tree. It is not used elsewhere in this book but it provides an uncluttered textbook example of a \( \diamond \) construction. A reader interested in learning the construction of a counterexample to Naimark’s problem given in §11.2 may want to use it as a warmup.

**Definition 8.4.3.** A tree (see Definition 8.4.1) is \( \aleph_1 \)-Suslin, or Suslin, or Souslin in French transliteration, if it has height \( \aleph_1 \), no uncountable chains, and no uncountable antichains.

---

\( ^8 \) Suslin’s Hypothesis, the assertion equivalent to the nonexistence of Suslin trees, has given a considerable impetus to the development of combinatorial set theory. See e.g., Notes to this chapter and the discussion in [166].
Lemma 8.4.4. Suppose that $T$ is a tree whose height $\kappa$ is a regular cardinal such that every $s \in T$ has $\preceq_T$-incomparable extensions. If $T$ has a chain of cardinality $\kappa$ then it has an antichain of cardinality $\kappa$.

Proof. Suppose $\mathcal{C} \subseteq T$ is a maximal chain of cardinality $\kappa$. For every $\alpha < \kappa$ there is a unique $s_\alpha \in \mathcal{C} \cap T_\alpha$. Since it has two incomparable extensions, $s_\alpha$ has an extension $t_\alpha$ that does not belong to $\mathcal{C}$. Fix $g : \kappa \to \kappa$ such that $t_\alpha \in T_{g(\alpha)}$ for all $\alpha < \kappa$. As $\kappa$ is a regular cardinal, the set $D := C_\kappa$ of all ordinals closed under $g$ is a club in $\kappa$.

We claim that $X := \{t_\alpha : \alpha \in D\}$ is an antichain. Fix $\alpha < \beta$ in $D$. Then $t_\alpha$ is an extension of $s_\alpha$ incomparable with $s_\beta$. Since $t_\beta$ extends $s_\beta$, $t_\alpha$ and $t_\beta$ are incomparable. Since $\alpha$ and $\beta$ were arbitrary distinct elements of $D$, $X$ is an antichain. Since $|X| = \kappa$ this completes the proof. \hfill \Box

A subtree of a tree $T$ is a subset $S \subseteq T$ such that for all $s \in S$ and all $t \preceq_T s$ we have $t \in S$.

Theorem 8.4.5. $\Diamond_{\aleph_1}$ implies that there exists a Suslin tree.

Proof. Let $S_\alpha$, for $\alpha < \aleph_1$, be a diamond sequence. We shall construct a Suslin subtree of the complete binary tree.

Since $\Diamond_{\aleph_1}$ implies the Continuum Hypothesis, every infinite level $\{0, 1\}^\alpha$ of $\{0, 1\}^{<\aleph_1}$ has cardinality equal to $\aleph_1$. Since there are $\aleph_1$ levels, $|\{0, 1\}^{<\aleph_1}| = \aleph_1$.

Let $\chi : \aleph_1 \to \{0, 1\}^{<\aleph_1}$ be a bijection. Writing $\{0, 1\}^{<\alpha} := \bigcup_{\beta < \alpha} \{0, 1\}^\beta$, the set

$$C := \{\alpha < \aleph_1 : \chi(\alpha) \subseteq \{0, 1\}^{<\alpha}\}$$

is a club. For $\alpha \in C$ let $R_\alpha := \chi[S_\alpha]$.

We shall find $T_\alpha \subseteq \{0, 1\}^\alpha$, for $\alpha < \aleph_1$, that satisfy the following for all $\alpha < \aleph_1$.

1. $T_0 = \{\emptyset\}$ and $T_\alpha$ is countable for all $\alpha > 0$.
2. For every $s \in T_\alpha$ both $s0$ and $s1$ belong to $T_{\alpha+1}$.
3. If $\alpha < \beta$ then $T_\alpha = \{s \upharpoonright \beta : s \in T_\beta\}$.
4. $T(\alpha) := \bigcup_{\beta < \alpha} T_\beta$ is a subtree of $\{0, 1\}^{<\aleph_1}$.

To start with, let $T_m = \{0, 1\}^m$ for $m < \omega$ (we clearly have no choice!).

The recursive construction of $T_\alpha$, for $\alpha < \aleph_1$, proceeds as follows. Suppose that $\alpha$ is a countable limit ordinal and $T(\alpha)$ has been defined to satisfy the requirements.

Suppose $\alpha \notin C$ or $\alpha \in C$ but $R_\alpha$ is not a maximal antichain of $T(\alpha)$. Since $\alpha$ is a countable ordinal, we can fix an increasing sequence of ordinals $\beta(j)$, for $j < \omega$, with supremum equal to $\alpha$. We claim that for every $s \in T(\alpha)$ there exists a chain $\mathcal{K}(s)$ in $T(\alpha)$ such that $s \in \mathcal{K}(s)$ and $\mathcal{K}(s)$ intersects every level of $T(\alpha)$. Fix $s$ and let $m(0)$ be such that $s \in T(\beta(m(0)))$. Using (3) recursively choose $s_j \in T_{\beta(m(0)+j)}$ for $j \geq 0$ such that $s_j$, for $j < \omega$, is a chain. Then $\mathcal{K}(s) = \{t : (\exists j)t \subseteq s_j\}$ is the required chain.
Since $T(\alpha)$ is countable, we can enumerate all chains constructed in the previous paragraph as $K_j$, for $j < \omega$. For $k < \omega$ let $t_k := \bigcup_{j < \omega} s_j$. This is the unique element of $\{0,1\}^\alpha$ that extends all elements of $K_j$. Let $T_\alpha := \{t_k : k < \omega\}$.

Now suppose $\alpha \in C$ is a limit ordinal and $R_\alpha$ is a maximal antichain in $T(\alpha)$. Since each level of $T(\alpha)$ is infinite, so is $R_\alpha$. Let

$$X := \{s \in T(\alpha) : (\exists t \in R_\alpha) t \sqsubseteq s\}.$$  

Since $R_\alpha$ is a maximal antichain of $T(\alpha)$, for every $r \in T(\alpha)$ such that $r \notin X$ there exists $s \in R_\alpha$ such that $r \sqsubseteq s$.

For $s \in X$ construct a chain $K(s)$ such that $s \in K(s)$ and $K(s) \cap T_\beta \neq \emptyset$ for all $\beta < \alpha$. Since $R_\alpha \subseteq X$, $X$ is infinite. Let $K_j$, for $j < \omega$, be an enumeration of all $K(s)$ for $s \in X$. Let $t_j := \bigcup K_j$ for $j < \omega$ and let $T_\alpha := \{t_j : j < \omega\}$.

If $\alpha$ is a successor ordinal, $\alpha = \beta + 1$, then let $T_{\alpha} := \{s0, s1 : s \in T_\beta\}$.

This describes the recursive construction. It remains to check that $T := T(\kappa_{\omega1})$ is a Suslin tree. By Lemma 8.4.4, it suffices to prove that $T$ has no uncountable antichains. Suppose otherwise. By Zorn’s lemma it has a maximal uncountable antichain, $A$. By the Löwenheim–Skolem Theorem applied to the structure $(T, \subseteq, A)$, the set $D := \{\alpha \in C : A \cap \{0,1\}^{<\alpha} \text{ is a maximal antichain in } T(\alpha)\}$ is a club and therefore there exists $\alpha \in D$ such that $S_\alpha = \chi^{-1}[A \cap \{0,1\}^{<\alpha}]$. By construction, every element of $T_\alpha$ extends an element of $A$. Since $A$ is an antichain, this implies that it is included in $T(\alpha)$, and therefore countable; contradiction. \(\square\)

### 8.5 A Very Weak Forcing Axiom

In this section we discuss the axiom that asserts that no Polish space can be covered by fewer than $\kappa$ nowhere dense sets, for a given cardinal $\kappa$. This assertion with $\kappa = \mathfrak{c}$ is used to construct a selective ultrafilter.

A half-decent introduction to forcing axioms would require substantially more than two pages of text. The Baire Category Theorem (Theorem B.2.1) asserts that a topological space $X$ cannot be covered by countably many nowhere dense sets if it satisfies an additional assumption of being completely metrizable or being compact and Hausdorff. (Some assumption on $X$ is needed since for example $\mathbb{Q}$ clearly can be covered by countably many nowhere dense sets.)

**Definition 8.5.1.** Given a class $\Omega$ of compact Hausdorff spaces and a cardinal $\kappa$, the forcing axiom $\text{FA}_\kappa(\Omega)$ asserts that if $X \in \Omega$ then $X$ cannot be covered by $\kappa$ nowhere dense sets.

**Definition 8.5.2.** Let $\text{cov}(\mathcal{M})$ (also denoted $\text{m}_{\text{countable}}$, see [106]) denote the minimal cardinality of a family of meager sets that cover $[0,1]$.

Therefore $\text{FA}_\kappa([0,1])$ is equivalent to $\text{cov}(\mathcal{M}) > \kappa$. Clearly $\text{cov}(\mathcal{M}) \leq \mathfrak{c}$, and the Baire Category Theorem implies $\mathfrak{c}_1 \leq \text{cov}(\mathcal{M})$. For a Polish space $X$ one could
denote the minimal cardinality of a family of meager sets that cover $X$ by $\text{cov}_X(\mathcal{M})$. The axiom $\text{cov}(\mathcal{M}) = \mathfrak{c}$ holds in Cohen’s original model for the negation of the Continuum Hypothesis. It also holds in every model of the negation of the Continuum Hypothesis obtained by finite support iteration of ccc forcing starting from a model of the Continuum Hypothesis.

**Lemma 8.5.3.** If $X$ and $Y$ are uncountable Polish spaces with no isolated points then $\text{cov}_X(\mathcal{M}) = \text{cov}_Y(\mathcal{M})$.

*Proof.* Every Polish space with no isolated points has a dense $G_\delta$ subspace homeomorphic to the Baire space (Corollary B.2.6). Removing a meager subset does not change the value of $\text{cov}_X(\mathcal{M})$, and $\text{cov}_X(\mathcal{M}) = \text{cov}_{\mathcal{M}}(\mathcal{N}) = \text{cov}_Y(\mathcal{M})$. \[\square\]

As an application we construct an interesting ultrafilter on $\mathbb{N}$. An ultrafilter $\mathcal{U}$ is *nonprincipal* (or *free*) if $\bigcap_{X \in \mathcal{U}} X = \emptyset$.

**Definition 8.5.4.** A nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is *selective* if for every $f \in \mathbb{N}^\mathbb{N}$ there exists $A \in \mathcal{U}$ such that the restriction of $f$ to $A$ is constant or one-to-one.

The Continuum Hypothesis implies the existence of a selective ultrafilter on $\mathbb{N}$. The reader may want work out a proof of this fact\(^9\) before proceeding further.

**Example 8.5.5.** Suppose $B \subseteq \mathbb{N}$ is infinite. Then $\mathcal{Z}_B := \{A \subseteq \mathbb{N} : A \cap B \text{ is infinite}\}$ is dense $G_\delta$ in the compact metric topology on $\mathcal{P}(\mathbb{N})$. Since $B$ is infinite, the set $\mathcal{Y}_n := \{A \subseteq \mathbb{N} : (A \cap B) \setminus n \text{ is nonempty}\}$ is dense and open in $\mathcal{P}(\mathbb{N})$ for all $n$. Clearly $\mathcal{Z}_B = \bigcap_n \mathcal{Y}_n$.

**Lemma 8.5.6.** If a directed poset $\mathbb{P}$ is partitioned into finitely many pieces, then at least one of them is cofinal.

*Proof.* Assume that $\mathbb{P}$ can be partitioned into two pieces, $X_0$ and $X_1$, none of which is cofinal. Fix $p_n$ with no upper bound in $X_n$ for $n < 2$. If $p$ is an upper bound for both $p_0$ and $p_1$, then $p \notin X_0 \cup X_1$; contradiction. The general case follows by induction on the number of pieces. \[\square\]

**Proposition 8.5.7.** If $\text{cov}(\mathcal{M}) = \mathfrak{c}$ then there exists a selective ultrafilter.

*Proof.* Let $f_\xi$, for $\xi < \mathfrak{c}$, enumerate $\mathbb{N}^\mathbb{N}$. It suffices to find $A_\xi \subseteq \mathbb{N}$, for $\xi < \mathfrak{c}$, such that for every $\alpha < \mathfrak{c}$ the following hold.

1. For every $F \subseteq \alpha$ the intersection $\bigcap_{\beta \in F} A_\beta$ is infinite, and
2. The restriction of $f_\alpha$ to $A_\alpha$ is either constant or one-to-one.

\(^9\) Hint: You only need to know that $p = \mathfrak{c}$.
We proceed to find sets $A_\alpha$ by recursion on $\alpha$. Suppose that $A_\beta$, for $\beta < \alpha$, have been constructed to satisfy the requirements. For every $F \in \alpha$ the set $B_F := \bigcap_{\beta \in F} A_\beta$ is infinite.

Suppose for a moment that there exists $n \in \mathbb{N}$ such that $f^{-1}_\alpha[n]$ has an infinite intersection with all $B_F$. Let $P$ denote the poset of all $F \in \alpha$, ordered by the inclusion. By Lemma 8.5.6 there exists $m < n$ such that $\{ F : f^{-1}_\alpha(m) \cap B_F \text{ is infinite} \}$ is cofinal in $P$. Therefore $A_\alpha := f^{-1}_\alpha(m)$ has an infinite intersection with all $B_F$ and it is as required.

We may therefore assume that for every $n$ there exists $F \in \alpha$ such that $f^{-1}_\alpha(n) \cap B_F$ is finite. Let $Y := \{ C \subseteq \mathbb{N} : f_\alpha \upharpoonright C \text{ is one-to-one} \}$. This is clearly a closed subset of $\mathcal{P}(\mathbb{N})$, and it is therefore Polish in the relative topology. Fix $F \in \alpha$. The set $Y_F := \{ C \in Y : C \cap B_F \text{ is infinite} \}$ is a dense $G_\delta$ subset of $Y$ by an argument analogous to Example 8.5.5. Since $\text{cov}(\mathcal{M}) > |\alpha|$, we can find $A_\alpha \in \bigcap_{F \in \alpha} Y_F$, and this set is clearly as required. \qed

### 8.6 Open Colourings

A rose by any other name would smell as sweet.

Shakespeare, Romeo and Juliet

In this section we introduce the Ramseyan axiom OCA$_\Gamma$ and prove some of its consequences.

**Definition 8.6.1.** For a set $X$ let $[X]^2 := \{ s \subseteq X : |s| = 2 \}$. If $X$ is equipped with a linear ordering, we write $\{ x, y \}^<_w$ to denote the element of $[X]^2$ whose elements are $x$ and $y$ and to simultaneously emphasize that we are assuming $x < y$.

If $L \subseteq [X]^2$ and $Y \subseteq X$, then $Y$ is said to be $L$-homogeneous if $[Y]^2 \subseteq L$. Some $Y \subseteq X$ is homogeneous for a partition $[X]^2 = L_0 \cup L_1$ if it is $L_0$-homogeneous or $L_1$-homogeneous.

For example, Ramsey’s Theorem asserts that for every partition $[\mathbb{N}]^2 = L_0 \cup L_1$ there exists an infinite homogeneous $Y \subseteq \mathbb{N}$ (see e.g., [120]). One can consider partitions into any number of ‘colours,’ or partitions of triples or $n$-tuples for any $n \geq 2$. The analog of Ramsey’s Theorem is true in this generality.

**Example 8.6.2.** A partition $[\mathbb{R}]^2 = L_0 \cup L_1$ with no uncountable homogeneous sets. Let $<$ be the usual ordering of the reals, fix a well-ordering of the reals, $<_w$, and define a partition $[\mathbb{R}]^2 = L_0 \cup L_1$ by $\{ x, y \}^<_w \in L_0$ if $x <_w y$. An $L_0$-homogeneous set defines a strictly increasing function from a well-ordered set into $((\mathbb{R}, <))$. Since $\mathbb{R}$ is separable, there are no uncountable $L_0$-homogeneous sets. Similarly, there are no uncountable $L_1$-homogeneous sets.
In particular, Ramsey’s Theorem fails for $\mathbb{R}_1$. Now think of the worst imaginable failure of Ramsey’s Theorem for $\mathbb{R}_1$. This is a theorem ([187, Theorem 1.1]).

If $X$ is a topological space, then the set $|X|^2$ of all unordered pairs of distinct elements of $X$ is identified with the quotient of $X^2$ in which $(x,y)$ and $(y,x)$ are identified for all distinct pairs $x,y$ in $X$. Alternatively, a subset of $|X|^2$ is open if the corresponding symmetric subset of $X^2 \setminus \Delta_X$ is open. (Here $\Delta_X := \{(x,x) : x \in X\}$, the diagonal of $X$.) If $X$ is a topological space, then a partition $|X|^2 = L_0 \sqcup L_1$ is an open colouring if $L_0$ is an open subset of $|X|^2$. Consider the following axiom.

OCA\(_T\) Whenever $X$ is a separable metrizable space and $|X|^2 = L_0 \sqcup L_1$ is an open colouring, one of the following alternatives applies.

a. There exists an uncountable $L_0$-homogeneous $Y \subseteq X$.

b. There are $L_1$-homogeneous sets $X_n$, for $n \in \mathbb{N}$, such that $\cup_n X_n = X$.

‘Open’ in the definition of OCA\(_T\) is something of a red herring (see Exercise 8.7.27 and the proof of Proposition 8.6.3). OCA\(_T\) is relatively consistent with ZFC (e.g., [246]) and it is a consequence of the Proper Forcing Axiom ([242, Theorem 8.1]).

Proposition 8.6.3. Assume OCA\(_T\). If $X \subseteq \mathbb{R}$ is uncountable and $g : X \to \mathbb{R}$, then there exists an uncountable $Y \subseteq X$ such that the restriction of $g$ to $Y$ is continuous.

Proof. Define $[\mathbb{R}]^2 = L_0 \sqcup L_1$ by $\{x,y\} \subseteq L_0$ if $g(x) < g(y)$. If $X \subseteq \mathbb{R}$ is homogeneous, then the restriction of $g$ to $X$ is monotonic and therefore has at most countably many points of discontinuity the removal of which results in an uncountable with the required property. In order to prove that an uncountable homogeneous set exists, identify $x \in \mathbb{R}$ with $(x,g(x)) \in \mathbb{R}^2$, and consider $\mathbb{R}$ with the subspace topology. This topology is separable and metrizable, and by OCA\(_T\) an uncountable homogeneous set exists.

Together with Example 8.1.5, Proposition 8.6.3 implies the following.

Corollary 8.6.4. OCA\(_T\) and the Continuum Hypothesis are incompatible.

Proposition 8.6.5. Assume OCA\(_T\). If $X$ and $Y$ are uncountable subsets of $\mathbb{R}$, then there exists an uncountable $X' \subseteq X$ and an increasing $f : X' \to Y$.

Proof. Define $[X \times Y]^2 = L_0 \sqcup L_1$ by $\{(x,y),(x',y')\} \subseteq L_0$ if $x < x'$ and $y > y'$, or $x > x'$ and $y < y'$. If $X \times Y$ is taken with the subspace topology inherited from $\mathbb{R}^2$, this is an open partition. If $Z \subseteq X \times Y$ is $L_0$-homogeneous, then $X' := \{x : (x,y) \in Z$ for some $y\}$ and $f(x) := y$ if $(x,y) \in Z$ are as required. By OCA\(_T\), it remains to prove that $X \times Y$ cannot be covered by $L_1$-homogeneous sets $Z_m$, for $m \in \mathbb{N}$. Assume otherwise. For every $x \in X$ there exists $m(x)$ such that $Z_{m(x)}(x) := \{y \in Y : (x,y) \in Z_{m(x)}\}$ is uncountable. Fix $q(x) \in \mathbb{Q}$ such that $y(x) < q(x) < y'(x)$ for some $y(x)$ and $y'(x)$ in $Z_{m(x)}(x)$. Since $X$ is uncountable, there are $x < x'$ in $X$ such that $m(x') = m(x')$ and $q(x) = q(x')$. But this implies $\{(x,y'(x)),(x',y'(x))\} \in L_0$; contradiction.

The following strengthening of OCA\(_T\) was used to produce liftings of $*$-homomorphisms between coronas of separable C*-algebras in [181], [182], and [251].
OCA\textsubscript{∞} Whenever X is a separable metrizable space and $|X|^2 = L_0^n \cup L_0^n$, for $n \geq 0$, are open colourings such that $L_0^n \supseteq L_0^{n+1}$ for all $n$, one of the following alternatives applies.

a. There exists an uncountable $Z \subseteq \{0,1\}^\mathbb{N}$ and a continuous $f: Z \to X$ such that $(f(a), f(b)) \in L_0^{(a,b)}$ for all distinct $a$ and $b$ in $Z$.

b. There are $X_n \subseteq X$, for $n \in \mathbb{N}$, such that $|X_n|^2 \subseteq L_0^n$ for all $n$.

This axiom was introduced in [76], where it was proved to hold in both standard models of OCA\textsubscript{T} (i.e., it is a consequence of the Proper Forcing Axiom and it can be forced by a ccc forcing starting from a model of $V = L$). The following elegant proof was found only recently by Justin T. Moore and it is included with his kind permission.

**Theorem 8.6.6.** OCA\textsubscript{T} implies OCA\textsubscript{∞}.

**Proof.** Fix $X$ and open colourings $|X|^2 = L_0^n \cup L_0^n$ for $n \geq 0$ such that $L_0^n \supseteq L_0^{n+1}$ for all $n$. Define a partition $|\{0,1\}^\mathbb{N} \times X|^2 = M_0 \cup M_1$ by

$$\{(a,x),(b,y)\} \in M_0 \text{ if and only if } a \neq b, x \neq y, \text{ and } \{x,y\} \in L_0^{(a,b)}.$$  

This set is clearly symmetric. To see that it is open, fix $\{(a,x),(b,y)\} \in M_0$ and let $n := \Delta(a,b)$. Since $L_0^n$ is open, there are open neighbourhoods $U_x$ and $U_y$ of $x$ and $y$, respectively, such that $U_x \times U_y \subseteq L_0^n$. If $V_a \ni a$ and $V_b \ni b$ are open and such that $\Delta(c,d) = n$ for all $(c,d) \subseteq V_a \times V_b$, then $(V_a \times U_x) \times (V_b \times U_y) \subseteq M_0$.

Suppose for a moment that $\{0,1\}^\mathbb{N} \times X$ contains an uncountable $M_0$-homogeneous set $\mathcal{H}$. Then $\mathcal{H}$ is the graph of a function $f$ with the domain

$$Z := \{a : (a,x) \in \mathcal{H} \text{ for some } x\},$$

and $\{(a,x),(b,y)\} \in M_0^{(a,b)}$ for all distinct $a$ and $b$ in $Z$.

Otherwise, OCA\textsubscript{T} implies that $\{0,1\}^\mathbb{N} \times X$ can be covered by $M_1$-homogeneous sets $Y_n$, for $n \in \mathbb{N}$. For $x \in X$ and $n \in \mathbb{N}$ let $Y_n(x) := \{a \in \{0,1\}^\mathbb{N} : (a,x) \in Y_n\}$. The sets $Y_n(x)$, for $n \in \mathbb{N}$, are closed and cover $\{0,1\}^\mathbb{N}$, hence by the Baire Category theorem at least one of them has a nonempty interior. Fix $s(x) \in \{0,1\}^<\mathbb{N}$ and $n(x) \in \mathbb{N}$ so that $s(x) \subseteq Y_{n(x)}$. For a pair $(s,n)$ let $X_{s,n} := \{x \in X : (s(x),n(x)) = (s,n)\}$. Then $[s] \times X_{s,n}$ is included in the closure of $Y_n$. Since $L_1^n$ is closed, we have $|X_{s,n}|^2 \subseteq L_1^n$. Since $L_1^n \subseteq L_1^{n+1}$ for all $n$, we can enumerate the sequence $(X_{s,n})_{s,n}$ so that $|X_{s,n}|^2 \subseteq L_1^n$ for all $n$.  

See also Exercise 8.7.27, Proposition 9.5.7, and of course Theorem 17.8.5.

### 8.7 Exercises

**Exercise 8.7.1.** Prove that the cardinality of any second-countable Hausdorff space is at most $c$.  

Exercise 8.7.2. Prove that the cardinality of any compact Hausdorff space with no isolated points is at least \( \mathfrak{c} \).

Exercise 8.7.3. An element of a Boolean algebra is an atom if it is nonzero and no nonzero element is strictly below it.

1. Use Cantor’s back-and-forth method to prove that any two countable atomless Boolean algebras are isomorphic.
2. Use Stone duality and Gelfand–Naimark duality (§1.3) to translate (1) into the categories of compact Hausdorff spaces and abelian, unital, \( \mathbb{C}^* \)-algebras, respectively.

Exercise 8.7.4. Suppose \( a_\alpha \), for \( \alpha < \aleph_1 \), is a Cauchy net in a metric space \( A \). (This means that \( (\forall \varepsilon > 0)(\exists \alpha)(\forall \beta > \alpha)d(a_\alpha, a_\beta) < \varepsilon \).) Prove that there exists \( \alpha < \aleph_1 \) and \( a \in A \) such that \( a_\beta = a \) for all \( \beta > \alpha \).

Exercise 8.7.5. 1. Prove that in the ordering \( (\{0, 1\}^{\aleph_1}, \text{Lex}) \) (Example 8.2.5) every subset has the supremum.
2. Prove that the subordering of \( (\{0, 1\}^{\aleph_1}, \text{Lex}) \) obtained by removing the endpoints and all eventually zero functions is an \( \eta_1 \) set (Definition 8.2.2).

Exercise 8.7.6. Prove that any two \( \eta_1 \) sets of cardinality \( \aleph_1 \) are order isomorphic.

Exercise 8.7.7. Prove that there are order-nonisomorphic \( \eta_1 \) sets of cardinality \( \max(\mathfrak{c}, \aleph_2) \). How many isomorphism classes can you construct?

Definition 8.7.8. Given an ordinal \( \alpha \), a linear ordering \( L \) is an \( \eta_\alpha \) set if it has no endpoints and for every two nonempty subsets \( A \) and \( B \) of \( L \) such that \( a < b \) for all \( a \in A \) and \( b \in B \) and \( \max(|A|, |B|) < \aleph_\alpha \) there exists \( c \) such that \( a < c \) and \( c < b \) for all \( a \in A \) and all \( b \in B \). The following two exercises are relatives of Example 8.1.5.

Exercise 8.7.10. Prove that the Continuum Hypothesis implies there are uncountable sets of reals \( X \) and \( Y \) such that for any uncountable \( X' \subseteq X \) there is no monotonic function \( f: X' \to Y \).

Exercise 8.7.11. Prove that there exists \( g: \mathbb{R} \to \mathbb{R} \) such that the restriction of \( g \) to \( X \) is discontinuous for every \( X \subseteq \mathbb{R} \) of cardinality \( \mathfrak{c} \).

Exercise 8.7.12. Suppose \( A \) is a nonseparable metric structure and \( F \) is a \( \sigma \)-complete back-and-forth system between \( A \) and itself. Prove that \( F_1 := \{(C, D, \Phi) \in F : C = D\} \) is a \( \sigma \)-complete back-and-forth system between \( A \) and itself such that every \( p \in F \) has an extension in \( F_1 \).
Exercise 8.7.13. Suppose that \( A \neq C \) is a separable and unital C*-algebra. Prove that \( \bigotimes_{\mathbb{R}} A \) and \( \bigotimes_{\mathbb{R}} A \) are not isomorphic, but there exists a \( \sigma \)-complete back-and-forth system between them. Then prove the analogous statement in which \( \mathbb{R}_1 \) and \( \mathbb{R}_2 \) are replaced by any two distinct uncountable cardinals.

Exercise 8.7.14. Suppose \( S_\alpha, \) for \( \alpha < \kappa, \) is a diamond sequence in \( \kappa \) and \( C \subseteq \kappa \) is a club. Let \( T_\alpha := S_\alpha \) if \( \alpha \in C \) and \( T_\alpha = \emptyset \) otherwise. Prove that \( T_\alpha, \) for \( \alpha < \kappa, \) is a diamond sequence.

Exercise 8.7.15. Suppose \( \kappa \) is a regular uncountable cardinal. Consider the following strengthening of \( \diamondsuit \kappa. \)

\[ \diamondsuit^{++}_\kappa: \text{There exists a family } S_\alpha, \text{ for } \alpha < \kappa, \text{ such that} \]

1. \( S_\alpha \subseteq \alpha \) for all \( \alpha, \) and
2. for every \( X \subseteq \kappa \) the set \( \{ \alpha : X \cap \alpha = S_\alpha \} \) includes a club.

Prove, in ZFC, that \( \diamondsuit^{++}_\kappa \) is false.

Exercise 8.7.16. Consider \( T = \omega^\epsilon_{\omega_1} \) as a tree with respect to the end-extension ordering. Show that \( \diamondsuit_{\mathbb{R}_1} \) is equivalent to the assertion that there is \( t_\alpha \) in \( T \) of length \( \alpha \) for every \( \alpha < \omega_1 \) such that for every \( \omega_1 \)-branch \( b \) of \( T \) the set of all \( \alpha \) such that \( b \upharpoonright \alpha = t_\alpha \) is stationary.

The conclusion of Theorem 6.5.10 can be considerably improved if \( \diamondsuit_{\mathbb{R}} \) holds.

Exercise 8.7.17. Suppose \( \kappa \) is a regular cardinal. Prove that \( \diamondsuit \kappa \) implies that there are stationary subsets \( S_\xi, \) for \( \xi < 2^\kappa, \) of \( \kappa \) such that \( S_\xi \cap S_\eta \) is nonstationary for all \( \xi \neq \eta. \)

A **level-by-level product** of trees \( S \) and \( T \) of the same height \( \alpha \) is

\[ S \otimes T := \bigcup_{\beta < \alpha} S_\alpha \times T_\alpha. \]

(Some authors choose to write \( S \otimes T. \))

Exercise 8.7.18. Suppose \( T \) is a Suslin tree. Prove that \( T \otimes T \) is not a Suslin tree.

Exercise 8.7.19. Prove that \( \diamondsuit_{\mathbb{R}_1} \) implies the following.

1. There exist Suslin trees \( S \) and \( T \) such that \( S \otimes T \) is a Suslin tree.
2. Prove the analogous statement about \( \otimes \)-products of \( n \) Suslin trees for all finite \( n \geq 2. \)
3. Prove that the \( \otimes \)-product of infinitely many Suslin trees is never a Suslin tree.

The following exercise uses the definition of the structure \( H_\kappa \) of all sets whose hereditary closure has cardinality smaller than \( \kappa \) (Definition A.7.1).

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10 There is room enough only to say that the answer to the question whether the conclusion of Exercise (8.7.17) can be proved in ZFC is nothing short of fascinating; see e.g., [149, §§16–17] and [264, §3.2].
Exercise 8.7.20. Consider the following assertion.

\(\lozenge \neg (H_{\kappa_1})\): There exists a sequence \(S_\alpha \in H_{\kappa_1}\), for \(\alpha < \kappa_1\), such that for every parameter \(p \in H_{\kappa_2}\) and every \(X \subseteq H_{\kappa_1}\) there exists a countable elementary submodel \(M \preceq H_{\kappa_2}\) such that \(p \in M\), and with \(\delta := M \cap \kappa_1\) (this is an ordinal, Exercise 6.7.18) we have \(X \cap M = S_\delta\).

1. Prove that \(\lozenge \neg (H_{\kappa_1})\) is equivalent to \(\lozenge \kappa_1\).
2. Formulate \(\lozenge \neg (H_{\kappa})\) for an uncountable regular cardinal \(\kappa\) and prove that it is equivalent to \(\lozenge \kappa\).

Exercise 8.7.21. Use \(\lozenge \neg (H_{\kappa_1})\) (without using the conclusion of Exercise 8.7.20) to prove Theorem 8.4.5 directly.

An Aronszajn tree is a tree of height \(\kappa_1\) with no uncountable chains all of whose levels are countable. Clearly every Suslin tree is Aronszajn, but unlike a Suslin tree an Aronszajn tree can be constructed in ZFC (see [166, Theorem III.5.9]). Neither this fact nor the conclusion of the following exercise have found an application to operator algebras—yet. Somewhat inaccurately, a branch of a tree is cofinal if it intersects all levels of the tree.

Exercise 8.7.22. Suppose that \(T\) is a tree of height \(\kappa_1\) all of whose levels are finite. Prove that \(T\) has a cofinal branch. More generally, suppose \(\kappa\) is an infinite cardinal. Prove that every tree of height \(\kappa^{++}\) all of whose levels has cardinality at most \(\kappa\) has a cofinal branch.

Exercise 8.7.23. Prove that the minimal cardinality of a subset of \(\mathbb{N}\) that generates a nonprincipal ultrafilter is at least \(\text{cov}(\mathcal{M})\).

Exercise 8.7.24. Prove that if \(\mathcal{U}\) is a selective ultrafilter and \(r_n\), for \(n \in \mathbb{N}\), is a sequence in a compact metric space, then there exists \(A = \{n(j) : j \in \mathbb{N}\} \in \mathcal{U}\) such that \(\lim_{j \to \infty} r_{n(j)}\) exists.

A function is finite-to-one if each of its fibres is finite.

Exercise 8.7.25. Define \(f : \mathbb{N} \to \mathbb{N}\) by \(f(2^m(2n + 1)) = m\). Let \(\mathscr{I}\) be the set of all \(A \subseteq \mathbb{N}\) that can be partitioned in two pieces \(A_0\) and \(A_1\) such that \(f[A_0]\) is finite and the restriction of \(f\) to \(A_1\) is finite-to-one. Prove that \(\mathscr{I}\) is a proper ideal on \(\mathbb{N}\) and that no selective ultrafilter is disjoint from \(\mathscr{I}\).

Exercise 8.7.26. Suppose \(X\) is a one-point compactification of an uncountable discrete topological space. Prove that the Tychonoff power \(X^{[\mathbb{N}]^\mathbb{N}}\) can be covered by \(\mathbb{R}_1\) nowhere dense sets.

Exercise 8.7.27. Prove that OCA\(_T\) is equivalent to the following statement:

For every \(X\) and every partition \(|X|^2 = L_0 \uplus L_1\) such that \(L_0\) is the union of countably many rectangles (i.e., \(L_0 = \bigcup_{m \in \mathbb{N}} A_m \times B_m\) for some \(A_m, B_m\)) one of the two alternatives of OCA\(_T\) applies.
Notes for Chapter 8

§8.1 The proof of Theorem 8.1.2 also shows that accepting the unlimited comprehension (see §A.1) leads to a contradiction. More precisely, $S := \{x : x \notin x\}$ is not well-defined. (Proof: $S \in S$ implies $S \notin S$ implies $S \in S$.) This argument that Cantor’s original set theory was inconsistent is Russell’s paradox. This issue was resolved by introducing the axiomatics of ZFC (see §A.1). There are other ways to resolve this issue and lay the foundations of mathematics, but we’ll stick with ZFC.

The Continuum Hypothesis is independent from ZFC by results of Gödel and Cohen (see e.g., [166]). In lieu of a philosophical discussion, here is a story once told by Paul Erdős and shared by Menachem Magidor.

When I (Erdős) die and get to see the God, I will ask him whether the Continuum Hypothesis is true. He may give me any of the following three answers. The first one, ‘Here is the answer and here is the proof’; the second one, ‘That is a meaningless question, go away.’ The third one would be the worst: ‘That is a great question with a beautiful answer, but you cannot understand it.’

§8.2 Readers familiar with Elliott’s approximate intertwining method (§6.1) may wonder whether the transfinite back-and-forth method can be spiced up by allowing the (naturally defined) approximate partial isomorphisms. This would, however, make no difference: In a metric space, every Cauchy net indexed by $\aleph_1$ is eventually constant (Exercise 8.7.4).

In the model-theoretic terms, a dense linear ordering without endpoints is an $\eta_1$ set if and only if it is countably quantifier-free saturated. Since the theory of linear orderings without endpoints admits elimination of quantifiers (see e.g., [178]), a linear ordering is an $\eta_1$ set if and only if it is countably saturated.

Hausdorff’s interest in $\eta_1$ drew from (in modern language) the study of Hardy fields; see [18] for the most recent progress.

Suppose $C$ and $D$ are nonseparable metric structures. By Lemma 8.2.9 (2), club many entries $(A, B, \Phi)$ of any $\sigma$-complete back-and-forth system between $C$ and $D$ have the property that $A$ is an elementary submodel of $C$ and $B$ is an elementary submodel of $D$. It is therefore easier to construct a $\sigma$-complete back-and-forth system if this condition is automatic, for example if the theories of $C$ and $D$ admit the elimination of quantifiers. This is the case with dense linear orderings without endpoints (see e.g., [178]), but very few theories of $C^*$-algebras—five out of continuum, to be precise—have this property ([69]). This is one of the reasons why finding an outer automorphism of the Calkin algebra took so long (see Exercise 17.9.1).

§8.3 In 1920 Suslin asked whether a compact linearly ordered topological space with no uncountable family of disjoint open intervals is necessarily order-isomorphic to the unit interval. Kurepa proved that the negative answer to Suslin’s question is equivalent to the existence of a Suslin tree and constructed an Aronszajn tree. While $\Diamond_{\mathbb{R}_1}$ implies that Suslin trees exist, and every set-theoretic axiom used in this text is compatible with the existence of Suslin trees, sufficiently strong forcing axioms (starting with Martin’s Axiom) imply that there are no Suslin trees. While $\Diamond_{\mathbb{R}_1}$

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11 Erdős used his own terminology at this point.
implies the Continuum Hypothesis, there are models of ZFC in which the Continuum Hypothesis holds and ♦₁ fails (see e.g., [224, Chapter V++]).

Ronald Jensen formulated ♦ and proved Theorem 8.3.3 and Theorem 8.4.5 ([135]). Although Proposition 8.3.6 and Proposition 8.3.8 are a part of the folklore of mathematical logic, they were apparently explicitly stated (only in the case of C*-algebras) for the first time in [92]. The equivalent reformulation of ♦₁ given in Exercise 8.7.16 was used in [5]. For strengthenings and other equivalent reformulations of ♦₁ see e.g., [166, §III.7]. Jensen constructed a κ-Suslin tree in L for every uncountable regular cardinal κ that is not weakly compact. The construction uses another combinatorial principle in addition to ♦₁, denoted □. There are consistent and useful strengthenings of ♦₁, denoted ♦²₁ and ♦⁺₁ (cf. Exercise 8.7.15). For these, and other, combinatorial principles that hold in Gödel’s constructible universe L (most of them isolated by Jensen) see e.g., [166] or [54].

A modern approach to diamonds can be found in [32], where Exercise 8.7.20 was taken from.

8.4 The construction of a Suslin tree from ♦₁ is due to Jensen. The arboreal branch of set theory has attracted a considerable attention. Trees provide what may be the cleanest examples of incompactness phenomena in uncountable cardinalities. They are also a powerful tool for constructions of combinatorially interesting large objects.

8.5 The assertion \( \text{cov}(\mathcal{M}) > \kappa \) barely deserves to be called a forcing axiom. The assertion \( \text{cov}(\mathcal{M}) = \kappa \) holds in models of ZFC obtained by finite support iteration of ccc forcings, starting from a model of the Continuum Hypothesis (see [166]). Hence the equality \( \text{cov}(\mathcal{M}) = \kappa \) holds in many ‘soft’ models of ZFC. This assertion is however independent from ZFC. Even the existence of selective ultrafilters is independent from ZFC, by a result of Kunen ([164]).

The earliest and most important (historically and mathematically) forcing axiom is Martin’s Axiom, or MA. A wealth of information on MA, including the proof that it implies there are no Suslin trees, can be found in [166]. As κ increases, the class Ω of compact Hausdorff spaces for which FAκ(Ω) cannot be refuted in ZFC shrinks (see Exercise 8.7.26 for the first step). With our present understanding, the axioms of the form FAκ₁(Ω) (just a notch above the bound given by the Baire Category Theorem) are the most useful ones. Forcing axioms of this sort, such as the Proper Forcing Axiom and its consequences, were successfully used in analyzing coronas of separable C*-algebras ([82], [81], [181], [182], [251]). The provably strongest forcing axiom of the form FAκ₁(Ω), Martin’s Maximum, or MM, was isolated in [105]. It implies the popular Proper Forcing Axiom, which in turn implies OCA_T. Martin’s Maximum can be strengthened in other directions, see [248].

Whether the relative commutant of \( \mathcal{B}(H) \) in an ultrapower associated to a non-principal ultrafilter \( \mathcal{U} \) on \( \mathbb{N} \) (§16.7) is trivial or not depends on the choice of \( \mathcal{U} \). If \( \mathcal{U} \) is selective then the relative commutant is trivial but it is nontrivial if \( \mathcal{U} \) is a so-called flat ultrafilter ([95]). Selective ultrafilters are also known as Ramsey ultrafilters, but that is an altogether different story.
§8.6 is largely based on [242], except Theorem 8.6.6 which is taken from [188]. OCA_T is known under other names, and a different axiom is also known under the name of OCA ([2]). (In both cases OCA stands for ‘Open Colouring Axiom.’)
Chapter 9
Set Theory and Quotients

Don’t let ideology or good taste stop you from proving a theorem. Saharon Shelah

In any sufficiently complicated category an increase in cardinality (or, in the case of metric structures, an increase in density character) of its objects results in the increase in the complexities of its objects and its morphisms alike. One instance of this phenomenon is the failure of the back-and-forth method to reach uncountable or nonseparable realm without some sort of a ‘crutch’ (our crutch of choice is discussed at length in §8.2 and §16.6). In a precisely defined set-theoretic sense, quotient structures have higher definable cardinality than the original structures.¹ The earliest instance of this phenomenon was probably the observation that if $G$ is a Polish group acting continuously on a Polish space $X$ with all orbits being both dense and meager, than the orbit equivalence relation is not smooth. (An equivalence relation $E$ on a Polish space $X$ is smooth if there exists a Borel-measurable function $f: X \to \mathbb{R}$ whose fibres are the equivalence classes.) Comparison of definable cardinalities from descriptive set-theoretic point of view resulted in the abstract classification theory. Notably, one of its ingredients comes from the theory of $C^*$-algebras. Glimm’s 1960 theorem (Theorem 3.7.2) can be used to show that the Borel structure of $\mathcal{P}(\mathbb{N})/\text{Fin}$ can be continuously embedded into the space of unitary equivalence classes of pure states on a non-type I separable $C^*$-algebra. The so-called Glimm–Effros Dichotomy in abstract classification theory was fuelled by the combinatorics borrowed from Glimm’s theorem (see e.g., [130] and [112]).

In the present chapter we introduce arguably the most elementary massive quotient structure. This is the quotient Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$.

The Stone duality (§1.3.1) associates $\mathcal{P}(\mathbb{N})$ with the Čech–Stone compactification $\beta\mathbb{N}$ and the associated quotient $\mathcal{P}(\mathbb{N})/\text{Fin}$ with the remainder (also called corona), $\beta\mathbb{N} \setminus \mathbb{N}$. The Gelfand–Naimark duality (Theorem 1.3.2) associates these two spaces with the $C^*$-algebras $\ell_\infty$ and $\ell_\infty/c_0$, respectively.

¹ This phenomenon can be viewed from the operator-algebraic point of view, see [47, §II].

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Much—but not all—of the relevance of set theory to the study of the Calkin algebra draws from the popularity of $\mathcal{P}(\mathbb{N})/\text{Fin}$ as a target for set-theoretic study. This resulted in a substantial body of deep results about this fascinating object. This algebra is possibly the most natural object whose theory is ‘set-theoretically malleable’ in the sense that few of its properties can be decided in ZFC without assuming additional set-theoretic axioms.

Many of these results transfer into the even more fascinating ‘quantized’ results about $\mathcal{Q}(\mathcal{H})$. Quantizations of some other results about $\mathcal{P}(\mathbb{N})/\text{Fin}$ are independent from ZFC, and for yet another group of results about $\mathcal{P}(\mathbb{N})/\text{Fin}$ it is not even clear what the correct quantization should be.

Ideals and filters are introduced in §9.1. Almost disjoint and independent families in $\mathcal{P}(\mathbb{N})$ are constructed in §9.2. The classical constructions of gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$ of Hausdorff and Luzin are given in §9.3. Later on, we will see that $\mathcal{P}(\mathbb{N})/\text{Fin}$ can be ‘injected’ into massive quotient $C^*$-algebras in a gap-preserving manner (Theorem 14.2.1). The Rudin–Keisler ordering on ultrafilters on $\mathbb{N}$ is studied in §9.4.

Small cardinal characteristics of the continuum and basics of the Tukey ordering on directed sets are studied in §9.6. This theory is put to use in §9.7 to study the directed set $\mathbb{N}^\omega$, as well as its lesser-known (but even more relevant to the corona algebras of $\sigma$-unital $C^*$-algebras) relatives Part$_{\mathbb{N}}$ and, in §9.8, Part$_{\mathbb{L}}$. These directed sets will be used to stratify and (to some extent) tame the Calkin algebra.

### 9.1 Ideals and Filters

In this section we review the basics of ideals and filters. We also briefly introduce Borel ideals on $\mathbb{N}$ and ideals associated with lower semicontinuous submeasures on $\mathbb{N}$. We review $F_\sigma$ ideals and analytic ideals on $\mathbb{N}$ and their relation to lower semicontinuous submeasures.

A family $\mathcal{U}$ of subsets of $X$ has the finite intersection property (sometimes abbreviated as the f.i.p.) if for every $\mathcal{U}_0 \in \mathcal{U}$ the set $\bigcap \mathcal{U}_0$ is nonempty.\footnote{Hence a more accurate, but not particularly catchy, name for this property would be ‘the nonempty intersection of all finite subfamilies property.’}

**Definition 9.1.1.** A filter on a set $X$ is a subset $\mathcal{F}$ of $X$ with the following properties.

1. It has the finite intersection property.
2. If $Y \in \mathcal{F}$ then $Y \cup Z \in \mathcal{F}$ for all $Z \subseteq X$.

It is proper if $\mathcal{F} \neq \mathcal{P}(X)$ (equivalently, if $\emptyset \notin \mathcal{F}$).

An ideal on a set $X$ is a subset $\mathcal{I}$ of $X$ satisfying the following.

3. The union of a finite subset of $\mathcal{I}$ belongs to $\mathcal{I}$.
4. If $Y \in \mathcal{I}$ then $Y \cap Z \in \mathcal{I}$ for all $Z \subseteq X$.

It is proper if $\mathcal{I} \neq \mathcal{P}(X)$ (equivalently, if $X \notin \mathcal{I}$). An ideal $\mathcal{I}$ on $\mathbb{N}$ is dense if every infinite subset of $\mathbb{N}$ has an infinite subset in $\mathcal{I}$.
A set $\mathcal{I}$ is an ideal on $X$ if and only if it is an ideal of the Boolean ring $\mathcal{P}(X)$.

**Definition 9.1.2.** The dual ideal of a filter $\mathcal{F}$ on $X$ is $\mathcal{F}_+ := \{X \setminus Z : Z \in \mathcal{F}\}$ and the coideal of $\mathcal{F}$-positive sets is $\mathcal{F}_* := \mathcal{P}(X) \setminus \mathcal{F}_+$. If $\mathcal{I} = \mathcal{F}_+$ then the $\mathcal{I}_* := \mathcal{F}$ is the dual filter and $\mathcal{I}_+ := \mathcal{I}_*$ is the coideal of $\mathcal{I}$-positive sets.

Note that $\mathcal{F}_+ = \mathcal{F}_+$ if and only if $\mathcal{I}$ is an ultrafilter.

**Example 9.1.3.** A variety of manifestations of the power set of the natural numbers, $\mathcal{P}(\mathbb{N})$, will play a role in our study. We’ll take a look at some of them (see also Example B.1.3).

First, $\mathcal{P}(\mathbb{N})$ is a concrete Boolean algebra. Identify $A \in \mathcal{P}(\mathbb{N})$ with its characteristic function defined by $\chi_A(n) := 1$ if $n \in A$ and $\chi_A(n) := 0$ if $n \notin A$. Then $A \mapsto \chi_A$ is a bijection between $\mathcal{P}(\mathbb{N})$ and $\{0, 1\}^\mathbb{N}$. The latter is a compact metrizable space with respect to the product topology. It is also naturally identified with the Cantor’s middle third space. A compatible metric on $\mathcal{P}(\mathbb{N})$ is defined by (writing $A \Delta B$ for the symmetric difference of sets $A$ and $B$)

$$d(A, B) := \frac{1}{\min(A \Delta B) + 1}.$$  

One can also identify $\{0, 1\}$ with the cyclic group $\mathbb{Z}/2\mathbb{Z}$ and $\mathcal{P}(\mathbb{N})$ with $\prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$. This is a compact metrizable abelian group, and its Haar measure coincides with the product of uniform probability measures on $\{0, 1\}^\mathbb{N}$.

An ideal $\mathcal{I}$ on $\mathbb{N}$ is *Borel* if it is a Borel subset of $\mathcal{P}(\mathbb{N})$ in the compact metric topology defined in Example 9.1.3.

It is sometimes convenient to identify $\mathcal{P}(\mathbb{N})$ with $\mathcal{P}(X)$ for a countably infinite set $X$. While any bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ lifts an isomorphism between $\mathcal{P}(\mathbb{N})$ and $\{0, 1\}^\mathbb{N}$. The latter is a compact metrizable space with respect to the product topology. It is also naturally identified with the Cantor’s middle third space. A compatible metric on $\mathcal{P}(\mathbb{N})$ is defined by (writing $A \Delta B$ for the symmetric difference of sets $A$ and $B$)

$$d(A, B) := \frac{1}{\min(A \Delta B) + 1}.$$  

An ideal $\mathcal{I}$ on $\mathbb{N}$ is *Borel* if it is a Borel subset of $\mathcal{P}(\mathbb{N})$ in the compact metric topology defined in Example 9.1.3.

It is sometimes convenient to identify $\mathcal{P}(\mathbb{N})$ with $\mathcal{P}(X)$ for a countably infinite set $X$. While any bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$ lifts an isomorphism between $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(X)$, additional structure of the set $X$ (e.g., if $X$ is $\mathbb{N}^\mathbb{N}$ or $\mathbb{Q}$) often makes constructions and arguments more transparent.

**Example 9.1.4.**  
1. By $\text{Fin}$ we denote the ideal of finite subsets of $\mathbb{N}$, also called the Fréchet ideal. Its dual filter is the filter of cofinite subsets of $\mathbb{N}$, also called the Fréchet filter. If $X$ is an infinite set then $\text{Fin}_X$ denotes the ideal of finite subsets of $X$. The subscript $X$ is dropped whenever $X$ is clear from the context.
2. Consider the ideal $\text{Fin} \times \emptyset$ on $\mathbb{N}^2$ consisting of all sets $X \subseteq \mathbb{N}^2$ whose projection to the first coordinate is finite. Equivalently,

$$\text{Fin} \times \emptyset = \{X \subseteq \mathbb{N}^2 : (\exists m)(\forall m \geq m)(\forall n)(m', n) \notin X\}.$$  

**Example 9.1.5.** Suppose $\mathbb{P}$ is a directed and $\sigma$-complete poset (Definition 6.2.3); e.g., an ordinal with uncountable cofinality, $|X|^\aleph_0$ for an uncountable set $X$, or $\text{Sep}(M)$ for a nonseparable metric structure $M$. Club subsets of $\mathbb{P}$ have the finite

---

3 In the literature $\chi_A$ is sometimes denoted $1_A$.

4 The reader can extrapolate the definition of a Fubini product $\mathcal{I} \times \mathcal{J}$ of ideals $\mathcal{I}$ and $\mathcal{J}$; it occurs implicitly in Exercise 8.7.25 and Example 9.3.9.
intersection property (Proposition 6.2.9). Therefore the club filter on $\mathbb{P}$, defined as the family of all $C \subseteq \mathbb{P}$ that include a club, is a filter. The dual ideal is the ideal of nonstationary sets. Stationary subsets of $\mathbb{P}$ comprise the coideal of positive sets.

The following definition will be used only in §14.1.

**Definition 9.1.6.** A function $\varphi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a submeasure if it satisfies the following requirements.

1. $\varphi(\emptyset) = 0$.
2. $\varphi(X) < \infty$ if $X$ is finite.
3. $\varphi$ is subadditive: $\varphi(X \cup Y) \leq \varphi(X) + \varphi(Y)$.
4. $\varphi$ is monotonic: $X \subseteq Y$ implies $\varphi(X) \leq \varphi(Y)$.

**Example 9.1.7.** 1. Every measure is a submeasure, and the supremum of a uniformly bounded family of measures is a submeasure.
2. If $f : \mathbb{R}_+ \to \mathbb{R}_+$ is concave and satisfies $f(0) = 0$, then the composition of a measure with $f$ is a submeasure.
3. There are more thought-provoking examples of submeasures (see [236]) some of which give rise to interesting examples of ideals on $\mathbb{N}$ (see [77, §11.1–1.13]).

Every ideal $\mathcal{J}$ on $\mathbb{N}$ is the null set of a submeasure $\varphi$ defined by setting $\varphi(X) = 0$ if $X \in \mathcal{J}$ and $\varphi(X) = 1$ if $X \not\in \mathcal{J}$; this observation can be useful (see [150]).

**Definition 9.1.8.** If $\varphi(X) = \lim_n \varphi(X \cap n)$ for all $X \in \mathcal{P}(\mathbb{N})$, a submeasure $\varphi$ is said to be lower semicontinuous.

(This is easily checked to be equivalent to $\varphi$ being lower semicontinuous with respect to the Cantor-set topology on $\mathcal{P}(\mathbb{N})$.) Given a lower semicontinuous submeasure $\varphi$ on $\mathcal{P}(\mathbb{N})$ we define

\[
\begin{align*}
\text{Fin}(\varphi) &:= \{ X \subseteq \mathbb{N} : \varphi(X) < \infty \} \\
\text{Exh}(\varphi) &:= \{ X \subseteq \mathbb{N} : \lim_n \varphi(X \cap n) = 0 \}.
\end{align*}
\]

**Definition 9.1.9.** An ideal $\mathcal{J}$ is a $P$-ideal if for every sequence $X_n$, $n \in \mathbb{N}$ of elements of $\mathcal{J}$ there exists $X \in \mathcal{J}$ such that $X_n \setminus X$ is finite for all $n$.

**Lemma 9.1.10.** If $\varphi$ is a submeasure on $\mathbb{N}$ then $\text{Exh}(\varphi)$ is a Borel $P$-ideal on $\mathbb{N}$ and $\text{Fin}(\varphi)$ is an $F_\sigma$ ideal on $\mathbb{N}$. If in addition $\lim_n \varphi(\{n\}) = 0$ then both of these ideals are dense.\(^5\)

**Proof.** For every $r \in (0, \infty)$ the set $\mathcal{Z}_r := \{A \subseteq \mathbb{N} : \varphi(A) \leq r\}$ is closed. Therefore $\text{Fin}(\varphi) = \bigcup_{n \in \mathbb{N}} \mathcal{Z}_n$ is $F_\sigma$ and $\text{Exh}(\varphi) = \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \{ A : A \setminus n \in \mathcal{Z}_{1/m}\}$ is $F_\sigma\delta$.

To see that $\text{Exh}(\varphi)$ is a $P$-ideal, fix $A_n \in \text{Exh}(\varphi)$ for $n \in \mathbb{N}$. If $k(n)$ is such that $\varphi(A_n \setminus k(n)) < 2^{-n}$ then $\bigcup_n (A_n \setminus k(n)) \in \text{Exh}(\varphi)$ and it includes all $A_n$ modulo finite. Now suppose $\lim_n \varphi(\{n\}) = 0$ and fix an infinite $X \subseteq \mathbb{N}$. Choose an infinite $Y \subseteq X$ such that $\sum_{n \in Y} \varphi(\{n\}) < \infty$; then $Y$ belongs to both ideals. \(\square\)

\(^5\) In the sense that every infinite $X \subseteq \mathbb{N}$ has an infinite subset in the ideal.
9.2 Almost Disjoint and Independent Families in \( \mathcal{P}(\mathbb{N}) \)

In this section we study the quotient Boolean algebra \( \mathcal{P}(\mathbb{N})/\text{Fin} \) of the power set of \( \mathbb{N} \) modulo the Fréchet ideal \( \text{Fin} \). We construct a family of almost disjoint subsets of \( \mathbb{N} \) and an independent family of subsets of \( \mathbb{N} \), each of cardinality \( c \). The latter family will be used to produce a simple graph CCR algebra with nonhomogeneous state space in §10.4.

Since \( \mathcal{P}(\mathbb{N}) \) carries a natural compact metric structure (Example 9.1.3) it is sometimes easier to work in \( \mathcal{P}(\mathbb{N}) \) than in the quotient \( \mathcal{P}(\mathbb{N})/\text{Fin} \).

**Definition 9.2.1.** Suppose \( A \) and \( B \) belong to \( \mathcal{P}(\mathbb{N}) \). We write \( A \subseteq^* B \) (or \( B \supseteq^* A \)) if \( A \setminus B \in \text{Fin} \) and \( A \perp B \) if \( A \subseteq^* \mathbb{N} \setminus B \). If \( A \subseteq^* B \) then \( A \) is *almost included* in \( B \) and if \( A \perp B \) then \( A \) and \( B \) are *almost disjoint*.

The relation \( \subseteq^* \) is transitive and reflexive but not antisymmetric; such relations are called *quasi-orderings*. If \( \rho \) is a quasi-ordering then the relation \( \sim_\rho \) defined by symmetrizing \( \rho \), \( a \sim_\rho b \) if and only if \( \rho a \rho b \), is an equivalence relation and \( \rho \) is compatible with \( \sim_\rho \). A family \( \mathcal{A} \subseteq \mathcal{P}(\mathbb{N}) \) is *almost disjoint* if every two elements of \( \mathcal{A} \) are almost disjoint.

**Proposition 9.2.2.** There exists an almost disjoint family \( \mathcal{A} \) in \( \mathcal{P}(\mathbb{N}) \) of cardinality \( c \). One can choose this family to be dense in the Cantor-set topology on \( \mathcal{P}(\mathbb{N}) \).

**Proof.** It suffices to find an almost disjoint family in \( \mathcal{P}(\mathbb{Q})/\text{Fin}_\mathbb{Q} \) (where \( \text{Fin}_\mathbb{Q} \) denotes the ideal of finite subsets of \( \mathbb{Q} \)) indexed by the irrationals. For every irrational \( x \) let (as usual \( \lceil q \rceil := \max \{ m \in \mathbb{Z} : m \leq q \} \))

\[
A_x := \{ \lceil 10^n x \rceil 10^{-n} : n \in \mathbb{N} \}.
\]

Then \( A_x \) is infinite for every \( x \) with an infinite decimal expansion and \( A_x \cap A_y \) is finite whenever \( x \neq y \). Since the space \( \mathcal{P}(\mathbb{N}) \) is second-countable, by making finite modifications to some \( \mathbb{R}_0 \) elements of this family we can obtain an uncountable almost disjoint family dense in \( \mathcal{P}(\mathbb{N}) \). \( \square \)

**Definition 9.2.3.** A family \( \mathcal{A} \subseteq \mathcal{P}(X) \) is *independent* if \( \bigcap F \setminus \bigcup G \) is nonempty for all nonempty \( F \in \mathcal{A} \) and \( G \in \mathcal{A} \setminus F \).

**Lemma 9.2.4.** An infinite family \( \mathcal{A} \) is independent if and only if if for all disjoint \( F \in \mathcal{A} \) and \( G \in \mathcal{A} \) such that \( F \) is nonempty the intersection \( \bigcap F \setminus \bigcup G \) is infinite.

**Proof.** For the nontrivial implication, suppose that \( \bigcap F \setminus \bigcup G \) is finite for some \( F \) and \( G \) as in the statement. Choose \( F \) and \( G \) so that \( n = | \bigcap F \setminus \bigcup G | \) is minimal possible. Since \( \mathcal{A} \) is infinite, there exists \( Y \in \mathcal{A} \setminus (F \cup G) \). Then at least one of \( \bigcap (F \cup Y) \setminus \bigcup G \) or \( \bigcap F \setminus (G \cup Y) \) has cardinality smaller than \( n \); contradiction. \( \square \)

**Proposition 9.2.5.** There exists an independent family of subsets of \( \mathbb{N} \) of cardinality \( c \). One can choose this family to be dense in the Cantor-set topology on \( \mathcal{P}(\mathbb{N}) \).
Proof. Our family will consist of subsets of the (countable) set
\[ Z := \{ F \subset \{0,1\}^{\mathbb{N}} : F \text{ is finite} \}. \]

An initial segment of length \( m \) of \( f \in \{0,1\}^{\mathbb{N}} \) is denoted \( f \upharpoonright m \), and the length of \( s \in \{0,1\}^{\mathbb{N}} \) is denoted \(|s|\). For \( f \in \{0,1\}^{\mathbb{N}} \) let
\[ X_f := \{ T \in Z : (\exists m)|s| = m \text{ for all } s \in T \text{ and } f \upharpoonright m \in T \}. \]

Assume \( f(0), \ldots, f(n-1) \) are distinct elements of \( \{0,1\}^{\mathbb{N}} \) and \( F \subseteq n \) is nonempty. Fix \( k \) large enough so that \( f(i) \neq f(j) \) \( \mid k \) for all distinct \( i \) and \( j \). Then
\[ s := \{ f(i) \mid k \in F \} \]
belongs to \( \bigcup_{i \in F} X_{f(i)} \setminus \bigcup_{j=m+1}^n X_{f(j)} \). Since \( n, f(0), \ldots, f(n-1), \) and \( F \) were arbitrary, \( \{ X_f : f \in \{0,1\}^{\mathbb{N}} \} \) is an independent family.

Since \( \mathcal{P}(\mathbb{N}) \) is second-countable, by making finite modifications to some \( \mathbb{R}_0 \) elements of this family we obtain a family \( \{ X_f : f \in \{0,1\}^{\mathbb{N}} \} \) dense in \( \mathcal{P}(\mathbb{N}) \). Any of the finite Boolean combinations \( \bigcap_j X_f^j \setminus \bigcup_k X_f^k \) is clearly still infinite. \( \square \)

The remainder of this section, and Lemma 9.2.8 in particular, will be used to construct Rudin–Keisler incomparable ultrafilters on \( \mathbb{N} \) in §9.4.

Definition 9.2.6. Suppose \( \mathcal{D} \) is a filter on \( \mathbb{N} \). A family \( \mathcal{A} \) is independent modulo \( \mathcal{D} \) if \( (\bigcap F \setminus G) \cap X \) is infinite for all nonempty \( F \in \mathcal{A} \), all \( G \in \mathcal{A} \setminus F \), and all \( X \in \mathcal{D} \).

If \( \mathcal{Y} \) is a family of subsets of a fixed set \( X \) then the filter generated by \( \mathcal{Y} \), denoted \( \langle \mathcal{Y} \rangle \), is the intersection of all filters on \( X \) that include \( \mathcal{Y} \). The following lemma is proved by parsing the definitions.

Lemma 9.2.7. If \( \mathcal{A} \) is independent modulo \( \mathcal{D} \) and \( X \in \mathcal{A} \) then \( \mathcal{A} \setminus \{X\} \) is independent modulo both \( \langle \mathcal{D} \cup \{X\} \rangle \) and \( \langle \mathcal{D} \cup \{\mathbb{N} \setminus X\} \rangle \). \( \square \)

Lemma 9.2.8. Suppose that \( \mathcal{A} \) is independent modulo \( \mathcal{D} \) and \( Y \subseteq \mathbb{N} \). Then there exists \( \mathcal{A}' \subseteq \mathcal{A} \) such that
1. The set \( \mathcal{A} \setminus \mathcal{A}' \) is finite, and
2. The family \( \mathcal{A}' \) is independent modulo at least one of the filters \( \langle \mathcal{D} \cup \{Y\} \rangle \) or \( \langle \mathcal{D} \cup \{\mathbb{N} \setminus Y\} \rangle \)

Proof. We may assume that \( \mathcal{A} \) is not independent modulo \( \langle \mathcal{D} \cup \{Y\} \rangle \). Then there exist finite disjoint subsets \( F \neq \emptyset \) and \( G \) of \( \mathcal{A} \) and \( X \in \mathcal{D} \) such that the set
\[ (\bigcap F \setminus \bigcup G) \cap X \cup Y \]
is finite. Therefore the set \( \bigcap F \setminus \bigcup G \) is, modulo \( \mathcal{D} \), included in \( \mathbb{N} \setminus Y \). Applying Lemma 9.2.7 successively to elements of \( F \cup G \), we can conclude that the family \( \mathcal{A}' := \mathcal{A} \setminus (F \cup G) \) is independent modulo \( \langle \mathcal{D} \cup \{\mathbb{N} \setminus Y\} \rangle \). \( \square \)
9.3 Gaps in $\mathcal{P}(\mathbb{N})$/Fin

In this section we study gaps in partially ordered sets, and in the quotient Boolean algebra $\mathcal{P}(\mathbb{N})$/Fin in particular. We define Borel gaps and construct Hausdorff gaps and Luzin families. Sup{\penalty0 (P, \leq) is a quasi-ordered set, possibly with the least element 0. On $\mathcal{P}$ define the orthogonality relation $\perp$ by

$$a \perp b \text{ if and only if } (\forall c)(c \leq a \land c \leq b \rightarrow c = 0).$$

(If $\mathcal{P}$ has no zero element then the right-hand side reads as $\neg(\exists c)(c \leq a \land c \leq b)$.)

We have already encountered gaps in the context of Hausdorff’s $\eta_1$ sets (Definition 8.2.2).

**Definition 9.3.1.** A pregap in $\mathcal{P}$ is a pair of subsets $(\mathcal{A}, \mathcal{B})$ of $\mathcal{P}$ such that $a \perp b$ for all $a \in A$ and all $b \in B$. A pregap is separated if there exists $c \in \mathcal{P}$ such that $a \leq c$ for all $a \in \mathcal{A}$ and $c \perp b$ for all $b \in \mathcal{B}$. If such $c$ exists we say that it separates the pregap $(\mathcal{A}, \mathcal{B})$. Some authors say that $c$ splits the pregap. A pregap is a gap if it is not separated. The sets $\mathcal{A}$ and $\mathcal{B}$ are sides of the (pre)gap $(\mathcal{A}, \mathcal{B})$. A gap $(\mathcal{A}, \mathcal{B})$ is linear of both of its sides are linearly ordered. A linear gap $(\mathcal{A}, \mathcal{B})$ is a $(\kappa, \lambda)$ gap, for a pair of cardinals $\kappa$ and $\lambda$ if $\mathcal{A}$ and $\mathcal{B}$ are order-isomorphic to $\kappa$ and $\lambda$, respectively.

Our definition of a pregap is symmetric, as $(\mathcal{A}, \mathcal{B})$ is a pregap if and only if $(\mathcal{B}, \mathcal{A})$ is a pregap. While it is not difficult to find an example of a poset $\mathcal{P}$ and a gap $(\mathcal{A}, \mathcal{B})$ in $\mathcal{P}$ such that $(\mathcal{B}, \mathcal{A})$ is not a gap, this will not be the case if $\mathcal{P}$ has sufficient separation properties satisfied in all instances of interest to us.

We follow the established terminology and consider gaps in the quasi-ordered set $\mathcal{P}(\mathbb{N})/\subseteq^*$ instead of gaps in the partially ordered set $\mathcal{P}(\mathbb{N})$/Fin. Thus pre-gaps and gaps in $\mathcal{P}(\mathbb{N})$/Fin are pairs of subsets of $\mathcal{P}(\mathbb{N})$ and not pairs of subsets of $\mathcal{P}(\mathbb{N})$/Fin. The orthogonal complement of $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is

$$\mathcal{A}^\perp := \{B \in \mathcal{P}(\mathbb{N}) : B \perp A \text{ for all } A \in \mathcal{A}\}.$$

As before, $\pi : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{P}(\mathbb{N})$/Fin denotes the quotient map.

**Lemma 9.3.2.** Suppose $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$.

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6 At this point, it may not be clear what gaps have to do with anything except a curiosity (Exercise 9.10.6) and a failed attempt to prove the Continuum Hypothesis. For now, we’ll leave this gun hanging on the wall, waiting for its moment to fire.

7 This encounter was only implicit since we haven’t used the terminology. It was also a ‘non-encounter’ since the $\eta_1$ sets were defined by postulating the absence of small gaps, small limits, and endpoints or ‘jumps.’
1. The family $\mathcal{A}^\perp$ is directed in both $(\mathcal{P}(\mathbb{N}), \subseteq)$ and $(\mathcal{P}(\mathbb{N}), \subseteq^*)$.

2. The pair $(\mathcal{A}, \mathcal{A}^\perp)$ is a gap if and only if $(\mathcal{A}^\perp, \subseteq^*)$ does not have a maximal element. □

Our main interest is in gaps both of whose sides are directed. A pregap $(\mathcal{A}_0, \mathcal{B}_0)$ is cofinal in a pregap $(\mathcal{A}_1, \mathcal{B}_1)$ if $\mathcal{A}_0$ is a cofinal subset of $\mathcal{A}_1$ and $\mathcal{B}_0$ is a cofinal subset of $\mathcal{B}_1$. If a gap is linear then it has a cofinal $(\kappa, \lambda)$-subgap, where $\kappa$ is the cofinality of $\mathcal{A}$ and $\lambda$ is the cofinality of $\mathcal{B}$.

**Lemma 9.3.3.** If $\mathcal{A}$ is an ideal on $\mathbb{N}$ such that $\text{Fin} \subseteq \mathcal{I}$ then there are no $(\mathcal{K}_0, \mathcal{K}_0)$-gaps in $\mathcal{P}(\mathbb{N})/\mathcal{I}$.

**Proof.** Suppose $A_n, B_n$, for $n \in \mathbb{N}$, are sets in $\mathcal{P}(\mathbb{N})$ that form two sides of a pregap in $\mathcal{P}(\mathbb{N})$. Let $A'_n := A_n \setminus \bigcup_{j<n} B_j$, and $C := \bigcup_n A'_n$. Then $A_n \setminus C \subseteq A \cap \bigcup_{j<n} B_j$ is finite for all $n$ and $B_j \cap C \subseteq B_j \cup \bigcup_{n<j} A_n$ is finite for all $j$. Therefore $C$ splits the pregap.

The case of Lemma 9.3.3 when $\mathcal{I} = \text{Fin}$ is an instance of the countable saturation of $\mathcal{P}(\mathbb{N})/\text{Fin}$, a property that will play an important role in subsequent chapters. It can be given an apparently (but only apparently) different proof: Use the Gelfand–Naimark duality (Theorem 1.3.1) and apply results from §15.1 to the countably degree-1 saturated C$^*$-algebra $C(\beta\mathbb{N}\setminus\mathbb{N})$.

Given a pregap $(\mathcal{A}, \mathcal{B})$ in $\mathcal{P}(\mathbb{N})$, the assertion that it is a gap is a second-order statement, since it requires quantification over $\mathcal{P}(\mathbb{N})$ (see §D). Using forcing, one can generically add subsets of $\mathbb{N}$ (see [166]). These ‘new’ subsets of $\mathbb{N}$ may split gaps and the assertion that a given pregap is a gap is not necessarily absolute between models of ZFC. We will introduce a first-order condition that can be imposed on a pregap $(\mathcal{A}, \mathcal{B})$ which implies that it is a gap whenever $\mathcal{A}$ and $\mathcal{B}$ are uncountable.

The proximity of a countable subset $\mathcal{B}$ of $\mathcal{P}(\mathbb{N})$ is the set

$$\text{prox}(\mathcal{B}) := \{ A \in \mathcal{B}^\perp : \{ B \in \mathcal{B} : A \cap B \subseteq s \} \text{ is finite for all } s \in \text{Fin} \}.$$ 

**Lemma 9.3.4.** Suppose $\mathcal{B} \subseteq \mathcal{P}(\mathbb{N})$.

1. If $\mathcal{B}$ is finite then $\text{prox}(\mathcal{B}) = \mathcal{B}^\perp$.

2. If $\mathcal{B} = \{ B_n : n \in \mathbb{N} \}$ and $B_n \setminus \bigcup_{j<n} B_j$ is infinite for all $n$ then $\text{prox}(\mathcal{B})$ is a $\subseteq$-cofinal subset of $\mathcal{B}^\perp$.

3. If $A \in \text{prox}(\mathcal{B})$, $C \in \mathcal{B}^\perp$, and $A \subseteq^* C$, then $C \in \text{prox}(\mathcal{B})$.

**Proof.** Only (2) may require a proof. Fix $C \in \mathcal{B}^\perp$ and for $n \in \mathbb{N}$ choose $m(n)$ in $(B_n \setminus \bigcup_{j<n} B_j) \setminus n$ and let $C' := C \cup \{ m(n) : n \in \mathbb{N} \}$. Then $C' \cap B_j \subseteq [0, n)$ implies $j < n$, and therefore $C' \in \text{prox}(\mathcal{B})$. Since $C \in \mathcal{B}^\perp$ was arbitrary, this completes the proof. □

The analysis of gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$ relies on the tension between the separability of $\mathcal{P}(\mathbb{N})$ and the uncountability of $\mathcal{K}_1$. The following is a textbook example of this phenomenon.
Lemma 9.3.5. Suppose $A_\alpha$, $B_\alpha$, for $\alpha < \mathbb{R}_1$, are subsets of $\mathcal{P}(\mathbb{N})$ such that for some $n \in \mathbb{N}$ the following conditions hold.

1. The families $\mathcal{A} := \{A_\alpha : \alpha < \mathbb{R}_1\}$ and $\mathcal{B} := \{B_\alpha : \alpha < \mathbb{R}_1\}$ form a gap.
2. $A_\alpha \cap B_\alpha \subseteq n$ for all $\alpha$.
3. For every $\gamma < \mathbb{R}_1$ the set $A_\alpha$ is in the proximity of $\{B_\beta : \beta < \gamma\}$ for all but countably many $\alpha < \mathbb{R}_1$.

Then $(\mathcal{A}, \mathcal{B})$ is a gap in $\mathcal{P}(\mathbb{N})$/Fin.

Proof. Suppose otherwise and fix $C$ that separates $\mathcal{A}$ from $\mathcal{B}$. For each $\alpha < \mathbb{R}_1$ there exists $m(\alpha) \in \mathbb{N}$ such that $A_\alpha \setminus C \subseteq m(\alpha)$ and $B_\alpha \cap C \subseteq m(\alpha)$. Fix $m \geq n \in \mathbb{N}$ such that $X := \{\alpha : m(\alpha) \leq m\}$ is uncountable. Then for all $\alpha$ and $\beta$ in $X$ we have

$$A_\alpha \cap B_\beta = ((A_\alpha \cap B_\beta) \setminus C) \cup (A_\alpha \cap B_\beta \cap C) \subseteq (A_\alpha \setminus C) \cup (B_\beta \cap C) \subseteq m.$$ 

For $\alpha \in X$ we have $\{\beta : B_\beta \cap A_\alpha \subseteq m\} \supseteq X \setminus \alpha$. Since $X$ is uncountable, $X \cap \gamma$ is infinite for some $\gamma < \mathbb{R}_1$. Fix $\alpha \in X \setminus \gamma$ such that $A_\alpha$ is in the proximity of $\{B_\beta : \beta < \gamma\}$. But $\{B_\beta : \beta \in X \cap \gamma\}$ is infinite and $A_\alpha \cap B_\beta \subseteq m$ for all $\beta \in X \cap \gamma$; contradiction. \square

A construction of an $(\mathbb{R}_1, \mathbb{R}_1)$ gap using the Continuum Hypothesis is a straightforward exercise in bookkeeping (Exercise 9.10.5). Notably, proofs of the following two theorems do not require any additional set-theoretic axioms.

Theorem 9.3.6. There exists an almost disjoint family $\mathcal{A}$ in $\mathcal{P}(\mathbb{N})$ of cardinality $\mathbb{R}_1$ such that every pair of disjoint uncountable subsets of $\mathcal{A}$ forms a gap.

Proof. We will construct an almost disjoint family $\mathcal{A} := \{A_\alpha : \alpha < \mathbb{R}_1\}$ such that for every $\alpha < \mathbb{R}_1$ the set $A_\alpha$ is in the proximity of $\{B_\beta : \beta < \alpha\}$. Since this requirement is vacuous for finite $\alpha$, we start by letting $A_n$, for $n < \omega$, be any partition of $\mathbb{N}$ into infinite sets.

The sets $A_\alpha$, for $\omega \leq \alpha < \mathbb{R}_1$, are chosen by recursion. Suppose $\alpha$ is a countable ordinal and $\mathcal{A}_\alpha := \{A_\beta : \beta < \alpha\}$ has already been chosen. Since $\alpha$ is countable, we can re-enumerate $\mathcal{A}_\alpha$ as $B_n$, for $n \in \mathbb{N}$. The sets $B_n \setminus \bigcup_{j<n} B_j$ are infinite and disjoint. Let $\alpha_\beta$ be such that the set $A_\alpha \cap (B_n \setminus \bigcup_{j<n} B_j)$ is nonempty and finite for all $n$. Then $A_\alpha$ belongs to the proximity of $\mathcal{A}_\alpha$. This describes the recursive construction of the family $\mathcal{A}$.

To prove that every pair of disjoint uncountable subsets of $\mathcal{A}$ forms a gap, suppose $X$ and $Y$ are two disjoint uncountable subsets of $\mathbb{R}_1$. Let $f : X \to Y$ be a bijection. If $m_\alpha := \max(A_\alpha \cap A_{f(\alpha)})$ for $\alpha \in X$, then there exist $m$ such that $X' := \{\alpha \in X : m_\alpha = m\}$ is uncountable. Then $\{A_\alpha : \alpha \in X'\}$ and $\{A_{f(\alpha)} : \alpha \in X\}$ satisfy the assumptions of Lemma 9.3.5 and therefore cannot be separated. \square

A family as in the conclusion of Theorem 9.3.6 is known as a Luzin family.

Theorem 9.3.7. There exists an $(\mathbb{R}_1, \mathbb{R}_1)$ gap in $\mathcal{P}(\mathbb{N})$/Fin.
Proof. We construct two $\subseteq^*$-strictly increasing chains $\mathcal{A} := \{A_\alpha : \alpha < \omega_1\}$ and $\mathcal{B} := \{B_\alpha : \alpha < \omega_1\}$ in $\mathcal{P}(\mathbb{N})$ such that for all $\alpha$ the following conditions hold.

1. The set $\mathbb{N} \setminus (A_\alpha \cup B_\alpha)$ is infinite,
2. The intersection $A_\alpha \cap B_\alpha$ is empty, and
3. The set $A_\alpha$ is in the proximity of $\{B_\beta : \beta < \alpha\}$.

All $A_n$ and $B_n$ for $n \in \mathbb{N}$ are chosen simultaneously as follows. Let $\mathbb{N} = \bigcup_n X_n$ be a partition of $\mathbb{N}$ into infinite sets and let $A_n := \bigcup_{m \leq n} X_{2m}$ and $B_n := \bigcup_{m \leq n} X_{2m+1}$. Then the above requirements are satisfied ((3) is vacuous since $A_\alpha$ is defined only for finite $\alpha$).

Suppose $\alpha < \omega_1$ is such that $A_\beta, B_\beta$ that satisfy the requirements have been defined for all $\beta < \alpha$. Suppose for a moment that $\alpha = \beta + 1$ for some $\beta$. Partition $\mathbb{N} \setminus (A_\beta \cup B_\beta)$ into three infinite sets $X_j$, for $j < 3$, and let $A_\alpha := A_\beta \cup X_0$ and $B_\alpha := B_\beta \cup X_1$. Then $A_\alpha \cap B_\gamma$ is finite and it includes $A_\beta \cap B_\gamma$ for every $\gamma < \alpha$, hence the inductive hypotheses are satisfied.

Now suppose $\alpha$ is a limit ordinal. Since it is countable, we can choose an increasing sequence of ordinals $\alpha(n)$, for $n \in \mathbb{N}$, with supremum $\alpha$. Let

$$C := \bigcup_n (A_{\alpha(n)} \setminus \bigcup_{j < n} B_{\alpha(j)}).$$

As in the proof of Lemma 9.3.3, we have $A_\gamma \subseteq^* C$ and $B_\gamma \perp C$ for all $\gamma < \alpha$.

For $k \in \mathbb{N}$ let

$$F_k := \{ \beta < \alpha : C \cap B_\beta \subseteq k\}.$$

Clearly $k \leq l$ implies $F_k \subseteq F_l$. Since $A_{\alpha(n)} \subseteq^* C$, the set $C$ is in the proximity of $\{B_\beta : \beta \leq \alpha(n)\}$ for every $n$. Therefore $F_l \cap \alpha(n)$ is finite for all $l$ and $n$.

Assume for a moment that $F_k$ is finite for all $k$. Then $C$ belongs to the proximity of $\{B_\beta : \beta \leq \alpha\}$. Let $A_\alpha := C$ and choose any $B_\alpha \subseteq \mathbb{N} \setminus A_\alpha$ such that $B_{\alpha(n)} \subseteq B_\alpha$ for all $n$ and $\mathbb{N} \setminus (A_\alpha \cup B_\alpha)$ is infinite.

Now assume $F_k$ is infinite for some $k$. Since $F_l \cap \alpha(n)$ is finite for all $l$ and $n$, the set $F_l$ has order type $\omega$ and supremum $\alpha$ for all $l \geq k$. This implies that

$$F := \bigcup_k (F_k \setminus \alpha(k))$$

has order type $\omega$. Let $\beta(n)$, for $n \in \mathbb{N}$, be the increasing enumeration of $F$. The sets $B_{\beta(n+1)} \setminus B_{\beta(n)}$ for $n \in \mathbb{N}$ are infinite, pairwise almost disjoint, and almost disjoint with $C$. Choose distinct $k(n) \in (B_{\beta(n+1)} \setminus B_{\beta(n)}) \setminus C$ for all $n$ and let

$$A_\alpha := C \cup \{k(n) : n \in \mathbb{N}\}.$$

Then $A_\alpha \cap B_{\beta(n)}$ is finite for all $n$, hence $A_\alpha \cap B_\beta$ is finite for all $\beta < \alpha$. Fix $k \in \mathbb{N}$. Then the intersection of $\{\beta : B_\beta \cap A_\alpha \subseteq k\}$ with $F$ is finite, and so is its intersection with $F_l$ for every $l$. Therefore $A_\alpha$ is in the proximity of $\{B_\beta : \beta < \alpha\}$. Let

$$B_\alpha := \bigcup_n (B_{\alpha(n)} \setminus \bigcup_{j < n} A_{\alpha(j)}) \setminus A_\alpha.$$
This completes the description of the recursive construction of the pregap \((\mathcal{A}, \mathcal{B})\).

By Lemma 9.3.5, it is a gap. \hfill \Box

The gap constructed in the proof of Theorem 9.3.7 is sometimes called Hausdorff gap. The reader should be warned that there is no consensus on the meaning of this term. For some authors, every \((\mathbb{R}_1, \mathbb{R}_1)\) gap is Hausdorff.

A subset of a Polish space is analytic if it is a continuous image of a Borel subset of a Polish space.

**Definition 9.3.8.** A gap in \(\mathcal{P}(\mathbb{N})\) is analytic if both of its sides are analytic subsets of \(\mathcal{P}(\mathbb{N})\) considered with its compact metrizable topology. It is Borel if both of its sides are Borel subsets of \(\mathcal{P}(\mathbb{N})\).

**Example 9.3.9.** To define a Borel gap in \(\mathcal{P}(\mathbb{N}^2)\)/Fin consider the \(F_\sigma\) set

\[ \mathcal{A} := \{ A : (\exists m)(\forall n)(\forall k)(m, k) \notin A \} . \]

Then \(\mathcal{A}^+ = \{ B : (\forall m)(\exists n)(\forall k)(m, k) \notin A \} \) is an \(F_\sigma\delta\) set. Since \(\mathcal{A}^+\) has no maximal element, the pair \((\mathcal{A}, \mathcal{A}^+)\) is a gap in \(\mathcal{P}(\mathbb{N}^2)\)/Fin. It is an easy exercise to prove that, up to a permutation of \(\mathbb{N}\), every gap in \(\mathcal{P}(\mathbb{N})\)/Fin one of whose sides is countably generated is of this form. See also Proposition 12.2.2.

Hausdorff’s and Luzin’s gaps are constructed by transfinite recursion and are therefore unlikely to be Borel or analytic.

**Theorem 9.3.10.** Suppose that \(\mathcal{B} \subseteq \mathcal{P}(\mathbb{N})\) is \(\sigma\)-directed under \(\subseteq^*\), \(\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})\) is analytic, and \(\mathcal{A} \perp \mathcal{B}\). Then \(\mathcal{A}\) and \(\mathcal{B}\) are separated by some \(X \subseteq \mathbb{N}\).

**Proof.** By removing a countable subset of \(\mathcal{A}\), we may assume that all sets in \(\mathcal{A}\) are infinite. By Theorem B.2.9, \(\mathcal{A} = p[T]\) for some tree \(T\) on \(\{0, 1\} \times \mathbb{N}\). Thus \(A \in \mathcal{A}\) if and only if there exists \(f \in \mathbb{N}^\mathbb{N}\) such that (writing \(\chi_A\) for the characteristic function of \(A\)):

\[ (\chi_A | n, f | n) \in T \tag{9.1} \]

for all \(n \in \mathbb{N}\). For \(A \in \mathcal{A}\) let \(f(A)\) be the lexicographically minimal element of \(\mathbb{N}^\mathbb{N}\) such that (9.1) holds for all \(n\). For \((s, t) \in T\) let (with \(n := |s| = |t|\))

\[ \mathcal{A}(s, t) := \{ A \in p[T] : \chi_A | n = s, f(A) | n = t \} \]

(for definiteness let \(\mathcal{A}(s, t) = \emptyset\) if \((s, t) \notin T\)). Let

\[ T_1 := \{ (s, t) \in T : \mathcal{A}(s, t) \) cannot be separated from \(\mathcal{B} \} . \]

By our assumption, the root of \(T\) belongs to \(T_1\), and clearly \(T_1\) is a (downwards closed) subtree of \(T\). Since \(\mathcal{B}\) is \(\sigma\)-directed, essentially by Lemma 9.5.2 we can conclude that \(\mathcal{B}\) is separated from a countable union of subsets of \(\mathcal{P}(\mathbb{N})\) if and only if it is separated from each one of them. Since \(\mathcal{A}(s, t) = \bigcup_{m, n \in \mathbb{N}} A(s^{-m,t-n})\) for all \((s, t) \in T_1\), this implies that \(T_1\) has no terminal nodes.
For \((s, t) \in T_1\) let \(C_{(s, t)} := \bigcup \mathcal{A}_{(s, t)}\). This set does not separate \(\mathcal{A}_{(s, t)}\) from \(\mathcal{B}\) and therefore there exists \(B_{(s, t)} \in \mathcal{B}\) whose intersection with \(C_{(s, t)}\) is infinite. Since \((\mathcal{B}, \subseteq)\) is \(\sigma\)-directed we can find \(B \in \mathcal{B}\) such that \(B_{(s, t)} \subseteq^* B\) for all \((s, t) \in T_1\).

Recursively choose \((s(n), t(n)) \in T_1\) such that the following holds for all \(n:\)

1. \((s(0), t(0))\) is the root of \(T_1\),
2. \(s(n) \subseteq s(n + 1)\) and \(t(n) \subseteq t(n + 1)\),
3. \(|s(n)| = |t(n)| = 1(n)|\) for some \(1(n)\), and
4. there exists \(j(n) \in [1(n), 1(n + 1)] \cap \mathcal{C}_{(s(n), t(n))}\) such that \(s(n)(j) = 1\).

Since \(T_1\) has no terminal nodes, every \((s(n), t(n))\) has an extension in \(T_1\). Since all sets in \(\mathcal{A}\) are infinite, we can choose this extension so that the condition (4) is satisfied. The sequence \((s(n), t(n))\) defines a branch of the form \((\chi_A, f)\) of \(T_1\), and \(A \in \mathcal{A}\). But \(j(n) \in A \cap B\) for all \(n\), hence this set is infinite; contradiction.

The proof of Theorem 9.3.10 gives a slightly more general result, but the fun of discovering it is left to the reader (Exercise 9.10.7).

### 9.4 Ultrafilters and the Rudin–Keisler Ordering

It is a lamentable habit, but operator algebraists tend to use the symbol \(\omega\) for an ultrafilter on \(\mathbb{N}\).

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In this section we study ultrafilters on \(\mathbb{N}\) and the Rudin–Keisler ordering, \(\leq_{\text{RK}}\), and construct a pair of \(\leq_{\text{RK}}\)-incompatible ultrafilters.

**Proposition 9.4.1.** Suppose \(\mathcal{U}\) is a filter on some set \(X\). The following are equivalent.

1. The filter \(\mathcal{U}\) is a maximal proper filter under the inclusion.
2. For every \(Y \subseteq X\) we have \(Y \in \mathcal{U}\) or \(X \setminus Y \in \mathcal{U}\).
3. For every compact Hausdorff space \(Z\) and \(f: X \to Z\) there is a unique \(z \in X\) such that \(f^{-1}(U) \in \mathcal{U}\) for every open neighbourhood of \(z\). We write \(z = \lim_{\mathcal{U}} f(x)\).

**Proof.** The equivalence of (1) and (2) is obvious. To see that (3) implies (1), take \(Z = \beta X\) and \(f = \text{id}_X\).

Finally assume (1) and fix \(f: X \to Z\). The sets \(\overline{Y}\), for \(Y \in \mathcal{U}\), have the finite intersection property and by compactness their intersection is nonempty. Since \(Z\) is Hausdorff, this intersection has exactly one point, \(\lim_{\mathcal{U}} f(x)\). \(\square\)

A filter that satisfies either of the equivalent conditions in Proposition 9.4.1 is called an *ultrafilter*. Using (1) one sees that the Axiom of Choice (in the form of Zorn’s Lemma) implies that every filter on any set can be extended to an ultrafilter.

The elements of the Čech–Stone compactification of \(\mathbb{N}\), \(\beta \mathbb{N}\), are in natural bijective correspondence with the set of ultrafilters on \(\mathbb{N}\) (see Example 1.3.7).
If $\mathcal{F}$ is a filter on a set $X$ and $g : Y \to X$ then
\[ g(\mathcal{F}) := \{ A \subseteq X : g^{-1}(A) \in \mathcal{F} \} \]
is a filter on $Y$. If $\mathcal{U}$ is an ultrafilter, then $g(\mathcal{U}) = (\beta g)(\mathcal{U})$, where $\beta g$ is the unique continuous extension of $g$ to a function from $\beta \mathbb{N}$ to $\beta \mathbb{N}$.

**Definition 9.4.2.** If $\mathcal{F}$ and $\mathcal{G}$ are filters such that $g(\mathcal{F}) = \mathcal{G}$ for some function $g$, then we say that $\mathcal{G}$ is Rudin–Keisler reducible to $\mathcal{F}$ and write $\mathcal{G} \leq_{RK} \mathcal{F}$. (Some authors say that $\mathcal{G}$ is Rudin–Keisler below $\mathcal{F}$.) Filters $\mathcal{F}$ and $\mathcal{G}$ are Rudin–Keisler isomorphic, $\mathcal{F} \equiv_{RK} \mathcal{G}$, if there are $A \in \mathcal{F}$, $B \in \mathcal{G}$, and a bijection $f : A \to B$ such that $C \in \mathcal{F}$ if and only if $f[C] \in \mathcal{G}$ for all $C \subseteq X$. Such function $f$ is called a Rudin–Keisler isomorphism between $\mathcal{F}$ and $\mathcal{G}$.

**Example 9.4.3.** Selective ultrafilters (Definition 8.5.4) are minimal among the non-principal ultrafilters in the Rudin–Keisler ordering. Suppose $\mathcal{U}$ and $\mathcal{V}$ are principal ultrafilters in the Rudin–Keisler ordering. Suppose $\mathcal{U} \leq_{RK} \mathcal{V}$. Then $\mathcal{U}$ is either one-to-one on a set in $\mathcal{V}$ and we have $\mathcal{V} \equiv_{RK} \mathcal{U}$, or $\mathcal{U}$ is constant on a set in $\mathcal{V}$ and $\mathcal{V}$ is a principal ultrafilter.

We have a variant of the Cantor–Schröder–Bernstein Theorem for the Rudin–Keisler ordering on the ultrafilters (cf. Exercise 9.10.10).

**Proposition 9.4.4.** If $\mathcal{U}$ is an ultrafilter and $\mathcal{V} \leq_{RK} \mathcal{W}$, then $\mathcal{V}$ is an ultrafilter. If in addition $\mathcal{V} \leq_{RK} \mathcal{U}$, then $\mathcal{U} \equiv_{RK} \mathcal{V}$.

The proof of Proposition 9.4.4 requires a combinatorial lemma.

**Lemma 9.4.5.** If a function $f : \mathbb{N} \to \mathbb{N}$ has no fixed points then there is a partition $\mathbb{N} = \bigcup_{j \leq 5} X_j$ such that $f[X_j] \cap X_j = \emptyset$ for all $j \leq 4$.

**Proof.** The $k$-th iterate of $f$ is denoted $f^k$. Let $Y := \{ n : f(n) < n \}$ and for $n \in Y$ let $k_n := \min\{ k : f^k(n) \notin Y \}$ (note that $k_n \leq n$). The sets
\[ X_0 := \{ n \in Y : k_n \text{ is even} \} \quad \text{and} \quad X_1 := \{ n \in Y : k_n \text{ is odd} \} \]
satisfy $f[X_1] \subseteq X_0$ and $f[X_0] \cap X_0 = \emptyset$. For $n \in \mathbb{N} \setminus Y$ we have $f(n) > n$ and we let (with $\min \emptyset = \infty$) $l_n := \min\{ l : f^l(n) \in Y \}$. The sets
\[ X_2 := \{ n \in \mathbb{N} \setminus Y : l_n \text{ is even} \} \quad \text{and} \quad X_3 := \{ n \in \mathbb{N} \setminus Y : l_n \text{ is odd} \} \]
satisfy $f[X_2] \subseteq X_3$ and $f[X_3] \subseteq X_2$. Fix $n \in \mathbb{N} \setminus \bigcup_{j \leq 4} X_j$ and let (with $f^0(d) := d$)
\[ d_n := \min\{ d : (\exists m)f^m(d) = n \}. \]
Fix $m_n$ such that $f^{m_n}(d_n) = n$. Such $m_n$ is unique because $f^j(n) \notin Y$ for all $j$. The sets $X_4 := \{ n : m_n \text{ is even} \}$ and $X_5 := \{ n : m_n \text{ is odd} \}$ satisfy $f[X_4] \subseteq X_5$ and $f[X_5] \subseteq X_4$, and the sets $X_j$, for $j < 6$, form a partition of $\mathbb{N}$ with the required property. \qed
Proof (Proposition 9.4.4). Suppose that $\mathcal{U} \leq_{\text{RK}} \mathcal{V}$ and $\mathcal{V} \leq_{\text{RK}} \mathcal{U}$. Fix $f: \mathbb{N} \to \mathbb{N}$ such that $f^{-1}[A] \in \mathcal{V}$ if and only if $A \in \mathcal{U}$ and $g^{-1}[B] \in \mathcal{U}$ if and only if $B \in \mathcal{V}$. Therefore the function $h := f \circ g$ satisfies $h^{-1}[B] \in \mathcal{V}$ if and only if $B \in \mathcal{V}$.

By Lemma 9.4.5, the complement of $X := \{n : h(n) = n\}$ can be partitioned into finitely many sets none of which can belong to $\mathcal{V}$. Therefore $X \in \mathcal{V}$. The restriction of $g$ to $X$ is a bijection between $X$ and a set in $\mathcal{U}$, and therefore $\mathcal{U} \cong_{\text{RK}} \mathcal{V}$. □

Theorem 9.4.6 below has no known direct applications to $C^*$-algebras. Its proof is included for three reasons. First, it demonstrates that the intuition ‘all nonprincipal ultrafilters on $\mathbb{N}$ are alike’ is wrong. Second, it showcases a use of transfinite diagonalization of length $\omega$ in ZFC. Third, it is a warmup for the (omitted) proof of Theorem 16.7.7.

**Theorem 9.4.6.** There are Rudin–Keisler incomparable ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$.

**Proof.** Let $f_\xi$, for $\xi < \omega$, be an enumeration of $\mathbb{N}^\mathbb{N}$. We will recursively construct filters $\mathcal{D}_\xi$ and $\mathcal{F}_\xi$, as well as independent families $\mathcal{A}_\xi$, for $\xi < \omega$, that satisfy the following conditions for all $\xi < \omega$.

1. The family $\mathcal{A}_0$ is a family of independent cardinality $\omega$.
2. Each of $\mathcal{D}_0$ and $\mathcal{F}_0$ is equal to the Fréchet filter.
3. Each $\mathcal{D}_\xi$ is a proper filter and if $\xi < \eta$ then $\mathcal{D}_\xi \subseteq \mathcal{D}_\eta$.
4. Each $\mathcal{F}_\xi$ is a proper filter and if $\xi < \eta$ then $\mathcal{F}_\xi \subseteq \mathcal{F}_\eta$.
5. The set $\mathcal{A}_\xi \setminus \mathcal{A}_{\xi + 1}$ is finite.
6. If $\xi$ is a limit ordinal then $\mathcal{D}_\xi = \bigcup_{\eta < \xi} \mathcal{D}_\eta$, $\mathcal{F}_\xi = \bigcup_{\eta < \xi} \mathcal{F}_\eta$ and $\mathcal{A}_\xi = \bigcap_{\eta < \xi} \mathcal{A}_\eta$.
7. The family $\mathcal{A}_\xi$ is independent modulo both $\mathcal{D}_\xi$ and $\mathcal{F}_\xi$.
8. There is $X \subseteq \mathbb{N}$ such that $\mathcal{D}_{\xi + 1} \supseteq \langle \mathcal{D}_\xi \cup \{X\} \rangle$ and $\mathcal{F}_{\xi + 1} \supseteq \langle \mathcal{F}_\xi \cup \{\mathbb{N} \setminus f_\xi^{-1}[X]\} \rangle$.
9. There is $X \subseteq \mathbb{N}$ such that $\mathcal{F}_{\xi + 1} \supseteq \langle \mathcal{F}_\xi \cup \{X\} \rangle$ and $\mathcal{D}_{\xi + 1} \supseteq \langle \mathcal{D}_\xi \cup \{\mathbb{N} \setminus f_\xi^{-1}[X]\} \rangle$.

Suppose for a moment that such families have been constructed. Then $\bigcup_{\xi < \omega} \mathcal{D}_\xi$ is, being equal to an increasing union of proper filters, a proper filter. We can therefore extend it to an ultrafilter, $\mathcal{U}$. Similarly, we can extend the family $\bigcup_{\xi < \omega} \mathcal{F}_\xi$ to an ultrafilter, $\mathcal{V}$.

Fix $f \in \mathbb{N}^\mathbb{N}$ and let $\xi < \omega$ be such that $f = f_\xi$. By (8) there exists $X \in \mathcal{D}_\xi \cup_{\xi + 1}$ such that $\mathbb{N} \setminus f_\xi^{-1}[X] \in \mathcal{F}_\xi$. Therefore $X \in \mathcal{U}$ and $\mathbb{N} \setminus f_\xi^{-1}[X] \in \mathcal{V}$, and $f(\mathcal{U}) \neq \mathcal{V}$. Similarly, (9) implies that $f(\mathcal{V}) \neq \mathcal{U}$.

Since $f$ was arbitrary, we conclude that $\mathcal{U}$ and $\mathcal{V}$ are $\leq_{\text{RK}}$-incomparable. It will therefore suffice to construct filters $\mathcal{D}_\xi$ and $\mathcal{F}_\xi$ for $\xi < \omega$, with the above properties. (The families $\mathcal{A}_\xi$ are auxiliary objects fostering the recursive construction.)

Before plunging into the recursive construction, observe that (6) and (5) together imply $|\mathcal{A}_0 \setminus \mathcal{A}_\xi| \leq |\xi| \cdot 2 < \omega$ for all $\xi < \omega$, and therefore $|\mathcal{A}_\xi| = \omega$.

Choose $\mathcal{A}_0$ by Proposition 9.2.5, and choose each $\mathcal{D}_0$ and $\mathcal{F}_0$ to obey (2). Suppose that $\xi < \omega$ and that $\mathcal{D}_\eta$, $\mathcal{F}_\eta$, and $\mathcal{A}_\eta$ as required were constructed for all $\eta < \xi$.

If $\xi$ is a limit ordinal then $\mathcal{D}_\xi$, $\mathcal{F}_\xi$, and $\mathcal{A}_\xi$ are uniquely determined by (6) and all conditions are still satisfied.
9.5 The Poset \((\mathbb{N}^\mathbb{N}, \leq^*)\)

Now suppose \(\xi = \eta + 1\) is a successor ordinal. Fix \(A \in \mathcal{A}_\eta\). By Lemma 9.2.8 there exists \(F \in \mathcal{A}_\eta\) such that \(\mathcal{A}_\eta \setminus F\) is independent modulo one of the filters \(⟨\mathcal{F}_\eta \cup \{f_\xi^{-1}[A]\}⟩\) or \(⟨\mathcal{F}_\eta \cup \{\mathbb{N} \setminus f_\xi^{-1}[A]\}⟩\).

If \(\mathcal{A}_\eta \setminus F\) is independent modulo \(⟨\mathcal{F}_\eta \cup \{\mathbb{N} \setminus f_\xi^{-1}[A]\}⟩\), let

\[
\mathcal{A}_\eta' := \mathcal{A}_\eta \setminus (F \cup \{A\}), \quad \mathcal{D}_\eta = ⟨\mathcal{D}_\eta \cup \{A\}⟩, \quad \mathcal{F}_\eta = ⟨\mathcal{F}_\eta \cup \{\mathbb{N} \setminus f_\xi^{-1}[A]\}⟩.
\]

If \(\mathcal{A}_\eta \setminus F\) is independent modulo \(⟨\mathcal{F}_\eta \cup \{f_\xi^{-1}[A]\}⟩\), then let

\[
\mathcal{A}_\eta' := \mathcal{A}_\eta \setminus (F \cup \{A\}), \quad \mathcal{D}_\eta = ⟨\mathcal{D}_\eta \cup \{\mathbb{N} \setminus A\}⟩, \quad \mathcal{F}_\eta = ⟨\mathcal{F}_\eta \cup \{f_\xi^{-1}[A]\}⟩.
\]

In each of these cases, Lemma 9.2.7 implies that \(\mathcal{A}_\eta', \mathcal{D}_\eta', \) and \(\mathcal{F}_\eta'\) satisfy all requirements except possibly (9). In order to satisfy (9), fix \(B \in \mathcal{F}_\eta\). An application of Lemma 9.2.7 to \(f_n^{-1}[B]\) analogous to the above produces a finite \(G \subseteq \mathcal{A}_\eta'\) such that \(\mathcal{A}_{\eta+1} := \mathcal{A}_\eta' \setminus G, \mathcal{D}_{\eta+1} := \mathcal{F}_\eta' \setminus \{B\}\) and \(\mathcal{D}_{\eta+1} \subseteq \mathcal{D}_\eta\) are as required.

This describes the recursive construction and completes the proof. \(\square\)

### Example 9.5.1

Every directed, \(\sigma\)-complete poset (Definition 6.2.3) is \(\sigma\)-directed, but not vice versa. Take for example \((\mathbb{N}^{1\mathbb{N}}, \leq^*),\) or \((\mathbb{A}^1_{+\mathbb{N}}, \leq)\) for a separable nonunital \(C^\ast\)-subalgebra \(A\) of the Calkin algebra (see Proposition 12.2.2). Each one of these posets is \(\sigma\)-directed and not \(\sigma\)-complete.

Here is an analog of Lemma 8.5.6.

#### Lemma 9.5.2

If a \(\sigma\)-directed poset is partitioned into countably many pieces, then at least one of them is cofinal.

**Proof.** Suppose \(P = \bigcup_{n \in \mathbb{N}} X_n\) and no \(X_n\) is cofinal. For each \(n\) pick \(p_n \in P\) with no upper bound in \(X_n\). An upper bound for \(\{p_j : j \in \mathbb{N}\}\) cannot belong to any \(X_n,\) and \(P\) is not \(\sigma\)-directed. \(\square\)

Let \(\mathbb{N}^\mathbb{N}\) denote the space of all functions \(f : \mathbb{N} \to \mathbb{N}\). It is quasi-ordered by

\[
f \leq^* g \text{ if and only if } (\forall^\mathbb{N}) f(n) \leq g(n).
\]
By a standard abuse of terminology, the quasi-ordering \((\mathbb{N}^\mathbb{N}, \leq^*)\) is routinely identified with the corresponding quotient partial ordering (similarly to the situation with \((\mathcal{P}(\mathbb{N}), \subseteq^*)\) vs. \(\mathcal{P}(\mathbb{N})/\mathrm{Fin}\)).

**Definition 9.5.3.** Suppose that \(\mathbb{P}\) is a directed quasi-ordered set. The *bounding number* of \(\mathbb{P}\), \(b_{\mathbb{P}}\), is the minimal cardinality of an unbounded subset of \(\mathbb{P}\). The bounding number, \(b\), is the bounding number of \((\mathbb{N}^\mathbb{N}, \leq^*)\).

The *dominating number* (or *cofinality*) of \(\mathbb{P}\), \(d_{\mathbb{P}}\), is the minimal cardinality of a cofinal subset of \(\mathbb{P}\). The dominating number, \(d\), is the dominating number of \(\mathbb{N}^\mathbb{N}\).

**Example 9.5.4.** 1. Clearly \(b_{\mathbb{P}} \leq d_{\mathbb{P}}\) for every \(\mathbb{P}\), and in particular \(b \leq d\).

2. We have \(b_{\mathbb{P}} \geq \kappa_1\) if and only if \(\mathbb{P}\) is \(\sigma\)-directed.

3. If \(\kappa\) is a cardinal, then \(b_{\kappa, \leq} = \text{cof}(\kappa)\) and \(d_{\kappa, \leq} = \kappa\). As a matter of fact, every subset of cardinality \(\kappa\) is cofinal.

4. Every countable directed set \(\mathbb{P}\) with no maximal element satisfies \(b_{\mathbb{P}} = d_{\mathbb{P}} = \kappa_0\).

For larger directed sets these two cardinals can differ. Take \(\mathbb{P} := \kappa_0 \times \kappa_1\) with the coordinatewise ordering. It satisfies \(b_{\mathbb{P}} = \kappa_0\) and \(d_{\mathbb{P}} = \kappa_1\).

5. Suppose \(\kappa\) is an infinite cardinal. The poset \([\kappa, \leq]^{\mathbb{P}}\) of all finite subsets of \(\kappa\) ordered by inclusion has dominating number \(\kappa\) and bounding number \(\kappa_0\). It is an exercise to check that each one of its infinite subsets is unbounded.

**Lemma 9.5.5.** The poset \((\mathbb{N}^\mathbb{N}, \leq^*)\) is \(\sigma\)-directed, and therefore \(\kappa_1 \leq b\).

**Proof.** Fix \(f_n \in \mathbb{N}^\mathbb{N}\), for \(n \in \mathbb{N}\), in \(\mathbb{N}^\mathbb{N}\). Then \(g(k) := \max_{n \leq k} f_n(k)\) defines a function in \(\mathbb{N}^\mathbb{N}\) such that \(f_n \leq^* g\) for all \(n\). \(\square\)

For the notation \([s]\) and \(\bar{s} \cap n\) see Definition B.1.2.

**Lemma 9.5.6.** Suppose \(\mathbb{X}\) is unbounded in \((\mathbb{N}^\mathbb{N}, \leq^*)\). Then for infinitely many \(m\) there exists \(s \in \mathbb{N}^m\) such that \(Z(s, \mathbb{X}) := \{k : [s \cap k] \cap \mathbb{X}\) is unbounded in \(\mathbb{N}^\mathbb{N}\}\) is infinite.

**Proof.** Clearly \(T := \{s \in \mathbb{N}^{<\mathbb{N}} : [s] \cap \mathbb{X}\) is unbounded in \(\mathbb{N}^\mathbb{N}\}\) is a nonempty subtree of \(\mathbb{N}^{<\mathbb{N}}\). By Lemma 9.5.2, \(T\) has no terminal nodes. It suffices to prove that every \(t \in T\) has an extension \(s \in T\) such that \(Z(s, \mathbb{X})\) is infinite. Otherwise, fix an offender \(t\). Let \(g(m) := 1 + \max \{Z(r, \mathbb{X}) : s \subseteq r, |r| = m\}\) for \(m \geq |t|\) and \(g(m) := t(m)\) for \(m < |t|\). If \(f \in T\) then \(f(k) < g(k)\) for all \(k\). Let \(\mathbb{Y} := \bigcup_{s \in T}[s] \cap \mathbb{X}\). This is a countable union of bounded sets, and therefore bounded by some \(g_1\) by Lemma 9.5.5. The pointwise maximum of \(g\) and \(g_1\) is a \(\leq^*\)-bound for \([t] \cap \mathbb{X}\); contradiction. \(\square\)

The following is much more than a proof that OCA\(_T\) and CH are incompatible (that was Corollary 8.6.4). See also Exercise 9.10.9.

**Proposition 9.5.7.** OCA\(_T\) implies \(b = \kappa_2\).
Proof. We will prove only what we will need later on, that \( b > \aleph_1 \). A proof that \( b \leq \aleph_2 \), with a hint, is Exercise 9.10.9. Assume \( b = \aleph_1 \). By Lemma 9.5.5 we can recursively build a \( \leq^* \)-increasing unbounded chain \( X \) of cardinality \( \aleph_1 \). We may also assume that all functions in \( X \) are nondecreasing. Define a partition \( [X]^2 = L_0 \sqcup L_1 \) by \( \{ f, g \} \in L_1 \) if and only if \( f(m) \leq g(m) \) for all \( m \) or \( g(m) \leq f(m) \) for all \( m \). Then \( L_1 \) is closed when the subspace topology inherited from the Baire space.

If \( Y \subseteq X \) is uncountable and \( L_0 \)-homogeneous, it is then cofinal in \( X \), and therefore unbounded. Fix a countable dense \( Y_0 \subseteq Y \). By Lemma 9.5.5 there is \( f \in Y \) such that \( g \leq^* f \) for all \( g \in Y_0 \). Lemma 9.5.2 implies that

\[
Y_1 := \{ g \in Y : (\forall n \geq m_0) f(n) \leq g(n) \}
\]

is cofinal in \( Y \) for some \( m_0 \), and by Lemma 9.6 there exists \( s \in \mathbb{N}^{<\mathbb{N}} \) such that \( Z(s, Y_1) \) is infinite. Choose \( g \in [s] \cap Y_0 \). Let \( m \geq m_0 \) be such that \( f(n) \geq g(n) \) for all \( n \geq m \). Fix \( k \geq f(m) \) such that \( [s^k] \cap Y_1 \) is unbounded and choose \( h \in [s^k] \cap Y_1 \). Then \( h(j) \geq g(j) \) for \( j < m \), and \( h(j) \geq f(j) \geq g(j) \) for \( j \geq m \). Therefore \( \{ h, g \} \in L_1 \); contradiction.

Therefore \( X \) has no unbounded, uncountable, \( L_0 \)-homogeneous subsets. OCA\( T \) implies that \( X \) can be covered by countably many \( L_1 \)-homogeneous sets. By Lemma 9.5.2, at least one of these \( L_1 \)-homogeneous subsets, denoted \( Y \), is unbounded in \( X \). It is therefore unbounded in \( \mathbb{N}^{\mathbb{N}} \). Use Lemma 9.5.6 to find \( s \sqsubset t \) such that both \( Z(s, Y) \) and \( Z(t, Y) \) are unbounded. Fix \( f \in [s] \cap Y \) such that \( f(|s|) > t(|s|) \). With \( g \in [t] \cap Y \) such that \( g(|t| + 1) > f(|t| + 1) \) we have \( \{ f, g \} \in L_0 \); contradiction. □

### 9.6 Cofinal Equivalence and Cardinal Invariants

In this section we introduce the Tukey ordering and cofinal equivalence of directed sets. We also study the bounding and dominating numbers of directed sets, and give a proof of Tukey’s characterization of cofinal equivalence.

The Tukey ordering is a rather coarse quasi-ordering on the category of directed sets, originally introduced to study the Moore–Smith convergence in topology.

We proceed to define two classes of functions between directed sets.

**Definition 9.6.1.** A function \( f : \mathbb{P} \to \mathbb{S} \) between directed sets is a **Tukey function** if for each \( s \in \mathbb{S} \) there exists \( p \in \mathbb{P} \) such that \( f(q) \leq s \) implies \( q \leq p \) for all \( q \in \mathbb{P} \).

In other words, \( f \) is Tukey if for every \( s \in \mathbb{S} \) the set \( \{ q : f(q) \leq s \} \) is bounded in \( \mathbb{P} \). This observation proves the following lemma.

**Lemma 9.6.2.** A function \( f : \mathbb{P} \to \mathbb{S} \) is Tukey if and only if the \( f \)-image of every unbounded subset of \( \mathbb{P} \) is unbounded in \( \mathbb{S} \). Therefore a composition of Tukey functions is a Tukey function. □

**Definition 9.6.3.** A function \( g : \mathbb{S} \to \mathbb{P} \) is **convergent** if for each \( p \in \mathbb{P} \) there is \( s \in \mathbb{S} \) such that \( t \geq s \) implies \( g(t) \geq p \) for all \( t \in \mathbb{S} \).
In other words, \( g \) is convergent if for every \( p \in \mathbb{P} \) the set \( \{ t : g(t) \not\leq p \} \) is not cofinal in \( \mathcal{S} \). This observation proves the following lemma.

**Lemma 9.6.4.** A function \( g : \mathcal{S} \to \mathbb{P} \) is convergent if and only if the \( g \)-image of every cofinal subset of \( \mathcal{S} \) is cofinal in \( \mathbb{P} \). Therefore a composition of convergent functions is convergent. \( \square \)

It needs to be emphasized that neither Tukey functions nor convergent functions are required to be monotonic or to possess any regularity properties other than those stated in the previous paragraphs. These concepts are therefore more general than the Galois correspondence in which \( f \) and \( g \) are required to be monotonic.

A progression of intuition-boosting examples is in order.

**Example 9.6.5.** The identity map on a set \( \mathbb{P} \) is denoted \( \text{id}_\mathbb{P} \).

1. If \( \mathbb{P} \) is a cofinal subset of a directed set \( \mathcal{S} \), then \( \text{id}_\mathbb{P} \) is both Tukey and convergent.
2. If \( \mathbb{P} \) is a subset of a directed set \( \mathcal{S} \) then \( \text{id}_\mathbb{P} \) is Tukey if and only if every unbounded subset of \( \mathbb{P} \) is unbounded in \( \mathcal{S} \).
3. Suppose that \( \preceq \) and \( \leq \) are orderings on a set \( \mathbb{P} \) such that both \( (\mathbb{P}, \preceq) \) and \( (\mathbb{P}, \leq) \) are directed, and \( p \leq q \) implies \( p \leq q \) for all \( p \) and \( q \) in \( \mathbb{P} \).
   Then \( \text{id}_\mathbb{P} : (\mathbb{P}, \preceq) \to (\mathbb{P}, \preceq) \) is Tukey and \( \text{id}_\mathbb{P} : (\mathbb{P}, \preceq) \to (\mathbb{P}, \leq) \) is convergent.
4. Suppose \( \lambda \) and \( \kappa \) are cardinals and consider \( \kappa \times \lambda \) with respect to the coordinatewise ordering. Then \( g : \kappa \times \lambda \to \kappa \) defined by \( g(\xi, \eta) = \xi \) is convergent, and \( f : \kappa \to \kappa \times \lambda \) defined by \( f(\xi) = (\xi, 0) \) is Tukey.
5. Suppose that \( \mathbb{P} \) is any directed set, \( \kappa \) is a cardinal, and \( h : \kappa \to \mathbb{P} \) is a surjection. If \( f : [\kappa]^{<\kappa_0} \to \mathbb{P} \) is such that \( f(s) \) is an upper bound for \( \{ h(\xi) : \xi \in s \} \) for all \( s \in [\kappa]^{<\kappa_0} \), then \( f \) is convergent.
6. Suppose that \( \mathbb{P} \) is a countable directed set with no maximal element. By recursion one can construct an increasing cofinal subset \( \{ p_j : j \in \mathbb{N} \} \) of \( \mathbb{P} \). Define \( f : \mathbb{N} \to \mathbb{P} \) by \( f(j) = p_j \) for all \( j \). Then \( f \) is both Tukey and convergent (see also Exercise 9.10.19).

A sequence of lemmas, culminating in Tukey’s characterization of cofinal equivalence (Theorem 9.6.10), follows.

**Lemma 9.6.6.** Suppose \( \mathbb{P} \) and \( \mathcal{S} \) are directed sets and \( f : \mathbb{P} \to \mathcal{S} \) and \( g : \mathcal{S} \to \mathbb{P} \) are such that \( f(p) \leq s \) implies \( p \leq g(s) \) for all \( p \in \mathbb{P} \) and all \( s \in \mathcal{S} \). Then \( f \) is Tukey and \( g \) is convergent.

**Proof.** Fix \( X \subseteq \mathbb{P} \) and \( s \in \mathcal{S} \). If \( f(p) \leq s \) for all \( p \in X \), then \( g(s) \geq p \) for all \( p \in X \); therefore if \( f \mid X \) is bounded by \( s \) then \( X \) is bounded by \( g(s) \). Since \( X \) and \( s \) were arbitrary, \( f \) is Tukey.

Now fix \( Y \subseteq \mathcal{S} \). For every \( p \in \mathbb{P} \), if \( p \not\leq g(t) \) for all \( t \in Y \) then \( f(p) \not\leq t \) for all \( t \in Y \). Therefore if \( g \mid Y \) is not cofinal in \( \mathbb{P} \) then \( Y \) is not cofinal in \( \mathcal{S} \). Since \( Y \) was arbitrary, \( g \) is convergent. \( \square \)

**Lemma 9.6.7.** Suppose \( \mathbb{P} \) and \( \mathcal{S} \) are directed sets. The following are equivalent.

1. There exists a Tukey function \( f : \mathbb{P} \to \mathcal{S} \).
2. There exists a convergent function \( g : S \rightarrow P \).
3. There exists \( f : P \rightarrow S \) and \( g : S \rightarrow P \) such that \( f(p) \leq s \) implies \( p \leq g(s) \) for all \( p \in P \) and all \( s \in S \).

**Proof.** By Lemma 9.6.6, condition (3) implies the other two conditions. Fix a well-ordering \( \prec_w \) of \( P \cup S \).

To prove that (1) implies (3), suppose \( f : P \rightarrow S \) is a Tukey function. For \( s \in S \) let \( g(s) = p \) where \( p \) is the \( \prec_w \)-minimal element of \( P \) such that \( f(q) \leq s \) implies \( q \leq p \) for all \( q \in P \). Then \( f(p) \leq s \) implies \( p \leq g(s) \) and (3) follows.

To prove that (2) implies (3), suppose \( g : S \rightarrow P \) is a convergent function. For \( p \in P \) let \( f(p) = s \) where \( s \) is the \( \prec_w \)-minimal bound of \( \{ t \in S : g(t) \leq p \} \). Then \( f(p) \leq s \) implies \( p \leq g(s) \) and (3) follows. \( \square \)

**Definition 9.6.8.** Suppose \( P \) and \( S \) are directed sets. We write \( P \leq_T S \) if either of the equivalent conditions in Lemma 9.6.7 holds. If \( P \leq_T S \) and \( S \leq_T P \) we write \( P \equiv_T S \) and say that \( P \) and \( S \) are cofinally equivalent.

Some authors call cofinal equivalence Tukey equivalence. The term ‘cofinal equivalence’ will be justified by Theorem 9.6.10.

**Proposition 9.6.9.** If \( P \leq_T S \) then \( b_P \geq b_S \) and \( \partial_P \leq \partial_S \).

**Proof.** Suppose \( P \leq_T S \) and fix a Tukey function \( f : P \rightarrow S \). Fix an unbounded \( X \subseteq P \) of the minimal cardinality. Then \( f[X] \) is unbounded in \( S \) and its cardinality is no larger than the cardinality of \( X \); therefore \( b_P \geq b_S \). The inequality \( \partial_P \leq \partial_S \) is proved analogously, using a convergent function \( g : S \rightarrow P \). \( \square \)

**Theorem 9.6.10.** If \( P \) and \( S \) are directed sets then \( P \equiv_T S \) if and only if there exists a directed set \( \mathcal{X} \) such that each one of \( P \) and \( S \) is isomorphic to cofinal subsets of \( \mathcal{X} \).

**Proof.** Only the direct implication requires a proof. If \( P \equiv_T S \), by Lemma 9.6.7 there are \( f_j : P \rightarrow S \) and \( g_j : S \rightarrow P \) for \( j < 2 \) such that \( f_0(p) \leq s \) implies \( p \leq g_0(s) \) and \( g_1(s) \leq p \) implies \( s \leq f_1(p) \). Fix a well-ordering \( \prec_w \) of \( \mathcal{X}_0 := P \sqcup S \). For \( p \in P \) let \( f(p) \in S \) be the \( \prec_w \)-minimal upper bound for \( f_0(p) \) and \( f_1(p) \). Similarly, for \( s \in S \) let \( g(s) \in P \) be the \( \prec_w \)-minimal upper bound for \( g_0(s) \) and \( g_1(s) \). Then for all \( p \in P \) and all \( s \in S \) the following holds.

1. \( f(p) \leq s \) implies \( p \leq g(s) \), and \( g(s) \leq p \) implies \( s \leq f(p) \).

On \( \mathcal{X}_0 \) define a relation \( \leq_0 \) as follows. The restriction of \( \leq_0 \) to \( P \) agrees with the ordering on \( P \), and the restriction of \( \leq_0 \) to \( S \) agrees with the ordering on \( S \). If \( x \in P \) and \( y \in S \), then

\[
\begin{align*}
  x &\leq_0 y \text{ if } (\exists z \in P) x \leq z \text{ and } f(z) \leq y \\
  y &\leq_0 x \text{ if } (\exists z \in S) y \leq z \text{ and } g(z) \leq x.
\end{align*}
\]

A verification that the following holds for all \( x \) and \( y \) in \( \mathcal{X} \) is left to the reader:

2. The relation \( \leq_0 \) is transitive.
3. If \( x \leq y, y \leq x \), and \( x \neq y \), then \( x \in \mathbb{P} \) if and only if \( y \in \mathbb{S} \).

Let \( \sim_0 \) denote the symmetrization of \( \leq_0 \) and let \((\mathcal{X}, \leq)\) denote the corresponding quotient of \((\mathcal{X}_0, \leq)\). By (3), \( x \sim_0 y \) implies that either \( x = y \) or \( \{x, y\} \) intersects both \( \mathbb{P} \) and \( \mathbb{S} \) nontrivially. Therefore \( \mathbb{P} \) and \( \mathbb{S} \) are naturally isomorphic to cofinal subsets of \( \mathcal{X} \).

Recall that for \( \mathcal{A} \subseteq \mathcal{P}(\mathbb{N}) \) we write \( \mathcal{A}^\perp := \{ B \subseteq \mathbb{N} : B \cap A \) is finite for all \( A \in \mathcal{A} \} \). The following proposition is related to a statement about \( \sigma \)-unital subalgebras of the Calkin algebra (Proposition 12.2.2).

**Proposition 9.6.11.**
1. If \( \mathcal{A} \) is a countable family in \( \mathcal{P}(\mathbb{N}) \) directed under \( \subseteq^* \) with no maximal element then \( (\mathcal{A}^\perp, \subseteq^*) \) is cofinally equivalent to \((\mathbb{N}^\mathbb{N}, \leq^*)\).
2. The bounding number \( b \) is equal to the minimal \( \kappa \) such that there exists an \( (\mathbb{R}_0, \kappa) \)-gap in \( \mathcal{P}(\mathbb{N})/\Fin \).

**Proof.** (1) By fixing a bijection \( h : \mathbb{N} \to \mathbb{N}^2 \), we may assume \( \mathcal{A} \) consists of \( n \times \mathbb{N} \) for \( n \in \mathbb{N} \) and therefore \( \mathcal{A}^\perp = \{ B \subseteq \mathbb{N}^2 : (\forall m)(\forall n)(m, n) \notin B \} \). Define \( F : \mathcal{A}^\perp \to \mathbb{N}^\mathbb{N} \) by \( F(B)(m) := \max \{ n : (m, n) \in B \} \). This is a strictly increasing function from \( (\mathcal{A}^\perp, \subseteq^*) \) into \( (\mathbb{N}^\mathbb{N}, \leq) \) where \( \leq \) denotes the pointwise ordering. Also, if \( B =^* C \), then for all but finitely many \( m \) we have \( F(B)(m) = F(C)(m) \) and \( F(B) =^* F(C) \). This implies that \( F \) lifts a monotonic function from \( (\mathcal{A}^\perp, \subseteq^*) \) into \( (\mathbb{N}^\mathbb{N}, \leq^*) \). The range of this function is cofinal, since the set \( \{ (m, n) : n \leq f(m) \} \) belongs to \( \mathcal{A}^\perp \) for every \( f \in \mathbb{N}^\mathbb{N} \). Therefore \( (\mathcal{A}^\perp, \subseteq^*) \) and \( (\mathbb{N}^\mathbb{N}, \leq^*) \) are cofinally equivalent.

(2) Theorem 9.6.9 and (1) together imply that \( b \) is equal to the bounding number of \( (\mathcal{A}^\perp, \subseteq^*) \) for any countable \( \mathcal{A} \subseteq \mathcal{P}(\mathbb{N})/\Fin \) with no maximal element. A moment of reflection shows that this is the minimal cardinal \( \kappa \) such that an \( (\mathbb{R}_0, \kappa) \)-gap exists in \( \mathcal{P}(\mathbb{N})/\Fin \), and (2) follows.

### 9.7 The posets \( \Part_\mathbb{N} \) and \( \mathbb{N}^\mathbb{N} \)

In this section we study the \( \sigma \)-directed poset \( (\mathbb{N}^\mathbb{N}, \leq^*) \) and its relative \( \Part_\mathbb{N} \). The elements of \( \Part_\mathbb{N} \) are partitions of cofinal subsets of \( \mathbb{N} \) into finite intervals. This poset is taken with an appropriate (albeit not the most obvious) ordering. The posets \( \mathbb{N}^\mathbb{N} \) and \( \Part_\mathbb{N} \) are proved to be Tukey equivalent.

Decompositions of \( \ell^2(\mathbb{N}) \) into a direct sum of finite-dimensional subspaces will play a significant role in our analysis of the Calkin algebra. Such decompositions compatible with a fixed basis of \( \ell^2(\mathbb{N}) \) are coded by elements of \( \Part_\mathbb{N} \), the set of all partitions of a cofinal subset of \( \mathbb{N} \) into finite intervals. The elements of \( \Part_\mathbb{N} \) have the form \( E = \langle E_j : j \in \mathbb{N} \rangle \) where \( E_j = [n(j), n(j+1)) \) (an interval in \( \mathbb{N} \)) and \( n(j) \), for \( j \in \mathbb{N} \), is a strictly increasing sequence in \( \mathbb{N} \).

**Lemma 9.7.1.** For \( E \) and \( F \) in \( \Part_\mathbb{N} \) the following are equivalent.

1. \( (\forall x)(\exists y)E_x \cup E_{x+1} \subseteq F_y \cup F_{y+1} \).
9.7 The posets $\text{Part}_\mathbb{N}$ and $\mathbb{N}^{[\mathbb{N}]}

2. $(\forall m)(\exists n)E_n \subseteq F_m$.

Proof. For the forward implication, we need to prove that for all large enough $m$ there exists $n$ such that $E_n \subseteq F_m$. Let $m$ be such that $E_m$ is disjoint from all $E_i \cup E_{i+1}$ not included in $F_j \cup F_{j+1}$ for any $j$. Let $k$ be such that $\min F_m \in E_k$. If $E_k \not\subseteq F_m$, let $n = k$. Otherwise, $E_k \cap F_{m+1} \neq \emptyset$ and necessarily $E_k \cup E_{k+1} \subseteq F_{m-1} \cup F_m$. Clearly $E_{k+1} \cap F_{m-1} = \emptyset$, and $n = k + 1$ is as required.

For the converse implication, fix $m$ for which $E_m \cup E_{m+1}$ is disjoint from every $F_j$ such that $\max E_m \not\subseteq F_n$. We need $n$ such that $E_m \cup E_{m+1} \subseteq F_n \cup F_n$. Suppose for a moment that $\max E_m = \max F_n$. There is $j$ such that $E_j \subseteq F_n$; clearly $l \leq m$ and we necessarily have $E_l \subseteq F_n$. Similarly, $E_{m+1} \subseteq F_n$ and $n$ is as required. We may therefore assume that $F_n \not\subseteq E_m$ intersects $E_{m+1}$ nontrivially. By considering $E_l$ such that $E_l \subseteq F_k$, we see that at least one of $E_m \subseteq F_k$ or $E_{m+1} \subseteq F_k$ must hold.

If $E_m \subseteq F_k$, then $l$ such that $E_l \subseteq F_{k+1}$ satisfies $l \geq m + 1$ and we must have $E_{m+1} \subseteq F_{k+1}$; therefore $n = k$ is as required. Otherwise, $E_{m+1} \subseteq F_k$ and an analogous argument shows that $E_m \subseteq F_{k-1}$, thus $n = k - 1$ is as required. □

Definition 9.7.2. The order on $\text{Part}_\mathbb{N}$ is defined by $E \leq^* F$ if either one of the equivalent conditions in Lemma 9.7.1 holds. The symmetrization of $\leq^*$ on the quasi-ordered set $(\text{Part}_\mathbb{N}, \leq^*)$ is denoted $\ast$.

Example 9.7.3. An ordering more obvious than $\leq^*$, defined by letting $E \preceq^* F$ if $(\forall i)(\exists j)E_i \subseteq F_j$, has the disadvantage of not being directed. Let $E_j^0 := [2j, 2j + 2)$ and $E_j^1 := [2j + 1, 2j + 3)$ for $j \geq 1$. Then $E_0^0$ and $E_1^1$ have no $\preceq^*$-upper bound.

9.7.1 Von Neumann Algebras of the Form $\mathcal{D}[E]$.

For $E \in \text{Part}_\mathbb{N}$ define two coarser partitions, $E^{\text{even}}$ and $E^{\text{odd}}$, by (with $E_{-1} := \emptyset$)

$$E^{\text{even}}_n := E_{2n} \cup E_{2n+1},$$

$$E^{\text{odd}}_n := E_{2n-1} \cup E_{2n}.$$

Lemma 9.7.4. For all $E$ and $F$ we have $E \leq^* F$ if and only if all but finitely many intervals in $E^{\text{even}} \cup E^{\text{odd}}$ are included in an interval in $F^{\text{even}} \cup F^{\text{odd}}$. □

Definition 9.7.5. Consider a Hilbert space $H$ with an orthonormal basis $\xi_n$, for $n \in \mathbb{N}$. For $E \in \text{Part}_\mathbb{N}$ and $X \subseteq \mathbb{N}$ let $p_X^E := \text{proj}_X(\bigcup_{n \in X} E_n)$, and let (see Fig. 9.1)

$$\mathcal{D}[E] := \{a \in \mathcal{B}(H) : (\forall m)(\exists n)((ae_m|e_n) \neq 0 \text{ implies } (\exists j)\{m, n\} \subseteq E_j\},$$

$$\mathcal{A}[E] := W^*\{a X^E : X \subseteq \mathbb{N}\}.$$  

These $C^*$-algebras are clearly WOT-closed, and are therefore von Neumann algebras. The algebra $\mathcal{D}[E]$ consists of ‘$E$-block-diagonal operators’ and if $m(j) := |E_j|$
then $\mathcal{D}[E] \cong \prod_i M_{m(i)}(\mathbb{C})$. We clearly have $\mathcal{D}[E] = \mathcal{Y}[E]'$ and $\mathcal{Y}[E] = \mathcal{D}[E]'$. Since $\mathcal{Y}[E] = \mathcal{D}[E] \cap \mathcal{D}[E]'$, it is the center of $\mathcal{D}[E]$. Our interest in $\mathcal{D}_{\mathbb{N}}$ is largely derived from Lemma 9.7.6 below. This lemma provides a ‘stratification’ of the Calkin algebra $\mathcal{D}(H)$ by a directed set of finite (in the operator-algebraic sense) subalgebras (see also the related Lemma 9.8.6).

**Fig. 9.1** With $E_j = [n(j), n(j + 1))$, the elements of $\mathcal{Y}[E]_{\text{even}}$ have their supports (i.e., the sets $\{\{m,n\} : (ae_m|e_n) \neq 0\}$) included in the solid line square regions while the elements of $\mathcal{Y}[E]_{\text{odd}}$ have their supports included in the dashed line square regions.

**Lemma 9.7.6.** Let $H$ be a Hilbert space with an orthonormal basis $\xi_n$, for $n \in \mathbb{N}$. For a sequence $a_n$, for $n \in \mathbb{N}$ in $\mathcal{B}(H)$ there are $E \in \mathcal{P}_{\mathbb{N}}$, $a_0 \in \mathcal{Y}[E]_{\text{even}}$ and $a_0 \in \mathcal{Y}[E]_{\text{odd}}$ such that $a_n - a_0 - a_1$ is compact for each $n$.

**Proof.** For $m \in \mathbb{N}$ we write $r_m := \text{proj}_{\text{span} \{\xi_j : j \leq m\}}$. Fix $m \in \mathbb{N}$ and $\varepsilon > 0$. For every $a \in \mathcal{B}(H)$ the operator $ar_m$ is compact and we can find $n > m$ depending on $a$, $m$, and $\varepsilon$, large enough so that $\|(1 - r_n)ar_m\| < \varepsilon$ and $\|(1 - r_n)a'r_m\| < \varepsilon$. This also implies $\|r_m(a(1 - r_n))\| < \varepsilon$ and $\|r_m(a^* (1 - r_n))\| < \varepsilon$. Recursively find a strictly increasing $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for all $m \leq n$ and all $i \leq n$ we have

$$\max(\|r_{f(n+1)}(1 - r_{f(n+1)})a_i r_{f(m)}\|, \|r_{f(m)}a_i (1 - r_{f(n+1)})\|) < 2^{-n}.$$ 

We claim $E := \langle f(n), f(n+1) : n \in \mathbb{N} \rangle$ is as required. Let $q_n := r_{f(n-1)}r_{f(n)}$ (with $f(1) = 0$). For $a = a_i$ define

$$a_0 := \sum_{n=0}^\infty (q_{2n}a_{2n} + q_{2n}a_{2n+1} + q_{2n+1}a_{2n}),$$
$$a_1 := \sum_{n=0}^\infty (q_{2n+1}a_{2n+1} + q_{2n+1}a_{2n+2} + q_{2n+2}a_{2n+1}).$$

Then $a_0 \in \mathcal{Y}[E]_{\text{even}}$ and $a_1 \in \mathcal{Y}[E]_{\text{odd}}$. Let $c := a - a_0 - a_1$. For every $n$ we have

$$\|(1 - r_{f(n)})c\| \leq \sum_{i=n}^\infty \|(1 - r_{f(n)})ar_{f(i-1)}\| + \sum_{i=n}^\infty (r_{i} - r_{n})a(1 - r_{f(i+1)})\|$$
$$\leq 2^{-n+2} + 2^{-n+1},$$
9.7 The posets $\text{Part}_N$ and $N^{\uparrow N}$

and therefore $c$ is compact. \hfill \Box

**Definition 9.7.7.** Consider

$$N^{\uparrow N} := \{f \in N^N : f \text{ is increasing and } f(0) > 0\}$$

with the pointwise ordering. For $f \in N^{\uparrow N}$ recursively define $f^+ \in N^{\uparrow N}$ by $f^+(0) := f(0)$ and $f^+(i + 1) := f(f^+(i))$ for $i \geq 0$.

**Theorem 9.7.8.** The directed sets $(N^N, \leq^*)$, $(N^{\uparrow N}, \leq^*)$, and $\text{Part}_N$ are cofinally equivalent.

**Proof.** As $N^{\uparrow N}$ is a $\leq^*$-cofinal subset of $N^N$, it suffices to prove that $(N^{\uparrow N}, \leq^*)$ and $\text{Part}_N$ are cofinally equivalent. Define $\Phi : \text{Part}_N \to N^{\uparrow N}$ and $\Psi : N^{\uparrow N} \to \text{Part}_N$ as follows. If $E \in \text{Part}_N$, define $g = \Phi(E)$ by

$$g(m) := \min E_{i+2}, \text{ if } m \in E_i.$$ 

If $h \in N^{\uparrow N}$, define $F = \Psi(h)$ by $F_0 := \{0\}$ and $F_k := [n_k, n_{k+1})$ for $k \geq 1$, with

$$n_{k+2} := \max h(\bigcup_{j \leq k} F_j).$$

Since $h(i) > i$ for all $i$, $F \in \text{Part}_N$.

**Claim.** If $\Psi(h) \leq^* E$ then $h \leq^* \Phi(E)$.

**Proof.** Let $F := \Psi(h)$ and $g := \Phi(E)$, and let $m$ be such that $F_n \subseteq E_m$ for some $n$. If $i \in E_{m-1}$ then $g(i) = \min E_{m+1}$. Since $i \leq \max F_{n-1}$, we have

$$h(i) \leq \min F_{n+1} \leq \min E_{m+1} = g(i).$$

Since this is true for an arbitrarily large $m$, $h \leq^* g$. \hfill \Box

**Claim.** If $\Phi(E) \leq^* h$ then $E \leq^* \Psi(h)$.

**Proof.** Again let $F := \Psi(h)$ and $g := \Phi(E)$. Writing $F_k = [n_k, n_{k+1})$, suppose $k$ is large enough to have $g(n_k) \leq h(n_k)$. If $j$ is such that $n_k \in E_{j-1}$, then

$$\min E_{j+1} \leq g(n_k) \leq h(n_k) = n_{k+1}.$$ 

Therefore $E_j \subseteq F_k$. Since this argument applies to any large enough $k$, $E \leq^* F$. \hfill \Box

The two claims prove that each of $\Phi$ and $\Psi$ is both Tukey and convergent, and therefore $N^{\uparrow N}$ and $\text{Part}_N$ are cofinally equivalent. \hfill \Box

For $s = \langle F_0, \ldots, F_{k-1}\rangle$ let $|s| = k$ and $[s] := \{E \in \text{Part}_N : E_j = F_j \text{ for } j < k\}$. In §17.8 we will need the following ‘twin’ of Lemma 9.5.6. Its proof is omitted, being analogous to the proof of the latter.

**Lemma 9.7.9.** Suppose that $X \subseteq \text{Part}_N$ is $\leq^*$-unbounded. Then for every $k$ there exists $s$ with $|s| = k$ such that the set $Z(s, Y') := \bigcup\{E_k : E \in [s] \cap Y'\}$ is infinite. \hfill \Box
9.8 The poset Part\(_{\ell^2}\)

In this section we study a \(\sigma\)-directed Part\(_{\ell^2}\), closely related to the poset Part\(_{\mathbb{N}}\) introduced in \S9.7 and cofinally equivalent to it. Its elements are decompositions of a subspace of \(\ell^2\) of co-finite dimension into a sequence of finite-dimensional orthogonal subspaces. The posets Part\(_{\mathbb{N}}\) and Part\(_{\ell^2}\) will be used to construct non-diagonalizable pure states of \(\mathcal{B}(H)\) in \S12.5.

Two technical issues need to be resolved before the ordering on Part\(_{\ell^2}\) is defined. The more obvious one is that we’ll have to work with approximate relations. For \(\varepsilon > 0\) and projections \(p\) and \(q\) let \(p \precsim_{\varepsilon} q\) if

\[
\| (1-q)p \| \leq \varepsilon.
\]

Then \(p \leq q\) if and only if \(p \precsim_0 q\), but for \(\varepsilon > 0\) this is not a transitive relation. The following weakening of transitivity will do.

**Lemma 9.8.1.** Suppose \(p, q, \) and \(r\) are projections, \(\varepsilon > 0, \delta > 0, \) \(p \precsim_{\varepsilon} q, \) and \(q \precsim_{\delta} r.\) Then \(p \precsim_{2\varepsilon + \delta} r.\)

**Proof.** We have

\[
\| (1-r)p \| = \| p - rp \| \leq \| p - qp \| + \| qp - rqp \| + \| rqp - rp \| \\
= \| (1-q)p \| + \| (1-r)qp \| + \| r(q-1)p \| \leq 2\varepsilon + \delta,
\]

as required. \(\square\)

The second issue is slightly more subtle, but we have already encountered it in Lemma 9.7.1. For \(E\) and \(F\) in Part\(_{\mathbb{N}}\) the conditions \((\forall m)(\exists j)(E_i \cup E_{i+1} \subseteq F_j \cup F_{j+1})\) and \((\forall m)(\exists n)(E_m \subseteq F_n)\) are equivalent. The analogous equivalence does not necessarily hold for partitions of \(\ell^2\) into finite-dimensional subspaces. To wit, \(\mathbb{N}\) is equipped with a canonical well-ordering, and \(\ell^2\) isn’t. We therefore cannot distinguish between finite-dimensional subspaces of \(\ell^2\) that are ‘intervals’ and those that are just ‘finite subsets.’ We will therefore impose both conditions by definition.

**Definition 9.8.2.** Let Part\(_{\ell^2}\) denote the set of all decompositions of a closed subspace of \(H = \ell^2(\mathbb{N})\) of finite co-dimension into an orthogonal sum of finite-dimensional subspaces. The elements of Part\(_{\ell^2}\) have the form \(K = \langle K_j : j \in \mathbb{N} \rangle\), where \(K_j\) is a nonzero finite-dimensional nonzero subspace of \(H, K_i\) and \(K_j\) are orthogonal for \(i \neq j\), and \((\bigoplus K_j) \cap H\) is finite-dimensional.

For \(K \in \text{Part}_{\ell^2}\) and \(X \subseteq \mathbb{N}\), we write\(^8\)

\[
p^K_X := \text{proj}_{\bigoplus_{j \in X} K_j}.
\]

The ordering is defined by \(K \leq^* L\) if there are \(f : \mathbb{N} \to \mathbb{N}\) and \(g : \mathbb{N} \to \mathbb{N}\) such that

\(^8\) Following von Neumann’s convention that \(j = \{0, \ldots, j-1\}\), one could write \(p^K_{\{0, \ldots, j-1\}}\); however this notation is avoided as \(p^K_j\) could easily be confused with \(p^K_{\{0\}}\).
9.8 The poset $\mathbb{P}_{\ell_2}$

1. $\sum_{n \in \mathbb{N}} \| P_{f_m}^K (1 - P_{m}^f) \| < \infty$, and
2. $\sum_{n \in \mathbb{N}} \| P_{(n,n+1)}^K (1 - P_{(g(n),g(n)+1)}^f) \| < \infty$.

**Lemma 9.8.3.** The structure $(\mathbb{P}_{\ell_2}, \leq^*)$ is a poset.

**Proof.** Only the transitivity of $\leq^*$ is not obvious. Suppose $K(0), K(1),$ and $K(2)$ in $\mathbb{P}_{\ell_2}$ satisfy $K(0) \leq^* K(1)$ and $K(1) \leq^* K(2)$. Let $f_i$ and $g_i$ for $i < 2$ be such that $\sum_{m \in \mathbb{N}} \| P_{f_i(m)}^{K(i)} (1 - P_{m}^{f_i}) \| < \infty$ and $\sum_{n \in \mathbb{N}} \| P_{n}^{K(i)} (1 - P_{(g(n),g(n)+1)}^{f_i}) \| < \infty$. Let $f := f_0 \circ f_1$. For $m \in \mathbb{N}$ Lemma 9.8.1 implies

$$\| P_{f(m)}^{K(0)} (1 - P_{m}^{K(2)}) \| \leq 2 \| P_{f_0(f_1(m))}^{K(0)} (1 - P_{f_1}^{K(1)}) \| + \| P_{f_1(m)}^{K(1)} (1 - P_{m}^{K(2)}) \|.$$

Since all terms in all three series are nonnegative, by rearrangement we obtain

$$\sum_{m \in \mathbb{N}} \| P_{f(m)}^{K(0)} (1 - P_{m}^{K(2)}) \| \leq 2 \sum_{m \in \mathbb{N}} \| P_{f_0(f_1(m))}^{K(0)} (1 - P_{f_1}^{K(1)}) \| + \sum_{m \in \mathbb{N}} \| P_{f_1(m)}^{K(1)} (1 - P_{m}^{K(2)}) \| < \infty.$$

An analogous argument shows that $g := g_1 \circ g_0$ satisfies

$$\sum_{n \in \mathbb{N}} \| P_{(n,n+1)}^{K(0)} (1 - P_{(g(n),g(n)+1)}^{K(2)}) \| \leq 2 \sum_{n \in \mathbb{N}} \| P_{(n,n+1)}^{K(0)} (1 - P_{(g(n),g(n)+1)}^{K(1)}) \| + \sum_{n \in \mathbb{N}} \| P_{(g(n),g(n)+1)}^{K(1)} (1 - P_{(g(n),g(n)+1)}^{K(2)}) \|.$$

These sums are finite and therefore $K(0) \leq^* K(2)$. \hfill $\Box$

The remainder of the present section is devoted to the proof of the following.

**Theorem 9.8.4.** The directed sets $(\mathbb{N}^\mathbb{N}, \leq^*), (\mathbb{N}^\mathbb{N}, \leq^*), (\mathbb{N}^\mathbb{N}, \prec^*), \mathbb{P},$ and $\mathbb{P}_{\ell_2}$ are cofinally equivalent.

**Lemma 9.8.5.** Suppose $H$ is a separable Hilbert space with an orthonormal basis $\xi_n$, for $n \in \mathbb{N}$. Define $\Psi : \mathbb{P} \to \mathbb{P}_{\ell_2}$ as follows. For $E \in \mathbb{P}$ let $K_j := \sup_{i \in E_j} \{ \xi_i : i \in E_j \}$ for $j \in \mathbb{N}$ and let $\Psi(E) := K$. Then $\Psi$ is an order-isomorphism.

**Proof.** Since $\| P(q - 1) \| \in \{0,1\}$ for commuting projections $p$ and $q$, this follows from the definitions of $\leq^*$ in $\mathbb{P}_{\ell_2}$. \hfill $\Box$

The following lemma gives an asymptotic comparison of the elements of $\mathbb{P}_{\ell_2}$. It is a ‘noncommutative’ variant of a simple and useful fact (Exercise 9.10.26; also compare Lemma 9.7.6).

**Lemma 9.8.6.** Suppose that $K$ and $L$ belong to $\mathbb{P}_{\ell_2}$. There are $g$ and $f$ in $\mathbb{N}^\mathbb{N}$ such that for every $j \in \mathbb{N}$ the following holds.

1. $g(f + 1) - g(j) \geq 2$,
2. $P_{(0,g(j))} \geq_{\ell_2} P_{(0,f(j))}$,
3. $P_{(0,f(j))} \geq_{\ell_2} P_{(0,g(j) + 1)}$ and
4. \( p^L_{(j),f(j+k)} \lesssim 2^{−j+1} p^K_{g(j),g(j+k+1)} \) for all \( k \geq 1 \).

**Proof.** The sequences \( g(j) \) and \( f(j) \), for \( j \in \mathbb{N} \), are chosen recursively. Suppose \( g(i) \) and \( f(i) \) as required have been chosen for \( i \leq j \). Since \( p^K_{0,0} \), for \( k \in \mathbb{N} \), is an approximate unit for \( \mathcal{K}(H) \), \( p^L_{0,f(j)} \lesssim 2^{−j} p^K_{0,g(j+1)} \) holds if \( g(j+1) \) is large enough. Similarly, since \( p^L_{0,0} \), for \( k \in \mathbb{N} \) is an approximate unit for \( \mathcal{K}(H) \) we can choose \( f(j+1) \) large enough so that \( p^K_{0,g(j+1)} \lesssim 2^{−j} p^L_{0,f(j)} \).

This completes the description of the recursive construction. The first three conditions are clearly satisfied.

To prove (4), fix \( j \) and let \( \delta = 2^{−j} \). For \( k \geq 1 \) we have \( p^L_{0,f(j+k)} \lesssim \delta p^K_{0,g(j+k+1)} \), and therefore \( p^L_{f(j),f(j+k)} \lesssim \delta p^K_{g(j),g(j+k+1)} \). On the other hand, \( p^K_{0,g(j)} \) is equivalent to \( p^L_{f(j),f(j)} \lesssim \delta p^K_{g(j),g(j+1)} \), which implies \( p^L_{f(j),f(j+1)} \lesssim \delta p^K_{g(j),g(j+1)} \). This implies \( p^L_{f(j),f(j+1)} \lesssim 2^{−j} p^K_{g(j),g(j+1)} \), as required. \( \square \)

We say that an \( M \in \text{Part}_2 \) is a *coarsening* of \( L \in \text{Part}_2 \) if there exists \( h \in \mathbb{N}^{\mathbb{N}} \) such that for all \( j \) we have \( M_j = \bigoplus_{i=h(j)}^{h(j+1)-1} L_i \).

**Lemma 9.8.7.** Suppose \( K \) and \( L \) are in \( \text{Part}_2 \).

1. If \( M \) is a coarsening of \( K \) then \( L \leq^* M \)
2. There exists a coarsening \( M \) of \( K \) such that \( L \leq^* M \).

**Proof.** The first statement is trivial. For the second, let \( f \) and \( g \) be as in the conclusion of Lemma 9.8.6. Consider the coarsening \( M \) of \( K \) given by \( h := g \). Then \( p^L_{f(n)} \lesssim 2^{−n} p^M_{g(n)} \) for all \( n \). Define \( g_1 \in \mathbb{N}^{\mathbb{N}} \) by \( g_1(j) := k \), where \( k \) is such that \( h(k) = j < h(k+1) \). Then \( p^L_{f(j),f(j+1)} \lesssim 2^{−j} p^M_{g_1(j),g_1(j+1)} \). Therefore \( L \leq^* M \) is witnessed by \( f \) and \( g_1 \). \( \square \)

**Proof (Theorem 9.8.4).** By Theorem 9.7.8 it suffices to prove that \( \text{Part}_\mathbb{N} \) and \( \text{Part}_2 \) are cofinally equivalent. We shall prove a stronger form of this fact, that \( \text{Part}_2 \) has a cofinal subset order-isomorphic to \( \text{Part}_\mathbb{N} \).

Let \( \Psi : \text{Part}_\mathbb{N} \to \text{Part}_2 \) be as defined in Lemma 9.8.5. Clearly \( \Psi[\text{Part}_\mathbb{N}] \) is closed under coarsenings, and Lemma 9.8.7 implies that it is \( \leq^* \)-cofinal in \( \text{Part}_2 \). \( \square \)

### 9.9 Meager Subsets of Product Spaces

In this section we provide a characterization of meager subsets of products of finite sets. This result will be used in §17.4 to handle Baire-measurable functions on discretizations.

Suppose \( D_n \), for \( n \in \mathbb{N} \), are finite sets and for \( X \subseteq \mathbb{N} \) let \( D_X := \prod_{n \in \mathbb{N}} D_n \). Then \( D_\mathbb{N} \) is compact with respect to the metric \( d(a,b) = 1/(\min\{n : a_n \neq b_n\} + 1) \). The basic open subsets of \( D_\mathbb{N} \) have the form \( [I,R] := \{a : a \in I = r \} \) for some \( I \subseteq \mathbb{N} \) and \( r \in D_I \). Two simple properties of such basic open sets will be used tacitly. First, if \( I \cap J = \emptyset \)
then \([I, r] \cap [J, s] = [I \cup J, rs]\) where \(rs \in D_{I,J}\) is defined as \((rs)(i) = r(i)\) if \(i \in I\) and \((rs)(i) = s(i)\) if \(i \in J\). In particular, in this case we have \([I, r] \cap [J, s] \neq \emptyset\). Second, \([I, r] \supseteq [J, s]\) if and only if \(I \subseteq J\) and \(s \cap I = r\).

**Theorem 9.9.1.** Some \(A \subseteq D_N\) is relatively comeager in \(D_N\) if and only if there are disjoint \(I(n) \in \mathbb{N}\), for \(n \in \mathbb{N}\), and \(s(n) \in D_{I(n)}\) such that \(\bigcap_{m \geq n} [I(n), s(n)] \subseteq A\).

**Proof.** For the converse implication, suppose \(I(n)\) and \(s(n)\) are as stated. Since every basic open set has the form \([I, s]\), the open set \(U_m := \bigcup_{n \geq m} [I(n), s(n)]\) is dense. Therefore \(\bigcap_{m} U_m\) is, by the Baire Category Theorem, comeager.

For the direct implication, assume \(A\) is comeager and fix dense open \(U_n \subseteq D_N\) such that \(\bigcap_{n} U_n \subseteq A\). By replacing \(U_n\) with \(\bigcap_{j \leq n} U_{j}\), we may assume that \(U_n \supseteq U_{n+1}\) for all \(n\). We will choose \(I(n)\) and \(s(n) \in D_{I(n)}\) such that \(I(n)\) are pairwise disjoint and \([I(n), s(n)] \subseteq U_n\) for all \(n\) by recursion. Fix a basic open \([I(0), s(0)] \subseteq U_0\).

Suppose that \(I(j)\) and \(s(j)\) have been chosen for \(j < n\). Let \(I := \bigcup_{j < n} I(j)\) and enumerate \(\prod_{j \leq n} D_J\) as \(t(i)\), for \(i < k\). We will choose \(J(j)\) and \(r(j) \in D_{I(J)}\) for \(j < k\) such that \(J(j)\) are pairwise disjoint and \([I, t(i)] \cap \bigcap_{j \leq n} [J(j), r(j)] \subseteq U_n\).

Assume \(i < k\) that \(I(j)\) and \(r(j)\) as required have been chosen for \(j < i\). Then \(W := [I, t(i)] \cap \bigcap_{j < i} [J(j), r(j)]\) is a basic open set. Since \(U_n\) is dense open, \(W\) has a further basic open subset included in \(U_n\). This subset is of the form \(W \cap [K, r]\) for \(K\) disjoint from \(I \cup \bigcup_{j < i} J(j)\). Then \(J(i) := K\) and \(r(i) := r\) are as required. Once \(J(i)\) and \(r(i)\) for \(i < k\) have been chosen, let \([I(n), s(n)] := \bigcap_{n} [J(i), r(i)]\).

This describes the construction of the sequence \(I(n), s(n)\). If \(a \cap I(n) = s(n)\) for infinitely many \(n\), then \(a \in U_n\) for infinitely many \(n\). Since the sets \(U_n\) are decreasing, \(a \in \bigcap_{n} U_n \subseteq A\), as required.

Here is a finer analog of the standard fact that a Baire-measurable function from a Polish space into a second-countable space is continuous on a dense \(G_δ\) set (Proposition B.2.10).

**Corollary 9.9.2.** If \(Y\) is a second countable space and \(f_n : D_N \to Y\), for \(n \in \mathbb{N}\), are Baire-measurable, then there are infinite \(X \subseteq \mathbb{N}\) and \(b \in D_{\mathbb{N},X}\) such that the function \(g_n : D_X \to Y\) defined by \(g_n(a) := f_n(a + b)\) is continuous for all \(n \in \mathbb{N}\).

**Proof.** Let \(U_j\), for \(j \in \mathbb{N}\), be an enumeration of a basis for \(Y\). For every \(n\) and \(j\) fix an open \(V_{n,j} \subseteq D_N\) such that \(A_{n,j} := f_n^{-1} [U_j] \Delta V_{n,j}\) is meager. Apply Theorem 9.9.1 to the complement of \(\bigcup_{n,j} A_{n,j}\) to obtain \(I(n)\) and \(s(n)\), for \(n \in \mathbb{N}\), such that with \(X := D \setminus \bigcup_n I(n)\) and \(b := \sum_n s(2n)\), \(g_n(a) := f_n(a + b)\) is continuous on \(D_X\).

### 9.10 Exercises

**Exercise 9.10.1.** 1. Prove that every automorphism \(\Phi\) of the Boolean algebra \(\mathcal{P}(\mathbb{N})\) is lifted by a permutation \(f\) of \(\mathbb{N}\), in the sense that \(\Phi(A) = f[A]\) for all \(A \in \mathcal{P}(\mathbb{N})\). Conclude that the Boolean algebra \(\mathcal{P}(\mathbb{N})\) has only \(c\) automorphisms.
Exercise 9.10.2. An element \(a\) of a Boolean algebra \(B\) is an atom if it is a minimal nonzero element of \(B\). A Boolean algebra is atomic if below every nonzero element of \(B\) there exists an atom. A Boolean algebra is complete if every subset has a supremum.

Prove that \(\mathcal{P}(\mathbb{N})\) is, up to the isomorphism, the unique complete and atomic Boolean algebra with a countably infinite set of atoms.

Exercise 9.10.3. Given \(A \subseteq \mathcal{P}(\mathbb{N})\), the ideal generated by \(A\) is the smallest ideal that includes \(A\). Suppose \(A\) and \(B\) are subsets of \(\mathcal{P}(\mathbb{N})\) and \(I, J\) are ideals generated by \(A\) and \(B\), respectively. Prove the following.

1. \((A, B)\) is a pregap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\) if and only if \((I, J)\) is a pregap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\).
2. \((A, B)\) is a gap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\) if and only if \((I, J)\) is a gap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\).

Exercise 9.10.4. A pair \((\mathcal{A}, p)\) is a \(\kappa\)-limit in a poset \(P\) if \(A \subseteq P\) is order-isomorphic to \(\kappa\) and \(p = \sup A\).

1. Prove that there are no \(\kappa_0\)-limits in \(\mathcal{P}(\mathbb{N})/\text{Fin}\).
2. The ideal of the sets of asymptotic density zero is

\[
\mathcal{Z}_0 := \{X \subseteq \mathbb{N} : \lim_n |X \cap n|/n = 0\}.
\]

If \(p \in \mathcal{P}(\mathbb{N})/\mathcal{Z}_0\) and \(p \neq \emptyset\), prove there is an \(\kappa_0\)-limit \((\mathcal{A}, p)\) in \(\mathcal{P}(\mathbb{N})/\mathcal{Z}_0\).

Exercise 9.10.5. Use the Continuum Hypothesis to give a simple construction of an \((\kappa_1, \kappa_1)\)-gap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\).

Exercise 9.10.6. Prove that the real line can be partitioned into \(\kappa_1\) \(G_\delta\) subsets.

Hint: First use a Hausdorff gap to partition the Cantor set into \(\kappa_1\) \(G_\delta\) subsets.

Exercise 9.10.7. Use the proof of Theorem 9.3.10 to prove the following. If \(A \subseteq \mathcal{P}(\mathbb{N})\) is analytic and \(B \subseteq \mathcal{P}(\mathbb{N})\) is such that \(A \perp B\), then one of the following applies.

1. \(A\) and \(B\) are countably separated: There are \(C_n \in \mathcal{P}(\mathbb{N})\), for \(n \in \mathbb{N}\), such that for all \(A \in A\) and \(B \in B\) there exists \(n\) such that \(A \subset C_n\) and \(C_n \perp B\).
2. There exist distinct \(n(s) \in \mathbb{N}\), for \(s \in \mathbb{N}^{\mathbb{N}}\), such that (i) \(\{n(f \upharpoonright k) : k \in \mathbb{N}\}\) is included in an element of \(A\) for every \(f \in \mathbb{N}^{\mathbb{N}}\) and \(\{n(s \upharpoonright j) : j \in \mathbb{N}\}\) is included in an element of \(B\) for all \(s \in \mathbb{N}^{\mathbb{N}}\).

Exercise 9.10.8. Suppose that OCA holds, and that \(\kappa\) and \(\lambda\) are uncountable regular cardinals.

1. If there exists a \((\kappa, \lambda)\)-gap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\), prove that \(\kappa = \lambda = \kappa_1\).
2. On \(\mathbb{N}^{\mathbb{N}}\) define the ordering by \(f \leq^* g\) if \(\lim_n (g(n) - f(n)) = \infty\). Prove that there exists a \((\kappa, \lambda)\)-gap in \(\mathbb{N}^{\mathbb{N}}/\leq^*\), prove that \(\kappa = \lambda = \kappa_1\).
Exercise 9.10.9. Use Exercise 9.10.8 (2) to prove that OCA\(_T\) implies \(b = \aleph_2\) and complete the proof of Proposition 9.5.7.

Exercise 9.10.10. Prove that there are filters \(\mathcal{F}\) and \(\mathcal{G}\) on \(\mathbb{N}\) such that \(\mathcal{F} \leq_{\text{RK}} \mathcal{G}\) and \(\mathcal{G} \leq_{\text{RK}} \mathcal{F}\) but \(\mathcal{F} \not\sim_{\text{RK}} \mathcal{G}\).

Exercise 9.10.11. Suppose that \(\mathcal{A}\) is an independent family on \(\mathbb{N}\).

1. Prove that the union of the Fréchet filter, \(\mathcal{A}\), and \(\{\mathbb{N}\setminus \bigcap X : X \in [\mathcal{A}]^{<\mathbb{N}}\}\) has the finite intersection property.
2. Prove that an ultrafilter that extends the family from (1) is not generated by a set of cardinality smaller than \(c\).

Exercise 9.10.12. Prove that there are \(2^c\) Rudin–Keisler isomorphism classes of ultrafilters on \(\mathbb{N}\).

Exercise 9.10.13. Prove that there is a family of \(c\) pairwise \(\leq_{\text{RK}}\)-incomparable ultrafilters on \(\mathbb{N}\).

Exercise 9.10.14. A family \(\mathcal{H} \subseteq \mathbb{N}^\mathbb{N}\) is independent if for every \(F \in \mathcal{H}\) and every \(g : F \to \mathbb{N}\) the set \(\{n \in \mathbb{N} : (\forall f \in F) g(f) = f(n)\}\) is infinite. Prove that there exists an independent family in \(\mathbb{N}^\mathbb{N}\) of cardinality \(c\).

A subset \(Z\) of a topological space \(X\) has the Property of Baire if there exists an open set \(U \subseteq X\) such that \(Z \Delta U\) is of the first category (i.e., meager).

Exercise 9.10.15. 1. Suppose that \(U\) is an ultrafilter on \(\mathbb{N}\) that is Haar-measurable as a subset of \(\mathcal{P}(\mathbb{N})\). Prove that it is principal.
2. Suppose that \(U\) is an ultrafilter on \(\mathbb{N}\) that has the Property of Baire as a subset of \(\mathcal{P}(\mathbb{N})\). Prove that it is principal.
   Hint: For both questions, \(\Phi : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})\) defined by \(\Phi(X) := \mathbb{N} \setminus X\) is a measure-preserving homeomorphism.

Exercise 9.10.16. Prove that \(d(\mathbb{N}^\mathbb{N}, \leq) = d\) but \(b(\mathbb{N}^\mathbb{N}, \leq) < b\).

Exercise 9.10.17. Prove that \(\mathfrak{K}_0 \times \mathfrak{K}_1\) and \(([\mathfrak{K}_1]^{<\mathfrak{K}_0}, \subseteq)\) have the same values for bounding and dominating numbers, but are not Tukey equivalent.

Exercise 9.10.18. Prove that \(\text{cov}(\mathcal{H}) \leq d\).

Exercise 9.10.19. Suppose that \(\mathbb{P}\) is a directed set and \(\kappa\) is a cardinal. Prove that the following are equivalent.
1. \(\kappa = b_{\mathbb{P}}\).
2. \(\kappa\) is the minimal cardinality of an increasing and unbounded chain in \(\mathbb{P}\).
3. \((\kappa, \leq) \leq^{\text{T}} \mathbb{P}\).

Exercise 9.10.20. Suppose that \(\mathbb{P}\) is a directed set. Prove that \(b_{\mathbb{P}} = d_{\mathbb{P}}\) if and only if \(\mathbb{P}\) has a cofinal subset well-ordered by \(\leq_{\mathbb{P}}\).
Exercise 9.10.21. Prove (using, of course, the Axiom of Choice) that every partial ordering has a cofinal well-founded subset.

Exercise 9.10.22. Suppose \( f: \varnothing \to \mathbb{P} \) is an order-preserving map between directed sets whose range is cofinal in \( \mathbb{P} \). Prove that \( \mathbf{S} \mapsto f[\mathbf{S}] \) sends cofinal subsets of \( \varnothing \) to cofinal subsets of \( \mathbb{P} \).

Exercise 9.10.23. A subset of a topological space is \( \sigma \)-compact if it is equal to the union of countably many compact sets. Let \( \mathcal{K}_0 \) be the ideal generated by \( \sigma \)-compact subsets of the Baire space \( \mathbb{N}^\mathbb{N} \). Prove that \( (\mathcal{K}_0, \subseteq) \) is Tukey-equivalent to \( (\mathbb{N}^\mathbb{N}, \leq) \).

Exercise 9.10.24. Prove that \( b \) is equal to the cardinal \( b_2 \) defined as follows. It is the minimal cardinal \( \kappa \) such that there are \( r_\xi \in (0, 1) \), for \( n \in \mathbb{N} \) and \( \xi < \kappa \), satisfying

1. \( \lim_n r_\xi^n = 0 \) for all \( \xi \).
2. There is no infinite increasing sequence \( n(i) \), for \( i \in \mathbb{N} \), of natural numbers such that \( \sum n(i) r_\xi^n < \infty \) for all \( \xi \).

Exercise 9.10.25. Prove that \( b \) is equal to the cardinal \( b_1 \) defined as follows. It is the minimal cardinal \( \kappa \) such that there are a unital C*-algebra \( A \) of density character \( \kappa \) and a sequence of unitaries \( u_n \), for \( n \in \mathbb{N} \), in \( A \) such that the following holds.

1. The sequence \( u_n \), for \( n \in \mathbb{N} \), is central: \( \lim_n \|[a, u_n]\| = 0 \) for every \( a \in A \).
2. For no subsequence \( u_{n(i)} \), for \( i \in \mathbb{N} \), with \( v_k : = u_{n(1)} u_{n(2)} \ldots u_{n(k)} \), the sequence of inner automorphisms \( \text{Ad} v_k \), for \( k \in \mathbb{N} \), converges pointwise on \( A \).

The following is a commutative version of Lemma 9.8.6.

Exercise 9.10.26. Suppose \( f: \mathbb{N} \to \mathbb{N} \) is finite-to-one. Prove that there exists a sequence \( m \) such that with \( E_j := [m(j), m(j + 1)) \) we have \( f[E_j] \subseteq E_{j-1} \cup E_j \cup E_{j+1} \) and \( f^{-1}[E_j] \subseteq E_{j-1} \cup E_j \cup E_{j+1} \) for all \( j \).

Exercise 9.10.27. Prove the following strengthening of Lemma 9.7.6. If \( |\mathbb{X}| < b \) then there are \( E \in \text{Part}_{\mathbb{N}} \), \( a^0 \in \mathcal{D}[E^{\text{even}}] \) and \( a^1 \in \mathcal{D}[E^{\text{odd}}] \) for all \( a \in \mathbb{X} \) such that \( a - a^0 - a^1 \) is compact for all \( a \in \mathbb{X} \).

Exercise 9.10.28. Prove that the following are equivalent for all \( E \) and \( F \) in \( \text{Part}_{\mathbb{N}} \).

1. \( E \leq^* F \)
2. \( (\forall^m n)(\exists n)E_n \cup E_{n+1} \subseteq F_m \) and \( (\forall^m n)(\exists n)E_n \subseteq F_m \cup F_{m+1} \).

Exercise 9.10.29. A subset \( \mathcal{J} \) of \( \mathcal{P}(\mathbb{N}) \) is hereditary if \( A \subseteq B \) and \( B \in \mathcal{J} \) implies \( A \in \mathcal{J} \). Suppose that \( \mathcal{J} \subseteq \mathcal{P}(\mathbb{N}) \) is hereditary and closed under making finite changes of its elements. Modify the proof of Theorem 9.9.1 to prove that \( \mathcal{J} \) is meager if and only if there are pairwise disjoint \( I(n) \subseteq \mathbb{N} \) such that \( \bigcup_{n \in A} I(n) \in \mathcal{J} \) if and only if \( A \) is finite.

Exercise 9.10.30. Stare at the proofs of Lemma 9.5.6 and Theorem 9.3.10 for a few moments. Then prove (in ZFC) the following. Suppose \( X \) is an analytic subset of \( \mathbb{N}^\mathbb{N} \) and \( |X|^2 = L_0 \cup L_1 \) is an open colouring. Then one of the following alternatives applies.
1. $X$ has an uncountable perfect $L_0$-homogeneous subset.
2. There are $L_1$-homogeneous sets $X_n$, for $n \in \mathbb{N}$, whose union covers $X$.

**Exercise 9.10.31.** A coherent family of functions indexed by an ideal $\mathcal{I}$ on $\mathbb{N}$ is a family $f_A : A \to \mathbb{N}$, for $A \in \mathcal{I}$, such that $\{m \in A \cap B : f_A(m) \neq f_B(m)\}$ is finite for all $A$ and $B$. It is trivial if there exists $f : \mathbb{N} \to \mathbb{N}$ such that $\{m \in A : f_A(m) \neq f(m)\}$ is finite for all $A$. Suppose $\mathcal{I} = \mathcal{I}^\perp$ where $\mathcal{I}$ is countably generated.

1. Prove that OCA implies every coherent family indexed by $\mathcal{I}$ is trivial.
2. Use Exercise 9.10.30 to prove that if $\mathcal{I}$ is a countable ideal then every analytic coherent family indexed by $\mathcal{I}^\perp$ is trivial.

**Notes for Chapter 9**

§9.1 Remarkably, the converse to Lemma 9.1.10 is a theorem of Solecki ([232]):

**Theorem 9.10.32.** 1. An ideal $\mathcal{I}$ on $\mathbb{N}$ is $F_\sigma$ if and only if $\mathcal{I} = \text{Fin}(\varphi)$ for a lower semicontinuous submeasure $\varphi$ on $\mathbb{N}$.
2. An ideal $\mathcal{I}$ on $\mathbb{N}$ is an analytic $P$-ideal if and only if $\mathcal{I} = \text{Exh}(\varphi)$ for a lower semicontinuous submeasure $\varphi$ on $\mathbb{N}$.

§9.3 Theorem 9.3.7 was proved by Hausdorff in 1908 (and republished in 1938 because the result was completely overlooked). Luzin proved Theorem 9.3.6 in 1942.

There is an alternative definition of pregaps and gaps. An (asymmetric) pregap in a quasi-ordered set $(P, \leq)$ is a pair $(A, B)$ of subsets of $P$ such that $a \leq b$ for all $a \in A$ and $b \in B$. Some $c \in P$ separates such asymmetric pregap if $a \leq c$ and $c \leq b$ for all $a \in A$ and $b \in B$. An asymmetric pregap that is not separated is an asymmetric gap.

A quasi-ordering $P$ that is downward directed (an important example is $(\mathbb{N}\uparrow, \leq^*)$ studied in §9.5) has no symmetric gaps. Sometimes the only difference between two types of gaps is in the vantage point. If $P$ is a Boolean algebra or a complemented lattice, then $(\mathcal{A}, \mathcal{B})$ is a ‘symmetric’ gap if and only if $(\mathcal{A}, \{b^\complement : b \in \mathcal{B}\})$ is an ‘asymmetric’ gap, and vice versa. All gaps in $(\mathbb{N}\uparrow, \leq^*)$ (see §9.5) are asymmetric. Gaps in $(\mathbb{N}\uparrow, \leq^*)$ play an important role in Woodin’s consistency proof of the automatic continuity for Banach algebra homomorphisms, see [51].

Theorem 9.3.10 and Exercise 9.10.7 were proved in [243].

§9.4 Theorem 9.4.6 is due to Kunen ([163]). See [61] for more information on ultrafilters and $\beta\mathbb{N}$.

§9.5 Proposition 9.5.7 comes from [242].

§9.6 The cardinals $b$ and $\d$ are two of over twenty six small uncountable cardinals associated to the continuum, such as $a, b, \d, e, f, g, h, \ldots$ and so on. Other cardinal characteristics making appearances in this book are $b^*, \d^*, p^*$, and $p$ (§12.2), as

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9 The letter $c$ has already been taken, for $2^{\aleph_0}$. 
well as cov(\mathcal{M}) (§8.5). Authoritative sources on cardinal characteristics of the continuum can be found in [28] and [19]. For a sample of the rich literature on the Tukey ordering, see [107] or [233]. The simple proof of Tukey’s Theorem 9.6.9 was taken from [241]. Proposition 9.6.11 was proved independently by Hausdorff and Rothberger. Exercise 9.10.25 and Exercise 9.10.24 were taken from [91, §2]. The converse to Proposition 9.6.9 fails even in the context of Borel directed sets; see [172, Theorem 7].

§9.7 The poset Part\mathbb{N} was introduced in [82] and used to ‘stratify’ the Calkin algebra and analyze its automorphisms (see §17.1). Lemma 9.7.1 connects it with the established poset denoted IP (‘interval partitions’) and used extensively in [28]. The proof of Theorem 9.7.8 is an adaptation of the proof of [28, Theorem 2.10]. Lemma 9.7.6 appears explicitly in [82, Lemma 1.2], but its proof (in a more general setting) is contained in the proof of [71, Theorem 3.1], as pointed out in [72].

The definable version of OCA in Exercise 9.10.30 comes from [101].

Exercise 9.10.31 essentially comes from [65], but it was recast as a consequence of OCA in [242].
Chapter 10
Constructions of Nonseparable C*-algebras, I:
Graph CCR Algebras

Words have very little to recommend them except as carriers of meaning. The shapes of written words are not especially interesting to look at. Even the sounds of sentences of spoken words are rarely engaging except when composed by those with extraordinary poetic gifts. If a sentence refuses to issue forth a fact, a request, a question, an assertion an explanation, it is nonsense, a mere grammatical shell.

Neil Postman, Amusing Ourselves to Death

This chapter is devoted to a family of twisted group algebras, called graph CCR algebras. It is richly illustrated (at least in comparison with the other chapters). In 1960, J. Glimm introduced the UHF algebras and provided three equivalent characterizations for separable UHF algebras: they are tensor products of full matrix algebras, unital inductive limits of full matrix algebras, and 'locally matricial' (LM) algebras (Theorem 6.1.3). This prompted Dixmier to ask whether these three classes of unital C*-algebras coincide in the nonseparable case. The simple graph CCR algebras form a large class of AM algebras many of which are not UHF.

The structure of graph CCR algebras is rather simple. Every such algebra has a Schauder basis consisting of finite products of generators (Lemma 10.2.6) and a tracial state that vanishes on all elements of this basis except the identity (Lemma 10.2.5). For club many separable C*-subalgebras B of a graph CCR algebra A the relative commutant of B in A has a simple description in terms of generators (Lemma 10.2.10). Every simple graph CCR algebra is an AM algebra with the same K-theoretic invariant as the CAR algebra, $M_{2^\infty}$. Thus all simple and separable graph CCR algebras are isomorphic to $M_{2^\infty}$. We prove that nonseparable AM algebras are not necessarily UHF, and even construct $2^\kappa$ nonisomorphic graph CCR algebras of density character $\kappa$ for every uncountable cardinal $\kappa$ (Theorem 10.3.4). An uncountable independent family in $\mathcal{P}(\mathbb{N})$ is used to define a simple graph CCR algebra whose pure state space is not homogeneous in §10.4. This algebra even has irreducible representations on Hilbert spaces of different density characters.
10.1 Graph CCR Algebras

One and one and one is three.

In this section we introduce and study a class of $C^*$-algebras associated with graphs, called graph CCR algebras. In §10.3 these algebras will be used to clarify the relation between UHF algebras and unital AM (approximately matricial) algebras.

Graph algebras are a well-studied generalization of Cuntz algebras (§2.3.10) and a source of attractive examples of separable $C^*$-algebras, but they are not what this section is about. We will introduce a different way of associating $C^*$-algebra to a graph, called graph CCR algebra.

A graph CCR algebra is a universal $C^*$-algebra given by generators and relations (see §2.3). This is a variant of the reduced group $C^*$-algebra modified by a cocycle coded by a graph. We use Boolean groups, i.e., the groups that satisfy the equation $1+1+1=1$.

For the set of unordered pairs $[X]^2$ and the notation $\{x,y\}_<$ see Definition 8.6.1.

We consider graphs of the form $G = (V,E)$, where $V$ is the set of vertices of $G$ and $E \subseteq [V]^2$ is the set of edges of $G$. A graph represented in this way has no loops or multiple edges. A graph with no loops and no multiple edges is said to be simple. Vertices connected by an edge are said to be adjacent to one another. There is no distinction between $\{x,y\}$ and $\{y,x\}$, hence $G$ is also undirected.

The unitaries

$v := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

in $M_2(\mathbb{C})$ are self-adjoint and anticommuting (i.e., $vw = -wv$). They also generate $M_2(\mathbb{C})$, and $M_2(\mathbb{C})$ is the universal $C^*$-algebra generated by such pair of unitaries (Example 2.3.6). Pairs of self-adjoint anticommuting unitaries are the building blocks of graph CCR algebras.

A proof of the following lemma is left to the reader.

**Lemma 10.1.1.** For $n \geq 1$ we have $M_{2^n}(\mathbb{C}) \cong \bigotimes_{i=1}^n M_2(\mathbb{C})$. Unitaries of the form $\bigotimes_{i=1}^n u_i$, with $u_i \in \{1,v,w,vw\}$, form a vector space basis for $M_{2^n}(\mathbb{C})$. The unique trace $\tau$ on $M_{2^n}(\mathbb{C})$ satisfies $\tau(\bigotimes_{i=1}^n u_i) = 0$ unless $u_i = 1$ for all $i < n$. \hfill $\Box$

**Definition 10.1.2.** Suppose $G = (V,E)$ is a simple, undirected graph. Let $\mathcal{G}(G)$ be the following set of relations in the generators $\mathcal{G}(G) := \{u_x : x \in V\}$.

$u_x^* u_x = u_x u_x^* = 1$ and $u_x = u_x^*$ for all $x$,

$u_x u_y = u_y u_x$ if $\{x,y\} \notin E$,

$u_x u_y = -u_y u_x$ if $\{x,y\} \in E$. 

The Beatles, Come Together
Proposition 10.1.3. Suppose $G$ is a graph.

1. The universal C*-algebra $C^*(\mathcal{G}(G)|\mathcal{R}(G))$ exists.
2. If $G$ is finite then $C^*(\mathcal{G}(G)|\mathcal{R}(G))$ is $2^n$-dimensional where $n = |V|$.
3. The algebra $C^*(\mathcal{G}(G)|\mathcal{R}(G))$ has a faithful tracial state $\tau$ such that $\tau(u_i) = 0$
   for all $x \in V$ and $\tau(v) = 0$ whenever $v$ is a product of generators distinct from 1.

Proof. (1) Suppose first that $G = (V, E)$ is a finite graph. We may assume $V = n$ for some $n \in \mathbb{N}$. For each pair $\{i, j\} \subseteq [n]^2$ fix a two-dimensional complex Hilbert space, denoted $H_{ij}$. Then $\mathcal{R}(H_{ij})$ can be identified with $M_2(\mathbb{C})$ for all $i < j$. We will find a representation of $\mathcal{R}(G)$ on $H = \bigotimes_{1 \leq i < j \leq n} H_{ij}$. Before continuing, note that $\mathcal{R}(H)$ is isomorphic to $M_{2^n}(\mathbb{C})$ with $k = n(n + 1)/2$. Let $v$ and $w$ be the generating unitaries of $M_{2^n}(\mathbb{C})$ as in (10.1). For $k < n$ and $\{i, j\} \subseteq [n]^2$ let

$$u_{i,j,k} := \begin{cases} 1 & \text{if } i \neq j, k \notin \{i, j\} \text{ or if } i \text{ is not adjacent to } j \\ v & \text{if } i \neq j, k = i \text{ and } i \text{ is adjacent to } j, \\ w & \text{if } i \neq j, k = j \text{ and } i \text{ is adjacent to } j, \text{ and} \\ v & \text{if } i = j. \end{cases}$$

Claim. For all $i, j, k,$ and $l$ the following are equivalent.

1. The unitaries $u_{i,j,k}$ and $u_{i,j,l}$ do not commute.
2. One has $i \neq j$, $i$ and $j$ are adjacent, and $\{i, j\} = \{k, l\}$.

Proof. For the converse implication, note that the assumption implies that one of $u_{i,j,k}$ is equal to $v$ and the other one is equal to $w$. For the direct implication, assume that $u_{i,j,k}$ and $u_{i,j,l}$ do not commute. Then one of them is equal to $w$ and the other is equal to $v$. This can happen only if $i \neq j$ and $\{k, l\} = \{i, j\}$. \qed

For $k < n$, let $u_k := \bigotimes_{1 \leq i < j \leq n} u_{i,j,k}$. This is a self-adjoint unitary in $\mathcal{R}(H)$. Fix distinct $k$ and $l$. By the Claim, $u_{i,j,k}$ and $u_{i,j,l}$ commute for all values of $i, j$ except possibly when $\{i, j\} = \{k, l\}$. Hence $u_k$ and $u_l$ commute if and only if $u_{i,j,k}$ and $u_{i,j,l}$ commute. Therefore $u_k$ and $u_l$ commute if $k$ is not adjacent to $l$ and they anticommute otherwise, and the generators $u_i$, for $i < n$, satisfy $\mathcal{R}(G)$.

Lemma 2.3.11 implies that $\mathcal{R}(G)$ is satisfiable for any graph $G$, and the universal C*-algebra $C^*(\mathcal{G}(G)|\mathcal{R}(G))$ exists.

(2) As in (1), assume $V = n$. Since $v_j^2 = 1$ and $v_i v_j = \pm v_j v_i$ for all $i$ and $j$, every word in $V$ is equivalent to a word in which each generator appears at most once and the generators appear in the original order. Therefore $C^*(\mathcal{G}(G)|\mathcal{R}(G))$ is the linear span of $v_S := \prod_{i \in S} v_i$, for $S \subseteq n$ (with $\prod_{i \in \emptyset} v_i := 1$). In the representation of $C^*(\mathcal{G}(G)|\mathcal{R}(G))$ constructed in the proof of (1) the interpretations of the unitaries $v_S$ for $S \subseteq n$ are linearly independent. The universality of $C^*(\mathcal{G}(G)|\mathcal{R}(G))$ implies that

$$C^*(\mathcal{G}(G)|\mathcal{R}(G)) \cong \text{span}\{ v_S : S \subseteq n \}$$

is $2^n$-dimensional.

\footnote{Of course, $n = \{0, 1, \ldots, n-1\}$.}
(3) Suppose for a moment that $G$ is finite. The algebra $A_G := C^*\big(\mathcal{G}(G_0)\big)$ constructed in the proof of (2) is a subalgebra of $M_{2^k} (\mathbb{C})$ for an appropriate $k$. Let $\tau_G$ be the restriction of the unique trace of $M_{2^k} (\mathbb{C})$ to $A_G$. Every product of the generators of $A_G$ is a unitary of the form $v = \bigotimes_{i \leq k} u_i$, where $u_i \in \{1, v, w, vw\}$. By Lemma 10.1.6, $\tau_G(v) = 0$ unless $v = 1$, and $\tau_G$ is as required. Since such unitaries span $A_G$, these equalities determine $\tau_G$ uniquely. We have

$$\tau_G\left(\sum \lambda_i u_i^* \sum \lambda_i u_i\right) = \sum \lambda_i \overline{\lambda}_j \tau_G(u_j^* u_j) = \sum |\lambda_i|^2,$$

and therefore $\tau_G$ is faithful.

If $G$ is infinite, then the algebra $C^*\big(\mathcal{G}(G)\big)$ is an inductive limit of finite-dimensional C*-subalgebras $A_K$ that correspond to finite induced subgraphs $K$ of $G$. Each of these C*-subalgebras has a trace $\tau_K$ as required. Being unique, these traces commute with the connecting maps in the inductive system. Let $\tau$ denote the limit trace. Then $J := \{a : \tau(a^* a) = 0\}$ is a two-sided, norm-closed ideal. Its intersection with $A_G$ for a finite set of generators $G$ is trivial, and Proposition 2.5.3 implies that $J$ is the inductive limit of $J \cap A_G$, for finite $G$. Therefore $J = \{0\}$ and $\tau$ is faithful.

**Definition 10.1.4.** Given a graph $G$ let $\text{CCR}(G) := C^*\big(\mathcal{G}(G)\big)$, the tracial state $\tau$ as guaranteed by Proposition 10.1.3 (3) is the standard tracial state of $\text{CCR}(G)$.

If $G = (V, E)$ and $X \subseteq V$ is nonempty then $G[X] := (X, E \cap [X]^2)$ is the induced subgraph of $G$ with the set of vertices $X$. A connected component of a vertex $w$ in $G$ is the induced subgraph $G[X]$ where $X$ is the set of all $v \in V$ that are connected to $w$ by a path.

**Lemma 10.1.5.** Suppose $G = (V, E)$ is a graph.

1. The algebra $\text{CCR}(G)$ is AF.
2. Suppose $V = V_0 \cup V_1$ is a partition into nonempty sets such that for any $v_0 \in V_0$ and $v_1 \in V_1$, the vertices $v_0$ and $v_1$ are not adjacent. Then

$$\text{CCR}(G) \cong \text{CCR}(G[V_0]) \otimes \text{CCR}(G[V_1]).$$

**Proof.** Since $\text{CCR}(G)$ is an inductive limit of $\text{CCR}(G')$, where $G'$ ranges over finite induced subgraphs of $G$, (1) is a consequence of Proposition 10.1.3 (2).

(2) Clearly the algebra $\text{CCR}(G)$ is generated by commuting copies of $\text{CCR}(G[V_0])$ and $\text{CCR}(G[V_1])$, and by the universality $\text{CCR}(G) \cong \text{CCR}(G[V_0]) \otimes_\alpha \text{CCR}(G[V_1])$ for some tensor product $\otimes_\alpha$. Since AF algebras are nuclear (Lemma 2.4.3), $\otimes_\alpha$ is the minimal tensor product.

Before delving deeper into the structure of graph CCR algebras in the following section, we conclude the present section with basic examples.

**Example 10.1.6.** Let us take a look at $\text{CCR}(G)$ for small finite graphs $G = (V, E)$.

1. If $|V| = n$ and $E = \emptyset$ then $\text{CCR}(G)$ is the universal algebra generated by $n$ commuting self-adjoint unitaries, $u_i$, for $i < n$. Each $p_i := (1 - u_i)/2$ is a projection, and the universality implies that, with $p^\wedge := 1 - p$ and $p^\vee := p$, for every $s \in \{\wedge, \vee\}^n$ the projection $\prod_{i < n} p_i^{x(i)}$ is nonzero. Therefore $\text{CCR}(G) \cong \mathbb{C}^n$. 

2. If \( G \) is a graph with two vertices, then CCR\( (G) \) is the universal C*-algebra generated by two anti-commuting self-adjoint unitaries. It is isomorphic to \( M_2(\mathbb{C}) \) (Example 2.3.6).

**Example 10.1.7.** Suppose \( \kappa \) is a cardinal, finite or infinite. The \( M_2\kappa \)-graph \( G_\kappa \) is defined as follows. Its vertex-set is \( \kappa \times \{0, 1\} \) and \( E = \{ (\xi, 0), (\xi, 1) : \xi < \kappa \} \) (see Fig. 10.1). Each connected component of \( G \) produces a copy of \( M_2(\mathbb{C}) \) and these copies are commuting. As in Lemma 10.1.5, this implies CCR\( (G_\kappa) \cong \bigotimes_\kappa M_2(\mathbb{C}) \).

Hence CCR\( (G_\kappa) \cong M_{2^\kappa}(\mathbb{C}) \) and CCR\( (G_\aleph_0) \cong M_{2^\aleph_0} \).

### 10.2 Structure Theory for Graph CCR Algebras

In this section we prove that every maximal independent set of vertices in a graph \( G_{<\aleph_0} \) corresponds to a subalgebra of CCR\( (G) \) which is a masa with the independence property. Products of the generating unitaries form a Schauder basis of a graph CCR algebra, and a graph CCR algebra is simple if and only if it is approximately matricial, if and only if it has a unique trace. We also construe these algebras as “twisted” versions of group C*-algebras associated with groups that are direct sums of copies of \( \mathbb{Z}/2\mathbb{Z} \).

**Definition 10.2.1.** To a graph \( G = (V, E) \) we associate the graph \( G_{<\aleph_0} = ([V]_{<\aleph_0}, E') \) defined as follows. The set of vertices of \( G_{<\aleph_0} \) is the set \([V]_{<\aleph_0}\) of all finite subsets of \( V \) (including the empty set). For \( s \) and \( t \) in \([V]_{<\aleph_0}\) let \( \{s, t\} \in E' \) if and only if the cardinality of the set \( \{x, y : x \in s, y \in t, \{x, y\} \in E\} \) is odd. We fix a linear ordering \( <_V \) on \( V \), and for \( s \in [V]_{<\aleph_0} \) define (the unitaries occurring in the product on the right-hand side are multiplied in the \( <_V \)-increasing order):

\[
u_s := \prod_{x \in s} u_x,
\]

with \( u_\emptyset := 1 \). The elements of \( V \) are identified with the singletons in \([V]_{<\aleph_0}\).

The adjacency relation in \( ([V]_{<\aleph_0}, E') \) was defined this way because for \( m \in \mathbb{N} \) we have \((-1)^m = -1\) if and only if \( m \) is odd. This, and mathematical induction, is all one needs in order to prove the following.

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\(^2\) This is known as the line graph \( L_2 \), but this book is too small for three \( L_2 \)'s.
Lemma 10.2.2. If \( s \) and \( t \) belong to \( [V]^{<\mathbb{R}_0} \) then \( \{s, t\} \not\in E' \) if and only if \( u_s \) and \( u_t \) commute and \( \{s, t\} \in E' \) if and only if they anticommute. \( \square \)

Lemma 10.2.3. If \( G \) is a graph and \( X \subseteq V \) then \( \text{span}\{u_s : s \in \langle X \rangle^{<\mathbb{R}_0}\} \) is a \( C^* \)-subalgebra of \( \text{CCR}(G) \). It is isomorphic to the graph \( \text{CCR} \) algebra \( \text{CCR}(G|X) \) associated with the induced subgraph \( G[X] \).

Proof. The set \( \{\pm u_s : s \in \langle X \rangle^{<\mathbb{R}_0}\} \) is a multiplicative group and therefore its linear span is a \( * \)-subalgebra of \( \text{CCR}(G) \). It is clearly isomorphic to \( \text{CCR}(G[X]) \). \( \square \)

Because of Lemma 10.2.3, whenever \( G_0 = G[X] \) is an induced subgraph of a graph \( G \), we identify \( \text{CCR}(G_0) \) with a subalgebra of \( \text{CCR}(G) \). We can now state and prove the analogs of Proposition 7.2.7 and Lemma 7.4.7.

Lemma 10.2.4. If \( G = (V, E) \) is an uncountable graph then \( \Phi(X) := \text{CCR}(G[X]) \) is an order-continuous order-isomorphism between \( [V]^{<\mathbb{R}_0} \) and a club in \( \text{Sep}(\text{CCR}(G)) \).

Proof. The function \( \Phi \) is well-defined by Lemma 10.2.3. It is clearly an order-continuous order-isomorphism with a cofinal range, and the conclusion follows from Lemma 7.2.4. \( \square \)

Lemma 10.2.5. Suppose \( G = (V, E) \) is a graph.

1. The map \( x \mapsto \{x\} \) is an isomorphism of \( G \) with the induced subgraph \( G^{<\mathbb{R}_0}[V] \).
2. If \( s \) and \( t \) belong to \( G^{<\mathbb{R}_0} \) then \( u_su_t = u_tu_s \) if \( \{s, t\} \not\in E' \) and \( u_su_t = -u_tu_s \) otherwise.
3. If \( \text{CCR}(G) \) is an AM algebra and \( \tau \) is its standard tracial state, then \( \tau \) is the unique tracial state on \( \text{CCR}(G) \) and \( \tau(u_s) = 0 \) for every \( s \in [V]^{<\mathbb{R}_0} \setminus \{\emptyset\} \).

Proof. Only (3) requires a proof. Since \( u_s = 1 \) if and only if \( s = \emptyset \), \( \tau \) vanishes on all \( u_s \) for \( s \neq \emptyset \) by Proposition 10.1.3. Since every full matrix algebra has a unique tracial state, the uniqueness of \( \tau \) follows from Lemma 4.1.9. \( \square \)

For the vertex-set \( V \) of a graph \( G \) consider \( [V]^{<\mathbb{R}_0} \) as a group with respect to the symmetric difference, \( \Delta \). It is isomorphic to \( \bigoplus_{s \in V} \mathbb{Z}/2\mathbb{Z} \), via the isomorphism that sends a nonempty \( s \in [V]^{<\mathbb{R}_0} \) to the sum of the generators of the copies of \( \mathbb{Z}/2\mathbb{Z} \) indexed by the elements of \( s \). We will see that \( \text{CCR}(G) \) is closely related to the reduced group \( C^* \)-algebra associated to this group. \(^3\)

The proof of the following lemma depends on the analysis of the relation between a \( C^* \)-algebra and the GNS Hilbert space.

Lemma 10.2.6. Suppose \( G = (V, E) \) is a graph.

1. The unitaries \( u_s \), for \( s \in [V]^{<\mathbb{R}_0} \), form a Schauder basis for \( \text{CCR}(G) \).
2. The density character of \( \text{CCR}(G) \) is equal to \( \max(|V|, \mathbb{R}_0) \).

\(^3\) It is really a ‘twisted group algebra’—see Exercise 10.6.5.
Proof. (1) On CCR\((G)\) consider the 2-norm associated with the canonical tracial state on CCR\((G)\),
\[
\|a\|_{2,\tau} := \tau(a^*a)^{1/2}.
\]
We claim that \(u_s \mapsto \delta_s\) extends to a linear isometry from CCR\((G)\) into a subspace of the Hilbert space \(\ell_2([V]^{<\aleph_0})\) with the orthonormal basis \(\delta_s\), for \(s \in [V]^{<\aleph_0}\).
We have \(\tau(u_s^*u_t) = 0\) if \(s \neq t\), every finite linear combination \(\sum_i \lambda_i u_{s(i)}\) satisfies
\[
\|\sum_i \lambda_i u_{s(i)}\|_{2,\tau} = \tau((\sum_i \lambda_i u_{s(i)})^*(\sum_i \lambda_i u_{s(i)})) = \sum_i \tau(\lambda_i^* \lambda_i) u_{s(i)} u_{s(i)}^* = \sum_i |\lambda_i|^2.
\]
The vectors \(\delta_s\), for \(s \in [V]^{<\aleph_0}\), form a Schauder basis for \(\ell_2([V]^{<\aleph_0})\) and on CCR\((G)\) we have \(\|\cdot\|_{2,\tau} \leq \|\cdot\|_2\). Therefore \(u_s\), for \(s \in [V]^{<\aleph_0}\), is a Schauder basis for CCR\((G)\).

(2) Without a loss of generality \(V\) is infinite, in which case we need to prove that the density character of CCR\((G)\) is \(|V|\). As the density of an infinite-dimensional Banach space with a Schauder basis is equal to the cardinality of this basis, the assertion follows from (1).

We’ll reuse the Hilbert space used in the proof of Lemma 10.2.6.

Lemma 10.2.7. 1. For all \(K \subseteq V\) there exists a unique \(\tau\)-preserving conditional expectation \(\theta_K : CCR(G) \rightarrow CCR(G[K])\).
2. The conditional expectations \(\theta_K\) as in (1) form a commuting family: If \(K \subseteq L \subseteq V\) then \(\theta_K = \theta_L \circ \theta_K\).

Proof. (1) Let \((\pi_G, H_G, \xi_G)\) be the GNS triplet associated with the canonical tracial state on CCR\((G)\). Since \(\tau\) is faithful, this is a faithful representation of CCR\((G)\). The norm \(\|\cdot\|_{2,\tau}\) used in the proof of Lemma 10.2.6 is exactly the norm associated to \(\tau\) by the GNS construction, and the space \(H_G\) is isomorphic to \(\ell_2([V]^{<\aleph_0})\). Since \(\tau\) is faithful, so is \(\pi_G\), and we can identify CCR\((G)\) with pi-GCCR\((G)\).

Fix \(K \subseteq V\) and write \(A := CCR(G)\) and \(B := CCR(G[K])\). Consider the subspace \(H_K := \overline{\text{span}} \{u_s : s \subseteq K\}\) of \(H_G\). It is invariant under \(\pi_G[B]\). Let \(p\) denote the orthogonal projection to \(H_K\). Then \(a \mapsto p\pi_G(a)p\) is spatially equivalent to the GNS-representation of \(B\) associated with the restriction of \(\tau\) to \(B\). We identify \(B\) with its image under this representation and define
\[
\Theta_K(a) := p a p.
\]
This map is completely positive and contractive since \(p\) is a positive contraction. Clearly \(\Theta_K(a) = a\) for \(a \in B\). Also, if \(s \not\subseteq K\) then \(\Theta_K(u_s) = 0\).
Since \(\{u_s\}\) form a Schauder basis for \(A\) by Lemma 10.2.6, \(\Theta_K\) is uniquely determined by its restriction to this basis and therefore \(\Theta_K\) is \(\tau\)-preserving. Since \(\Theta_K\) vanishes on the generators that do not belong to \(B\), the linearity implies that the bimodule condition
\[
\Theta_K(xyz) = x \Theta_K(y) z
\]
holds for all \(x, z\) in \(B\) and \(y \in A\). Therefore \(\Theta_K\) is a conditional expectation.
(2) The conditional expectations $\Theta_K$ are uniquely determined by $\Theta_K(\mu_s) = \mu_s$ if $s \subseteq K$ and $\Theta_K(\mu_s) = 0$ otherwise. This immediately implies that they commute. □

Still considering $[V]^{<\mathbb{R}_0}$ as a group with respect to $\Delta$, a nonempty $X \subseteq [V]^{<\mathbb{R}_0}$ is a subgroup if and only if it is closed under taking symmetric differences of its elements. If $X$ and $Y$ are subsets of $[V]^{<\mathbb{R}_0}$ we let $\langle X, Y \rangle$ denote the subgroup of $[V]^{<\mathbb{R}_0}$ generated by $X$ and $Y$.

Definition 10.2.8. Suppose $G = (V, E)$ is a graph and $X \subseteq [V]^{<\mathbb{R}_0}$. Let

$$\text{CCR}(G, X) := \overline{\text{span}}\{\mu_s : s \in X\}.$$  

For a general $X$ this is a Banach subspace of $\text{CCR}(G)$.

The straightforward proof of the following lemma (and all other proofs of this lemma) is omitted.

Lemma 10.2.9. If $G = (V, E)$ is a graph and $X$ and $Y$ are subgroups of $[V]^{<\mathbb{R}_0}$ then we have the following.

1. $\langle X, Y \rangle = \{s \Delta t : s \in X, t \in Y\}$.
2. $\text{CCR}(G, X)$ is a $C^*$-subalgebra of $\text{CCR}(G)$.
3. $C^*(\text{CCR}(G, X) \cup \text{CCR}(G, Y)) = \text{CCR}(G, \langle X, Y \rangle)$. □

The “relative commutant” of $X \subseteq [V]^{<\mathbb{R}_0}$ is defined as

$$X' := \{s \in [V]^{<\mathbb{R}_0} : (\forall x \in X\} \{s, x\} \notin E'\}.$$  

It is not difficult to see that $X'$ is a subgroup of $[V]^{<\mathbb{R}_0}$. Recall that a $C^*$-subalgebra $B$ of $A$ is complemented in $A$ if $A = C^*(B, B' \cap A)$ (Definition 7.4.1).

Lemma 10.2.10. Suppose $G = (V, E)$ is a graph, identify $[V]^{<\mathbb{R}_0}$ with the group $\bigoplus_{x \in V} \mathbb{Z}/2\mathbb{Z}$, and suppose $X \subseteq [V]^{<\mathbb{R}_0}$ is a subgroup.

1. Then $\text{CCR}(G, X)' \cap \text{CCR}(G) = \text{CCR}(G, X')$.
2. The $C^*$-subalgebra $\text{CCR}(G, X)$ is complemented in $\text{CCR}(G)$ if and only if for every $x \in V$ there exist $s \in X$ and $t \in X'$ such that $s \Delta t = \{x\}$.

Proof. (1) Since $\mu_s \in \text{CCR}(G, X)' \cap \text{CCR}(G)$ for every $s \in X'$, we have

$$\text{CCR}(G, X)' \cap \text{CCR}(G) \supseteq \text{CCR}(G, X').$$  

For the converse inclusion fix $a \in \text{CCR}(G, X)' \cap \text{CCR}(G)$. By Lemma 10.2.6 (1) there are scalars $\lambda_s$, for $s \in [V]^{<\mathbb{R}_0}$, such that $a = \sum \lambda_s \mu_s$ (meaning, as usual, that the finite partial sums of the sum on the right-hand side uniformly converge to $a$).

In order to prove $a \in \text{CCR}(G, X')$, it suffices to prove that $s \notin X'$ implies $\lambda_s = 0$. Assume otherwise and fix an offender, $s$. Let $x \in X$ be such that $\{x, s\} \in E'$. Then

$$au_s - u_xa = 2\lambda_s \mu_s u_x + c,$$  

where $c = u_xa - au_s$. Thus $u_x(au_s - u_xa) = c$. Since $\mu_s$ is an orthoprojector, $u_x(\mu_s u_x - au_s) = c$, which is a contradiction.
with $c \in \text{span}\{u_t : t \neq s \cup x\}$. Therefore $a \notin \text{CCR}(G, X) \cap \text{CCR}(G)$; contradiction.

(2) By (1), $\text{CCR}(G, X)$ is complemented in $\text{CCR}(G)$ if and only if together with $\text{CCR}(G, X')$ it generates $\text{CCR}(G)$. Lemma 10.2.9 implies that the $C^*$-algebra generated by $\text{CCR}(G, X)$ and $\text{CCR}(G, X')$ is equal to $\text{CCR}(\langle X, X' \rangle)$. But this is exactly what (2) asserts.

It is not always easy to identify the CCR algebra associated with a given graph, or which graphs correspond to the full matrix algebras. The following lemma gives a sufficient condition for the latter.

**Lemma 10.2.11.** Suppose that $G = (V, E)$ is a graph with $V = m \times \{0, 1\}$ such that (see Fig. 10.2)

1. $\{(k, 0), (k, 1)\} \in E$ for all $k < m$, and
2. $\{(k, 0), (l, 1)\} \in E$ implies $k \leq l$.

Then $\text{CCR}(G) \cong M_{2^m}(\mathbb{C})$.

**Proof.** By recursion on $l < m$ define (with $\prod_{k \notin s} v_k := 1$)

$$s(l) := \{k < l : \{(k, 0), (l, 1)\} \in E\},$$
$$u_l := u_{(l, 0)}, \quad \text{and}$$
$$v_l := u_{(l, 1)} \prod_{k \in s(l)} v_k.$$

We claim that $\text{CCR}(G) = C^*(u_l, v_l : l < m)$. Since all $u_l$ and all $v_l$ belong to $\text{CCR}(G)$, only the direct inclusion requires a proof. Clearly $u_{(l, 0)} \in C^*(u_l, v_l : l < m)$. Also, $u_{(l, 1)} = v_l \prod_{k \notin s(l)} v_k$, and by induction on $l$ all generators of $\text{CCR}(G)$ belong to $C^*(u_l, v_l : l < m)$, and our claim follows.

Since each $v_l$ is a product of commuting self-adjoint unitaries, it is a self-adjoint unitary.

**Claim.** For distinct $l < m$ and all $j < m$ we have $u_l v_j = -v_j u_l$.

**Proof.** If $k < l$ then $u_{(k, 1)}$ commutes with $u_l = u_{(l, 0)}$. By induction one proves that $v_j$ is a product of $u_{(j, 1)}$ and some of $u_{(k, 1)}$ for $k < j$. This implies that $u_l$ and $v_j$ commute if $j < l$, and that they anticommute if $j = l$. 

**Fig. 10.2** An example of a graph as in Lemma 10.2.11 with $m = 5$. Any of the dashed lines may (but need not) be an edge.
For a fixed $l$, we prove that $u_{j}v_{j} = v_{j}u_{l}$ for all $j > l$ by induction on $j \geq l+1$. If $j = l+1$, then $v_{l+1} = u_{(j, 1)}\prod_{k \in \mathbb{N}} v_{k}$. If $\{l, l+1\}$ is not an edge, then all factors of $v_{l+1}$ commute with $u_{l}$. If $\{l, l+1\}$ is an edge, then the factors of $v_{l+1}$ that do not commute with $u_{l}$ are $v_{l}$ and $u_{(l+1, 1)}$. Since $(-1)^{2} = 1$, $v_{l+1}$ commutes with $u_{l}$. The proof of the inductive step is almost identical, as the only factors of $v_{j}$ that may not commute with $u_{l}$ are $v_{l}$ and $u_{(j, 1)}$.

Therefore $A_{l} := C^{*}(u_{l}, v_{l})$ for $l < m$ are commuting unital copies of $M_{2}(\mathbb{C})$ that generate $CCR(G)$. Since $M_{2}(\mathbb{C})$ is nuclear, tensor product norm is unique and $CCR(G) \cong \bigotimes_{l \in \mathbb{N}M_{2}(\mathbb{C})}$.

A subset $X$ of the set of vertices of a graph $G = (V, E)$ is independent if $\{x, y\} \notin E$ for all pairs $x, y$ of elements of $X$.

**Definition 10.2.12.** A $C^{*}$-subalgebra $D$ of a $C^{*}$-algebra $A$ has the extension property if every pure state $\varphi$ of $D$ has a unique extension to a pure state on $A$.

**Proposition 10.2.13.** Suppose $G = (V, E)$ is a graph and $X \subseteq |V|^{<\aleph_{0}}$ is a maximal independent set in $G^{<\aleph_{0}}$. Then $CCR(G, X)$ is a masa in $CCR(G)$, and it has the extension property.

**Proof.** Since $X$ is independent, $CCR(G, X)$ is abelian. Lemma 10.2.10 and the maximality of $X$ together imply $CCR(G, X)^{'} \cap CCR(G) = \mathbb{UP}(u_{s} : s \in X^{'}).$ Therefore $CCR(G, X)$ is a masa.

To prove that it has the extension property, suppose $\varphi$ is a pure state on $CCR(G, X)$. Fix an extension $\psi$ of $\varphi$ to a state on $CCR(G)$. Fix $t \notin X$. Since $X$ is maximal independent, there exists $s \in X$ such that $\{t, s\} \notin E$, and therefore $u_{s}u_{t} = -u_{t}u_{s}$. Being a self-adjoint unitary, $u_{s}$ is equal to $1 - 2p$ for a projection $p \in CCR(G, X)$. Since $CCR(G, X)$ is abelian and $\varphi$ is a pure, $\varphi(p) \in \{0, 1\}$ and therefore $\varphi(u_{s}) \in \{1, -1\}$. In particular $\psi(u_{s}) = \varphi(u_{s})$ is an extreme point of the convex closure of $sp(u_{s})$, and Proposition 1.7.8 implies

$$\psi(u_{t})\psi(u_{s}) = \psi(u_{s}u_{t}) = -\psi(u_{t}u_{s}) = -\psi(u_{t})\psi(u_{s}).$$

Therefore $0 = \psi(u_{t})\psi(u_{s}) = \psi(u_{t})$.

Since $t \in |V|^{<\aleph_{0}} \setminus X$ was arbitrary and $u_{t}$, for $t \in |V|^{<\aleph_{0}}$ is a Schauder basis of $CCR(G)$ (Lemma 10.2.6), $\varphi$ has a unique state extension to $CCR(G)$.

### 10.3 Many Examples of AM Algebras that are not UHF

In this section we first construct an AM $C^{*}$-algebra that is not UHF. Next, for every regular uncountable cardinal $\kappa$ we construct $2^{\kappa}$ non-isomorphic AM $C^{*}$-algebras, none of which are UHF.

The $M_{2}\kappa$-graph $G_{\kappa}$ (Example 10.1.7) will be modified in various ways to obtain appealing and counterintuitive examples of AM algebras. Here is the simplest one.
Theorem 10.3.1. There exists a family of AM algebras $B_\kappa$ indexed by infinite cardinals $\kappa$ with the following properties.

1. The algebra $B_\kappa$ is AM, and $B_{\aleph_0}$ is the CAR algebra.
2. The density character of $B_\kappa$ is $\kappa$.
3. If $\aleph_0 \leq \lambda < \kappa$ then $B_\kappa$ has club many $C^*$-subalgebras of density character $\lambda$ isomorphic to $B_\lambda$.
4. If $\kappa$ is uncountable then $B_\kappa$ is not a UHF algebra.

Proof. This algebra will be associated to a graph that has the $M_2 \cdot \kappa$-graph $G_\kappa$ as an induced subgraph and a single additional vertex denoted $*$ adjacent to "half" of the vertices in $G_\kappa$. More precisely, let $G^+_{\kappa} := (V,E)$ where $V := (\kappa \times \{0,1\}) \cup \{\ast\}$ and $E := \{(\xi,0), (\xi,1), (\xi,0), * \} : \xi < \kappa \}$ (see Fig. 10.3).

![Fig. 10.3 The graph $G^+_{\kappa}$ when $\kappa > \aleph_0$.](image)

(1) Let $B_\kappa := \text{CCR}(G^+_{\kappa})$. For a nonempty $F = \{\xi_j : j < n\} \subseteq \kappa$ let (Fig. 10.4)

$$F^+ := \{(\xi,0), (\xi,1) : \xi \in F \setminus \{\xi_{n-1}\}\} \cup \{(\xi_{n-1},0), *\}$$

and $C(F^+) := \text{CCR}(G^+_{\kappa}[F^+])$. The family $\{F^+ : F \subseteq \kappa\}$ is directed and cofinal

![Fig. 10.4 The induced subgraph $G^+_{\kappa}[F^+]$.](image)

in $|V|^{\leq \aleph_0}$. Therefore $\text{CCR}(G^+_{\kappa}) = \lim_{\kappa \subseteq \aleph_0} C(F^+)$, and it suffices to prove $C(F^+)$ is isomorphic to $M_{2^n}(\mathbb{C})$ for $n = |F|$. The $C^*$-subalgebra

$$C := C^*(\{u(\xi,i,j) : i < n-1, j < 2\})$$
of $C(F^+)$ is isomorphic to $M_{2n-1}(\mathbb{C})$ by Example 10.1.7. With $v := u_n \prod_{i < n - 1} u_{(\xi_i, 1)}$, since $u_{(\xi, 1)}$ are commuting and self-adjoint we have $u_n = v \prod_{i < n - 1} u_{(\xi_i, 1)}$ and therefore

$$C(F^+) = C^*\left(\{u_\xi : x \in F^+ \setminus \{\ast\} \cup \{v\}\right).$$

Since $(-1)^2 = 1$, $v$ commutes with all $u_x$ for $x \in F^+$ except $\{(\xi_{n-1}, 0)\}$.

\begin{equation}
\begin{array}{ccccccc}
(\xi_0, 1) & (\xi_1, 1) & (\xi_2, 1) & \cdots & (\xi_{n-2}, 1) & \cdots & (\xi_{n-1}, 1) \\
(\xi_0, 0) & (\xi_1, 0) & (\xi_2, 0) & \cdots & (\xi_{n-2}, 0) & \cdots & (\xi_{n-1}, 0) \\
\end{array}
\end{equation}

Fig. 10.5 The anticommutation graph associated with the new set of generators.

Hence CCR$(F^+)$ is generated by $C \cong M_{2n-1}(\mathbb{C})$ and a copy of $M_2(\mathbb{C})$ commuting with $C$. Since $M_2(\mathbb{C})$ is nuclear, CCR$(F^+) \cong M_{2e}(\mathbb{C})$. This proves that $B_\kappa$ is AM for every infinite cardinal $\kappa$. Since $B_{\mathbb{R}_0}$ is an inductive limit of full matrix algebras of the form $M_{2e}(\mathbb{C})$ and all unital *-homomorphisms between two fixed full matrix algebras are unitarily equivalent, it is isomorphic to the CAR algebra.

(4) The density character of $B_\kappa$ is $\kappa$ by Lemma 10.2.6 (1).

(2) Assume $\mathbb{R}_0 \leq \lambda < \kappa$. For $X \in [\kappa]^\lambda$ let

$$X^\ast := [(X \times \{0, 1\}) \cup \{\ast\}]^{<\mathbb{R}_0}.$$

Then

$$B_X := \text{CCR}(G^+_\kappa, X^\ast)$$

is a C*-subalgebra of $B_\kappa$ of density character $\lambda$. If $\lambda = \mathbb{R}_0$ then the induced subgraph of $G^+_\kappa$ with the vertex-set $X^\ast$ is isomorphic to $G^+_\mathbb{R}_0$ and $B_X$ is AM by the proof of (1). Lemma 10.2.4 implies that

$$\{B_X : X \in [\kappa]^\lambda\}$$

is a club in the poset of all C*-subalgebras of $B_\kappa$ of density character $\lambda$.

(4) Suppose $B_\kappa$ is isomorphic to a UHF algebra $A$. By Example 7.4.2, $A$ has club many complemented C*-subalgebras. Lemma 7.4.4 implies that the set

$$\{C \in \text{Sep}(B_\kappa) : C \text{ is complemented in } B_\kappa\}$$

includes a club. By Lemma 10.2.4 the set $\{X \in [\kappa]^{\mathbb{R}_0} : B_X \text{ is complemented in } B_\kappa\}$ includes a club.

Claim. If $X \in [\kappa]^{\mathbb{R}_0}$ then
There are nonisomorphic AM algebras $D$ such that all elements of $D$ are isomorphic to the CAR algebra.

Proof. The reverse inclusion is clear. By Lemma 10.2.10 (1) it suffices to prove that if $|s \cap X^*| \neq \emptyset$ or $|s \cap (\kappa \times \{0\})|$ is odd then there exists $x \in X$ such that $\{x, s\} \notin E'$.

We'll go by cases. Assume first that $* \in s$. As $*$ is the only element of $V$ of infinite degree and $X$ is infinite, there is $\xi \in X$ such that $\{(\xi, 0), y\} \notin E$ for all $y \in X^*$. Then $\{(\xi, 0), s\} \in E'$. Now assume $* \notin s$ and $s \cap X^* \neq \emptyset$. Fix $s \in s$ with $\xi \in X$. As $* \notin s$, we have $\{(\xi, 1 - j), s\} \in E'$. Finally, if $s \cap X^*$ is empty but $|s \cap (\kappa \times \{0\})|$ is odd, then $\{*, s\} \in E'$. This exhausts the cases and completes the proof.

Fix a nonempty $X \subseteq [\kappa]^{<\kappa_0}$ such that $B_X$ is complemented in $B_\kappa$. Since $\kappa$ is uncountable we can find $\xi \in \kappa \setminus X$. Claim implies that $C^*(B_X, B'_X \cap B_\kappa)$ does not contain $u(\xi, 0)$ for any $\xi \in \kappa \setminus X$, and therefore $B_X$ is not complemented in $B_\kappa$.

We can now prove that the converse to Lemma 8.2.9 does not hold.

**Corollary 10.3.2.** There are nonisomorphic $C^*$-algebras $C$ and $D$ of density character $\kappa_1$ and clubs $C$ in $\text{Sep}(C)$ and $D$ in $\text{Sep}(D)$ such that all elements of $C$, as well as all elements of $D$, are isomorphic to the CAR algebra.

Proof. Take $C := \bigotimes_{\kappa_1} M_2(\mathbb{C})$ and $D := B_{\kappa_1}$, as in Theorem 10.3.1. By Theorem 10.3.1 (3), $D$ has many $C^*$-subalgebras isomorphic to $B_{\kappa_0}$, and the latter is by Theorem 10.3.1 (1) isomorphic to the CAR algebra.

Also, $C := \{\bigotimes_{\kappa_1} M_2(\mathbb{C}) : X \subseteq [\kappa_1]^{<\kappa_0}\}$ is naturally identified with a club in $\text{Sep}(C)$. All elements of $C$ are clearly isomorphic to $\bigotimes_{\kappa_0} M_2(\mathbb{C})$.

By Exercise 7.5.11, there are at most $2^\kappa$ nonisomorphic $C^*$-algebra in an infinite density character $\kappa$.

**Theorem 10.3.3.** There exist $2^{\kappa_1}$ nonisomorphic AM algebras $D_\gamma$, for $\gamma \subseteq \kappa_1$, of density character $\kappa_1$.

Proof. For a set $S$ of countable limit ordinals we define a graph $G_S = (V,E)$ as follows. Its vertex-set is

$$V := (\kappa_1 \times \{0, 1\}) \cup (S \times \{2\}).$$

The induced graph on $\kappa_1 \times \{0, 1\}$ is the $M_2$-$\kappa_1$-graph $G_{\kappa_1}$. The only additional edges are defined by letting $\{((\xi, 0), (\eta, 2)) \in E \text{ if and only if } \xi < \eta\}$ (see Fig. 10.6).

Claim. Each $G_S := \text{CCR}(G_S)$ is an AM algebra.

Proof. As in the proof of Theorem 10.3.1, $G_S$ is the inductive limit of algebras of the form $\text{CCR}(K) := C^*(u_x : x \in K)$ for $K \subseteq V$ and it suffices to prove that the set

$$\{K \subseteq V : \text{CCR}(K)\text{ is a full matrix algebra}\}$$

is cofinal in $|V|^{<\kappa_0}$.
For $F \subseteq \mathbb{R}_1$ let $\eta_i$, for $i < n$ be the increasing enumeration of $F \cap S$. Since all $\eta_i$ are limit ordinals, for $i < n$ the interval $(\eta_{i-1}, \eta_i)$ is infinite and $\xi_i := \max(F \cap \eta_i) + 1$ lies in $(\eta_{i-1}, \eta_i)$ (with $\eta_{-1} := 0$, used only to define $\xi_0$). Let

- $F_0^+ := \{(\xi, 0), (\xi, 1) : \xi \in F\}$,
- $F_1^+ := \{\eta, 2 : i < n\}$,
- $F_2^+ := \{\zeta, 0 : i < n\}$.

If $F^+ := F_0^+ \cup F_1^+ \cup F_2^+$ then $\{F^+ : F \subseteq \mathbb{R}_1\}$ is a directed and cofinal family of sub-

sets of $V$ (see Fig. 10.7). Lemma 10.2.11 implies that $C^*(u_x : x \in F^+)$ is isomorphic to a full matrix algebra, and this completes the proof. □

With $S$ and $G_S = (V, E)$ suppose $\xi$ is a countable ordinal. Let

$$X_S,\xi := \{(\eta, j) \in V : \eta < \xi, j < 3\}.$$ 

Since the mapping $\xi \mapsto X_S,\xi$ is order-continuous and cofinal in $[V_S]^{\mathbb{R}_0}$, the set $\{X_S,\xi : \xi < \mathbb{R}_1\}$ is a club in $[V]^{\mathbb{R}_0}$ and the $C^*$-subalgebras

$$C_S,\xi := \text{CCR}(G_S, X_S,\xi),$$

for $\xi < \mathbb{R}_1$, comprise a club in Sep(CCR($G_S$)).
10.3 Many Examples of AM Algebras that are not UHF

Claim. If $S$ is a set of countable limit ordinals and $\xi$ is a countable limit ordinal then $\text{CCR}(G_S, X_{S, \xi}) = (\text{CCR}(G_S, X_{S, \xi}) \cap \text{CCR}(G_S))'$ if and only if $\xi \notin S$.

Proof. We suppress writing $S$ in the subscripts and write $G$ and $X_\xi$ for $G_S$ and $X_{S, \xi}$, respectively. Fix a countable limit ordinal $\xi$ and let $C := \text{CCR}(G, X_\xi)$. By Lemma 10.2.10 (1), $C' \cap \text{CCR}(G) = \text{CCR}(G, X_\xi')$ and $(C' \cap \text{CCR}(G))' = \text{CCR}(G, X''_\xi)$.

We claim that if $\xi < \omega_1$ is a limit ordinal then the following hold:

\[ X'_\xi = \{ s : s \cap X_\xi = \emptyset \text{ and } |\{ \eta : (\eta, 2) \in s \}| \text{ is even} \}, \tag{10.2} \]
\[ X''_\xi = \{ s : s \subseteq X_\xi \cup \{(\xi, 2)\} \}. \tag{10.3} \]

The reverse inclusion in (10.2) is clear.

For the direct inclusion, fix $s$ such that $s \cap X_\xi \neq \emptyset$ or $|\{ \eta : (\eta, 2) \in s \}|$ is odd. Let

\[ Z := \{ \eta : (\eta, 2) \in s \}, \]
\[ s_0 := \{ (\eta, j) \in s : j \neq 2 \}. \]

As in the proof of Theorem 10.3.1, we proceed by cases.

Assume for a moment that $Z \cap \xi \neq \emptyset$. Let $\tilde{\eta} := \max(Z \cap \xi)$. Since both $\tilde{\eta}$ and $\xi$ are limit ordinals, there are $\zeta_0 \in (\tilde{\eta}, \xi)$ and $\zeta_1 \in (\max(Z \cap \tilde{\eta}), \tilde{\eta})$ such that $\{x, (\zeta_j, 0)\} \notin E$ for all $x \in s_0$ such that $j < 2$. Then the cardinalities of $Z \cap \zeta_0$ and $Z \cap \zeta_1$ differ by one, and exactly one of $\{(\zeta_0, 0), s\} \in E'$ and $\{(\zeta_1, 0), s\} \in E'$ holds. Either way, we have $s \notin X'_\xi$. We may therefore assume $Z \cap \xi = \emptyset$.

If $|Z|$ is odd, then since $\xi$ is a limit there is $\zeta < \xi$ such that $\{x, (\zeta, 0)\} \notin E$ for all $x \in s_0$. Then $Z \setminus \xi$ is odd and $\{(\zeta, 0), s\} \notin E'$. Finally, assume $|Z|$ is even. If $s \cap X_\xi \neq \emptyset$, then $s_0 \cap X_\xi \neq \emptyset$ and we can fix $\{(\zeta, j), s\} \in E'$, and $s \notin X'_\xi$. This completes the proof of (10.2).

For (10.3), the reverse inclusion is again clear. For the direct inclusion, fix $s$ such that $t := s \setminus (X_\xi \cup \{(\xi, 2)\})$ is nonempty.

Suppose that $(\eta, 2) \in t$ for some $\eta$; let $\eta$ be the maximal such ordinal. Since $\eta$ is a limit, we can find $\xi \in (\xi, \eta)$ such that $\{(\zeta, 0), x\} \notin E$ for all $x \in t \setminus \{(\eta, 2)\}$. Then $\{(\zeta, 0), x\} \notin E$ for all $x \in s$, and $\{x, s\} \in S'$. Therefore $(\eta, 2) \notin t$ for all $\eta$. Fix $(\xi, j) \in t$. Then $\{(\zeta, 1-j), x\} \notin E$ for all $x \in s \setminus \{(\xi, j)\}$, and $\{(\xi, 1-j), s\} \in E'$; contradiction. This concludes the proof of (10.3).

Lemma 10.2.9, (10.2), and (10.3) together imply that $(C' \cap \text{CCR}(G))' = C$ if and only if $(\xi, 2) \notin S$, as promised. □

The property $(A' \cap C')' = A$ of a separable $C^*$-subalgebra $A$ of a $C^*$-algebra $C$ is preserved under isomorphisms (Definition 7.4.3).

Claim. If $S$ and $T$ are subsets of $\mathbb{R}_1$ such that $S \Delta T$ is stationary then $C_S \cong C_T$.

Proof. Towards a contradiction, suppose $\Phi : \text{CCR}(G_S) \to \text{CCR}(G_T)$ is an isomorphism and $S \Delta T$ is nonstationary. Fix $g : \mathbb{R}_1 \to \mathbb{R}_1$ such that $\Phi[C_S, \xi] \subseteq C_{T, g(\xi)}$ and $\Phi^{-1}[C_{T, \xi}] \subseteq C_{S, g(\xi)}$ for all $\xi$. The set of all $\xi$ such that $g[\xi] \leq \xi$ includes a
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club (Example 6.2.8). Since \( \Delta T \) is stationary, there exists \( \xi \in \cap (\Delta T) \) such that \( g(\xi) \leq \xi \). Therefore \( \Phi[C_S, \xi] = C_T, \xi \). By the previous claim, one of the statements \( (C'_{S, \xi} \cap \text{CCR}(G_S))' = C_S, \xi \) and \( (C_T, \xi_\cap \text{CCR}(G_T))' = C_T, \xi \) is true and the other is false. As \( \Phi \) is an isomorphism, this contradicts Lemma 7.4.4 (3).

Fix sets \( T(Y) \subseteq \kappa_1 \), for \( Y \subseteq \kappa_1 \), such that \( T(Y) \Delta T(Z) \) is stationary whenever \( Y \neq Z \) (Corollary 6.5.11). Since the limit ordinals form a club, the intersections of these sets with the set of limit ordinals retain the latter property. Then each of the algebras \( D_Y := C_T(Y) \) has density character equal to \( \kappa_1 \). These algebras are pairwise nonisomorphic by the previous claim.

Theorem 10.3.4. For every regular uncountable cardinal \( \kappa \) there exist \( 2^\kappa \) nonisomorphic AM algebras of density character \( \kappa \).

Proof. Corollary 6.5.11 implies that there are subsets of \( \kappa \), \( T(X) \), for \( X \subseteq \kappa \), such that \( T(Y) \Delta T(Z) \) is stationary whenever \( X \neq Y \). For \( S \subseteq \kappa \) define graph \( G_S \) and C* -algebra \( C_S \) exactly as in the first paragraph of the proof of Theorem 10.3.3. Then the proof of this theorem taken almost verbatim shows that the \( C_T(X) \), for \( X \subseteq \kappa \), are nonisomorphic AM algebras.

The conclusion of Theorem 10.3.4 is true even for singular uncountable cardinals, but the proof in this case uses Shelah’s non-structure theory ([226], see also [94]) and is beyond the scope of this book.

10.4 Nonhomogeneity of the Pure State Space, II

In this section we study a simple graph CCR algebra \( A \) constructed from a bipartite graph associated with an independent family of subsets of \( \mathbb{N} \). The pure state space of \( A \) is nonhomogeneous in the sense that it does not satisfy the conclusion of Theorem 5.6.1. Moreover, the algebra \( A \) has faithful irreducible representations on Hilbert spaces of different density characters, including the separable Hilbert space.

None of the AM algebras from \( \S 10.1 \) can be represented on a separable Hilbert space. An AM algebra that is not UHF is necessarily nonseparable, but can it be separably represented?

Proposition 10.4.1. Suppose \( A_j \), for \( j \in J \), are noncommutative unital C* -algebras. Then \( A := \bigotimes_{j \in J} A_j \) cannot be faithfully represented on \( \ell_2(\kappa) \) for \( \kappa < |J| \).

Proof. Suppose the contrary and fix a representation \( \pi: A \to \mathcal{B}(\ell_2(\kappa)) \) for some cardinal \( \kappa < |J| \). Without a loss of generality, we may assume \( \kappa \) is infinite and \( |J| = \kappa^+ \). Then \( |J| \) is an uncountable regular cardinal. We may also assume that \( A \) is C*-subalgebra of \( \mathcal{B}(\ell_2(\kappa)) \).

In \( A_j \) choose non-commuting contractions \( a_j \) and \( b_j \). Let \( \xi_j \) be a vector such that \( a_j b_j \xi_j \neq b_j a_j \xi_j \). By the pigeonhole principle and replacing \( J \) with a subset of the

By Exercise 3.10.26 we could assume \( ||a_j, b_j|| = 2 \), but this would not change the proof.
same cardinality, we may assume there exists $\epsilon > 0$ such that $\|a_j b_j \xi_j - b_j a_j \xi_j\| > \epsilon$ for all $j$.

Let $\delta = \epsilon / 7$. By Lemma 6.6.5, the set of all triplets $(\xi_j, a_j \xi_j, b_j \xi_j)$ has a complete accumulation point, $(\xi, \eta_a, \eta_b)$. By shrinking $\delta$ (while keeping $\|J\| = \kappa^+$) we may assume $\max(\|\xi_j - \xi\|, \|a_j \xi_j - \eta_a\|, \|b_j \xi_j - \eta_b\|) < \delta$ for all $j$. Similarly, find $\eta_{ab}$ and $\eta_{ba}$ and shrink $\delta$ so that $\max(\|b_j \eta_a - \eta_{ba}\|, \|a_j \eta_b - \eta_{ab}\|) < \delta$ for all $j$.

Fix distinct $i$ and $j$ in $J$. Since $a_i$ and $b_j$ are contractions, we now have

$$\eta_{ab} \approx \delta a_i \eta_b \approx \delta a_i b_j \xi_j \approx \delta a_i b_j \xi_j = b_j a_i \xi_j \approx \delta b_j a_i \xi_j \approx \delta b_j \eta_a \approx \delta \eta_{ba}$$

but $\|\eta_{ab} - \eta_{ba}\| \geq \epsilon$; contradiction. $\square$

**Corollary 10.4.2.** Neither $\otimes_{\mathbb{R}_1} M_2(\mathbb{C})$ nor any of the $C^*$-algebras constructed in Theorem 10.3.1 and Theorem 10.3.3 have nonzero representations on a separable Hilbert space.

**Proof.** For $\otimes_{\mathbb{R}_1} M_2(\mathbb{C})$, this is a special case of Proposition 10.4.1. Each of the $C^*$-algebras constructed in Theorem 10.3.1 and Theorem 10.3.3 contains an isomorphic copy of $\otimes_{\mathbb{R}_1} M_2(\mathbb{C})$. As all of these algebras are simple, their nonzero representations are faithful and the conclusion follows. $\square$

If a $C^*$-algebra $A$ has a faithful representation on a separable Hilbert space $H$, then its density character is at most $c$, the density character of $\mathcal{B}(H)$.

**Theorem 10.4.3.** For any cardinal $\kappa$ such that $\mathbb{R}_1 \leq \kappa \leq c$ there exists an AM algebra of density character $\kappa$ with the following properties.

1. The algebra $A$ has an irreducible representation on a separable Hilbert space and an irreducible representation on a Hilbert space of density character $\kappa$.
2. There are pure states $\varphi$ and $\psi$ of $A$ such that $\varphi \circ \Phi \neq \psi$ for every $\Phi \in \text{Aut}(A)$.
3. The algebra $A$ has a separable masa with the extension property and a masa of density character $\kappa$ with the extension property.

The proof of Theorem 10.4.3 occupies the remainder of this section. Its final product (the algebra $A$) will be revisited, reused, and recycled in Proposition 10.5.5.

**Definition 10.4.4.** For a set $Y$ and $\mathcal{A} \subseteq \mathcal{P}(Y)$ consider the bipartite graph $G_{\mathcal{A}} = (V, E)$ with $V := Y \times \{0\} \cup \mathcal{A} \times \{1\}$ and

$$E := \{(y, 0), (X, 1) : y \in X\}.$$  

We identify $\mathcal{P}(Y)$ with $\{0, 1\}^Y$ and equip it with the product topology.

Recall that a family $\mathcal{A} \subseteq \mathcal{P}(Y)$ is independent if for every pair of finite disjoint subsets $F \neq \emptyset$ and $G$ of $\mathcal{A}$ the set $\bigcap F \setminus \bigcup G$ is infinite (see Proposition 9.2.5).

**Lemma 10.4.5.** Suppose $Y$ is an infinite set and $\mathcal{A} \subseteq \mathcal{P}(Y)$ is an independent and dense family in $\mathcal{P}(Y)$. Then $\text{CCR}(G_{\mathcal{A}})$ is an AM algebra.
Lemma 10.4.7. Suppose \( \eta \in B \) belongs to \( \mathcal{A} \) as defined in Definition 10.4.6. Given an infinite family \( \eta \) with a distinguished unit vector \( \xi \), if \( F \subseteq \mathcal{A} \) was arbitrary, CCR assumptions of Lemma 10.2.11, and hence CCR \( \xi F \) for \( \eta \) we can choose \( Y \). For \( \eta \) let \( v = 1 \) and \( v \in \mathbb{H} \) and \( \xi = \eta \) for all but finitely many \( y \).

Lemma 10.4.7. Suppose \( \mathcal{A} \) is an infinite set and \( \mathcal{A} \subseteq \mathcal{P}(\mathcal{A}) \) is an independent and dense family in \( \mathcal{P}(\mathcal{A}) \). Then CCR \( (G_{\mathcal{A}}) \) has an irreducible representation on a Hilbert space of density character \( |\mathcal{A}| \).

Proof. This proof is an infinitary version of the proof of Proposition 10.1.3. Since CCR \( (G_{\mathcal{A}}) \) is, being AM, simple, it suffices to find an irreducible representation of the generators on a Hilbert space of density \( |\mathcal{A}| \) that satisfies the relations.

For \( \eta \in \mathcal{A} \) let \( H_{\eta} := \ell_2(2) \) and \( \eta_y := (1, 0) \). Let \( H := \bigotimes_{\eta \in \mathcal{A}} (H_{\eta}, \eta_y) \). For \( F \subseteq \mathcal{A} \) the space

\[
H_F := \bigotimes_{\eta \in \mathcal{A}} H_{\eta}
\]

is identified with the subspace \( H_F \otimes \bigotimes_{\eta \in \mathcal{A} \setminus F} \eta_y \) of \( H \). As an inductive limit of \( |\mathcal{A}| \) finite-dimensional Hilbert spaces, the space \( H \) has density character \( |\mathcal{A}| \). The UHF algebra \( \bigotimes_{\eta \in \mathcal{A}} \mathcal{B}(H_{\eta}) \) is, as the inductive limit of \( \mathcal{B}(H_F) \) for \( F \subseteq \mathcal{A} \), isomorphic to the \( \mathbb{C} \)-factor of an inductive limit of \( \mathcal{A} \)-subalgebra of \( \mathcal{B}(H) \).

With \( v := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( w := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), for \( y \in \mathcal{A} \) and \( X \subseteq \mathcal{A} \) let

\[
w_{yz} := \begin{cases} 1 & \text{if } y \neq z \\ w & \text{if } y = z, \end{cases} \quad v_{Xz} := \begin{cases} 1 & \text{if } z \notin X \\ v & \text{if } z \in X. \end{cases}
\]

In order to distinguish generators associated with vertices in two parts of the bipartite graph \( G_{\mathcal{A}} \), we write \( w_y := u_{(y,1)} \) for \( y \in \mathcal{A} \) and \( v_X := u_{(X,1)} \) for \( X \in \mathcal{A} \). If \( y \in \mathcal{A} \), then \( w_{y,z} \neq 1_2 \) for exactly one \( z \in \mathcal{A} \), and therefore

\[
w_y := \bigotimes_{z \in \mathcal{A}} w_{y,z}
\]

belongs to \( \mathcal{B}(H) \). If \( X \in \mathcal{A} \), then \( v_{X,z}(\eta_z) = \eta_z \) for all \( z \in \mathcal{A} \), and therefore

\[
v_X := \bigotimes_{z \in \mathcal{A}} v_{X,z}
\]

belongs to \( \mathcal{B}(H) \). Clearly, each of \( \{w_y : y \in \mathcal{A}\} \) and \( \{v_X : X \in \mathcal{A}\} \) is a commuting family of self-adjoint unitaries. Since \( v_{X,z} \) and \( w_{y,z} \) anticommute if \( y = z \) and \( y \in X \),
and commute otherwise, we have \( w_y v_X = v_X w_y \) if \( y \notin X \) and \( w_y v_X = -v_X w_y \) if \( y \in X \). The \( C^* \)-algebra \( C^*(\{w_y : y \in Y\} \cup \{v_X : X \in \mathcal{A}\}) \) is, by the universality of \( \text{CCR}(G_{\mathcal{A}}) \), a quotient of \( \text{CCR}(G_{\mathcal{A}}) \). As the latter algebra is (being AM) simple, we have defined a faithful representation \( \pi : \text{CCR}(G_{\mathcal{A}}) \to \mathcal{B}(H) \).

It remains to prove that \( \pi \) is irreducible. Fix \( F \subseteq Y \). As \( \mathcal{A} \) is dense in \( \mathcal{P}(Y) \), for \( y \in F \) we can choose \( X(y) \in \mathcal{A} \) so that \( X \cap F = \{y\} \). For all \( y \in F \), both \( v_{X(y)} \) and \( w_y \) have \( H_F \) as an invariant subspace. The algebra \( A_0 := C^*(w_y, v_{X(y)} : y \in F) \) is isomorphic to \( M_{2^n}(\mathbb{C}) \), with \( n := |F| \).

Fix \( y \in F \). For all \( z \in Y \setminus F \) we have \( w_{y,z} = 1_2 \) and \( v_{X(y), z} \in \{1_2, v\} \). All vectors in \( H_F \) are of the form \( \xi \otimes \bigotimes_{y \in Y \setminus F} \eta_y \) for some \( \xi \), and both \( w_y \) and \( v_{X(y)} \) send a vector of this form to a vector of this form. Therefore \( H_F \) is an invariant subspace for the action of \( A_0 \). The representation of \( A_0 \) on \( H_F \) obtained in this way is spatially isomorphic to the standard representation of \( M_{2^n}(\mathbb{C}) \) on \( \ell_2(2^n) \), and therefore transitive. Since the directed union \( \bigcup_{F \subseteq Y} H_F \) is a dense linear subspace of \( H \), we conclude that \( \pi(\text{CCR}(G_{\mathcal{A}})) \) acts transitively on \( H_0 \). By Lemma 3.6.8, \( \pi \) is irreducible. \( \Box \)

Fix \( \mathcal{A} \subseteq \mathcal{P}(Y) \). For \( y \in Y \) let \( \mathcal{A}(y) := \{X \in \mathcal{A} : y \in X\} \). The \textit{dual} of \( \mathcal{A} \) is a subset of \( \mathcal{P}(\mathcal{A}) \) defined as

\[ \mathcal{A}^* := \{\mathcal{A}(y) : y \in Y\}. \]

In the following lemma we identify \( \mathcal{P}(Y) \) with \( \{0,1\}^Y \) and \( \mathcal{P}(\mathcal{A}) \) with \( \{0,1\}^\mathcal{A} \). Each of these spaces is equipped with the product topology.

**Lemma 10.4.8.** If \( \mathcal{A} \subseteq \mathcal{P}(Y) \) then the following assertions hold.

1. The bipartite graphs \( G_{\mathcal{A}} \) and \( G_{\mathcal{A}}^* \) are isomorphic.
2. \( \mathcal{A}^* = \mathcal{A} \).
3. \( \mathcal{A} \) is dense in \( \mathcal{P}(Y) \) if and only if \( \mathcal{A}^* \) is independent.
4. \( \mathcal{A} \) is independent if and only if \( \mathcal{A}^* \) is dense in \( \{0,1\}^Y \).

**Proof.** (1) Since for \( X \in \mathcal{A} \) and \( y \in Y \) we have \( y \in X \) if and only if \( X \in \mathcal{A}(y) \), the function that sends \( y \in Y \) to \( z(y) \) and \( X \in \mathcal{A} \) to itself is an isomorphism between \( G_{\mathcal{A}} \) and \( G_{\mathcal{A}}^* \).

(2) is trivial.

(3) A family \( \mathcal{A} \) is dense in \( \mathcal{P}(Y) \) if and only if for every two disjoint finite subsets \( F \) and \( G \) or \( Y \) there exists \( X \in \mathcal{A} \) such that \( F \subseteq X \) and \( G \cap X = \emptyset \). This is equivalent to \( X \in \bigcap_{y \in F} z(y) \setminus \bigcup_{y \in G} z(y) \). Since all basic open subsets of \( \{0,1\}^Y \) are of this form and \( F \) and \( G \) were arbitrary, (3) follows.

(4) is a consequence (2) and (3). \( \Box \)

**Proof (Theorem 10.4.3).** By Proposition 9.2.5 there is a dense independent family in \( \mathcal{P}(\mathbb{N}) \) of cardinality \( \kappa \). Since \( \mathcal{P}(\mathbb{N}) \) is separable, we can choose a dense subfamily \( \mathcal{A} \) of cardinality \( \kappa \); it is clearly still independent. Then \( \mathcal{A} := \text{CCR}(G_{\mathcal{A}}) \) is an AM algebra by Lemma 10.4.5. Since \( |G_{\mathcal{A}}| = |\mathcal{A}| = \kappa \), \( \mathcal{A} \) has density character \( \kappa \). It has an irreducible representation on a separable Hilbert space by Lemma 10.4.7. Since \( G_{\mathcal{A}} \cong G_{\mathcal{A}}^* \), we have \( \text{CCR}(G_{\mathcal{A}}) \cong \text{CCR}(G_{\mathcal{A}}^*) \). Lemma 10.4.8 implies that
is independent and Lemma 10.4.7 implies that CCR(\(G_{\mathcal{J}}\)) has an irreducible representation on a Hilbert space of density character \(\kappa\). Pure states on CCR(\(G_{\mathcal{J}}\)) corresponding to these two irreducible representations clearly cannot be conjugate by an automorphism.

Let \(Z_0 := \{ (y, 0) : y \in \mathbb{N} \} < \mathbb{R}_0 \) and \(Z_1 := \{ (X, 1) : X \in \mathcal{J} \} < \mathbb{R}_0\). Proposition 10.2.13 implies that \(D_j := CCR(\mathcal{J}, Z_j)\) is a masa with the extension property for \(j < 2\). Also \(|Z_0| = \mathbb{R}_0\) and \(|Z_1| = \kappa\), and therefore the density characters of these masas are \(\mathbb{R}_0\) and \(\kappa\), as required.

\[\square\]

10.5 Characters of States and Quantum Filters

In this section we prove that the graph CCR algebra constructed in §10.4 has pure states of both countable and uncountable character. It also has pure states \(\varphi\) and \(\psi\) such that whenever \(A\) is embedded into a C*-algebra \(B\) and both \(\varphi\) and \(\psi\) have unique state extensions to \(B\), these unique extensions are not equivalent.

**Definition 10.5.1.** The character of an ultrafilter \(U\) on a set \(X\), denoted \(\chi(U)\), is the minimal cardinality of a set that generates it. It is equal to the minimal cardinality of a local neighbourhood basis for \(U\) in the Čech–Stone compactification \(\beta X\).

The character of a quantum filter \(\mathcal{F}\) (Definition 5.3.3), denoted \(\chi(\mathcal{F})\), is the minimal cardinality of a generating subset \(\mathcal{A}\) of \(\mathcal{F}\). (Recall that \(\mathcal{A} \subseteq \mathcal{F}\) generates \(\mathcal{F}\) if \(\mathcal{F}\) is the intersection of all quantum filters including \(\mathcal{A}\).)

The character of a state \(\varphi\), denoted \(\chi(\varphi)\), is the minimal cardinality of a family that excises \(\varphi\) (see Exercise 5.7.4), if such a family exists.

A state \(\varphi\) can be excised (Definition 5.4.2) if and only if it is pure or a weak*-limit of pure states (by Theorem 5.2.1 and Exercise 5.7.4). Glimm’s lemma (Theorem 5.2.7) implies that every state on a primitive C*-algebra is excised. In particular, if \(A\) is infinite-dimensional and simple then \(\chi(\varphi)\) is defined for every state on \(A\).

**Lemma 10.5.2.** Suppose \(\varphi\) is a state on a C*-algebra \(A\) such that \(\chi(\varphi)\) is defined. Then every family \(\mathcal{F} \subseteq A_{+1}\) that excises \(\varphi\) has a subfamily \(\mathcal{F}' \subseteq \mathcal{F}\) of cardinality \(\chi(\varphi)\) which excises \(\varphi\). Therefore some subset of \(\mathcal{F}_\varphi\) of cardinality \(\chi(\varphi)\) excises \(\varphi\).

**Proof.** Fix a family \(\mathcal{F}_1\) that excises \(\varphi\) and has cardinality \(\chi(\varphi)\). For every \(m \geq 1\) and \(a \in \mathcal{F}_1\) we can find \(b = b(a, m) \in \mathcal{F}\) such that \(\|b a b - b^2\| < 1/m\). Then \(\mathcal{F}' := \{ b(a, m) : a \in \mathcal{F}_1, m \in \mathbb{N} \}\) has cardinality no greater than \(|\mathcal{F}_1|\). It also has the property that if \(\psi\) is a state such that \(\psi(b) = 1\) for all \(b \in \mathcal{F}'\) then by Proposition 1.7.8 \(\psi(a) = 1\) for all \(a \in \mathcal{F}_1\). Since \(\mathcal{F}_1\) excises \(\varphi\), this implies \(\psi = \varphi\) and therefore \(\mathcal{F}'\) is as required. \[\square\]

**Proposition 10.5.3.** If \(\varphi\) is a pure state on a C*-algebra \(A\), then \(\chi(\varphi) = \chi(\mathcal{F}_\varphi)\).

**Proof.** Suppose \(\mathcal{U} \subseteq \mathcal{F}_\varphi\) generates \(\mathcal{F}_\varphi\). Since \(\varphi\) is pure it is excised by \(\mathcal{F}_\varphi\) by Theorem 5.2.1, hence \(\chi(\mathcal{F}_\varphi) \geq \chi(\varphi)\). If \(\mathcal{U} \subseteq A_{+1}\) excises \(\varphi\) then Lemma 10.5.2
implies that there exists \( \mathcal{G}' \subseteq \mathcal{F}_\varphi \) of cardinality \( \leq |\mathcal{G}| \) that excises \( \varphi \). Then \( \mathcal{G}' \) generates \( \mathcal{F}_\varphi \) and \( \chi(\mathcal{F}_\varphi) \leq \chi(\varphi) \).

**Proposition 10.5.4.** Assume \( A \) and \( B \) are simple \( C^* \)-algebras and \( B \) is a unital \( C^* \)-subalgebra of \( A \). If \( \psi \) is a pure state on \( B \) with a unique extension \( \varphi \) to a state on \( A \), then \( \chi(\varphi) = \chi(\psi) \).

**Proof.** Proposition 10.5.3 implies that \( \mathcal{F}_\psi \) is \( \chi(\psi) \)-generated and Lemma 5.4.3 implies that \( \mathcal{F}_\psi \) excises \( \varphi \) on \( A \). Therefore \( \chi(\varphi) \leq \chi(\psi) \). For the converse inequality, if \( \mathcal{G} \) excises \( \varphi \) on \( A \) then it excises \( \psi \) on \( B \), and Lemma 10.5.2 implies that there exists \( \mathcal{F} \subseteq \mathcal{F}_\psi \) of cardinality \( \chi(\varphi) \) that excises \( \psi \) on \( B \). Therefore \( \chi(\psi) \leq \chi(\varphi) \). \( \square \)

The uniqueness of the state extension of \( \psi \) cannot be dropped from the assumptions of Proposition 10.5.4 (see Exercise 10.6.10).

**Proposition 10.5.5.** For every cardinal \( \aleph_1 \leq \kappa \leq \aleph \) there exists an AM algebra \( A \) of density character \( \kappa \) with pure states \( \varphi \) and \( \psi \) such that the following holds.

1. \( \chi(\varphi) = \aleph_0 \) and \( \chi(\psi) = \kappa \).
2. If a \( C^* \)-algebra \( B \) has \( A \) as a \( C^* \)-subalgebra and \( \varphi \) and \( \psi \) have unique state extensions \( \tilde{\varphi} \) and \( \tilde{\psi} \) to \( B \), then \( \tilde{\varphi} \neq \tilde{\psi} \circ \Phi \) for every automorphism \( \Phi \) of \( B \).

**Proof.** Let \( A := \text{CCR}(G_{\Delta}) \) be the AM algebra of density character \( \aleph_1 \leq \kappa \leq \aleph \) constructed in Theorem 10.4.3. As in the proof of Theorem 10.4.3 (3), \( \mathcal{A} \subseteq \mathcal{P}(\mathbb{N}) \) is an independent family of cardinality \( \kappa \), \( G_{\Delta} \) is the bipartite graph as in Definition 10.4.4, \( Z_0 := \{ \{ (y, 0) : y \in \mathbb{N} \} \}^{< \kappa} \), and \( Z_1 := \{ \{ (X, 1) : X \in \mathcal{A} \} \}^{< \mathcal{A}} \).

By Theorem 10.4.3 (3), \( D_0 := \text{CCR}(G_{\Delta}, Z_0) \) and \( D_1 := \text{CCR}(G_{\Delta}, Z_1) \) are masas in \( A \) with the extension property and density characters \( \aleph_0 \) and \( \kappa \), respectively.

Let \( \varphi \) be a pure state on \( A \) whose restriction to \( D_0 \) is pure. Since \( D_0 \) has the extension property, \( \mathcal{F}_\varphi \) is generated by \( \mathcal{F}_\varphi \cap D_0 \) and therefore countably generated. Let \( \psi \) be a pure state on \( A \) whose restriction to \( D_1 \) is pure. Since \( D_1 \) has the extension property, \( \mathcal{F}_\psi \) is generated by \( \mathcal{F}_\psi \cap D_1 \). Since \( D_1 \) has density character \( \kappa \), \( \mathcal{F}_\psi \) is \( \kappa \)-generated. We claim that it is not \( \lambda \)-generated for any \( \lambda < \kappa \).

Suppose otherwise, fix \( \lambda < \kappa \) such that \( \mathcal{F}_\psi \) is \( \lambda \)-generated. Since \( \text{CCR}(G_{\Delta}, Z_1) \) is the inductive limit of \( \text{CCR}(G_{\Delta}, Z) \) for \( Z \in [Z_1]^\Delta \), \( \psi \) is uniquely determined by its restriction to \( \text{CCR}(G_{\Delta}, Z) \) for some \( Z \in [Z_1]^\Delta \). The restriction of \( \psi \) to the abelian algebra \( \text{CCR}(G_{\Delta}, Z) \) is a character. If \( w = w_{(X, 1)} \) for some \( X \in Z \) then it is a unitary whose spectrum is equal to \( \{-1, 1\} \) and therefore \( \psi(w) = 1 \) or \( \psi(w) = -1 \). Since \( \pm 1 \) are extreme points of \( \text{sp}(w) \), Proposition 1.7.8 implies that \( \psi(aw) = \psi(wa) = \psi(a) \) for all \( a \in A \).

Since the restriction of \( \psi \) to \( \text{CCR}(G_{\Delta}, Z) \) is a pure state, every projection \( p \) in \( \text{CCR}(G_{\Delta}, Z) \) satisfies \( \psi(p) \in \{ 0, 1 \} \). For \( X \in Z \) let

\[
p_X \in \{ \frac{1}{2}(1 + w_{(X, 1)}), \frac{1}{2}(1 - w_{(X, 1)}) \}
\]

be the projection such that \( \psi(p_X) = 1 \). Then \( \psi(a) = \psi(ap_X) = \psi(p_X a) \) for all \( a \in A \) and all \( X \in Z \), and \( \mathcal{F}_\psi \) is generated by \( \{ p_X : X \in Z \} \). Choose \( X \in \mathcal{A} \setminus Z_2 \) and let \( p := \frac{1}{2}(1 + w_{(X, 1)}) \). Since \( \mathcal{A} \) is independent, for every \( F \in Z \) we have
Therefore the restriction of $\psi$ to $\text{CCR}(G_{\mathbb{Z}}, \mathbb{Z})$ can be extended to pure states $\psi_0$ and $\psi_1$ such that $\psi_0(p) = 0$ and $\psi_1(p) = 1$; contradiction. We conclude that the character of $\psi$ is equal to $\kappa$.

Let $B$ be a $C^*$-algebra having $A$ as a unital $C^*$-subalgebra. If the pure states $\varphi$ and $\psi$ have unique state extensions to $B$, denoted $\tilde{\varphi}$ and $\tilde{\psi}$, respectively, then by Proposition 10.5.4 we have $\chi(\tilde{\varphi}) = \chi(\varphi) = R_0$ and $\chi(\tilde{\psi}) = \chi(\psi) = \kappa$ and therefore there is no automorphism $\Phi$ of $C$ such that $\tilde{\varphi} = \psi \circ \Phi$.

No separable $C^*$-algebra satisfies Proposition 10.5.5 (2). This is quite important for the construction of a counterexample to Naimark’s problem (Proposition 11.2.1).

## 10.6 Exercises

**Exercise 10.6.1.** 1. Prove that every graph CCR algebra associated to a non-null graph with exactly three vertices is isomorphic to $M_2(\mathbb{C}) \otimes \mathbb{C}^2$.

2. Prove that there are only two nonisomorphic graph CCR algebras of the form $\text{CCR}(G)$ where $G$ is a non-null graph with four vertices.

3. Classify all separable graph CCR algebras: Writing $M_{2^n}(\mathbb{C})$ for $M_{2^n}$ and $\mathbb{C}^c$ for $C((0,1)^{\mathbb{N}})$, prove that every separable graph CCR algebra is isomorphic to $M_{2^n}(\mathbb{C}) \otimes \mathbb{C}^{2d}$ for some $m$ and $n$ in $\mathbb{N} \cup \{0\}$.

**Exercise 10.6.2.** Find a representation of the CAR algebra $\pi: A \to \mathcal{B}(H)$ and an approximately inner automorphism $\Phi$ of $A$ such that no unitary $u$ in $\mathcal{B}(H)$ satisfies $(\text{Ad} u) \downarrow \pi[A] = \Phi$.

**Exercise 10.6.3.** Prove that the graph CCR constructed in Theorem 10.3.1 and Theorem 10.3.3 are crossed products of $\bigotimes_{\Gamma_1} M_2(\mathbb{C})$ with $\mathbb{Z}/2\mathbb{Z}$ and $\bigoplus_{\Gamma_1} \mathbb{Z}/2\mathbb{Z}$, respectively.

**Exercise 10.6.4.** Fix $n \geq 2$ and a primitive $n$th root of unity $\gamma$. Prove that $M_n(\mathbb{C})$ is generated by unitaries $u$ and $v$ such that $u^n = v^n = 1$ and $uv = \gamma vu$.

Graph CCR algebras are a special case of the construction described in Exercise 10.6.5 when $\Gamma$ is a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$.

**Exercise 10.6.5.** Suppose $\Gamma$ is a discrete group and $b: \Gamma^2 \to \mathbb{T}$ satisfies the following for all $f$, $g$, and $h$ in $\Gamma$.

1. $b(f, g) = b(g, f)$.
2. $b(fgh) = b(f, g)b(f, h)$.
3. If $f^n = 1$ then $b(f, g)^n = 1$ for all $g$.

Let $\mathcal{G}(\Gamma) := \{u_g : g \in \Gamma\}$ and

$$\mathcal{G}(\Gamma, b) := \{u_f u_g = u_g u_f, u_f u_g = b(f, g) u_g u_f : f \in \Gamma, g \in \Gamma\}.$$
Prove that there exists a universal C*-algebra given by $\mathcal{G}(\Gamma)$ and $\mathcal{R}(\Gamma, b)$.

**Exercise 10.6.6.** Let $\kappa \geq \aleph_1$ and let $B_\kappa$ be the non-UHF AM algebra constructed in Theorem 10.3.1. Prove that $A \otimes B_\kappa$ is not UHF for any separable C*-algebra $A$.

**Exercise 10.6.7.** Prove that every graph CCR algebra $A$ is isomorphic to its opposite algebra $A^{\text{op}}$.

**Exercise 10.6.8.** Suppose $Y$ and $Z$ are distinct subsets of $\aleph_1$ and let $D_Y$ and $D_Z$ be nonisomorphic AM algebras of density character $\aleph_1$ constructed in Theorem 10.3.3.  
1. Prove that $A \otimes D_Y \not\cong A \otimes D_Z$ for every separable C*-algebra $A$.
2. Prove that $A \otimes D_Y \cong A \otimes D_Z$ for some AM algebra $A$ of density character $\aleph_1$.

The following exercise should be compared to the fact that every unital and separable inductive limit of copies of $O_2$ is isomorphic to $O_2$ ([208, Corollary 5.1.5]).

**Exercise 10.6.9.** Using $O_2 \otimes M_2 \cong O_2$ prove that there are $2^{\aleph_1}$ nonisomorphic C*-algebras of density character $\aleph_1$ each of which is a unital inductive limit of separable C*-algebras isomorphic to $O_2$.

**Exercise 10.6.10.** Find an example of a C*-algebra $A$, a C*-subalgebra $B$ of $A$, a state $\psi$ of $B$, and a state $\phi$ of $A$ that extends $\psi$, with the property that $\chi(\phi) > \chi(\psi)$. Repeat this exercise, this time with $\chi(\phi) < \chi(\psi)$.

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**Notes for Chapter 10**

§10.1 Graph CCR algebras were explicitly introduced in [80], but they appear implicitly already in [93].

§10.3 Theorem 10.3.1 and Theorem 10.3.3 were proved in [93] and [94], respectively. Group CCR algebras were introduced in [231] in the case of locally compact groups (see Exercise 10.6.5 for a discrete version).

In addition to the algebras in Theorem 10.3.1 and Theorem 10.3.3, three other different families of AM C*-algebra that are not UHF were constructed in [93] using classical C*-algebraic methods. All of these C*-algebras are indistinguishable from the CAR algebra by any of the classical C*-algebraic invariants *(such as $K_0$ or the Cuntz semigroup).

The second part of Dixmier’s problem, whether every LM algebra is AM, was resolved by a cocycle crossed product construction in [93, §6]. We omitted the elaborate machinery used in this construction due to the lack of space and the fact that the set-theoretic content of this proof is essentially Lemma 11.1.1.

The results of Exercise 6.7.11, Theorem 10.3.4 and the paragraph following it, and Exercise 7.5.11 are summarized in Figure 10.6. It is therefore not only true that some AM algebras are not UHF. In a sufficiently large density character, very few AM algebras are UHF!  

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5 This is a special case of a deep theorem of Kirchberg; see e.g., [208, Theorem 7.1.2].

6 The Generalized Continuum Hypothesis is used only to simplify the notation.
### §10.4

The question whether an AM algebra that is not UHF can be represented on a separable Hilbert space was asked by Takesaki. Proposition 10.4.1 is taken from [93, Proposition 7.6].

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**Fig. 10.8** The number of isomorphism classes of UHF algebras, AM algebras, and all $C^*$-algebras in given infinite density character assuming the Generalized Continuum Hypothesis.

<table>
<thead>
<tr>
<th>density character</th>
<th>$\aleph_0$</th>
<th>$\aleph_1$</th>
<th>$\aleph_2$</th>
<th>$\ldots$</th>
<th>$\aleph_\omega$</th>
<th>$\ldots$</th>
<th>$\aleph_{\omega_1}$</th>
<th>$\ldots$</th>
<th>$\aleph_{\omega_2}$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UHF</td>
<td>$\aleph_1$</td>
<td>$\aleph_1$</td>
<td>$\aleph_1$</td>
<td>$\ldots$</td>
<td>$\aleph_1$</td>
<td>$\ldots$</td>
<td>$\aleph_2$</td>
<td>$\ldots$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AM</td>
<td>$\aleph_1$</td>
<td>$\aleph_2$</td>
<td>$\aleph_3$</td>
<td>$\ldots$</td>
<td>$\aleph_{\omega+1}$</td>
<td>$\ldots$</td>
<td>$\aleph_{\omega_1+1}$</td>
<td>$\ldots$</td>
<td>$\aleph_{\omega_2+1}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>All $C^*$-algebras</td>
<td>$\aleph_1$</td>
<td>$\aleph_2$</td>
<td>$\aleph_3$</td>
<td>$\ldots$</td>
<td>$\aleph_{\omega+1}$</td>
<td>$\ldots$</td>
<td>$\aleph_{\omega_1+1}$</td>
<td>$\ldots$</td>
<td>$\aleph_{\omega_2+1}$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>
Perhaps it is worth pointing out that in general, heroic attempts to get rid of separability hypotheses for problems in operator algebras can force one to look carefully at the fundamentals of set theory.


This chapter is short, but fully comprehensive. To start with, we have a construction of a $C^*$-algebra with no commutative approximate unit. The main dishes are the construction of a counterexample to Naimark’s problem and of an example that Glimm’s dichotomy does not hold for nonseparable $C^*$-algebras given in §11.2. As a dessert, we prove that an uncountable free product of nontrivial countable groups has homogeneous pure state space, only separable masas, and that its automorphism group acts transitively on the pure state space.

11.1 A $C^*$-algebra With no Commutative Approximate Unit

In this section we construct a $C^*$-algebra with no commutative approximate unit.

**Lemma 11.1.1.** The following are equivalent for every cardinal $\kappa$.

1. $\kappa \geq \aleph_2$.
2. For every $f : \kappa \to [\kappa]^{\aleph_0}$ there are $\xi < \eta < \kappa$ such that $\eta \notin f(\xi)$ and $\xi \notin f(\eta)$.

**Proof.** Fix $\kappa$ and suppose (1) fails. (2) trivially fails if $\kappa = \aleph_0$. To show that it fails for $\kappa = \aleph_1$, let $f : \aleph_1 \to [\aleph_1]^{\aleph_0}$ be defined by $f(\xi) = \{ \eta : \eta < \xi \}$ (since we identify $\xi$ with the set of smaller ordinals, $f$ is really equal to the identity on $\aleph_1$). Then $\xi < \eta$ implies $\xi \in f(\eta)$, hence (2) fails.

Now assume (1) and fix $f : \kappa \to [\kappa]^{\aleph_0}$. If $\kappa > \aleph_2$ define $f' : \aleph_2 \to [\aleph_2]^{\aleph_0}$ by $f'(\xi) := f(\xi) \cap \aleph_2$. It will suffice to find $\xi < \eta < \aleph_2$ such that $\xi \notin f'(\eta)$ and $\eta \notin f'(\xi)$.

Since $\text{cof}(\aleph_2)$ is uncountable, $|\bigcup f'(\xi)| \leq \aleph_0|\xi|$ for all $\xi < \aleph_2$ and therefore $C_f := \{ \xi < \aleph_2 : \bigcup f'(\xi) \subseteq \xi \}$ is a club (Lemma 6.2.2). If $\eta \in C_f$ then $\eta \notin f'(\xi)$
for all $\xi < \eta$. If $\eta \in C_f$ is in addition uncountable then $\eta \setminus f'(\eta)$ is nonempty and we can choose $\xi \in \eta \setminus f'(\eta)$. Then $\xi < \eta$ are as required. \hfill \Box

**Theorem 11.1.2.** There exists a C$^*$-algebra $A$ with no abelian approximate unit. It can be chosen to be AF and of density character $\aleph_2$.

**Proof.** Fix $\kappa \geq \aleph_2$. For $\xi \leq \eta < \kappa$ define the C$^*$-algebra $A_{\xi\eta}$ to be isomorphic to $M_2(\mathbb{C})$ if $\xi < \eta$ and isomorphic to $\mathbb{C}$ if $\xi = \eta$. We will find a C$^*$-subalgebra $A$ of $\prod_{\xi \leq \eta < \kappa} A_{\xi\eta}$ with the required properties.

If $\xi < \eta$ let $p_{\xi\eta}^0 := \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ and $p_{\xi\eta}^1 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. These are projections in $A_{\xi\eta}$ such that the self-adjoint unitaries $u := 1 - 2p_{\xi\eta}^0$ and $v := 1 - 2p_{\xi\eta}^1$ satisfy $uv = -vu$.

Also, $\| [p_{\xi\eta}^0, p_{\xi\eta}^1] \| = 1/2$. For $\xi < \kappa$ define $q_{\xi} \in \prod_{\xi \leq \eta < \kappa} A_{\xi\eta}$ as follows:

$$q_{\xi}(\xi, \eta) := \begin{cases} 1_{A_{\xi\eta}} & \text{if } \xi = \xi = \eta, \\ p_{\xi\eta}^0 & \text{if } \xi = \xi \neq \eta, \\ p_{\xi\eta}^1 & \text{if } \xi = \eta \neq \xi, \\ 0 & \text{if } \xi \neq \xi \text{ and } \xi \neq \eta. \end{cases}$$

Clearly, $q_{\xi}$ is a projection. For $X \subseteq \kappa$ let $A_X := C^*(q_{\xi} : \xi \in F)$ and let $A := A_\kappa$.

**Claim.** If $F \subseteq \kappa$ then $A_F$ is finite-dimensional.

**Proof.** Let $r_F := \sum_{\xi \leq \eta \leq F} 1_{A_{\xi\eta}}$ (we are not claiming that $r_F \in A$). Since $r_F$ is a central projection in $\prod_{\xi \leq \eta < \kappa} A_{\xi\eta}$, the C$^*$-algebra $A_F$ is isomorphic to a C$^*$-subalgebra of $r_F A_F \oplus (1 - r_F) A_F$.

We have that $r_F(\xi, \eta) \neq 0$ implies $\{ \xi, \eta \} \subseteq F$. Since $F$ is finite, $r_F A_F$ is isomorphic to a C$^*$-subalgebra of a direct sum of finitely many copies of $M_2(\mathbb{C})$ and $\mathbb{C}$, and therefore finite-dimensional. Also, $1_{\xi\eta} q_{\xi} \neq 0$ if and only if $\xi \in \langle \xi, \eta \rangle$. Therefore $r_F q_{\xi} \neq 0$ implies $\xi \in F$, and $(1 - r_F) A_F$ is isomorphic to a C$^*$-subalgebra of $\mathbb{C}^k$ for $k = |F|$. This proves that $A_F$ is finite-dimensional. \hfill \Box

Since $A = \lim_{F \subseteq \kappa} A_F$, $A$ is approximately finite.

We claim that $A$ does not have an abelian approximate unit. Suppose otherwise, and let $D$ be the algebra generated by an abelian approximate unit of $A$. For all $\xi < \kappa$ and $\eta < \kappa$ define $\Phi_{\xi\eta}$: If $\xi \leq \eta$ then $\Phi_{\xi\eta} : A \rightarrow A_{\xi\eta}$ and $\Phi_{\xi\eta}(a) := a(\xi, \eta)$. If $\eta < \xi$ then $\Phi_{\xi\eta} : A \rightarrow A_{\xi\eta}$ and $\Phi_{\xi\eta}(a) := a(\eta, \xi)$.

For $\alpha < \kappa$ we have $\Phi_{\xi\eta}(q_{\alpha}) \neq 0$ if and only if $\alpha \in \{ \xi, \eta \}$, and if $\alpha \neq \beta$ then $\Phi_{\xi\eta}(q_{\alpha} q_{\beta}) \neq 0$ if and only if $\{ \alpha, \beta \} = \{ \xi, \eta \}$. Since $\bigcap_{\xi, \eta} \ker(\Phi_{\xi\eta}) = \{ 0 \}$, the product of any three distinct generators $q_{\alpha} q_{\beta} q_{\gamma}$ is equal to 0.

Fix $0 < \varepsilon < 1/8$. Since $\Phi_{\xi\eta}(q_{\varepsilon}) = \delta_{\xi\eta}$ and $D$ contains an approximate unit for $A$, there is $a_{\varepsilon} \in D_+$ such that $\| \Phi_{\xi\eta}(a_{\varepsilon}) - 1 \| < \varepsilon$. Then $b_{\varepsilon} := a_{\varepsilon} - q_{\varepsilon}$ satisfies $\| q_{\xi\eta} (b_{\varepsilon}) \| < \varepsilon$. Let $X(\xi) \in [\kappa]^{\aleph_3}$ be such that $b_{\xi} \in A_{X(\xi)}$. Then $b_{\xi}$ is the sum of a series with entries $zq_{\alpha}$ and $zq_{\alpha}q_{\beta}$ for $z \in \mathbb{C}$ and $\alpha, \beta \in X(\xi) \cup \{ \xi \}$. Suppose $\eta \notin X(\xi) \cup \{ \xi \}$. Then $\Phi_{\xi\eta}$ and $\Phi_{\eta\xi}$ (whichever is defined) sends all such entries except $zq_{\xi}$ to 0.
11.2 Consistency of a Counterexample to Naimark’s Problem

By Lemma 11.1.1 there are \( \xi < \eta < \kappa \) such that \( \xi \notin X(\eta) \) and \( \eta \notin X(\xi) \). Then \( \|\Phi_{\xi\eta}(b_\xi)\| < \varepsilon \) and \( \|\Phi_{\xi\eta}(b_\eta)\| < \varepsilon \). Therefore \( \Phi_{\xi\eta}(a_\xi) \approx_\varepsilon \Phi_{\xi\eta}(q_\xi) = p^0_{\xi\eta} \) and \( \Phi_{\xi\eta}(a_\eta) \approx_\varepsilon \Phi_{\xi\eta}(q_\eta) = p^1_{\xi\eta} \). This implies \( \|a_\xi, a_\eta\| \geq \|\{p^0_{\xi\eta}, p^1_{\xi\eta}\}\| - 4\varepsilon > 0 \); contradiction. \( \square \)

11.2 Consistency of a Counterexample to Naimark’s Problem

In this section we construct a counterexample to Naimark’s problem using \( \diamondsuit_{K_1} \). We also prove that \( \diamondsuit_{K_1} \) implies the failure of Glimm’s Dichotomy for simple \( C^* \)-algebras of density character \( \aleph_1 \).

In addition to \( \diamondsuit_{K_1} \) (§8.3, §8.4), the proofs in this section use the Kishimoto–Ozawa–Sakai Theorem (Theorem 5.6.1) and the analysis of pure states in crossed products from §5.4.

A counterexample to Naimark’s problem (Problem 5.5.2) is a \( C^* \)-algebra all of whose pure states are unitarily equivalent but it is not isomorphic to the algebra of compact operators on any Hilbert space. It will be constructed as an inductive limit of a continuous increasing family of separable \( C^* \)-algebras \( A_\alpha \), for \( \alpha < K_1 \).

The key to many \( \diamondsuit \)-construction is in devising the single step: the construction of \( A_{\alpha+1} \) from \( A_\alpha \). This one is no exception, and the construction is propelled by the following proposition.

**Proposition 11.2.1.** Assume \( A \) is a separable, simple, unital, non-type I \( C^* \)-algebra. Also assume \( X \sqcup Y \) is a countable set of inequivalent pure states on \( A \) and \( E \) is an equivalence relation on \( X \). Then there is an asymptotically inner automorphism \( \Phi \) of \( A \) such that the reduced crossed product \( B = A \rtimes_\Phi \mathbb{Z} \) has the following properties:

1. Every \( \varphi \in X \) has a unique extension \( \hat{\varphi} \) to \( B \).
2. For all \( \varphi \) and \( \psi \) in \( X \) we have \( \hat{\varphi} \sim \hat{\psi} \) if and only if \( \varphi \text{ E } \psi \).
3. Every \( \varphi \in Y \) has multiple state extensions to \( B \).

**Proof.** Since \( A \) is non-type I and separable, by Proposition 5.5.6 it has continuum many inequivalent pure states and therefore we may assume that both \( X \) and \( Y \) are infinite, that every \( E \)-equivalence class is infinite, and that there are infinitely many \( E \)-equivalence classes. Let \( \theta_{0,j} \), for \( j \in \mathbb{Z} \), be an enumeration of \( Y \) and let \( \theta_{i+1,j} \), for \( j \in \mathbb{Z} \), be an enumeration of the \( i \)-th \( E \)-equivalence class.

We re-enumerate \( \{\theta_{i,j}\} \) as \( \varphi_m \) and \( \psi_m \) for \( m \in \mathbb{Z} \), as follows. Every \( m \in \mathbb{N} \) is equal to \( 2^i(2j+1) \) for a unique \( i \in \mathbb{N} \) and \( j \in \mathbb{Z} \). For \( i \geq 1 \) let \( \varphi_{2^i(2j+1)} := \theta_{i,j} \) and \( \psi_{2^i(2j+1)} := \theta_{i-1,j} \). Also, if \( j \geq 2 \) let \( \varphi_{2j+1} := \theta_{0,j} \) and \( \psi_{2j+1} := \theta_{0,j} \).

Since no two pure states in \( X \sqcup Y \) are equivalent, Theorem 5.6.1 applies and there is an automorphism \( \Phi \) of \( A \) such that \( \varphi_j \sim \psi_j \circ \Phi \) for all \( j \).

This means that for \( i \geq 1 \) we have \( \theta_{j+k} \circ \Phi^i \sim \theta_{j+k} \) for all \( j \) and \( k \). Therefore the \( \Phi \)-orbit of a pure state \( \varphi \in X \) is exactly its \( E \)-equivalence class. Hence any two pure states \( \varphi \) and \( \psi \) in \( X \) satisfy \( \varphi \sim \psi \circ \Phi^n \) for some \( n \in \mathbb{Z} \) if and only if \( \varphi \text{ E } \psi \). Also, if \( \varphi \in X \) then \( \varphi \circ \Phi^n \in X \) for all \( n \in \mathbb{Z} \). Since all states in \( X \) are inequivalent, this implies
that for every \( \varphi \in X \) we have \( \varphi \sim \varphi \circ \Phi^n \) if and only if \( n = 0 \). Proposition 5.4.7 implies that every \( \varphi \in X \) has a unique extension \( \tilde{\varphi} \) to a pure state on \( B = A \otimes \varphi \mathbb{Z} \). Since \( \varphi_{2j+1} = \psi_{2j+1} \) for all \( j \), we have \( \varphi = \varphi \circ \Phi \) for all \( \varphi \in Y \). By Proposition 5.4.7, every \( \varphi \in Y \) has multiple state extensions to \( B \).

It remains to prove (2). Theorem 5.4.8 implies that \( \varphi \) and \( \psi \) in \( X \) have equivalent pure state extensions to \( B \) if and only if there exists \( n \in \mathbb{Z} \) such that \( \varphi \sim \psi \circ \Phi^n \), and (2) follows.

By Proposition 10.5.5 (2), the conclusion of Proposition 11.2.1 does not necessarily hold if the separability assumption on \( A \) is dropped.

**Theorem 11.2.2.** Suppose \( \diamond \mathcal{R}_1 \) holds. If \( 1 \leq n \leq \aleph_0 \) then there exists a simple \( C^* \)-algebra \( A \) of density character \( \varepsilon \) which is not isomorphic to the algebra of compact operators on any Hilbert space and which has exactly \( n \) unitarily inequivalent irreducible representations.

**Proof.** Let \( \mathcal{L} \) be the language of \( C^* \)-algebras expanded by relation symbols \( R_{k,j} \), for \( k < n + 1 \) and \( j < 2 \). The intended interpretations of \( R_{k,0} \) and \( R_{k,1} \) are the real and imaginary parts of a state \( \varphi_k \) of the underlying \( C^* \)-algebra \( A \). Let \( \mathcal{L}_0 \) denote the reduct of \( \mathcal{L} \) to the language obtained by removing \( R_{n,0} \) and \( R_{n,1} \).

Codes for \( \mathcal{L} \)-structures were described in §7.1.2. If \( A \) is a \( C^* \)-algebra with distinguished states \( \psi_j \), for \( j < n + 1 \), and it has an ordinal \( \alpha \) as a dense subset, then the corresponding codes for \( (A, \psi_j : j < n + 1) \) in \( \text{Struct}(\mathcal{L}, \alpha) \) and for \( (A, \psi_j : j < n) \) in \( \text{Struct}(\mathcal{L}_0, \alpha) \) will be denoted \( \mathfrak{A}(\psi_j : j < n + 1) \) and \( \mathfrak{A}(\psi_j : j < n) \), respectively. If \( \mathfrak{A} \) is a code in \( \text{Struct}(\mathcal{L}_0, \alpha) \) and \( \psi \) is a state on the corresponding \( C^* \)-algebra \( (A, \psi_j : j < n) \), then \( \mathfrak{A}(\psi) \) will denote the code in \( \text{Struct}(\mathcal{L}, \alpha) \) of the expansion \( (A, \psi_j : j < n + 1) \).

By Proposition 8.3.8 \( \diamond \mathcal{R}_1(\mathcal{L}) \) holds (Definition 8.3.7). We therefore have a sequence \( R_\alpha \in \text{Struct}(\mathcal{L}, \alpha) \), for \( \alpha < \aleph_1 \), such that the following holds.

For every \( C^* \)-algebra \( A \) of density character \( \aleph_1 \) with distinguished pure states \( \psi_j \), for \( j \leq n \), if \( \mathfrak{A} \in \text{Struct}(\mathcal{L}, \aleph_1) \) is a code for \( (A, \psi) \), then the set \( \{ \alpha : \mathfrak{A}[\alpha] \leq \aleph_1 \text{ and } \mathfrak{A}[\alpha] = R_\alpha \} \) is stationary.

We construct \( C^* \)-algebras \( A_\alpha \), for \( \alpha < \aleph_1 \), and codes \( \mathfrak{A}_\alpha \in \text{Struct}(\mathcal{L}_0, \alpha) \), for limit \( \alpha < \aleph_1 \), such that the following statements hold for all \( \alpha < \beta < \aleph_1 \).

1. The \( C^* \)-algebra \( A_\alpha \) has \( n \) distinguished pure states \( \varphi_\alpha^j \), for \( j < n \).
2. The pure states \( \varphi_\alpha^j \), for \( j < n \), are not unitarily equivalent.
3. If \( \alpha < \beta \) then \( \varphi_\beta^j \) is a unique extension of \( \varphi_\alpha^j \) for all \( j < n \).
4. If \( \alpha < \beta \) then \( A_\alpha \) is identified with a \( C^* \)-subalgebra of \( A_\beta \).
5. If \( \alpha \) is a limit ordinal, then \( \mathfrak{A}_\alpha \) codes \( A_\alpha \) and \( \varphi_\alpha^j \), for \( j < n \).
6. If \( \beta \) is a limit ordinal, then \( A_\beta = \lim_{\alpha < \beta} A_\alpha \).
7. If \( \alpha < \beta \) are limit ordinals, then \( \mathfrak{A}_\beta \upharpoonright \alpha = \mathfrak{A}_\alpha \).
8. If \( \alpha \) is a limit ordinal and \( \mathfrak{A}_\alpha(\varphi) = R_\alpha \) for some pure state \( \varphi \) of \( A_\alpha \) then \( \varphi \) has a unique extension to a state unitarily equivalent to \( \varphi_0^{\alpha + 1} \) on \( A_{\alpha + 1} \).
For a successor ordinal $\alpha < \mathbb{R}_1$ we will not specify a code for $A_\alpha$. Since the limit ordinals form a club, this will not affect our construction. Similarly, we will let $A_\alpha = A_\omega$ for all finite $\alpha$. Let $A_\omega$ be the CAR algebra, or any other simple, unital, separable, non-type I C$^*$-algebra. By Glimm’s Theorem we can choose inequivalent pure states $\varphi_j^{a_0}$, for $j < n$. Identify a countable dense subset of $A_\omega$ with $\omega$ and let $A_\omega$ be the corresponding code.

The recursive construction proceeds as follows. Suppose $\gamma < \mathbb{R}_1$ and $A_\alpha$, $\varphi_1^{a_\alpha}$, and $A_\beta$, for $j < n$, $\alpha < \gamma$, and limit $\beta < \gamma$, have been constructed and satisfy the requirements.

If $\gamma$ is a limit, let $A_\gamma := \lim_{\alpha < \gamma} A_\alpha$ and let $\varphi_j^{\gamma}$, for $j < n$, be the extension of states $\varphi_j^{a_\alpha}$, for $\alpha < \gamma$. This state is uniquely determined by Lemma 5.5.3.

If in addition $\gamma$ is a limit of limit ordinals, then the code $A_\gamma \in \text{Struct}(L_0, \alpha)$ has been defined for cofinally many limit ordinals $\alpha < \gamma$. Let $A_\gamma$ be the limit (this is literally the union) of these codes. It is straightforward to check that $A_\gamma$ codes $(A_\gamma, \varphi_j^{\gamma} : j < n)$.

Otherwise, let $\beta$ be the maximal limit ordinal below $\gamma$. This is the maximal ordinal such that the code $A_\beta$ has been defined. Since $A_\beta$ is separable and the interval $[\beta, \gamma)$ is infinite, we can extend the distinguished dense subset of $A_\beta$ given by $A_\beta$ to $\gamma$ and find a code $A_\gamma \in \text{Struct}(L_0, \gamma)$ as required.

Now consider the case when $\gamma = \alpha + 1$ is a successor ordinal. If $\alpha$ is a successor ordinal, apply Lemma 11.2.1 to obtain a nontrivial unital extension $A_\gamma$ of $A_\alpha$ such that all pure states $\varphi_j^{a_\alpha}$, for $j < n$, have unique pure state extensions.

Otherwise, if $\alpha$ is a limit, we consult $\hat{\omega}_{\mathbb{R}_1}$ and distinguish two cases. If there does not exist a pure state $\psi$ of $A_\alpha$ such that $R_\alpha = A_\alpha(\psi)$, again use Lemma 11.2.1 to obtain a nontrivial unital extension $A_\gamma$ of $A_\alpha$ such that all pure states $\varphi_j^{a_\alpha}$, for $j < n$, have unique pure state extensions.

The remaining case is when $R_\alpha = A_\alpha(\psi)$ for some pure state $\psi$ of $A_\alpha$. Apply Lemma 11.2.1 to obtain a unital extension $A_\gamma$ of $A_\alpha$ such that $\varphi_j^{a_\alpha}$ has a unique pure state extension $\varphi_j^{a_\alpha + 1}$ to $A_{\alpha + 1}$ for all $j < n$, $\varphi_j^{a_\alpha + 1}$ are inequivalent, and $\psi$ has a unique pure state extension to $A_{\alpha + 1}$ equivalent to $\varphi_0^{a_\alpha + 1}$.

This describes the recursive construction. Let $A := \lim_{\alpha < \mathbb{R}_1} A_\alpha$. This is a code for the C$^*$-algebra $A := \lim_{\alpha < \mathbb{R}_1} A_\alpha$ and the states $\varphi_j$, for $j < n$, on $A$. By the continuity of the construction at limit ordinals, $\{A_\beta : \beta < \mathbb{R}_1\}$ is a $\sigma$-complete family of separable C$^*$-subalgebras of $A$ and the states $\varphi_j$, for $j < n$, are pure by Lemma 5.5.3.

We claim that the states $\varphi_j$, for $j < n$, are not unitarily equivalent. Suppose otherwise. If $j < k < n$ and $u \in U(A)$ is such that $\varphi_j \circ \text{Ad} u = \varphi_k$, then since $A = \bigcup_{\alpha < \mathbb{R}_1} A_\alpha$ we have $u \in A_\alpha$ for some $\alpha$ and therefore $\varphi_j^{a_\alpha} \circ \text{Ad} u = \varphi_k^{a_\alpha}$, contradicting the assumptions. Therefore $A$ has at least $n$ inequivalent pure states.

Suppose that $A$ has more than $n$ inequivalent pure states and let $\psi$ be a pure state on $A$ inequivalent to $\varphi_j$, for $j < n$. Consider the code $\mathfrak{A}(\psi) \in \text{Struct}(L', \mathbb{R}_1)$. By Proposition 7.3.11 and the fact that $\{\alpha < \mathbb{R}_1 : \varphi \mid A_\alpha \text{ is a pure state on } A_\alpha\}$ includes a club. Since $R_\alpha$ is a $\hat{\omega}_{\mathbb{R}_1}(L')$ sequence there exists $\alpha < \mathbb{R}_1$ such that $\mathfrak{A}(\psi) \mid \alpha = R_\alpha$. By construction $\psi \mid A_\alpha$ has a unique extension to a pure state on $A_{\alpha + 1}$, and this unique extension is unitarily
equivalent to $\phi|^{\mathbb{R}^+}$. Since the latter state has a unique state extension to $A$, so does $\psi| A_\alpha$ and therefore $\psi$ is unitarily equivalent to $\phi|_\alpha$; contradiction. \hfill \Box

**Corollary 11.2.3.** Suppose that $\diamondsuit_{\mathcal{R}_\mathcal{I}}$ holds. Then Naimark’s problem has a negative solution and the conclusion of Glimm Dichotomy (Corollary 5.5.8) fails for nonseparable simple $C^*$-algebras. \hfill \Box

### 11.3 Reduced Free Group $C^*$-Algebras

In this section we prove that the reduced group $C^*$-algebra of an uncountable free product of nontrivial countable groups has only separable masas and its automorphism group acts transitively on its pure state space.

We use the terminology introduced in Definition 6.2.3 and the results from §7.2, §7.3, and §4.3.

**Proposition 11.3.1.** Suppose that $\Gamma = *_{j \in J} G_j$ is a free product of nontrivial countable groups. Then $\Sigma := \{C^*_r(\sigma G_j) : \sigma \in [J]^{\mathbb{R}_0}\}$ is a $\sigma$-complete family of separable $C^*$-algebras with $C^*_r(\Gamma)$ as its limit. For every $A$ in $\Sigma$ the following conditions hold.

1. Every automorphism of $A$ can be extended to an automorphism of $C^*_r(\Gamma)$.
2. Every pure state on $A$ has a unique extension to a state on $C^*_r(\Gamma)$.

**Proof.** The group $\Gamma$ is the direct limit of the family of countable subgroups of $\Gamma$, defined by $\Gamma_\sigma := *_{j \in J} G_j$, for $\sigma \in [J]^{\mathbb{R}_0}$. This family is clearly closed under increasing countable unions. As in Example 7.2.5, $\Sigma$ is a $\sigma$-complete family of separable $C^*$-subalgebras of $C^*_r(\Gamma)$ whose inductive limit is $C^*_r(\Gamma)$.

By Lemma 4.3.5, $\Gamma_\sigma$ is simple and has a unique tracial state for every infinite $\sigma$. For $\sigma \subseteq \tau$ we have $\Gamma_\sigma \cong \Gamma_\tau \wr J_{\sigma \setminus \tau}$, and Lemma 4.3.6 implies that every automorphism of $C^*_r(\Gamma_\sigma)$ extends to an automorphism of $C^*_r(\Gamma)$.

Since every nontrivial free product of groups has an element of infinite order, Lemma 4.3.8 implies that for $\sigma \in [J]^{\mathbb{R}_0}$ every pure state on $C^*_r(\Gamma_\sigma)$ has a unique extension to a pure state on $C^*_r(\Gamma)$. \hfill \Box

If $A$ is a $C^*$-algebra, $\varphi$ is a pure state on $A$, and $B$ is a $C^*$-subalgebra of $A$, we say that $\varphi$ *drops* to $B$ if the restriction of $\varphi$ to $B$ has a unique extension to a state on $A$. Such unique extension is necessarily equal to $\varphi$.

**Theorem 11.3.2.** Suppose that $\Gamma$ is the free product of an infinite, possibly uncountable, family $\{G_i : i \in J\}$ of nontrivial countable groups.

1. Every pure state on $C^*_r(\Gamma)$ drops to a separable subalgebra.
2. If $n \geq 1$ and $\{\varphi_i : i < n\}$ and $\{\psi_i : i < n\}$ are two $n$-tuples of inequivalent pure states on $C^*_r(\Gamma)$, then there is $\Phi \in \text{Aut}(C^*_r(\Gamma))$ such that $\varphi_i \circ \Phi = \psi_i$ for all $i$.
3. If $\{\varphi_i : i \in \mathbb{N}\}$ and $\{\psi_i : i \in \mathbb{N}\}$ are two families of inequivalent pure states on $C^*_r(\Gamma)$, then there is $\Phi \in \text{Aut}(C^*_r(\Gamma))$ such that $\varphi_i \circ \Phi$ is unitarily equivalent to $\psi_i$ for all $i$. 

4. Every abelian subalgebra in the group von Neumann algebra $L(\Gamma)$ with a separable predual.

5. Every abelian $C^*$-subalgebra of $C_r^*(\Gamma)$ is separable.

Proof. If $\mathbb{J}$ is countable then $C_r^*(\Gamma)$ is separable. In this case (5) and (1) are vacuous, (3) is a consequence of Theorem 5.6.1, and (2) is Exercise 5.7.27.

If $\mathbb{J}$ is uncountable, then Proposition 11.3.1 implies that $\Sigma := \{C_r^*(I_\mathbb{J}) : I \in [\mathbb{J}]^{\mathbb{R}_0}\}$ is a $\sigma$-complete family of separable $C^*$-subalgebras of $C_r^*(\Gamma)$ whose inductive limit is $C_r^*(\Gamma)$. It has the property that for every $A \in \Sigma$ every automorphism of $A$ extends to an automorphism of $C_r^*(\Gamma)$ and every pure state on $A$ has a unique extension to a pure state on $C_r^*(\Gamma)$.

(1) For a pure state $\varphi$ on $C_r^*(\Gamma)$ the set $\{A \in \Sigma : \varphi|_A \text{ is pure}\}$ includes a club in $\text{Sep}(C_r^*(\Gamma))$ by Proposition 7.3.10. By the first paragraph of this proof, $\varphi$ drops to every such $A$.

(2) Since the intersection of countably many clubs is a club, there is $A \in \Sigma$ such that $\varphi_i$ and $\psi_i$ drops to $A$ for all $i$. Denote the restriction of $\varphi_i$ to $\mathbb{J}$ by $\bar{\varphi}_i$ and the restriction of $\psi_i$ to $\mathbb{J}$ by $\bar{\psi}_i$. By Theorem 5.6.1, $A$ has an automorphism $\Phi$ such that $\bar{\varphi}_i \circ \Phi$ is unitarily equivalent to $\bar{\psi}_i$ for all $i$. As $A \in \Sigma$, by the universal property of the free product $\Phi$ has an extension to an automorphism $\Phi$ of $C_r^*(\Gamma)$ as required.

The proof of (3) is similar, using Exercise 5.7.27 in place of Theorem 5.6.1.

(4) Lemma 4.3.5 and Lemma 4.3.4 together imply that $C_r^*(\Gamma)$ has a unique tracial state, $\tau$. It has a unique extension to $L(\Gamma) = \pi_r[C_r^*(\Gamma)]$ by Lemma 4.2.5. It will suffice to prove that every masa has a separable predual. Suppose $D \subseteq L(\Gamma)$ is a a masa. Then Proposition 4.4.2 implies that $\tau$ is diffuse on $D$. Since $D$ is a von Neumann algebra, for every $n \in \mathbb{N}$ we can choose a partition of unity in $D$ into $n$ projections each of which has trace $1/n$. If $D_0$ is a masa of $D$ containing all of these projections, then it is countably generated and the restriction of $\tau$ to $D_0$ is diffuse. Fix a countable $\mathbb{K} \subseteq I$ such that $M := L(\ast_{\mathbb{K}} G_i)$ contains all of these projections. Since $D \cap M$ is diffuse, Theorem 4.4.6 (2) with $G := \ast_{\mathbb{K}} G_i$, $H := G$, and $B := D \cap M$ implies that the normalizer of $B$ is included in $M$. Since the commutant is included in the normalizer, $D = B$ and it therefore has a separable predual.

(5) It will suffice to prove that every masa is separable. Suppose $D \subseteq C_r^*(\Gamma)$ is a masa and extend its WOT-closure in $L(\Gamma)$ to a masa. By the previous paragraph, the WOT-closure of $D$ is WOT-separable, hence countably generated. Therefore $D$ is countably generated. \qed

11.4 Exercises

Exercise 11.4.1. Prove that if $A$ is a separable $C^*$-algebra then it has a masa that contains an approximate unit for $A$.

Prove that there exists a separable $C^*$-algebra $A$ and a masa $D$ of $A$ such that no approximate unit for $D$ is an approximate unit for $A$. 

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Exercise 11.4.2. Suppose that a group $\Gamma$ has the following weakening of the Powers approximation property:

For every $g \in \Gamma \setminus \{e\}$ and $\varepsilon > 0$ there exists $F \subseteq \Gamma$ such that

$$\left\| \frac{1}{|F|} \sum_{h \in F} \lambda(h g^{-1}) \right\| < \varepsilon.$$ 

Prove that $C^*_r(\Gamma)$ has a unique tracial state.

Exercise 11.4.3. Prove that there exist a Hilbert space $H$, a $C^*$-algebra $A \subseteq B(H)$, and a masa $D$ of $A$ such that the WOT-closure of $D$ is not a masa in the WOT-closure in $A$. Then prove that there are separable $H$, $A$, and $D$ with these properties.

Exercise 11.4.4. Suppose $A$ is a counterexample to Naimark’s problem.

1. Prove that $A \otimes \mathcal{K}(H)$ is a counterexample to Naimark’s problem for any Hilbert space $H$.
2. Prove that every nontrivial hereditary $C^*$-subalgebra of $A$ is a counterexample to Naimark’s problem.

Definition 11.4.5. A pure state $\varphi$ of a $C^*$-algebra $A$ is determined by a contraction $a \in A_{+1}$ if $\varphi$ is a unique state on $A$ that satisfies $\varphi(a) = 1$.

Exercise 11.4.6. Suppose $A$ is a counterexample to Naimark’s problem of density character $\aleph_1$ and $\varphi$ is a pure state on $A$.

1. Prove that there exists a separable $C^*$-subalgebra $B$ of $A$ such that $\varphi$ is a pure state on $B$ and it is the unique extension of $\varphi \upharpoonright B$ to a state on $A$.
2. Use Exercise 5.7.5 to conclude that $\varphi$ is determined by a positive contraction.
3. Generalize (1) to the case when the density character of $A$ is greater than $\aleph_1$.

Exercise 11.4.7. Suppose $A$ is a $C^*$-algebra every pure state on which is determined by a positive contraction. Then the following are equivalent.\(^1\)

1. All pure states on $A$ are equivalent.
2. There is a faithful irreducible representation $\pi : A \to B(H)$ such that for every normal $b \in A$ and every $\lambda \in \text{sp}(b)$ there exists a $\lambda$-eigenvector for $\pi(a)$ in $H$.

Exercise 11.4.8. Suppose that a simple $C^*$-algebra $A$ has only one irreducible representation up to the unitary equivalence, $\pi : A \to B(H)$. Prove that for every normal $a \in A$ we have $\text{sp}(a) = \{\lambda : \pi(a) \text{ has a } \lambda\text{-eigenvector}\}$.

In [160] Kishimoto proved that for every outer automorphism $\Phi$ of a separable $C^*$-algebra $A$ there exists a pure state $\varphi$ of $A$ such that $\varphi$ is not unitarily equivalent to $\varphi \circ \Phi$. Kishimoto’s argument can be modified to produce $c$ inequivalent pure states with this property (see [92]).

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\(^1\) One direction is Exercise 5.7.21.
Exercise 11.4.9. Use the variant of Kishimoto’s result stated in the previous paragraph and ♦\(\mathcal{G}_1\) to prove that there exists a counterexample to Naimark’s problem without outer automorphisms.

Exercise 11.4.10. Use ♦\(- (H_{\mathcal{G}_1})\) (without using the conclusion of Exercise 8.7.20) to prove Theorem 11.2.2 directly.

The following two exercises should be compared to Exercise 5.7.24. They require familiarity with the method of forcing (see e.g., [166]).

Exercise 11.4.11. Use ♦\(\mathcal{G}_1\) (or ♦\(- (H_{\mathcal{G}_1})\)) to construct a counterexample to Naimark’s problem and a Suslin tree \(T\) such that \(T\) forces that \(A\) remains a counterexample to Naimark’s problem.

Exercise 11.4.12. Suppose \(A\) is a counterexample to Naimark’s problem of density character \(\aleph_1\). Prove that \(A\) remains a counterexample in every forcing extension by a \(\sigma\)-closed poset.

Notes for Chapter 11

\[\S\]11.1 Lemma 11.1.1 may be the simplest instance of the free set mapping theorem; see [74] for more.

Theorem 11.1.2 is based on [3, Example 2.1]. I am indebted to Xin Li for alerting me to this result. Akemann’s algebra has density character \(2^\mathfrak{c}\). The algebra constructed in Theorem 11.1.2 has density character \(\mathfrak{K}_2\), a cardinal not greater, and possibly much smaller, than \(2^\mathfrak{c}\) (see e.g., [166, Corollary IV.7.18]). This can be further improved to \(\mathfrak{K}_1\) (the least possible density character of a C\(^*\)-algebra with no abelian approximate unit); see Exercise 14.6.11.

I originally planned to include a construction of a prime C\(^*\)-algebra that is not primitive (see [256] and [151]). In the meantime, this result appeared as Proposition 6.9.6 in the revised version of [194], making this plan redundant.

\[\S\]11.2 A counterexample to Naimark’s problem was constructed from ♦\(\mathcal{G}_1\) in [5]. Theorem 11.2.2 is a special case of the main result of [92], where an AM counterexample has been constructed.

The minimal density character to a counterexample to Naimark’s problem (if there are any) is equal to \(\mathfrak{c}\) (Corollary 5.5.6, see also Exercise 5.7.24). Since it is relatively consistent with ZFC that the Continuum Hypothesis fails, the existence of a counterexample to Naimark’s problem of density character \(\mathfrak{K}_1\) is independent from ZFC. It is a major open problem whether a counterexample to Naimark’s problem, or a counterexample to Glimm’s Dichotomy, can be constructed in ZFC. All known counterexamples to these problems were constructed using ♦\(\mathcal{G}_1\), but there is quite a variety of them. One counterexample is AM and it is not isomorphic to its opposite algebra \(A^{\text{op}}\) ([92]) and there are counterexamples with nontrivial tracial simplices ([244]).
§11.3 Theorem 11.3.2 (5) and (1)–(3) are taken from [204] and [4], respectively. Another remarkable property of $C^*_r(F_\kappa)$ (for any $\kappa$) is that it is projectionless ([201]). The proof of this fact uses K-theory and I am not aware of the existence of an elementary proof.
Part III
Massive Quotient $C^*$-algebras
In Part III we study massive quotient C*-algebras. Unlike the counterexamples constructed in chapters 10 and 11, these are canonically defined C*-algebras of central importance. Their properties are highly sensitive to the set theoretic axioms. These quotient C*-algebras, such as coronas, asymptotic sequence algebras, and ultraproducts, have played a role in the theory of C*-algebras since its very conception. A prime example is the Calkin algebra $\mathcal{Q}(H)$, the quotient of the algebra of bounded linear operators on $H$ modulo the ideal of compact operators. The first (implicit) appearance of quotient operator algebras was perhaps the Weyl–von Neumann theorem. It asserts that two self-adjoint operators on a separable and infinite-dimensional complex Hilbert space $H$ have unitarily conjugate compact perturbations if and only if their essential spectra coincide. In other words, the spectral measure is trivialized by modding out the compact operators. Reformulating again, the unitary equivalence of self-adjoint elements of $\mathcal{Q}(H)$ coincides with the equality of spectra, and is therefore smooth (if construed as a relation on a Polish space such as the self-adjoint part of the unit ball of $B(H)$). In the 1970s Brown, Douglas and Fillmore introduced methods of homological algebra and algebraic topology to operator algebras and extended the Weyl–von Neumann theorem to the essentially normal operators (see the introduction to §12).

2 The unitary equivalence of self-adjoint elements of $\mathcal{B}(H)$ is much more complicated: It is not smooth, and not even classifiable by countable structures (see [153]).
Chapter 12
The Calkin Algebra

Apart from its intrinsic interest, the analysis of the ring $M$ [the Calkin algebra] yields in a very simple way theorems of considerable depth concerning $B [\mathcal{B}(H)]$. Of results of this sort, we will here mention only a generalization of Weyl’s classical theorem comparing the spectra of two self-adjoint operators with totally continuous [compact] difference $[259]$.

J.W. Calkin, [38]

In 1909, when considering the stability of differential operators under boundary conditions, Weyl proved that two self-adjoint operators on $H$ with compact difference have the same spectrum, except for the isolated points of finite multiplicity $([259])$. In 1935 von Neumann proved a converse: if two self-adjoint operators on $H$ have the same spectrum, except for the isolated points of finite multiplicity, then they are unitarily equivalent up to a compact difference. The quotient of $\mathcal{B}(H)$ over $\mathcal{K}(H)$—‘ring $M$,’ nowadays known as the Calkin algebra and denoted $Q(H)$, was explicitly introduced in 1941 by Calkin ([38]).

All of these developments predate the introduction of C∗-algebras by Gelfand–Naimark and Segal, making the Calkin algebra the earliest example of a non-type I C∗-algebra that is not a von Neumann algebra.

Last, but not least, the Calkin algebra is a quotient object of monstrous proportions, some of whose most basic properties are independent from ZFC.

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1 This was the year when Hausdorff’s gap (Theorem 9.3.7) celebrated its first birthday.

2 The Calkin algebra was the earliest example of an abstract C∗-algebra, and therefore arguably the first ‘genuine’ example of a C∗-algebra. Earlier examples of C∗-algebras were the von Neumann algebras and the algebra of compact operators. In [38] Calkin constructed the relevant instance of the GNS representation and proved that it is isometric. He also proved the relevant instance of Lemma 3.1.13, hinted at taking quotient of a II∞ factor by its Breuer ideal (Example 4.1.5 (2)), and gave an early construction of invariant means.
12.1 Basic Properties of the Calkin algebra

In this section we prove that every separable C*-algebra is isomorphic to a C*-subalgebra of \( \mathcal{D}(H) \), that \( \mathcal{D}(H) \) does not have a representation on any Hilbert space of density character less than \( c \), and that any countable decreasing family of nonzero projections in \( \mathcal{D}(H) \) has a lower bound.

The Calkin algebra associated to an infinite-dimensional Hilbert space \( H \) was defined in §2.5 as the quotient (\( K(H) \) is the ideal of compact operators of \( B(H) \))

\[
\mathcal{Q}(H) := B(H)/K(H).
\]

The quotient map is denoted \( \pi: B(H) \to \mathcal{Q}(H) \).

Although the case when \( H \) is separable is most interesting, the proofs of many of the properties of \( \mathcal{Q}(H) \) do not require this assumption. The following example illustrates why is the Calkin algebra the ‘noncommutative’ analog of \( \ell_\infty/c_0 \), and therefore (via the Gelfand–Naimark and Stone dualities, see Theorem 1.3.2 and §1.3.1) the ‘noncommutative’ analog of \( \mathcal{P}(\mathbb{N})/\text{Fin} \).

**Example 12.1.1.** In a separable, infinite-dimensional Hilbert space \( H \) fix an orthonormal basis \( \xi_n \), for \( n \in \mathbb{N} \). All operators in \( B(H) \) that have each \( \xi_n \) as an eigenvector comprise von Neumann algebra \( A \). This is an atomic masa of \( B(H) \) (see Example 3.1.20). The lattice of its projections consists of all projections diagonalized by the basis \( (\xi_n) \) and is therefore isomorphic to the Boolean algebra \( \mathcal{P}(\mathbb{N}) \).

Thus \( A \) is isomorphic to \( \ell_\infty \) via the map that sends \( a \) to the sequence of its eigenvalues associated to the basis vectors. This isomorphism sends \( A \cap \mathcal{K}(H) \to c_0 \), and hence \( A/(A \cap \mathcal{K}(H)) \cong \ell_\infty/c_0 \). The poset of projections of \( A/(A \cap \mathcal{K}(H)) \) is isomorphic to \( \mathcal{P}(\mathbb{N})/\text{Fin} \).

**Lemma 12.1.2.** Suppose \( H \) is an infinite-dimensional Hilbert space. Then the Calkin algebra \( \mathcal{D}(H) \) has real rank zero. If \( H \) is separable, then any two nonzero projections in \( \mathcal{D}(H) \) are Murray–von Neumann equivalent and \( \mathcal{D}(H) \) is simple.

**Proof.** The Calkin algebra has real rank zero since it is a quotient of the von Neumann algebra \( B(H) \) (Corollary 2.7.2). Therefore every hereditary C*-subalgebra of \( \mathcal{D}(H) \) is generated by projections (Theorem 2.7.1). If \( p \) is a nonzero projection in \( \mathcal{D}(H) \) then it can be lifted to a projection \( p_1 \) in \( B(H) \) (Lemma 3.1.13). The range of \( p_1 \) is infinite-dimensional, and if \( H \) is separable then there exists an isometry \( v \in B(H) \) such that \( vv^* = p \). Then \( \pi(v) \) is an isometry in \( \mathcal{D}(H) \) such that \( \pi(vv^*) = p \). Since \( p \) was an arbitrary nonzero projection in \( \mathcal{D}(H) \), all nonzero projections in \( \mathcal{D}(H) \) are Murray–von Neumann equivalent to \( 1_{\mathcal{D}(H)} \).

An extension of the following lemma is in Exercise 12.6.1.

**Lemma 12.1.3.** Every separable C*-algebra is isomorphic to a C*-subalgebra of the Calkin algebra.
Proof. Let $A$ be a separable C*-algebra. By Corollary 1.10.4 there exists a faithful representation $\sigma : A \to \mathcal{B}(H)$, for a separable Hilbert space $H$. Let $\sigma'$ denote the the amplification $\bigoplus_{\mathbb{N}} \sigma : A \to \mathcal{B}(\bigoplus_{\mathbb{N}} H)$ (note that $\bigoplus_{\mathbb{N}} H \cong H \otimes \ell_2(\mathbb{N}) \cong \ell_2(\mathbb{N})$). For every $a \in A \setminus \{0\}$ the image $\sigma'(a)$ is not compact, and therefore $\sigma'$ is a faithful representation of $A$ on a separable Hilbert space such that $\sigma'(a)$ is compact only if $a = 0$. Therefore the composition of $\sigma'$ with the quotient map is the required embedding of $A$ into the Calkin algebra. \qed

Proposition 12.1.4. The Calkin algebra $\mathcal{Q}(H)$ has density character $\kappa$. It has a representation on a Hilbert space $K$ if and only if the density character of $K$ is at least $\kappa$.

Proof. Since $\mathcal{Q}(H)$ is the quotient of $\mathcal{B}(H)$, its density character is at most $\kappa$. Corollary 1.10.4 now implies $\mathcal{Q}(H)$ can be represented on $\ell_2(\kappa)$ by taking amplifications, one obtains a representation of $\mathcal{Q}(H)$ on $\mathcal{B}(\ell_2(\kappa))$ for every $\kappa \geq \epsilon$ (see Exercise 12.6.1). Example 12.1.1 can be used to prove that the density character of $\mathcal{Q}(H)$ is at least $\kappa$, and therefore equal to $\kappa$, but by Corollary 1.10.4 it will suffice to prove that there is no representation of $\mathcal{Q}(H)$ on $\ell_2(\kappa)$ for $\kappa < \epsilon$. Let $\mathcal{A}$ be an almost disjoint family in $\mathcal{P}(\mathbb{N})$ (Proposition 9.2.2). Suppose $e_j$, for $j \in \mathbb{N}$, is an orthonormal basis for $H$. As before, for $X \subseteq \mathbb{N}$ let $p_X$ be the projection to the closed linear span of $\{e_j : j \in X\}$. Then the projections $\{p(p_X : X \in \mathcal{A})$ are orthogonal. Since $\mathcal{Q}(H)$ is simple, every representation is faithful, and any family of orthogonal projections in $\mathcal{Q}(\ell_2(\kappa))$ has cardinality at most $\kappa$. \qed

12.2 Projections in the Calkin Algebra

In this section we study the poset of projections of the Calkin algebra and the associated cardinal characteristics $b^*$, $\mathfrak{d}^*$, and $\text{p}^*$. This section relies on §9.5 and §9.6.

As in §1.5, the set of projections in a C*-algebra is considered as a partially ordered set with the ordering inherited from the set of self-adjoint elements. For projections $p$ and $q$ we have $p \leq q$ if and only if $pq = p$ (Lemma 1.5.1).

Lemma 12.2.1. For projections $p$ and $q$ in $\mathcal{B}(H)$, $\pi(p) \leq \pi(q)$ if and only if for every $\epsilon > 0$ some finite-rank projection $p_0 \leq p$ satisfies $\|\pi(p) - \pi(p_0)(1 - q)\| < \epsilon$.

Proof. Clearly $\pi(p) \leq \pi(q)$ is equivalent to $p(1 - q)$ being compact. Since finite-rank projections $p_0 \leq p$ form an approximate unit for the corner $p\mathcal{K}(H)$, the statement follows. \qed

We write $p \leq^X q$ if either of the equivalent conditions in Lemma 12.2.1 is satisfied. Since Lemma 3.1.13 implies that $\pi(\text{Proj}(\mathcal{B}(H))) = \text{Proj}(\mathcal{Q}(H))$, the poset $(\text{Proj}(\mathcal{Q}(H)), \leq)$ is isomorphic to the quotient of $\text{Proj}(\mathcal{B}(H))$ by the relation obtained by symmetrizing $\leq^X$. If $H$ is separable then $\text{Proj}(\mathcal{Q}(H))$ is a Polish space in the strong operator topology (Example B.2) and Lemma 12.2.1 implies that the graph of $\leq^X$ is a Borel subset of $\text{Proj}(\mathcal{Q}(H)) \times \text{Proj}(\mathcal{Q}(H))$. 

The annihilator of a C*-subalgebra $A$ of a C*-algebra $C$ is

$$A^\perp := \{ c \in C : ac = 0 \text{ for all } a \in A \}.$$ 

It is a hereditary C*-subalgebra of $C$ (Exercise 12.6.7). Cofinal equivalence of directed sets was defined in Definition 9.6.8.

**Proposition 12.2.2.** If $A$ is a nonunital, separable C*-subalgebra of the Calkin algebra then the posets $((A^\perp)_{+,1}, \leq)$ and $(\mathbb{N}^\mathbb{N}, \leq^*)$ are cofinally equivalent.

**Proof.** The proof is closely related to the proof of Proposition 9.6.11. The lift of $A$, $\{ a \in \mathcal{B}(H) : \pi(a) \in A \}$ is $\sigma$-unital. Let $A_0$ be the hereditary subalgebra of $\mathcal{B}(H)$ generated by this lift. It is $\sigma$-unital and it has real rank zero, therefore by Theorem 2.7.1 $A_0$ has a sequential approximate unit consisting of projections, $p_m$, for $m \in \mathbb{N}$. Since $A$ is not unital, $p_{m+1} - p_m$ is not compact for infinitely many $m$ and by refining the sequence we may assume that it is not compact for all $m$. Let $p_{m,n}$ for $m \in \mathbb{N}$ and $n \in \mathbb{N}$, be orthogonal rank-1 projections such that $\bigvee_n p_{m,n} = p_m$ for all $m$. Define $T : \mathbb{N}^\mathbb{N} \to \mathcal{B}(H)$ by

$$T(f) := \bigvee \{ p_{m,n} : n \leq f(m) \}.$$ 

Clearly, if $f \leq g$ then $T(f) \leq T(g)$. Since $p_m T(f)$ has finite rank for all $m$ and $f$, and $p_m$ is an approximate unit for $A_0$, the range of $T$ is included in the lift of $A^\perp$. This lift is equal to

$$B := \{ b \in \mathcal{B}(H) : ab \in \mathcal{K}(H) \text{ for all } a \in A_0 \}$$

and we need to prove that the poset $(B_{+,1}, \leq)$ is Tukey equivalent to $(\mathbb{N}^\mathbb{N}, \leq^*)$. Fix $b \in B$. Then $p_m b$ is compact for all $m$, and there exists $f \in \mathbb{N}^\mathbb{N}$ such that

$$\|(1 - \bigvee_{n \leq f(m)} p_{m,n})b\| < 2^{-m-1} \varepsilon$$

for all $m$. Then $\|(1 - T(f))b\| < \varepsilon$ and $(1 - T(f))b$ is compact. Therefore the range of $T$ is an approximate unit of $B$ which is cofinal in $B_{+,1}$. Since $T$ is monotonic, this proves that $(\mathbb{N}^\mathbb{N}, \leq^*)$ is order-isomorphic to a cofinal subset of $B_{+,1}$ and therefore the two posets are cofinally equivalent.

Two cofinally equivalent directed sets have the same bounding number and dominating numbers (Definition 9.5.3) by Theorem 9.6.9, and we have the following.

**Corollary 12.2.3.** Suppose $A$ is a nonunital, separable C*-subalgebra of the Calkin algebra. The cardinals $b^* := b^{(A^\perp)_{+,1}, \leq}$ and $\vartheta^* := \vartheta^{(A^\perp)_{+,1}, \leq}$ do not depend on the choice of $A$. In addition, $b = b^*$ and $\vartheta = \vartheta^*$.

A quantum filter (Definition 5.3.3) $\mathcal{F}$ on $\mathcal{B}(H)$ is diagonalized by a projection $p$ if $p$ is not compact and $p \leq^\mathcal{K} q$ for all $q \in \mathcal{F}$. A quantum filter $\mathcal{F}$ is nonprincipal if $\mathcal{F} \cap \mathcal{K} = \emptyset$. 

Definition 12.2.4. Writing $\text{Proj}(\mathcal{B}(H))$ for the poset of projections in $\mathcal{B}(H)$, let
\[ p^* := \min\{|F| : F \subseteq \text{Proj}(\mathcal{B}(H)) \text{ generates a quantum filter which is not diagonalized} \}. \]

This is the ‘quantized’ analog of the standard cardinal invariant
\[ p := \min\{|F| : F \subseteq \mathcal{P}(\mathbb{N}) \text{ generates a filter which is not diagonalized} \}. \]

The following lemma will be used in §12.5.

Lemma 12.2.5. Assume $p_n$, for $n \in \mathbb{N}$, is a decreasing sequence of projections in $\mathcal{L}(H)$. Then there is a nonzero projection $p$ in $\mathcal{L}(H)$ such that $p \leq p_n$ for all $n$. Therefore $\aleph_1 \leq p^* \leq c$, and the Continuum Hypothesis implies $p^* = \aleph_1 = c$.

Proof. Recursively find $q_n$ in $\text{Proj}(\mathcal{B}(H))$ that lift $p_n$ for $n \in \mathbb{N}$. Since $p_{n+1} \leq p_n$ we can choose $q_{n+1} \in \mathcal{B}(q_nH)$ so that $q_n \leq q_{n+1}$ for all $n$. Recursively choose vectors $\xi_n$ in the range of $q_n$ so that $\xi_m$ is orthogonal to $\xi_n$, for $n < m$. Let $p$ be the projection to the closed linear span of $\xi_n$, for $n \in \mathbb{N}$. Then $\pi(p) \leq \pi(q_n) = p_n$ for all $n$. \qed

12.3 Maximal Abelian $C^*$-subalgebras of $\mathcal{B}(H)$ and $\mathcal{L}(H)$

In this section we classify maximal abelian $C^*$-subalgebras of $\mathcal{B}(H)$. We also prove the Johnson–Parrott theorem: the image of a masa in $\mathcal{B}(H)$ under the quotient map is a masa in $\mathcal{L}(H)$. The existence of an abelian $C^*$-subalgebra of $\mathcal{L}(H)$ with no abelian lift is proved by a counting argument. This result will be improved in §12.4. The section concludes with a discussion of the role of $\mathcal{B}(\ell_2(\kappa))$ and $\mathcal{L}(\ell_2(\kappa))$ as noncommutative analogs of $\mathcal{P}(\kappa)$ and $\mathcal{P}(\kappa)/\text{Fin}_\kappa$.

Proposition 12.3.1. Suppose $H$ is a separable Hilbert space. Then there are $\mathfrak{c}$ masas in $\mathcal{B}(H)$ and $\mathfrak{R}_0$ unitary equivalence classes of masas in $\mathcal{B}(H)$.

Proof. By the Spectral Theorem (Theorem C.6.11), for every masa $A$ in $\mathcal{B}(H)$ there exists an isomorphism between $H$ and $L_2(X, \mu)$ for a probability measure space $(X, \mu)$ that sends $A$ to $L_\infty(X, \mu)$. Therefore every masa $D$ in $\mathcal{B}(H)$ is isomorphic (as a von Neumann algebra) with $L_\infty(X, \mu)$ for some separable measure space $(X, \mu)$. Let $m_D$ be the number of atoms in this space. Proposition C.6.11 implies that $m_D$ is a complete invariant for the unitary equivalence of masas. Since $m_D \in \mathbb{N} \cup \{\mathfrak{R}_0\}$, there are $\mathfrak{R}_0$ unitary equivalence classes of masas.

Since the unit ball of $\mathcal{B}(H)$ is a Polish space, it has $\mathfrak{c}$ closed subsets. The intersection of a masa is such a closed set and it uniquely determines a masa, and therefore there are at most $\mathfrak{c}$ masas in $\mathcal{B}(H)$. \qed

Theorem 12.3.2 (Johnson–Parrott). If $A$ is a masa in $\mathcal{B}(H)$ then $\pi[A]$ is a masa in the Calkin algebra.
Proof. Let \( b \in \mathcal{B}(H) \) be such that \( ba - ab \in \mathcal{K}(H) \) for all \( b \in A \). We will find \( b_1 \in A \) such that \( b - b_1 \in \mathcal{K}(H) \). By Theorem 3.1.14, with \( \text{tr} \) denoting the trace on the algebra \( \mathcal{B}_1(H) \) of trace class operators, the bilinear map

\[
(c|d) := \text{tr}(cd)
\]

implements the duality between \( \mathcal{B}(H) \) and the ideal \( \mathcal{B}_1(H) \) of trace class operators.

For a trace class operator \( c \) define \( f_c : U(A) \to \mathbb{C} \) by \( f_c(u) := \langle ubu^*, c \rangle \). Clearly \( f \) is norm-continuous and its norm is bounded by \( ||b|| ||c|| \). Therefore \( f_c \in \ell_\infty(U(A)) \).

With \( \mu \) denoting an invariant mean on \( U(A) \), let

\[
b_1 := \int_{U(A)} ubu^* \, d\mu(u).
\]

In other words, with the conditional expectation \( \Theta : \mathcal{B}(H) \to A' \) defined in Proposition 3.3.10 (see also Example 3.3.2), \( b_1 := \Theta(b) \). Since \( A \) is a masa, it satisfies \( A' = A \) and \( b_1 \in A \). Therefore every \( u \in U(A) \) satisfies

\[
b_1 - ub_1 u^* = (b_1 u - ub_1) u^* \in \mathcal{K}(H).
\]

We aim to prove that \( b - b_1 \in \mathcal{K}(H) \).

Claim. If \( b - b_1 \notin \mathcal{K}(H) \) then there are nonzero orthogonal projections \( p_n \in A \), for \( n \in \mathbb{N} \), such that \( \inf_n ||p_n(b - b_1)p_n|| = \varepsilon > 0 \).

Proof. For a projection \( p \in A \) let \( \varepsilon_p := ||\pi(p(b - b_1)p)|| \). Then \( \varepsilon_1 = ||\pi(b - b_1)|| > 0 \). As \( pq = 0 \) implies \( \varepsilon_{p+q} = \max(\varepsilon_p, \varepsilon_q) \),

\[
\mathcal{J} := \{ p \in \text{Proj}(A) : \varepsilon_p < \varepsilon_1 \}
\]

is an ideal of the Boolean algebra \( \text{Proj}(A) \). Let \( \mathcal{V} \) be an ultrafilter of \( \text{Proj}(A) \) disjoint from \( \mathcal{J} \). Then \( \varepsilon_p = \varepsilon_1 \) for all \( p \in \mathcal{V} \). Since \( A \) is a von Neumann algebra and \( \mathcal{V} \) is directed, Lemma 3.1.3 implies that \( q := \inf \{ p : p \in \mathcal{V} \} \) is a projection in \( A \).

Suppose for a moment that \( \varepsilon_q = \varepsilon_1 \). This implies \( q \notin \mathcal{J} \). If \( r \in qAq \) is a nonzero projection then \( \varepsilon_q = \max(\varepsilon_r, \varepsilon_{q-r}) \) and the minimality of \( q \) implies \( q = r \). Since \( A \) is a masa, \( qAq = Cq \). Therefore for every \( u \in U(A) \) we have \( uq = \lambda q \) for some \( \lambda \in \mathbb{T} \), and \( q(b - ubu^*)q = 0 \). This contradicts \( ||q(b - b_1)q|| = \varepsilon_1 > 0 \).

We may therefore assume \( \varepsilon_q < \varepsilon_1 \), hence \( q \notin \mathcal{V} \). Recursively choose a decreasing sequence of projections \( q_n \in \mathcal{V} \) such that \( \varepsilon_{q_n-q_{n+1}} > \varepsilon_1/2 \) for all \( n \). Then the projections \( p_n := q_n - q_{n+1} \) are as required.

Fix orthogonal projections \( p_n \), for \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) as guaranteed by the Claim. Since \( b_1 = \int_{U(A)} ubu^* \, d\mu(u) \), for every \( n \) there is a unitary \( u_n \) in \( p_nA \) such that

\[
||p_n(b - u_nbu_n^*)p_n|| = ||p_n(bu_n - u_n b)p_n|| > \varepsilon/2.
\]

Since \( A \) is a von Neumann algebra, it contains \( a := \sum_n p_n u_n \). However, we have

\[
||p_n(ba - ab)p_n|| > \varepsilon/2.
\]
for all \( n \); this is a contradiction since \( ba - ab \in \mathcal{K}(H) \). Therefore \( b - b_1 \in \mathcal{K}(H) \).

Since \( b_1 \in A \) and \( b \) was an arbitrary element of \( \mathcal{B}(H) \) that commutes with \( A \) modulo \( \mathcal{K}(H) \), \( \pi[A] \) is a masa. \( \square \)

See also Exercise 12.6.15. Theorem 12.3.2 justifies the terminology of the following definition.

**Definition 12.3.3.** The image of the atomic masa in \( \mathcal{B}(H) \) (Example 3.1.20) under the quotient map is known as the atomic masa in \( \mathcal{Q}(H) \). The image of the atomless masa in \( \mathcal{B}(H) \) (Example 3.1.21) under the quotient map is known as the atomless masa in \( \mathcal{Q}(H) \).

Since \( \mathcal{K}(H) \cap \ell_\infty \cong c_0 \), the atomic masa in \( \mathcal{Q}(H) \) is isomorphic to \( \ell_\infty / c_0 \).

If \( (X, \mu) \) is an atomless measure space, then basic spectral analysis shows that \( L_\infty(X, \mu) \) does not contain any nonzero compact operators. Therefore the restriction of the quotient map to the atomless masa is the identity map. This implies that the poset \( \text{Proj}(\mathcal{D}(H)), \leq \) has a maximal commuting subset isomorphic to the Lebesgue measure algebra that lifts to a commuting set of projections in \( \mathcal{B}(H) \).

### 12.3.1 \( \mathcal{B}(\ell_2(\kappa)) \) and \( \mathcal{D}(\ell_2(\kappa)) \) as Noncommutative Analogs of the Boolean Algebras \( \mathcal{P}(\kappa) \) and \( \mathcal{P}(\kappa) / \text{Fin}_\kappa \)

Throughout this subsection we suppose \( \kappa \) is an infinite cardinal. As in Example 6.2.4, for an infinite cardinal \( \lambda \leq \kappa \) let

\[
\mathcal{P}_\lambda(\kappa) := \{ X \subseteq \kappa : |X| < \lambda \}
\]

(pronounced P-kappa-lambda). These are the simplest (but by no means the only) ideals in the Boolean algebra \( \mathcal{P}(\kappa) \).

Let \( H := \ell_2(\kappa) \). For an infinite cardinal \( \lambda \leq \kappa \) let

\[
\mathcal{K}_\lambda(\ell_2(\kappa)) := \{ a \in \mathcal{B}(H) : \text{the density character of } a[H] \text{ is smaller than } \lambda \}.
\]

Hence \( \mathcal{K}_0(\ell_2(\kappa)) \) is the ideal of finite-rank operators, \( \mathcal{K}_{\aleph_1}(\ell_2(\kappa)) \) is the ideal of operators with separable range, and \( \mathcal{K}_{\lambda^+}(\ell_2(\kappa)) \) is the ideal of operators whose range has density character not greater than \( \lambda \). The case when \( \kappa = \lambda = \aleph_0 \) of the following lemma is Proposition C.6.2.

**Proposition 12.3.4.** If \( H = \ell_2(\kappa) \), then we have the following.

1. \( \mathcal{K}_\lambda(\ell_2(\kappa)) \) is a two-sided, self-adjoint, ideal of \( \mathcal{B}(H) \) for any \( \lambda \leq \kappa \)
2. If \( \lambda \) is uncountable then \( \mathcal{K}_\lambda(\ell_2(\kappa)) \) norm-closed.
3. Every norm-closed ideal in \( \mathcal{B}(H) \) is equal to \( \mathcal{K}_\lambda(\ell_2(\kappa)) := \mathcal{K}^0_\lambda(\ell_2(\kappa)) \) for some cardinal \( \lambda \leq \kappa \).
Proof. The proofs of (1) and (2) are straightforward.

(3) Suppose \( J \) is an ideal of \( H \). For a projection \( p \in J \) let \( \theta(p) \) denote the density character of \( p[H] \), and let \( \lambda \) be the least cardinal strictly greater than \( \theta(p) \) for all \( p \in J \). We claim that \( J = \mathcal{K}_\lambda(H) \).

Since \( \mathcal{B}(H) \) has real rank zero, \( J \) has an approximate unit, \( e' \), consisting of projections ((1) of Theorem 2.7.1). If \( p \in e' \) then for every \( a \in \mathcal{B}(\ell_2(\kappa)) \) the density character of \( ap[H] \) is at most \( \theta(p) \). Since \( \theta(p) < \lambda \), we have \( J \subseteq \mathcal{K}_\lambda(H) \).

For the converse inclusion it suffices to prove that every self-adjoint \( a \in \mathcal{K}_\lambda(H) \) belongs to \( J \). Since \( \mathcal{K}_\lambda(H) \) has real rank zero (Corollary 2.7.2), every self-adjoint element can be approximated by a linear combination of projections and it suffices to prove that every projection \( p \in \mathcal{K}_\lambda(H) \) belongs to \( J \). Fix a projection \( p \in \mathcal{K}_\lambda(H) \). Then \( \theta(p) < \lambda \) and some \( q \in J \) satisfies \( \theta(p) \leq \theta(q) \). This implies that there exists a partial isometry \( v \in \mathcal{B}(H) \) such that \( v^*v = p \) and \( vv^* \leq q \). Therefore \( qv \in J \) and \( p = v^*qv \in J \). Since \( p \) was arbitrary, \( \mathcal{K}_\lambda(H) \subseteq J \). \( \square \)

Let \( e := (\xi_j : j \in \kappa) \) be an orthonormal basis of \( H = \ell_2(\kappa) \). By \( \text{proj}_K \) we denote the projection to a closed subspace \( K \) of \( H \). For every \( X \subseteq \kappa \) define the projection

\[
p_X := \text{proj}_\mathcal{K}\{\xi_j : j \in X\}.
\]

We omit the superscript and write \( p_X \) whenever the basis \( e \) is clear from the context. A proof of the following lemma is left as an exercise.

**Proposition 12.3.5.** If \( H = \ell_2(\kappa) \), then we have the following.

1. The correspondence \( X \mapsto p_X \) is an isomorphism from the Boolean algebra \( \mathcal{P}(\kappa) \) onto a maximal Boolean subalgebra of the lattice \( \text{Proj}(\mathcal{B}(H)) \) of projections on \( H \).
2. The projections \( p_X \), for \( X \in \mathcal{P}(\kappa) \), form an approximate unit for \( \mathcal{K}_\lambda(H) \).
3. An \( a \in \mathcal{B}(H) \) belongs to \( \mathcal{K}_\lambda(H) \) and only if \( \inf_{|X| < \lambda} \| a - p_X a p_X \| = 0 \). \( \square \)

**Corollary 12.3.6.** The Boolean algebra \( \mathcal{P}(\kappa)/\mathcal{P}(\kappa) \) is isomorphic to the lattice of projections in the atomic masa \( \mathcal{B}(\ell_2(\kappa))/\mathcal{K}_\lambda(\kappa) \). \( \square \)

### 12.3.2 Lifting Masas

Does every masa in \( \mathcal{B}(H) \) lift to a masa in \( \mathcal{B}(H) \)? This question deserves being answered three times.

**Theorem 12.3.7.** If \( H \) is a separable Hilbert space then some commuting family of projections in \( \mathcal{B}(H) \) does not lift to a commuting family of projections in \( \mathcal{B}(H) \).

**Proof.** By Proposition 12.3.1 there are \( \kappa \) masas in \( \mathcal{B}(H) \). Fix an almost disjoint (modulo finite) family \( \mathcal{A} \) of infinite subsets of \( \mathbb{N} \) of cardinality \( \kappa \) (Proposition 9.2.2). By Corollary 12.3.6 there exists a Boolean algebra homomorphism of \( \mathcal{P}(\mathbb{N})/\text{Fin} \) into the algebra of projections in the atomic masa in \( \mathcal{B}(H) \). The image of \( \mathcal{A} \) under
this homomorphism consists of orthogonal projections \( p_X \), for \( X \in \mathcal{A} \), in \( \mathcal{D}(H) \). For every \( X \in \mathcal{A} \) the corner \( p_X \mathcal{D}(H) p_X \) is isomorphic to \( \mathcal{D}(H) \) and we fix non-commuting projections \( q_{X,0} \) and \( q_{X,1} \) in \( p_X \mathcal{D}(H) p_X \).

To each \( f : \mathcal{A} \to \{0, 1\} \) associate a family of orthogonal projections

\[
\mathcal{G}_f := \{ q_{X, f(X)} : X \in \mathcal{A} \}.
\]

We thus obtain \( 2^\mathfrak{c} \) families of commuting projections. If \( f \neq g \) and \( X \in \mathcal{A} \) is such that \( f(X) \neq g(X) \), then \( \mathcal{G}_f \cup \mathcal{G}_g \) contains non-commuting projections, \( q_{X,0} \) and \( q_{X,1} \). There are only \( \mathfrak{c} \) masas in \( \mathcal{B}(H) \), and the image of any of these masas under the quotient map is abelian. It can therefore include at most one \( \mathcal{G}_f \). By a counting argument, \( \mathcal{G}_f \) does not have a commutative lift for some \( f \). \( \square \)

**Corollary 12.3.8.** Some masas in \( \mathcal{D}(H) \) do not lift to masas in \( \mathcal{B}(H) \). \( \square \)

We will sharpen the result of Theorem 12.3.7 in Theorem 14.3.2 by showing in ZFC that there exists a family of \( \mathfrak{r}_1 \) projections in \( \mathcal{D}(H) \) that cannot be lifted to commuting projections. In Proposition 12.4.3 we will find a separable abelian subalgebra of \( \mathcal{D}(H) \) with no abelian lift. Heading in the opposite direction, by Lemma 12.4.4 (and its proof), every countable family of commuting projections in \( \mathcal{D}(H) \) lifts to a commuting family of projections in \( \mathcal{B}(H) \).

### 12.4 Lifting Separable Abelian C*-subalgebras of \( \mathcal{D}(H) \). The Weyl–von Neumann Theorem

Second, the mathematician must risk frustration. Most of the time, in fact, he finds himself, after weeks or months of ceaseless searching, with exactly nothing: no results, no ideas, no energy. Since some of this time, at least, has been spent in total involvement, the resulting frustration is very nearly total. Certainly it seriously affects his attitude toward all other affairs.

Donald R. Weidman, *Emotional Perils of Mathematics*

In this (uplifting!) section we study the existence of diagonalizable lifts of self-adjoint and normal elements of the Calkin algebra. We prove the Weyl–von Neumann–Berg–Sikonia theorem on diagonalizing normal operators modulo the ideal of compact operators. Continuing the discussion of the lifting problem for abelian subalgebras of \( \mathcal{D}(H) \) started in §12.3, we isolate two additional obstructions to the existence of abelian liftings: a Fredholm index/K-theoretic obstruction (Proposition 12.4.3), and incompactness of \( \mathfrak{r}_1 \) (Theorem 14.3.2).

**Definition 12.4.1.** An operator \( b \in \mathcal{B}(H) \) is **diagonalizable** if \( H \) has an orthonormal basis consisting of eigenvectors of \( b \). A set of operators is **simultaneously diagonalizable** if all of its elements are diagonalized by a single orthonormal basis of \( H \). A C*-subalgebra \( B \) of \( \mathcal{B}(H) \) is **diagonalizable** if there exists an orthonormal basis that diagonalizes every \( b \in B \).
A few observations:

1. If \( a \) is diagonalizable then \( a^* \) is diagonalized by the same orthonormal basis as \( a \) because a \( \lambda \)-eigenvector for \( a \) is a \( \bar{\lambda} \)-eigenvector for \( a^* \) for all \( \lambda \in \mathbb{C} \). Since operators diagonalized by the same basis commute, every diagonalizable operator is normal.

2. Every diagonalizable subalgebra of \( \mathcal{B}(H) \) is included in an isomorphic copy of \( \ell_\infty \) and therefore abelian.

3. If \( \kappa \) is an infinite cardinal then a subalgebra \( B \) of \( \mathcal{B}(\ell_2(\kappa)) \) is diagonalized by the standard basis if and only if \( B \subseteq \ell_\infty(\kappa) \). Therefore a subalgebra \( B \) of \( \mathcal{B}(H) \) is diagonalizable if and only if there exists a unitary \( u : H \to \ell_2(\kappa) \) (where \( \kappa \) is the density character of \( H \)) such that \( uBu^* \subseteq \ell_\infty(\kappa) \).

As in §2.5, a lift of a C* -subalgebra \( A \) of a quotient \( M/J \) is a C* -subalgebra \( B \) of \( M \) such that \( \pi[B] = A \), where \( \pi : M \to M/J \) denotes the quotient map. The algebra \( \{ b \in M : \pi(b) = A \} \) is the largest lift of \( A \). This section is about the existence of abelian lifts of abelian C* -subalgebras of \( \mathcal{L}(H) \).

**Lemma 12.4.2.** If a C* -subalgebra \( A \) of \( \mathcal{L}(H) \) is generated by a single self-adjoint element, then it has an abelian lift.

**Proof.** By Lemma 2.5.4 the self-adjoint generator of \( A \) has a self-adjoint lift, and a self-adjoint operator generates an abelian C* -algebra. \( \square \)

The situation is more exciting in the case of normal elements of the Calkin algebra. Recall that \( s \) denotes the unilateral shift of a fixed basis \( \xi_n \), for \( n \in \mathbb{N} \) of \( H = \ell_2(\mathbb{N}) \) (Example 1.1.1). Atkinson’s theorem (Theorem C.6.3) provides characterization of invertible elements of \( \mathcal{L}(H) \) as those with a Fredholm lift.

**Proposition 12.4.3.** The algebra \( C^*(\pi(s)) \) is an abelian C* -subalgebra of \( \mathcal{L}(H) \) isomorphic to \( C(T) \) with no abelian lift.

**Proof.** Since \( s^*s = 1 \) and \( 1 - ss^* \) is a rank-one projection, \( \pi(s) \) is a unitary in \( \mathcal{L}(H) \) and therefore \( C^*(\pi(s)) \cong C(\text{sp}(\pi(s))) \) is abelian. Proposition C.6.5 implies that every lift \( a \) of \( s \) has a nonzero Fredholm index. Such an \( a \) is not normal, and no C* -algebra containing it is abelian.

It remains to prove that \( C^*(\pi(s)) \cong C(T) \). It is not difficult to prove this by showing that \( \text{sp}(s) = T \), but an indirect proof may be more illuminating.\(^3\) By the continuous functional calculus, a unitary \( v \) whose spectrum is a proper subset of \( T \) can be represented as \( \exp(ia) \) for a self-adjoint \( a \). Such \( a \) has a self-adjoint lift (Lemma 2.5.4), \( \tilde{a} \), and \( \exp(i\tilde{a}) \) is a unitary lift for \( v \). Since \( s \) has no unitary lift, we have \( \text{sp}(\pi(s)) = T \) and \( C^*(\pi(s)) \cong C(T) \). \( \square \)

An alternative proof of the following lemma is sketched in Exercise 12.6.13.

**Lemma 12.4.4.** Suppose \( H \) is a separable Hilbert space and \( A \) is a separable abelian C* -subalgebra of \( \mathcal{L}(H) \).

\(^3\) Or rather, it illuminates the ideas relevant to the present context.
12.4 Lifting Separable Abelian $C^*$-subalgebras of $\mathcal{B}(H)$. The Weyl–von Neumann Theorem 309

1. If $A$ has real rank zero then it has a diagonalizable lift to $\mathcal{B}(H)$.

2. If $A$ has an abelian lift then it has a diagonalizable lift.

Proof. (1) Let $p_n$, for $n \in \mathbb{N}$, be a set of projections dense in $\text{Proj}(A)$ and let $\xi_n$, or $n \in \mathbb{N}$, be a dense subset of the unit ball of $H$. We will recursively choose projections $q_n$, for $n \in \mathbb{N}$, in $\mathcal{B}(H)$, an orthonormal basis $\eta_n$, for $n \in \mathbb{N}$, of $H$, and an increasing sequence $k(n) \in \mathbb{N}$ so that for all $m$, all $n < m$, and all $j < k(m)$ we have

3. $\pi(q_n) = p_n$.

4. $[q_n, q_m] = 0$.

5. $\eta_j$ is an eigenvector for $q_n$, and

6. $\text{dist}(\xi_n, \overline{\text{span}}(\eta_j : j < k(m))) < 1/m$.

Assume $q_j$, $\eta_j$, and $k(j)$ were chosen to satisfy these requirements for all $j < m$. By the instances of (5) for $n < m$, the projection $r$ to the orthogonal complement of $\{\eta_j : j < k(m)\}$ commutes with all $q_n$, for $n < m$. For $s \in \{1, \bot\}^m$ let

$$r_s := r \prod_{j=m}^s q_j^{s(j)}.$$

This is an atom in the Boolean algebra generated by $r$ and $q_j$, for $j < m$. For each $s \in \{1, \bot\}^m$ the projections $p_m$ and $\pi(r_s)$ commute. By Lemma 3.1.13 applied in $\mathcal{B}(r_n[H])$ there is a projection $q_s$ in $\mathcal{B}(r_n[H])$ such that $\pi(q_s) = p_m \pi(r_s)$. Therefore

$$q_m := \sum \{q_s : s \in \{0, \bot\}^m\}$$

lifts $p_m$ and (3) holds for $n = m$. Also, $q_m \eta_j = 0$ for all $j < k(m)$ and (5) holds. Fix $n < m$. Since $q_m q_n q_n = q_m q_n q_s$ for all $s$, we have $[q_n, q_m] = 0$ and (4) holds as well.

It remains to choose $k(m)$ and $\eta_j$, for $j < k(m)$, so that (6) holds. For $s \in \{0, \bot\}^m$ find a finite orthonormal system $\eta_s$ in $q_s[H]$ such that

$$\text{dist}(q_s \xi_m, \overline{\text{span}}(\eta_s)) < m^{-1} 2^{-m}.$$

Let $\{\eta_j : k(m-1) \leq j < k(m)\}$ be an enumeration of $\bigcup_s \eta_s$. Then (6) holds because

$$\text{dist}(\xi_n, \overline{\text{span}}(\eta_j : j < k(m))) < m^{-1}.$$

Each $\eta_j$ is an eigenvector of every $q_s$, hence (5) holds for all $n$ and $j < k(m)$.

This describes the recursive construction. The vectors $\eta_n$, for $n \in \mathbb{N}$, form an orthonormal basis of $H$ since every $\xi_n$ is in their closed linear span. As each $q_n$ is diagonalized by this basis, $B := C^*(q_n : n \in \mathbb{N})$ is a diagonalizable lift of $A$.

(2) Let $M$ be the WOT-closure of a separable abelian lift $A_0$ of $A$. Being a von Neumann algebra, $M$ has real rank zero. Let $C$ be a separable elementary submodel of $M$ which includes $A_0$ (Theorem 7.1.3). Since having real rank zero is axiomatizable ([187, Theorem 2.5.1]), $C$ has real rank zero.

The $C^*$-subalgebra $\pi(C)$ of $\mathcal{B}(H)$ is abelian, includes $A$, and has real rank zero by Corollary 2.7.2. Additionally, (1) implies that $\pi(C)$ has a diagonalizable lift $D$, and $\pi^{-1}[A] \cap D$ is the required diagonalizable lift of $A$. □
By Lemma 12.4.4, if $H$ is a separable Hilbert space and $p_n$, for $n \in \mathbb{N}$, are commuting projections in $\mathcal{B}(H)$ then there are projections $q_n$, for $n \in \mathbb{N}$, diagonalized by a single basis of $H$ such that $p_n - q_n$ is compact for all $n$. However, not every countable collection of commuting projections in $\mathcal{B}(H)$ is diagonalized by a single basis of $H$ (Exercise 12.6.11).

**Corollary 12.4.5.** A separable abelian $C^*$-subalgebra of $\mathcal{B}(H)$ has an abelian lift if and only if it is included in an abelian $C^*$-subalgebra of $\mathcal{B}(H)$ of real rank zero.

**Proof.** By Lemma 12.4.4 it suffices to prove the direct implication. Suppose $A$ is a $C^*$-algebra of $\mathcal{B}(H)$ with an abelian lift $B$. Then $B''$ is an abelian von Neumann algebra and it therefore has real rank zero. Therefore $\pi[B'']$ has real rank zero (Corollary 2.7.2). As it includes $A$, this concludes the proof.

Can the assumption of separability be dropped from Lemma 12.4.4 or from Corollary 12.4.5? One could consider the least cardinal $\kappa$ such that some abelian $C^*$-subalgebra of $\mathcal{B}(H)$ of density character $\kappa$ does not have an abelian lift. We will see later on that it is equal to a well-studied small cardinal (Theorem 14.3.2). Meanwhile, we return to lifting problems similar to those briefly considered in §2.5.

**Corollary 12.4.6.** Suppose $H$ is a Hilbert space and $a \in \mathcal{B}(H)$.

1. If $a$ is self-adjoint then it has a diagonalizable lift.
2. If $a$ has a normal lift then it has a diagonalizable lift.

**Proof.** If $a$ is self-adjoint then it has a self-adjoint lift by Lemma 2.5.4, and it suffices to prove (2).

(2) If $b$ is a normal lift of $a$ then $C^*(b)$ is an abelian lift of $C^*(a)$. Lemma 12.4.4 implies that $C^*(a)$ has a diagonalizable lift, and therefore so does $a$. □

The essential spectrum of $a \in \mathcal{B}(H)$ is

$$sp_{ess}(a) := sp(a) \setminus \{ \lambda \in sp(a) : \lambda \text{ is an isolated point of finite multiplicity} \}.$$  

The pure point spectrum of a normal $a \in \mathcal{B}(H)$ is the set of its eigenvalues. It is a subset of the spectrum of $a$. Since the eigenvectors corresponding to distinct eigenvalues are orthogonal, the pure point spectrum has cardinality no greater than the density character of $H$. It is therefore typically a proper (and possibly empty) subset of $sp(a)$.

**Theorem 12.4.7.** Suppose $a$ in $\mathcal{B}(H)$ is normal and $D \subseteq sp_{ess}(a)$ is countable and dense. Then there exists a normal and diagonalizable $a_1 \in \mathcal{B}(H)$ such that $a_1 - a$ is compact, and the pure point spectrum of $a_1$ is equal to $D$.

**Proof.** By Corollary 12.4.6 (2) there exists a diagonalizable $b \in \mathcal{B}(H)$ such that $\pi(b) = \pi(a)$. We may assume $b \in \ell_\infty$, and write $b = (\lambda_j)_j$. Then $\{ \lambda_j : j \in \mathbb{N} \}$ is the pure point spectrum of $b$ and it is a dense subset of $sp(b)$. Since $\pi(a) = \pi(b)$ we have $sp(b) \supseteq sp_{ess}(a)$. Since $sp_{ess}(a)$ is compact, for each $j$ we can choose $\lambda_j' \in sp(a)$ so that $|\lambda_j - \lambda_j'| = \text{dist}(\lambda_j, sp(a))$. Then $a_1 = (\lambda_j')_j$ satisfies $sp(a_1) \subseteq sp_{ess}(a)$. Also, for every $\epsilon > 0$ the set $\{ j : |\lambda_j - \lambda_j'| > \epsilon \}$ is finite, and therefore $b - a_1$ is compact. Thus $a_1$ is a diagonalizable lift of $\pi(a)$, and $sp(a_1) = sp_{ess}(a)$. □
Corollary 12.4.8. Suppose $A$ is a unital abelian singly generated $C^*$-algebra and that $\Phi_j : A \to \mathcal{B}(H)$, for $j < 2$, are injective unital $*$-homomorphisms such that $\Phi_j[A]$ has an abelian lift for $j < 2$. Then there exists a unitary $u \in \mathcal{B}(H)$ such that $\Phi_0 = \text{Ad} \pi(u) \circ \Phi_1$. $\square$

12.5 The Other Kadison–Singer Problem

What I tell you three times is true.

Lewis Carroll, The Hunting of the Snark

In this section we provide three constructions of a pure state on $\mathcal{B}(H)$ whose restriction to any atomic masa is not pure. Each of the constructions uses maximal quantum filters and an additional set-theoretic axiom: Continuum Hypothesis, the assertion that the real line cannot be covered by fewer than $c$ closed nowhere dense sets, and the cardinal inequality $\mathcal{d} \leq \mathfrak{p}^*$, respectively. This section relies on §5, §8, and §9.

Suppose $B$ is a $C^*$-subalgebra of a $C^*$-algebra $A$. The set of extensions of a pure state $\varphi$ of $B$ to a state on $A$ is a face of the state space $S(A)$ of $A$ (Lemma 5.3.5). In particular, if $\varphi$ has a unique state extension to $A$ then this extension is pure. If every pure state on $B$ has a unique state extension to $A$ then $B$ is said to have the extension property (Definition 10.2.12).

In [56, p. 74–75] Dirac asserted that a representation of a maximal set of commuting observables uniquely determines the representation of the space of all observables. Since observables in quantum mechanics are self-adjoint operators ([126]), and since an abelian $C^*$-algebra is generated by its self-adjoint part, Dirac’s statement directly translates into the assertion that masas in $\mathcal{B}(H)$ have the extension property.

Every vector state $\omega_\xi$ whose restriction to a masa in $\mathcal{B}(H)$ is pure has a unique state extension to $\mathcal{B}(H)$. This is because the projection $p$ to $\mathbb{C}\xi$ belongs to the masa and $\omega_\xi(a)$ is a unique $\lambda$ such that $pap = \lambda a$, for every $a \in \mathcal{B}(H)$. (In other words, the maximal quantum filter associated with $\omega_\xi$ is principal.) In [148] Kadison and Singer proved that he atomless masa does not have the extension property. They achieved this by constructing different conditional expectations from $\mathcal{B}(H)$ onto the atomless masa and composing them with its pure states. Kadison and Singer proved that there is a unique conditional expectation from $\mathcal{B}(H)$ onto the atomic masa but the question whether the atomic masa $\ell_\infty(\mathbb{N})$ has the extension property in $\mathcal{B}(\ell^2(\mathbb{N}))$ remained elusive for almost sixty years. It gradually gained prominence as the Kadison–Singer Problem, finally solved in the affirmative in [177]. The following conjecture ([148, p. 399]) is in a sense dual to the more famous Kadison–Singer problem.

Conjecture 12.5.1 (Kadison–Singer). For every pure state $\varphi$ of $\mathcal{B}(H)$ there is a masa (perhaps many) $A$ such that $\varphi | A$ is multipliciative.
Definition 12.5.2. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and let $e := (\eta_n)_n$ be an orthonormal basis for $H$. Define a functional on $\mathcal{B}(H)$ by
\[ \varphi^e_{\mathcal{U}}(a) := \lim_{{n \to \mathcal{U}}} (a\eta_n|\eta_n). \]
Being a weak*-limit of states, $\varphi^e_{\mathcal{U}}$ is a state. We say that it is diagonalized by $e$, and that a state on $\mathcal{B}(H)$ is diagonalizable if it is diagonalized by some basis.

The positive solution to the Kadison–Singer problem, together with the fact that a pure state on a $C^*$-subalgebra has a unique extension if and only if it has a unique pure state extension (Lemma 5.4.1), implies that every state on $\mathcal{B}(H)$ with pure restriction to some atomic masa is pure. In particular every diagonalizable state is pure (this was first proved in [12]).

A confirmation of the following conjecture is equivalent to the assertion that for every pure state on $\mathcal{B}(H)$ there exists an atomic masa $A$ such that $\varphi \upharpoonright A$ is multiplicative.

Conjecture 12.5.3 (Anderson). Every pure state on $\mathcal{B}(H)$ is diagonalizable.

We will prove that Conjecture 12.5.3 consistently fails in three different classes of models of ZFC in Proposition 12.5.4, Proposition 12.5.9, and Theorem 12.5.10. These constructions are given in the order of the increasing complexity and the last two together cover many (but not all) ‘common’ models of ZFC. It is not known whether Conjecture 12.5.3 can be refuted in ZFC.

Take One

Our first refutation of Conjecture 12.5.3 uses the strongest of the three assumptions and accordingly has the simplest proof.

Proposition 12.5.4. For any collection $\mathcal{A}$ of $\aleph_1$ masas of the Calkin algebra $\mathcal{B}(H)$ there exists a pure state on $\mathcal{B}(H)$ that is not multiplicative on any masa in $\mathcal{A}$.

We postpone the proof of Proposition 12.5.4 in order to show that its conclusion refutes both Conjecture 12.5.3 and Conjecture 12.5.1.

Corollary 12.5.5. The Continuum Hypothesis implies that there exists a pure state on $\mathcal{B}(H)$ that is not multiplicative on any masa.

Proof. Proposition 12.3.1 implies that there are only $\mathfrak{c}$ masas in $\mathcal{B}(H)$. By the Continuum Hypothesis, Proposition 12.5.4 implies that there exists a pure state $\varphi$ of $\mathcal{B}(H)$ not multiplicative on any masa in $\mathcal{A}$. The composition of the quotient map with $\varphi$ is the required pure state on $\mathcal{B}(H)$. $\square$

To a pure state $\varphi$ of a $C^*$-algebra $A$ we associated the maximal quantum filter (Definition 5.3.1) $\mathcal{F}_\varphi := \{a \in A^{\mathbb{N}} : \varphi(a) = 1\}$. General maximal quantum filters were defined in Definition 5.3.3.
Lemma 12.5.6. Let $\mathcal{F}$ be a maximal quantum filter on a C*-algebra $B$ and let $A$ be an abelian C*-subalgebra of $B$ of real rank zero. The following are equivalent.

1. The pure state $\varphi_{\mathcal{F}}$ is multiplicative on $A$.
2. For all $q \in \mathcal{F}$ and all projections $r$ in $A$ we have $\max(\|rq\|, \|(1-r)q\|) = 1$.
3. $\mathcal{F} \cap \text{Proj}(A)$ is an ultrafilter of the Boolean algebra $\text{Proj}(A)$.

Proof. Let $X$ be the compact Hausdorff space such that $A \cong C(X)$. The multiplicative states on $C(X)$ are the evaluation functionals (Example 1.7.2). Since $A$ has real rank zero, $X$ is zero-dimensional and therefore (1) is equivalent to (3).

Since $\varphi(pq) = \varphi(q)$ for all $p \in \mathcal{F}$ (Proposition 1.7.8), $\text{Proj}(A) \cap \mathcal{F}$ is a filter. Therefore (2) is also equivalent to (3).

The following terminology is borrowed from the theory of forcing. Given a C*-algebra $A$, a set $\mathcal{D} \subseteq \text{Proj}(A) \setminus \{0\}$ is order-dense and open in $\text{Proj}(A)$ if the following two conditions hold.

1. For every $p \in \text{Proj}(A) \setminus \{0\}$ there exists $q \in \mathcal{D}$ such that $q \leq p$.
2. For every $p \in \mathcal{D}$, every nonzero projection $q \leq p$ is in $\mathcal{D}$.

These two requirements correspond to $\mathcal{D}$ being 'dense' and 'open,' respectively, in the topology whose basic open sets are $U_p := \{q \neq 0 : q \leq p\}$.

Since every vector state is multiplicative on some atomic masa, we mod out the compact operators and work with $\mathcal{D}(H)$.

Definition 12.5.7. If $A$ is a masa in $\mathcal{B}(H)$ let

$$\mathcal{D}(A) := \{q \in \text{Proj}(\mathcal{D}(H)) \setminus \{0\} : (\forall \varphi \in S(\mathcal{D}(H))) \varphi(q) = 1 \text{ implies } \varphi \text{ is not multiplicative on } \pi[A]\}.$$  

Lemma 12.5.8. The set $\mathcal{D}(A)$ is order-dense and open in $\text{Proj}(\mathcal{D}(H))$ for every masa $A$ of $\mathcal{B}(H)$.

Proof. Fix $p \in \text{Proj}(\mathcal{D}(H)) \setminus \{0\}$. Proposition 3.6.6 implies that for a C*-algebra $C$ there is a homeomorphic copy of $\mathbb{T}$ in the space $P(C)$ of pure states (considered with respect to the norm topology) if and only if $C$ is nonabelian, if and only if $P(C)$ is nondiscrete. Therefore $P(A)$ is discrete, while $B := p\mathcal{D}(H)p$ is isomorphic to $\mathcal{D}(H)$ and therefore $P(B)$ contains a homeomorphic copy of $\mathbb{T}$. Since $B$ is hereditary, every state on $B$ has a unique extension $\tilde{\varphi}$ to a state on $\mathcal{D}(H)$ (Lemma 1.8.5). Since the restriction map from $S(\mathcal{D}(H))$ to $S(A)$ is norm-continuous, there exists a pure state $\varphi$ of $B$ whose restriction to $A$ is not multiplicative.

By Lemma 12.5.6, there are $q_0 \in \mathcal{F}_\varphi$ and $r \in \text{Proj}(A)$ such that

$$\max(\|rq_0\|, \|(1-r)q_0\|) < 1.$$  

Since $\varphi(q_0) = 1$, we can choose $q \leq q_0$. Lemma 12.5.6 implies $q \in \mathcal{D}(A)$. Since $q \in p\mathcal{D}(H)p$ and $p$ was arbitrary, $\mathcal{D}(A)$ is dense.

To see that $\mathcal{D}(A)$ is open, fix $q \in \mathcal{D}(A)$. If $p \leq q$ is a nonzero projection then $p \in \mathcal{D}(A)$ since $\varphi(p) = 1$ implies $\varphi(q) = 1$ for all states $\varphi$.  \qed
Proof (Proposition 12.5.4). Enumerate $\mathbb{A}$ as $A_\alpha$, for $\alpha < \mathfrak{c}$. We recursively choose projections $p_\alpha$, for $\alpha < \mathfrak{c}$, in $\mathcal{B}(H)$ so that $\pi(p_\alpha) \geq \pi(p_\beta)$ and $\pi(p_\beta) \in \mathcal{D}(A_\alpha)$ for all $\alpha < \beta$.

The recursive construction of $p_\alpha$, for $\alpha < \mathfrak{c}$, proceeds as follows. Suppose $\beta$ is a countable ordinal such that $p_\alpha$ were chosen to satisfy the conditions for all $\alpha < \beta$. If $\beta$ is a limit ordinal, then it has countable cofinality and we can choose a projection $q$ in $\mathcal{D}(H)$ such that $q \leq \pi(p_\alpha)$ for all $\alpha < \beta$. Let $p_\beta$ be a lift of $q$ to $\mathcal{B}(H)$. Since $\mathcal{D}(A_\alpha)$ is order-dense and open, $\pi(p_\beta) = q \in \mathcal{D}(A_\alpha)$ for all $\alpha < \beta$.

Now suppose that $\beta = \alpha + 1$ for some $\alpha$. By Lemma 12.5.6 we can find $q \leq \pi(p_\alpha)$ in $\mathcal{D}(A_\alpha)$ and take $p_\beta$ to be a lift of $q$. This describes the recursive construction.

Let $\mathcal{F}$ be the quantum filter generated by $p_\alpha$, for $\alpha < \mathfrak{c}$. If $\varphi$ is a pure state such that $\mathcal{F}_0 \supseteq \mathcal{F}$, then $\varphi(p_\alpha) = 1$ for all $\alpha$, and the restriction of $\varphi$ to $A_\alpha$ is not multiplicative for any $\alpha$. This completes the proof. \qed

The proof of Proposition 12.5.4 gives a stronger statement (Exercise 12.6.18).

Take Two

Our second refutation of Anderson’s conjecture uses a weakening of the Continuum Hypothesis, the axiom $\text{cov} (\mathcal{M}) = \mathfrak{c}$ asserting that no Polish space can be covered by fewer than $\mathfrak{c}$ nowhere dense subsets (see $\S 8.5$).

**Proposition 12.5.9.** If the Baire space cannot be covered by fewer than $\mathfrak{c}$ nowhere dense sets then there exists a pure state on $\mathcal{B}(H)$ whose restriction to any atomic masa is not multiplicative.

**Proof.** Let $e_\alpha$, for $\alpha < \mathfrak{c}$, be an enumeration of all orthonormal bases of $H$. Let $A_\alpha$ be the masa diagonalized by $e_\alpha$. We will recursively find quantum filters $\mathcal{F}_\alpha$, for $\alpha < \mathfrak{c}$, in $\mathcal{B}(H)$ such that the following conditions hold.

1. Each $\mathcal{F}_\alpha$ is generated by $|1 + \alpha|$ sets.
2. $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ if $\alpha < \beta$.
3. $\mathcal{F}_{\alpha + 1}$ contains a projection $p$ such that $\pi(p) \in \mathcal{D}(A_\alpha)$ (see Definition 12.5.7).

Suppose $\mathcal{F}_\alpha$, for $\alpha < \beta$, were chosen to satisfy these conditions. If the quantum filter generated by $\mathcal{G} := \bigcup_{\alpha < \beta} \mathcal{F}_\alpha$ is not diagonalized by $e_\beta$, let $\mathcal{F}_\beta := \mathcal{G}$. Otherwise, let $q_\gamma$, for $\gamma < \beta$, be an enumeration of a downwards directed generating set for $\mathcal{G}$ and for each $q_\gamma$ choose $X(\gamma) \subseteq \mathbb{N}$ such that $q_\gamma - p^{e_\beta}_{X(\gamma)}$ is compact (where $p^{e_\beta}_{X(\gamma)}$ denotes the projection to $\text{span}(e_\beta(n) : n \in X(\gamma))$). The set

$$\mathcal{Z} := \{ f \in \mathbb{N}^\mathbb{N} : f^2 = \text{id}_\mathbb{N}, f(n) \neq n \text{ for all } n \}$$

is a closed subset of the Baire space and it is therefore Polish in the subspace topology. Suppose $X \subseteq \mathbb{N}$ is infinite. Then the set $X \setminus k$ is infinite for every $k \in \mathbb{N}$ and
\( \mathcal{V}_k := \{ f \in \mathcal{Z} : (\exists n \geq k) \{n, f(n)\} \subseteq X \} \) is dense in \( \mathcal{Z} \) (considered with the subspace topology inherited from the Baire space). It is clearly open and

\[ \mathcal{Y}_k := \bigcap_k \{ f \in \mathcal{Z} : (\exists n \geq k) \{n, f(n)\} \subseteq X \}. \]

is a dense \( G_\delta \) subset of \( \mathcal{Z} \). Since the Baire space cannot be covered by \( |\alpha| \) meager sets, neither can \( \mathcal{Z} \) (Lemma 8.5.3). Fix \( f \in \bigcap_{\gamma < \beta} \mathcal{Y}_k(\gamma) \). Let \( Y := \{ n : n < f(n) \} \), let

\[ \xi(n) := \frac{\sqrt{2}}{\pi} (\eta_{\beta}(n) + \eta_{\beta}(f(n))) \]

for \( n \in Y \), and let \( q := \text{proj}_{\mathcal{Z}(n) = \mathcal{Z}(n) \cap \mathcal{Y}(n) \subseteq \mathcal{Y}(n)} \). For every \( \gamma < \beta \) there are infinitely many \( n \in \mathbb{N} \) such that \( \{n, f(n)\} \subseteq Y_{\gamma} \). Then \( \zeta(n) \) is in the range of \( p_{\gamma} \) for every such \( n \), and therefore \( \|\pi(q)p_{\gamma}\| = 1 \). Since the set \( \{p_{\gamma} : \gamma < \beta\} \) is directed, \( \pi(q) \prod_{\gamma \neq F} p_{\gamma} = 1 \) for every \( F \in \beta \), and \( \mathcal{F} \cup \{\pi(q)\} \) generates a quantum filter; this is our \( \mathcal{F}_\beta \).

It is generated by \( |\beta| < c \) sets. Let \( p := p_{\mathcal{F}_\beta} \). Then \( p_{\mathcal{F}_\beta}(n) = \frac{\sqrt{2}}{\pi} \eta_{\beta}(n) \) for all \( n \).

Therefore \( \|pq\| = \|(1-p)q\| = \frac{\sqrt{2}}{\pi} \), and (with \( A_e \) denoting the atomic masa diagonalized by \( e \)) Lemma 12.5.6 implies that \( \pi(q) \in \mathcal{D}(A_e) \).

This describes the recursive construction of \( \mathcal{F}_\beta \), for \( \beta < c \). Then \( \mathcal{F} := \bigcup_{\beta < c} \mathcal{F}_\beta \) is a quantum filter not diagonalized by any orthonormal basis of \( H \). We can extend it to a maximal quantum filter if necessary. The corresponding pure state is not diagonalized by any atomic masa in \( \mathcal{Z}(H) \).

The reader may have noticed that the proof of Proposition 12.5.9 resembles the construction of a selective ultrafilter (Proposition 8.5.7).

Take Three

The assumption of our third refutation of Anderson’s Conjecture is \( d \leq p^* \) (see Definition 9.5.3 and Definition 12.2.4). Since \( K_1 \leq d \leq c \) and \( K_1 \leq p^* \leq c \), the inequality \( d \leq p^* \) is a weakening of the Continuum Hypothesis.

The poset \( \text{Part}_{\ell_2} \) of all decompositions \( K = \langle K_n : n \in \mathbb{N} \rangle \) of a closed subset of \( H = \ell_2(\mathbb{N}) \) of finite co-dimension into a direct sum of finite-dimensional subspaces was introduced in Definition 9.8.2. For \( K \in \text{Part}_{\ell_2} \) and \( X \subseteq \mathbb{N} \) we write

\[ p^K_X := \text{proj}_{\bigoplus \{X\}} K_j. \]

Since these projections commute, the von Neumann algebra

\[ \mathcal{A}[K] := W^* \{ p^K_j : j \in \mathbb{N} \} \]

is abelian and the Boolean algebra of its projections is equal to \( \{ p^K_X : X \subseteq \mathbb{N} \} \). If all spaces \( K_j \) are one-dimensional and \( H = \bigoplus K_j \) then \( \mathcal{A}[K] \) is an atomic masa.

The poset Part_{\ell_2} is ordered by \( K \leq^* L \) if there is \( f : \mathbb{N} \rightarrow \text{Fin} \) such that

1. \( (\forall m) (\exists n) \{j, j + 1\} \subseteq f(n) \), and
2. \( \Sigma_n \|p^K_{f(n)}(1 - p^K_{(n,n+1)})\| < \infty. \)
A proof of the following theorem is given at the end of this section.

**Theorem 12.5.10.** Suppose that $d \leq p^*$. Then for a separable Hilbert space $H$ there exists a pure state on $\mathcal{B}(H)$ that is not diagonalized by $\mathcal{A}[K]$ for any $K \in \mathcal{P}_N$. In particular it is not diagonalized by any atomic masa.

If $\mathcal{G}$ is a quantum filter in $\mathcal{A}[E]$, the quantum filter of $\mathcal{B}(H)$ generated by $\mathcal{G}$ is (Definition 5.3.12)

$$\mathcal{F}(\mathcal{G}) := \bigcap \{ \mathcal{F} : \mathcal{F} \text{ is a maximal quantum filter in } \mathcal{B}(H) \text{ and } \mathcal{G} \subseteq \mathcal{F} \}.$$ 

To $L \in \mathcal{P}_2$, we associate two coarser elements of $\mathcal{P}_2$: 

$L_{\text{even}}$, whose $n$th element is $L_{2n} \oplus L_{2n+1}$, and 

$L_{\text{odd}}$, whose $n$th element is $L_{2n-1} \oplus L_{2n+1}$ (with $L_{-1} = \{0\}$).

The state $\phi$ in the following lemma is not required to be pure.

**Lemma 12.5.11.** Suppose $K \preceq L$. If a singular state $\phi$ is multiplicative on $\mathcal{A}[K]$, then it is multiplicative on at least one of $\mathcal{A}[L_{\text{even}}]$ or $\mathcal{A}[L_{\text{odd}}]$.

**Proof.** If $\phi$ is a state multiplicative on $\mathcal{A}[K]$, then $\mathcal{U} := \{ X : \phi(p^K_X) = 1 \}$ is an ultrafilter on $\mathbb{N}$ because $\text{Proj}(\mathcal{A}[K]) = \{ p^K_X : X \subseteq \mathbb{N} \}$. Since $\phi$ is singular, $\mathcal{U}$ is nonprincipal. Fix $f$ and $g$ that witness $K \preceq L$ and let $\mathcal{V} := \{ X : g^{-1}(X) \in \mathcal{U} \}$. This is an ultrafilter $g(\mathcal{U})$ (§9.4). We have $\sum_{n \in \mathbb{N}} \| p^K_{X(n)} (1 - p^L_{X(g(n)+1)}) \| < \infty$ and therefore $\pi(p^K_X) \leq \pi(p^L_{X(g(n)+1)})$ for all $X \subseteq \mathbb{N}$.

Since $\phi(p^K_X) = 1$ implies $\phi(p^L_{X(g(n)+1)}) = 1$ for all $X \subseteq \mathbb{N}$, we have (writing $\tilde{X} := \{ n, n+1 : n \in X \}$) $\phi(p^L_{X}) = 1$ for all $X \in \mathcal{V}$. In other words,

$$\mathcal{F}_\phi \supseteq \{ p^L_X : X \in \mathcal{V} \}.$$ 

Suppose for a moment that $Z := \{ n : g(n) \text{ is even} \} \in \mathcal{U}$. Then $p^L_X \in \mathcal{A}[L_{\text{even}}]$ for every $X \in \mathcal{V}$. Projections of this form form an ultrafilter in $\text{Proj}(\mathcal{A}[L_{\text{even}}])$. Therefore the restriction of $\mathcal{F}_\phi$ to $\text{Proj}(\mathcal{A}[L_{\text{even}}])$ is an ultrafilter. Since $\mathcal{A}[L_{\text{even}}]$ is an abelian C*-algebra of rank zero, $\phi$ is multiplicative on $\mathcal{A}[L_{\text{odd}}]$. \hfill $\square$

The following is the analog of $\mathcal{G}(A)$ (Definition 12.5.7).

**Definition 12.5.12.** For $L \in \mathcal{P}_2$, let

$$\mathcal{E}(L) := \{ q \in \text{Proj}(\mathcal{B}(H)) \setminus \{0\} : (\forall \phi \in \mathcal{S}(\mathcal{B}(H)) \phi(q) = 1 \text{ implies } \phi \text{ is not multiplicative on } \mathcal{A}[L] \}.$$ 

**Lemma 12.5.13.** If $H$ is an infinite-dimensional separable Hilbert space then for every $L \in \mathcal{P}_2$ the set $\mathcal{E}(L)$ is order-dense and open in $\text{Proj}(\mathcal{B}(H)) \setminus \{0\}$.
12.6 Exercises

Exercise 12.6.1. Prove that for infinite cardinals $\lambda \leq \kappa$ every $C^*$-algebra of density character $\kappa$ is isomorphic to a subalgebra of $\mathcal{B}(\ell_2(\kappa))/\mathcal{K}_\lambda(\ell_2(\kappa))$.\(^4\)

Exercise 12.6.2. Suppose that $\kappa$ is an infinite cardinal. Prove that the quotient $\mathcal{B}(\ell_2(\kappa))/\mathcal{K}_\kappa(\ell_2(\kappa))$ is simple.

Exercise 12.6.3. Prove that every $C^*$-algebra is isomorphic to a subalgebra of a simple $C^*$-algebra of the same density character.

Exercise 12.6.4. Prove that the Calkin algebras associated to $\ell_2(\mathbb{K}_0)$ and $\ell_2(\mathbb{K}_1)$ are not isomorphic.\(^5\)

\(^{4}\) Used in the proof of Proposition 12.1.4.

\(^{5}\) This is a rare example of a noncommutative analog of a problem which is easier than the original, ‘commutative,’ problem. Although the Boolean algebras $\mathcal{P}(\mathbb{K}_0)/\text{Fin}$ and $\mathcal{P}(\mathbb{K}_1)/\text{Fin}$ were
Exercise 12.6.5. Suppose \( \kappa \) and \( \lambda \) are infinite cardinals. Prove that the Calkin algebras associated to \( \ell_2(\kappa) \) and \( \ell_2(\lambda) \) are isomorphic if and only if \( \kappa = \lambda \).

Exercise 12.6.6 is a continuation of Exercise 9.10.15. The assertion that all sets of reals have the Property of Baire contradicts the Axiom of Choice, but it is relatively consistent with ZF (\([224]\)).

Exercise 12.6.6. Suppose that all sets of reals have the property of Baire and let \( H \) denote the separable, infinite dimensional, Hilbert space. Prove the following.

1. The Calkin algebra has no states and no representations on any Hilbert space.
2. All irreducible representations of \( \mathcal{B}(H) \) are unitarily equivalent.\(^6\)

Exercise 12.6.7. Suppose \( B \) is a \( C^* \)-subalgebra of a \( C^* \)-algebra \( A \). Prove that the annihilator of \( B \) is a hereditary \( C^* \)-subalgebra of \( A \).

Exercise 12.6.8. With \( \pi : \mathcal{B}(H) \to \mathcal{Q}(H) \) denoting the quotient map, find projections \( p \) and \( q \) in \( \mathcal{B}(H) \) such that \( \pi(p) = \pi(q) \neq 0 \) but \( p \land q = 0 \).

Solving the following two exercises requires some familiarity with measure theory, and Maharam’s Theorem in particular; see \([108]\).

Exercise 12.6.9. Suppose \( \gamma \) is an ordinal. Classify masas in \( \mathcal{B}(\ell_2(\mathbb{R}_\gamma)) \) up to the unitary equivalence and prove that there are exactly \( \aleph_0 + (|\gamma| + 1)^{\aleph_0} \) unitary equivalence classes.

Exercise 12.6.10. Suppose \( X \) is a compact Hausdorff space. Prove that if \( X \) is separable then \( C(X) \) has a faithful representation on \( \ell_2(\mathbb{N}) \). Is it true that if \( C(X) \) has a faithful representation on \( \ell_2(\mathbb{N}) \) then \( X \) is separable?

Exercise 12.6.11. Find a countable collection of commuting projections in \( \mathcal{B}(H) \) that cannot be simultaneously diagonalized by an orthonormal basis.

Exercise 12.6.12. Prove that every unital \( C^* \)-algebra generated by a countable family of projections is generated by a single self-adjoint element.

Exercise 12.6.13. Use Corollary 12.4.6 (1) and Exercise 12.6.12 to provide a short proof of Lemma 12.4.4.

Exercise 12.6.14. Prove that every maximal chain of projections in \( \mathcal{Q}(H) \) is a countably saturated linear ordering. Equivalently, prove that the linear ordering obtained by removing its minimal and maximal elements is an \( \eta_1 \)-set.

Conclude that the Continuum Hypothesis implies all maximal linearly ordered sets of projections in \( \mathcal{Q}(H) \) are isomorphic.

\(^6\) It is therefore relatively consistent with ZF that \( \mathcal{B}(H) \) is a counterexample to Naimark’s problem. Of course this is but a curiosity; there is little doubt that Naimark was assuming the Axiom of Choice.
Exercise 12.6.15. Find a C*-subalgebra $B$ of $\mathcal{B}(H)$ and a masa $A$ in $B$ such that $\pi[A]$ is not a masa in $B/(B \cap \mathcal{K}(H))$.

Exercise 12.6.16. Suppose that $B$ is a C*-subalgebra of $A$ with the extension property. Prove that $B' \cap A = Z(B)$ (where $Z(B)$ denotes the center of $B$).

Exercise 12.6.17. For each $n$ let $D_{2^n}$ be the algebra of diagonal matrices in $M_{2^n}(\mathbb{C})$. Let $D$ be the inductive limit of $D_{2^n}$. Prove that $D$ is a masa of the CAR algebra that has the extension property.

Exercise 12.6.18. Recall Definition 12.2.4,

$$p^* := \min \{|\mathcal{F}| : \mathcal{F} \subseteq \text{Proj}(\mathcal{B}(H)) \text{ generates a nonprincipal quantum filter which is not diagonalized} \}.$$ 

Prove that for any collection $\mathcal{A}$ of $p^*$ many masas of the Calkin algebra $\mathcal{B}(H)$ there exists a pure state on $\mathcal{B}(H)$ that is not multiplicative on any of the masas in $\mathcal{A}$.

Exercise 12.6.19. Suppose $B$ is a C*-subalgebra of $A$ and $\varphi$ is a state on $B$ that has a unique extension to a state $\psi$ of $A$. Suppose moreover that $\psi$ is a pure state on $A$. Prove that $\varphi$ is a pure state on $B$.

Exercise 12.6.20. Suppose $\varphi$ is a diagonalizable state on $\mathcal{B}(H)$. Prove that there are $c$ distinct atomic masas each of which diagonalizes $\varphi$.

Exercise 12.6.21. Suppose $A$ is a separable and unital C*-subalgebra of the Calkin algebra, and that $f : A \to \mathcal{B}(H)$ is such that $\pi(f(a)) = a$ for all $a \in A$.

Prove that for every $\varphi \in P(A), F \in A, \epsilon > 0$, and every finite-dimensional subspace $H_0$ of $H$ there is a unit vector $\xi \in H_0 \cap H$ such that $|\varphi(a) - \omega_\xi(f(a))| < \epsilon$ for all $a \in F$.

The last two exercises show that one ‘obvious simplification’ of the proof of Lemma 12.5.13 does not work.

Exercise 12.6.22. Prove that for every $m \geq 1$ there exist a finite-dimensional Hilbert space $K$, projections $p_j$, for $j < m$, and $q$ in $K$ that satisfy the following: $\sum_{j<m} p_j = 1$, $\|p_j q\| = 1/\sqrt{2}$ for all $j < m$, and $\|(p_j + p_k)q\| = 1$ for all $j < k < m$.

Exercise 12.6.23. Use Exercise 12.6.22 to prove that the set of all $f \in \mathbb{N}^{\mathbb{N}}$ satisfying the following conditions is $\leq^*$-cofinal in $\mathbb{N}^{\mathbb{N}}$: There are $L \in \text{Part}_\epsilon$, and a noncompact projection $q \in \mathcal{B}(L_2(\mathbb{N}))$ such that $f(m) \leq j < k < f(m+1)$ implies $\|\pi(p_{(j,k)}^L q)\| = 1$ for all $j, k$, and $m$, but $\|p_{(j)}^L q\| = 2^{-1/2}$ for all $j$.

For the following exercise see Problem 12.6.25, Theorem B.2.12, and the discussion preceding it.

Exercise 12.6.24. Prove that the assertion ‘There exist a separable C*-algebra $A$ and its proper C*-subalgebra $B$ such that $B$ separates the pure states of $A$ and $0$ is $\Sigma_1^1$ and therefore absolute between models of a large enough fragment of ZFC that contain all countable ordinals.’ 

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There are no other assumptions on $f$—it is not assumed to be continuous, linear, multiplicative, self-adjoint, etc.
Notes for Chapter 12

Who is General Stone–Weierstrass?

A whisper overheard during a seminar.

§ 12.1 Not surprisingly, the analogy between $\mathcal{D}(H)$ and $\mathcal{P}(\mathbb{N})/\text{Fin}$ comes with a disclaimer. The atomless masa embeds into $\mathcal{D}(H)$ (Definition 12.3.3 and the paragraph following it), and therefore the Lebesgue measure algebra embeds into the poset of projections of $\mathcal{D}(H)$. While the Continuum Hypothesis implies the commutative analog of this statement, that the Lebesgue measure algebra embeds into $\mathcal{P}(\mathbb{N})/\text{Fin}$, OCA implies that the Lebesgue measure algebra does not embed into $\mathcal{P}(\mathbb{N})/\text{Fin}$ ([64]).

§ 12.2 Lemma 12.2.5 is a special case of a result of [125], where it was proved that every maximal chain in the poset of projections in the Calkin algebra is a countably saturated linear ordering (that is, an $\eta_1$ set—see Definition 8.2.2). This is Exercise 12.6.14. We shall return to saturation in the context of metric structures in § 15.1.

Proposition 12.2.2 was taken from [266]. The cardinals $t$ and $p$ are equal ([175]), but nothing nontrivial is known about the relation between $p^*$ and $t^*$ ($p^* \leq t^*$ is trivial).

§ 12.3 Theorem 12.3.2 is a special case of a result about derivations of von Neumann algebras from [136]. Theorem 12.3.7 was taken from [6].

§ 12.4 Corollary 12.4.8 has been vastly generalized. The assumption that $A$ be singly generated is not necessary; this is a very special case of the BDF theory [34]. On the other hand, the conclusion of Corollary 12.4.8 may fail without the assumption on the existence of lifts $\Phi_j$. As an example, take the algebra generated by the unilateral shift (Proposition 12.4.3) and a unital copy of $C(\mathbb{T})$ in $\mathcal{D}(H)$ with an abelian lift (we will return to this in Example 16.1.1).

Corollary 12.4.8 has a noncommutative version. If $A$ is a separable and unital $C^*$-algebra and $\Phi_j: A \to \mathcal{D}(H)$, for $j < 2$, are unital $^*$-homomorphisms with the same kernel such that there exists a unital $^*$-homomorphism $\Psi_j: A \to \mathcal{D}(H)$ that satisfies $\Phi_j = \pi \circ \Psi_j$ for $j < 2$, then $\Phi_0 = \text{Ad} \pi(u) \circ \Phi_1$ for some unitary $u \in \mathcal{D}(H)$. This is Voiculescu’s Theorem ([253], see also the discussion in [35, §1.7]).

§ 12.5 Anderson’s Conjecture 12.5.3 first appeared in [13]. In [6] it was proved that the Continuum Hypothesis refutes both Anderson’s conjecture and the Kadison–Singer Conjecture 12.5.1. Proposition 12.5.4 is this result, recast in the language of maximal quantum filters. It is not known whether either of these conjectures can be refuted in ZFC or whether [6] is ‘one-half of a consistency result’ (cf. [257]). Theorem 12.5.10 was proved by Weaver and the author. Its proof was sketched in [99]. The assumption $\text{cov}(\mathcal{M}) = \mathfrak{c}$ (used in Proposition 12.5.9) is arguably the simplest situation in which the assumption of Theorem 12.5.10. $\mathfrak{d} \leq p^*$, fails. Both $\text{cov}(\mathcal{M}) = \mathfrak{c}$ and $\mathfrak{d} \leq p^*$ are consequences of the Martin’s Axiom ([166]). Probably the most prominent models of ZFC which satisfy both $\mathfrak{d} > p^*$ and $\text{cov}(\mathcal{M}) < \mathfrak{c}$ (and which are therefore the most likely candidates for a model in which Conjecture 12.5.3 may have a positive answer) are the Laver model and the Mathias model (see e.g., [20]). It is not known whether Conjecture 12.5.3 is true in either of these
models, or any other model of ZFC. I conjecture that in both Laver and Mathias models every pure state on $\mathcal{D}(H)$ is multiplicative when restricted to $\mathcal{A}[K]$ for some $K \in \mathcal{P}_\ell$.

The uniqueness of a distinguished extension of a state from a C*-subalgebra to the crossed product has played a key role in the construction of a counterexample to Naimark’s problem (Theorem 11.2.2). If $B \subseteq A$ are C*-algebras we say that $B$ separates pure states on $A$ if for all pure states $\psi \neq \varphi$ of $A$ there is $a \in B$ such that $\varphi(a) \neq \psi(a)$. If $A$ is a nonunital C*-algebra then it separates pure states of its unitization $\tilde{A}$, but clearly $A \neq \tilde{A}$. However $A$ does not separate the zero functional from the lift of the quotient map from $\tilde{A}$ to $\tilde{A}/A$, which is in this case a pure state.

**Problem 12.6.25 (The General Stone-Weierstrass problem).** Assume a C*-subalgebra $B$ of $A$ separates pure states of $A$ and 0. Can we conclude that $A = B$?

By the (complex) Stone–Weierstrass Theorem, the answer is positive if $A$ is abelian. It is not known whether the answer to Problem 12.6.25 is positive even for separable C*-algebras (see also Exercise 12.6.24). For more information see e.g., [215], [170], and [205].

Another analog of the Stone–Weierstrass theorem is false in the noncommutative context: There is a C*-algebra $A$ with a proper unital C*-subalgebra $B$ such that every pure state on $B$ has a unique extension to a state on $A$. Take $A := M_2(\mathbb{C})$ and let $B$ be the diagonal masa. It is however not difficult to see that $B$ does not separate pure states on $A$.

Exercise 12.6.22 is a relative of lemmas used in [266] to construct gaps in $\mathcal{D}(H)$ (see Exercise 14.6.4). The short proof of Lemma 12.4.4 given as Exercise 12.6.13 was suggested by Andrea Vaccaro,
Chapter 13
Multiplier Algebras and Coronas

As we saw in Chapter 12, $B(H)$ is a noncommutative analog of the Boolean algebra $\mathcal{P}(\mathbb{N})$ and the Calkin algebra $Q(H)$ is a noncommutative analog of the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$. Via the Stone duality, the Boolean algebras $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})/\text{Fin}$ are associated with their Stone spaces, $\beta\mathbb{N}$ and $\beta\mathbb{N}\setminus\mathbb{N}$, respectively. These Stone spaces are also the Čech–Stone compactification of $\mathbb{N}$ and the Čech–Stone remainder of $\mathbb{N}$. In this Chapter we will describe noncommutative analogs of these two constructions. They generalize the passage from $K(H)$ to $B(H)$ and $Q(H)$.

13.1 The Strict Topology

In this section we introduce the strict topology induced by a family of seminorms on a C$^*$-algebra $A$ and define the multiplier algebra of $A$, $\mathcal{M}(A)$, as its strict completion.

The one-point compactification $X \cup \{\infty\}$ of a locally compact Hausdorff space $X$ is the minimal compactification of $X$ in the sense that for every other compactification $\gamma X$ of $X$ the identity map on $X$ extends continuously to a surjection from $\gamma X$ onto $X \cup \{\infty\}$. Since $C(X \cup \{\infty\})$ is isomorphic to the unitization $C_0(X)$ of $C_0(X)$, the unitization of a nonunital C$^*$-algebra is a non-commutative analog of the minimal compactification.

The maximal compactification of a completely regular topological space $X$, denoted $\beta X$, is uniquely determined by the requirement that for every other compactification $\gamma X$ the identity function on $X$ extends continuously to a surjection from $\beta X$ onto $\gamma X$. Such a compactification exists and it is unique. It is called the Čech–

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1 By the Gelfand–Naimark duality these two spaces correspond to the C$^*$-algebras $C(\beta\mathbb{N}) \cong C_0(\mathbb{N})$ and $C(\beta\mathbb{N}\setminus\mathbb{N}) \cong C_0(\mathbb{N})/C_0(\mathbb{N})$, respectively.

2 In addition, if $A$ is an essential ideal in $B$ then the identity map on $A$ uniquely and trivially extends to a unital $^*$-homomorphism from $A$ to $B$. 
Stone compactification (or the Stone–Čech compactification, in languages in whose alphabets the letter ‘S’ precedes the letter ‘Č’). The construction of the multiplier algebra is a generalization of the construction of \( \beta X \). The definition, and even examples, of multiplier algebras will have to wait. An impatient reader may want to peek ahead at Example 13.2.4.

**Definition 13.1.1.** Suppose \( A \) is a \( C^* \)-subalgebra of a \( C^* \)-algebra \( M \). To every \( h \in A \) we associate two seminorms on \( M \), 
\[
\lambda_h(b) := \|hb\| \quad \text{and} \quad \rho_h(b) := \|bh\|
\]
The weak topology induced by these seminorms (see Definition C.4.1) is called the \( A \)-strict topology, or just the strict topology if \( A \) is clear from the context.

**Lemma 13.1.2.** Suppose \( A \) is a \( C^* \)-subalgebra of a \( C^* \)-algebra \( M \).

1. Then \( M \) is \( A \)-strictly Hausdorff if and only if \( A^\perp \cap M = \{ 0 \} \).
2. Every \( A \)-strictly open subset of \( M \) is open in the norm topology.
3. If \( A \) is a unital subalgebra of \( M \) then the \( A \)-strict topology coincides with the norm topology.

**Proof.** (1) follows from the definitions.

(2) The basic \( A \)-strict open sets are finite intersections of sets of the form
\[
U_{h,a} = \{ b : \|b(b-a)\| < \epsilon \} \cap \{ b : \|(b-a)h\| < \epsilon \}
\]
for a nonzero \( h \in A_+ \). To see that a nonempty intersection of finitely many neighbourhoods of \( a \) of this form includes a nonempty ball centered at \( a \), note that
\[
\|b-a\| \geq \max(\|h(b-a)\|, \|(b-a)h\|)\|h\|^{-1}.
\]

(3) Clearly \( \lambda_1(\cdot) = \rho_1(\cdot) = \|\cdot\| \), hence if \( A \) is unital then every norm-open subset of \( M \) is \( A \)-strictly open. \( \Box \)

**Example 13.1.3.** 1. In the case when \( A \) is an ideal in \( M \), (1) from Lemma 13.1.2 is equivalent to \( A \)'s being an essential ideal (Definition 2.5.5).
2. Similarly, if \( \pi : A \to \mathcal{B}(H) \) is a nondegenerate representation then \( \mathcal{B}(H) \) is Hausdorff in the \( \pi[A] \)-strict topology.

A net \( (b_\lambda) \) in \( M \) is \( A \)-strictly Cauchy (or strictly Cauchy if \( A \) is clear from the context) if it is Cauchy with respect to both \( \lambda_h \) and \( \rho_h \) for every \( h \in A \). The \( C^* \)-algebra \( M \) is said to be \( A \)-strictly complete (or strictly complete if \( A \) is clear from the context) if every bounded \( A \)-strictly Cauchy net in \( M \) converges.

**Lemma 13.1.4.** Suppose \( A \) is a \( C^* \)-subalgebra of \( M \).

1. If \( \mathcal{E} \) is an approximate unit for \( A \) then on every bounded subset of \( M \) the \( A \)-strict topology is equivalent to the weak topology \( \tau \) induced by \( \{ \rho_e, \lambda_e : e \in \mathcal{E} \} \).
2. If some \( h \in A \) is strictly positive in \( M \) then the \( A \)-strict topology on \( M \) is induced by a single seminorm, \( \rho_h + \lambda_h \).

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\(^3\) Čech–Stone compactification can be easily constructed using the Gelfand–Naimark theorem. See Exercise 13.4.1 and Exercise 13.4.2. This is a somewhat contentious issue; see the amusing discussion in the introduction to [62].
Proof. We prove (1) and leave the similar (2) as Exercise 13.4.4. Clearly every bounded net that converges $A$-strictly also converges in the weaker topology $\tau$. To prove the converse, fix a bounded net $(a_\lambda)$ in $M$ and $a \in M$ such that $a_\lambda$ does not $A$-strictly converge to $a$. We need to prove that $a_\lambda$ does not converge to $a$ in $\tau$. By taking adjoints and linear combinations, without a loss of generality we may assume that each $a_\lambda$ is self-adjoint and that $a = 0$. Fix $h \in A$ such that $\varepsilon := \limsup_\lambda \|he\lambda\| > 0$. We may assume $\|h\| \leq 1$ and $\sup_\lambda \|a_\lambda\| \leq 1$. Fix $e \in \mathcal{E}$ such that $\|h - he\| < \varepsilon/2$. Then $\|ea_\lambda\| \geq \|he\lambda\| = \|ha\lambda\| - \varepsilon/2$, and $\|ea_\lambda\| \not\to 0$. An analogous proof shows that if $\|a_\lambda h\| \not\to 0$ for some $h \in A$ then $\|a_\lambda e\| \not\to 0$ for some $e \in \mathcal{E}$. Since $(a_\lambda)$ was an arbitrary bounded net in $A$, this concludes the proof. \qed

Lemma 13.1.5. The completion $\mathcal{M}(A)$ of $A$ in the strict topology is equipped with a unital $C^*$-algebra structure such that $A$ is an essential ideal in $\mathcal{M}(A)$.

Proof. On the space of all $A$-strict Cauchy nets in $A$ define $(a_\mu) \sim (b_\mu)$ if and only if $(a_\mu - b_\mu) \to 0$ $A$-strictly. This is an equivalence relation and $\mathcal{M}(A)$ is the quotient space. Routine computations show that $+, \cdot$, and $^*$ are congruences with respect to $\sim$ and therefore naturally extend to $\mathcal{M}(A)$. The extensions of $+$ and $^*$ are strictly continuous (with $+$ being jointly continuous), and the extension of $\cdot$ is separately continuous.

Fix an approximate unit $\mathcal{E}$ of $A$.

Claim. For a Cauchy net $(b_\mu)$, $\| (b_\mu) \| := \sup_{e \in \mathcal{E}} \lim_\mu \| \lambda e (b_\mu) \|$ defines a norm on $\mathcal{M}(A)$.

Proof. Since $(b_\mu)$ is strictly Cauchy, $\| c \|_* := \lim_\mu \| cb_\mu \|$ exists for all $c \in A$, and by the submultiplicativity of the norm on $A$ we have $\| cd \|_* \leq \| c \| \| d \|_*$ for all $c$ and $d$ in $A$. This is evidently a seminorm.

We will prove that $\| (b_\mu) \| < \infty$. Assume otherwise, and fix $e_n \in \mathcal{E}$ such that $\| e_n \|_* \geq 2^{2n+1}$ for all $n$. By Corollary 1.6.11 there exist $d \in A_{+1}$ and contractions $c_n$ such that $2^{-n} e_n = c_n d$ for all $n$. Then $2^n \leq \| 2^{-n} e_n \|_* \leq \| c_n \| \| a \|_* \leq \| a \|_*$ for all $n$; contradiction.

The verification that $\| \cdot \|$ is strictly positive, subadditive and homogeneous is straightforward. \qed

It is straightforward to check that $\mathcal{M}(A)$ is a Banach algebra with respect to the norm defined in the Claim. Considering $\mathcal{E}$ as a net with the natural ordering, we have $\| eb_\mu \| = \lim_{f \to \mathcal{E}} \| eb_\mu f \|$. By taking adjoints one sees that $\| (b_\mu) \| = \lim_{e \to \mathcal{E}} \lim_{\mu} \| eb_\mu \|$, and therefore the involution is an isometry of $\mathcal{M}(A)$. By using the $C^*$-equality in $A$ we have

$$\| (b_\mu) \|^2 = \lim_{e \to \mathcal{E}} \lim_{\mu} \| eb_\mu \|^2 = \lim_{e \to \mathcal{E}} \lim_{\mu} \| eb_\mu b_\mu^* e \|^2 \geq \lim_{e \to \mathcal{E}} \lim_{\mu} \| b_\mu b_\mu^* e \| = ||(b_\mu)(b_\mu)^*||.$$
Since \( \|(b_N)(b_N)^*\| \geq \|(b_N)\|(b_N)^*\| = \|(b_N)\|^2 \), and since \((b_N) \in M(A)\) was arbitrary this shows that the \(C^*\)-equality holds in \(\mathcal{M}(A)\) and completes the proof that \(\mathcal{M}(A)\) is a \(C^*\)-algebra.

Finally, to see that \(\mathcal{M}(A)\) is unital, note that any approximate unit of \(A\) is a Cauchy net whose limit is the multiplicative unit of \(\mathcal{M}(A)\). \(\square\)

**Definition 13.1.6.** The \(A\)-strict completion of \(A\) is called the *multiplier algebra* of \(A\) and denoted \(\mathcal{M}(A)\).

We can now give the general definition of a corner (cf. §2.1.6).

**Definition 13.1.7.** The *corner* of a \(C^*\)-algebra \(A\) is a subalgebra of the form \(pAp\) for some projection \(p\) in \(\mathcal{M}(A)\).

If \(A\) is unital then the \(A\)-strict topology on \(A\) is complete and \(\mathcal{M}(A) = A\).

**Lemma 13.1.8.** Suppose that \(A\) is a separable \(C^*\)-algebra.

1. The strict topology on the multiplier algebra \(\mathcal{M}(A)\) is Polish (Definition B.0.1).
2. The unit ball \(\mathcal{M}(A)_1\) of \(\mathcal{M}(A)\) is a closed subspace and therefore Polish itself.
3. \(A\) is a \(G_{\delta\sigma}\) subset of \(\mathcal{M}(A)\).

**Proof.** Choose a countable dense subset \(a(n)\) for \(n \in \mathbb{N}\), of \(A\). Then the strict topology is induced by the seminorms \(\lambda_{a(n)}, \rho_{b(n)}\), for \(n \in \mathbb{N}\). It is therefore metrizable, for example by \(d(x,y) := \sum_n 2^{-n} \|a(n)\|^{-1}(\lambda_{a(n)}(x-y) + \rho_{b(n)}(x-y))\). This metric is complete by the definition and any countable norm-dense subset of \(A\) is strictly dense in \(\mathcal{M}(A)\).

The verification that the unit ball of \(\mathcal{M}(A)\) is closed is left to the reader.

We have yet to prove that \(A\) is a \(G_{\delta\sigma}\) subset of \(\mathcal{M}(A)\). Let \(e_n\), for \(n \in \mathbb{N}\), be an approximate unit in \(A\). Then some \(c \in \mathcal{M}(A)\) belongs to \(A\) if and only if

\[
(\forall m)(\exists n)(\forall k \geq n) \rho_{e_n}(c) + \lambda_{e_n}(c) \leq \frac{1}{m},
\]

i.e., \(A\) is equal to the intersection of a sequence of \(F_{\sigma}\) subsets of \(\mathcal{M}(A)\). \(\square\)

**Lemma 13.1.9.** If \(N\) is a von Neumann algebra and \(A \subseteq N\) is a \(C^*\)-algebra such that \(A^+ \cap N = \{0\}\) then \(N\) is \(A\)-strictly complete.

**Proof.** The assumption that \(A^+ \cap N = \{0\}\) only assures that the \(A\)-strict topology is Hausdorff (Lemma 13.1.2). Let \(\mathcal{E}\) be an approximate unit for \(A\). Since every element of \(N\) is a linear combination of four positive elements, it will suffice to show that every bounded Cauchy net \((b_N)\) consisting of positive elements of \(N\) converges to an element of \(N\). Fix such \((b_N)\). Fix \(e \in \mathcal{E}\). On the corner \(eNe\) each of the seminorms \(\rho_e\) and \(\lambda_e\) is equivalent to the uniform metric. Therefore there exists \(b_e \in e\mathcal{A}e\) such that \(\lim_k \|eb_ne - b_ne\| = 0\).

If \(e \leq f\) in \(\mathcal{E}\) are such that \(ef = e\) then \(eb_fe = b_fe\). Therefore \(b_fe\), for \(e \in \mathcal{E}\), is a bounded increasing net of positive elements in \(N\). Since \(N\) is a von Neumann algebra, Lemma 3.1.3 implies that it contains \(b := \sup_e b_e\). For every \(e \in \mathcal{E}\) we have \(ebe = b_e\), hence \(\lim_k b_kb = b\). \(\square\)
This section concludes with two handy lemmas.

**Lemma 13.1.10.** Suppose \((e_\mu)_\mu\) is an approximate unit in \(A\) and \((a_\mu)_\mu\) is a bounded net in \(A\) indexed by the same directed set. If for every \(\mu\) the net \((e_\mu a_\nu)_\nu\) is norm-convergent, then the net \((e_\mu a_\mu)_\mu\) converges to an element of \(\mathcal{M}(A)\) strictly.

*Proof.* By writing \(a_\nu = e_\nu + id_\nu\), we may assume all \(a_\nu\) are self-adjoint and \(\lim_\nu a_\nu e_\mu = b_\mu^*\). It will suffice to prove that for every \(\mu\) both nets \((e_\mu e_\nu a_\nu)_\nu\) and \((e_\mu a_\nu e_\mu)_\nu\) are norm-convergent. Let \(M := \sup_\nu ||a_\nu||\). Fix \(\mu\). Then
\[
||e_\mu e_\nu a_\nu - b_\mu|| \leq ||(e_\mu e_\nu - e_\mu)a_\nu|| + ||e_\mu a_\nu - b_\mu|| \leq M||e_\mu e_\nu a_\mu|| + ||e_\mu a_\nu - b_\mu||
\]
and both expressions on the right-hand side converge to 0. Also,
\[
||e_\nu a_\nu e_\mu - b_\mu^*|| \leq ||e_\nu (a_\nu e_\mu - b_\mu^*)|| + ||e_\nu b_\mu^* - b_\mu^*||
\]
and again both expressions on the right-hand side converge to 0. \(\square\)

**Lemma 13.1.11.** If \((e_n)_n\) is a sequential approximate unit in \(A\), a sequence \((e_n)_n\) in \(A\) is norm-bounded, and \(f_n := (e_{n+1} - e_n)^{1/2}\) for all \(n\), then the series \(\sum_n f_n c_n f_n\) converges to an element of \(\mathcal{M}(A)\) strictly.

*Proof.* Let \(a_n := \sum_{j<n} f_j c_j f_j\). The sequence \(e_m a_n\), for \(n \in \mathbb{N}\), is norm-convergent for every \(m\) and \(||a_m|| \leq \sup_j ||c_j||\) and the conclusion follows by Lemma 13.1.10. \(\square\)

### 13.2 The Multiplier Algebra

In this section we prove that the multiplier algebra of \(A\) is isomorphic to the idealizer of the image of \(A\) under any faithful nondegenerate representation.

The idealizer of a nondegenerate \(C^*\)-subalgebra \(A\) of \(\mathcal{B}(H)\) is
\[
\mathcal{M} := \{b \in \mathcal{B}(H) : bA \cup Ab \subseteq A\}.
\]

**Proposition 13.2.1.** Suppose \(A\) is a \(C^*\)-algebra and \(\pi : A \to \mathcal{B}(H)\) is a nondegenerate faithful representation. Then \(\pi\) has a unique extension to a representation \(\tilde{\pi} : \mathcal{M}(A) \to \mathcal{B}(H)\), and \(\tilde{\pi} | \mathcal{M}(A)\) is equal to the idealizer of \(\pi[A]\) in \(\mathcal{B}(H)\).

*Proof.* Since \(\pi\) is nondegenerate, the \(\pi[A]\)-strict topology on \(\mathcal{B}(H)\) is Hausdorff. Since \(\pi\) is an isomorphism, it sends \(A\)-strictly Cauchy nets in \(A\) to \(\pi[A]\)-strictly Cauchy nets.

Fix \(b \in \mathcal{M}(A)\) and let \(a_\lambda\), for \(\lambda \in \Lambda\), be an \(A\)-strictly Cauchy net converging to \(b\). Then \(\pi(a_\lambda)\), for \(\lambda \in \Lambda\), is a \(\pi[A]\)-strictly Cauchy net in \(\mathcal{B}(H)\). It is convergent by Lemma 13.1.9. Let \(\tilde{\pi}(b)\) be its limit. This defines \(\tilde{\pi} : \mathcal{M}(A) \to \mathcal{B}(H)\). Trivially \(\tilde{\pi}\) extends \(\pi\). By Lemma 13.1.5 and uniqueness of a completion of a metric space, \(\tilde{\pi}\) is an isomorphism of \(\mathcal{M}(A)\) onto its image.
It remains to prove that $\pi[A]$ is equal to the idealizer $M$ of $\pi[A]$ in $\mathcal{B}(H)$. Since $A$ is an ideal in $\mathcal{M}(A)$, we have $\pi[\mathcal{M}(A)] \subseteq M$. Fix an approximate unit $e_\lambda$, for $\lambda \in \Lambda$ for $A$. Then SOT-\lim_{\lambda} \pi(e_\lambda) = 1_{\mathcal{B}(H)}$ (Exercise 3.10.3). Therefore for every $b \in M$ the net $b\pi(e_\lambda)$, for $\lambda \in \Lambda$, is $\pi[A]$-strictly Cauchy and it converges to $b$. This implies $b \in \hat{\pi}[\mathcal{M}(A)]$. Since $b \in M$ was arbitrary, this completes the proof. \hfill \Box

Proposition 13.2.1 gives an equivalent characterization of the multiplier algebra.

Corollary 13.2.2. Suppose $A$ is a $C^*$-algebra. Then $\mathcal{M}(A)$ isomorphic to the idealizer of the image of $A$ under any nondegenerate faithful representation $\pi$ of $A$ on a Hilbert space via an isomorphism that extends $\pi$. \hfill \Box

The multiplier algebra $\mathcal{M}(A)$ shares the universality property of the Čech–Stone compactification (note that the arrows are reversed).

Corollary 13.2.3. Suppose that a $C^*$-algebra $A$ is an essential ideal in a unital $C^*$-algebra $C$. Then there is an injective *-homomorphism $\Psi: C \to \mathcal{M}(A)$ whose restriction to $A$ is equal to the identity.

Proof. Let $\pi: C \to \mathcal{B}(H)$ be a faithful and unital representation. If $(e_\lambda)$ is an approximate unit for $A$ then WOT-\lim_{\lambda} \pi(e_\lambda) = 1$, and therefore the restriction of $\pi$ to $A$ is nondegenerate. Since $A$ is an ideal in $C$, $\pi[C]$ is contained in the idealizer of $\pi[A]$. The latter is, by Proposition 13.2.1, isomorphic to $\mathcal{M}(A)$ via $\Psi := \hat{\pi}$. \hfill \Box

We can finally list some examples of multiplier algebras.

Example 13.2.4. 1. If $X$ is a locally compact Hausdorff space then $\mathcal{M}(C_0(X))$ is isomorphic to $C(\beta X)$ (Exercise 13.4.1).
2. Since $\mathcal{K}(H)$ is an essential ideal of $\mathcal{B}(H)$ for any Hilbert space $H$, uniqueness of the strict completion and Lemma 13.1.9 together imply $\mathcal{M}(\mathcal{K}(H)) = \mathcal{B}(H)$.
3. More generally, if $\lambda < \kappa$ are infinite cardinals then the ideal $\mathcal{K}_\lambda(\ell_2(\kappa))$ of operators whose range has density character $\lambda$ (Proposition 12.3.4) has $\mathcal{B}(\ell_2(\kappa))$ as its multiplier algebra.
4. Suppose $M$ is a $\mathcal{I}_\alpha$ factor and $J$ is its Breuer ideal (Example 4.1.5). Then $J$ is an essential ideal in $M$ and $M$ is $J$-strictly complete by Lemma 13.1.9. Therefore $M$ can be identified with the multiplier algebra of $J$.

As in §2.5, for $C^*$-algebras $B_j$, for $j \in \mathbb{J}$, and an ideal $\mathcal{J}$ on $\mathbb{J}$ we define

$$\bigoplus \mathcal{J} B_j := \{ \tilde{b} \in \prod_{j \in \mathbb{J}} B_j : \limsup_{j \to \mathcal{J}} \| b_j \| = 0 \}.$$ 

This is an essential ideal in $\prod_{j \in \mathbb{J}} B_j$. The following lemma provides us with a large nested family of $C^*$-algebras with the same multiplier algebra.

Lemma 13.2.5. Suppose $\{ B_j : j \in \mathbb{J} \}$ is a family of unital $C^*$-algebras and $\mathcal{J}$ is an ideal on the index set $\mathbb{J}$ that includes the Fréchet ideal. The algebra $\prod_{j \in \mathbb{J}} B_j$ is isomorphic to $\mathcal{M}(\bigoplus \mathcal{J} B_j)$ via an isomorphism that fixes $\bigoplus \mathcal{J} B_j$. 
Proof. Let $A := \bigoplus_{j \in J} B_j$. Since $A$ is an essential ideal in $\prod_j B_j$, it suffices to prove that $\prod_j B_j$ is $A$-strictly complete.

For every $j \in J$, define $p_j \in \prod_j B_j$ by $p_j(i) := \delta_{ij} 1_B$. The assumption that $J$ includes the Fréchet ideal implies $p_j \in A$. Therefore $A$ is an essential ideal in $\prod_j B_j$ and the $A$-strict topology on $\prod_j B_j$ is Hausdorff. If $(b_\lambda)$ is a bounded Cauchy net in $\prod_j B_j$, then $\lim_{\lambda} p_j b_\lambda p_j$ exists for every $j$ and it is equal to some $c(j) \in B_j$. We have $\|c(j)\| \leq \sup_{\lambda} \|b_\lambda\|$. Therefore this defines $c \in \prod_j B_j$ and it is straightforward to show that $\lim_{\lambda} b_\lambda = c$. □

We conclude this section with a discussion on functoriality (or lack thereof) of the multiplier algebra construction. Not every $*$-homomorphism between $C^*$-algebras extends to a $*$-homomorphism between their respective multiplier algebras. For example, take the embedding of $K(H)$ into $\widetilde{K(H)}$. The multiplier algebras are $\mathcal{B}(H)$ and $\mathcal{H}(H)$, respectively, and $\mathcal{B}(H)$ has no nontrivial ideals other than $\mathcal{H}(H)$. A more surprising fact is that the identity map on $A$ sometimes does extend to a $*$-homomorphism from $\mathcal{M}(A)$ into $\mathcal{A}$, because $\mathcal{M}(A)$ is equal to $\mathcal{A}$. See notes to this section. For a sufficient condition that a $*$-homomorphism of $A$ into $B$ extends to a $*$-homomorphism of $\mathcal{M}(A)$ into $\mathcal{M}(B)$, see Exercise 13.4.9 and Exercise 13.4.10.

As always, the real fun starts after passing to the quotient; but this deserves a whole new section.

### 13.3 Introducing Coronas

In this section we begin the study of the corona $\mathcal{M}(A)/A$, also known as the outer multiplier algebra and denoted $\mathcal{Q}(A)$, of a $C^*$-algebra $A$. The section concludes with a proof that the set of projections in the corona of a stabilization of a unital $C^*$-algebra is not a lattice.

One of the friendliest presentations of $\beta\mathbb{N}$ is as the (still rather mind-boggling) space of all ultrafilters on $\mathbb{N}$.\footnote{Another friendly representation of $\beta\mathbb{N}$ is as the Gelfand spectrum of $\ell_\infty(\mathbb{N})$ (Theorem 1.3.1), and many a functional analyst will agree that this is the natural representation of $\beta\mathbb{N}$ but see footnote 3 on page 324.} To find a structure more mind-boggling than $\beta\mathbb{N}$, or $\beta X$ in general, one need not go far. The Čech–Stone remainder (also known as corona), $\beta\mathbb{N}\setminus\mathbb{N}$ (the space of non-principal ultrafilters on $\mathbb{N}$) is one of the most important esoteric mathematical objects, and many of its fundamental properties are independent from ZFC ([184]). It is not surprising that the noncommutative analog of the Čech–Stone remainder is at least as unruly.

**Definition 13.3.1.** The corona (also known as the outer multiplier algebra) of a nonunital $C^*$-algebra $A$ is the quotient $\mathcal{D}(A) := \mathcal{M}(A)/A$.

**Example 13.3.2.** 1. Since the multiplier algebra of $\mathcal{K}(H)$ is isomorphic to $\mathcal{B}(H)$, its corona is the Calkin algebra ($\mathcal{K}(\mathcal{H}), \mathcal{K}(\mathcal{H})$).
2. If \( X \) is a locally compact Hausdorff space, then its multiplier algebra is isomorphic to \( C(\beta X) \) and its corona is \( C(\beta X) / C_0(X) \cong C(\beta X \setminus X) \).

3. If \( \lambda < \kappa \) are infinite cardinals then the multiplier algebra of \( \mathcal{K}(\ell_2(\kappa)) \) has the generalized Calkin algebra \( \mathcal{B}(\ell_2(\kappa)) / \mathcal{K}(\ell_2(\kappa)) \) as its multiplier algebra.

4. The multiplier algebra of \( \bigoplus_n M_n(\mathbb{C}) \) is isomorphic to \( \prod_n M_n(\mathbb{C}) \) and its corona is \( \prod_n M_n(\mathbb{C}) / \bigoplus_n M_n(\mathbb{C}) \).

5. More generally, if \( B_j \), for \( j \in \mathbb{J} \), are unital C*-algebras and \( \mathcal{J} \) is an ideal on \( \mathbb{J} \), then the corona of \( \bigoplus_{j \in \mathbb{J}} B_j / \bigoplus_{j \in \mathcal{J}} B_j \).

An overarching model-theoretic property of coronas that implies many of their remarkable properties will be studied in Chapter 15.

Recall that the stabilization of a C*-algebra \( B \) is the tensor product \( B \otimes \mathcal{K}(H) \) for a separable, infinite-dimensional Hilbert space \( H \) (§2.1). A C*-algebra \( A \) is stable if \( A \cong A \otimes \mathcal{K}(H) \). The operation of stabilization provides means for producing a nonunital C*-algebra that has given unital C*-algebra \( B \) as a hereditary subalgebra.

Now that we introduced coronas, let’s see how ill-behaved they can be. Since both \( \ell_\infty \) and \( \mathcal{B}(H) \) are von Neumann algebras, both \( \text{Proj}(\ell_\infty) \cong \mathcal{P}(\mathbb{N}) \) and \( \text{Proj}(\mathcal{B}(H)) \) are complete lattices. On the other hand, \( \text{Proj}(\ell_\infty / c_0) \cong \mathcal{P}(\mathbb{N}) / \text{Fin} \) is not a complete Boolean algebra (see §9.3). At least it is a lattice.

**Proposition 13.3.3.** If \( A \) is the stabilization of a unital C*-algebra then the poset of projections in \( \mathcal{L}(A) \) is not a lattice.

The conclusion of Proposition 13.3.3 is not a bug. It is a feature of the rich structure of coronas. Its proof relies on combinatorics and a soft application of compactness of \( \text{Proj}(M_2(\mathbb{C})) \).

**Lemma 13.3.4.** For every \( m \geq 1 \) and \( \varepsilon > 0 \) there exists \( \delta = \delta(m, \varepsilon) > 0 \) such that if \( p \) and \( q \) are rank-1 projections in \( M_2(\mathbb{C}) \), \( x, y \) are scalars in \([0,1]\), \( \|p - q\| \geq 1/m \), and \( \|xp - yq\| \leq \delta \), then \( \max(|x|, |y|) \leq \varepsilon \).

**Proof.** Suppose otherwise and fix \( m \) and \( \varepsilon > 0 \) for which the statement fails. By the compactness of \( \mathcal{K} := \text{Proj}(M_2(\mathbb{C})) \) and \( [\varepsilon, 1] \) there exist \((x, y, p, q) \in [\varepsilon, 1]^2 \times \mathcal{K}^2 \) such that \( \|p - q\| \geq 1/m \) and \( \|xp - yq\| = 0 \); contradiction. \( \square \)

**Proof (Proposition 13.3.3).** Suppose \( B \) is a unital C*-algebra such that \( A = B \otimes \mathcal{K} \). Let \( p_{m,n} \), for \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \), be an enumeration of a maximal orthogonal family of projections in \( A \) of the form \( 1_B \otimes e \) for a minimal projection \( e \) of \( \mathcal{K}(H) \). For \( X \subseteq \mathbb{N}^2 \) the series \( \sum_{(m,n) \in X} p_{m,n} \) is strictly convergent and we let \( p_X := \sum_{(m,n) \in X} p_{m,n} \) in \( \mathcal{M}(A) \). The projections \( p_X \), for \( X \in \text{Fin} \), form an approximate unit of \( A \) and \( p_X \) belongs to \( A \) if and only if \( X \) is finite.

Since \( p_{m,2n} \) and \( p_{m,2n+1} \) are Murray-von Neumann equivalent, Example 2.3.5 implies that \( A_{m,n} := (p_{m,2n} + p_{m,2n+1})A(p_{m,2n} + p_{m,2n+1}) \) has a unital copy of \( M_2(\mathbb{C}) \) for all \( m \) and \( n \). Hence some projection \( q_{m,2n} \in A_{m,n} \) satisfies

\[
\|p_{m,2n} - q_{m,2n}\| = \frac{1}{m+1}.
\]
Each one of \( r_m := \sum_0^\infty p_{m,n} \), \( p_m := \sum_0^\infty p_{m,2n} \), and \( q_m := \sum_0^\infty q_{m,2n} \), for \( m \in \mathbb{N} \), belongs to \( \mathcal{A}(A) \). Let \( p := \sum_m p_m \) and \( q := \sum_m q_m \). Each of these series is strictly convergent, and both \( p \) and \( q \) belong to \( \mathcal{A}(A) \).

We will prove that \( \pi(p) \) and \( \pi(q) \) have no greatest lower bound in \( \mathcal{D}(A)_+ \). This will imply that they have no greatest lower bound in \( \text{Proj}(\mathcal{D}(A)) \). Fix \( a \in \mathcal{A}(A)_+ \) such that \( \pi(a) \leq \pi(p) \) and \( \pi(a) \leq \pi(q) \). Fix \( m \). Since \( p_m \leq r_m \) and \( q_m \leq r_m \), we have \( \pi(p_m a p_m) = \pi(r_m a r_m) = \pi(q_m a q_m) \). As \( p_X \notin A \) for an infinite \( X \subseteq \mathbb{N}^2 \),

\[
\lim_n \| p_{m,2n} a p_{m,2n} - q_{m,2n} a q_{m,2n} \| = 0.
\]

By Lemma 13.3.4 we can choose \( f(m) \) so that

\[
\max(\| p_{m,2f(m)} a p_{m,2f(m)} \|, \| q_{m,2f(m)} a q_{m,2f(m)} \|) < \frac{1}{m}.
\]

Then \( r := P_{\{f(m): m \in \mathbb{N}\}} \) is a projection such that \( \pi(r) \leq \pi(p) \), \( \pi(r) \leq \pi(q) \), and \( \pi(r) \) is nonzero and orthogonal to \( \pi(a) \). Since \( a \) was an arbitrary positive element such that \( \pi(a) \leq \pi(p) \) and \( \pi(a) \leq \pi(q) \), this proves that there is no greatest lower bound for \( \pi(p) \) and \( \pi(q) \) in \( \mathcal{D}(A) \). \( \square \)

A C*-algebra is an AW*-algebra if it possesses the following two properties of von Neumann algebras: every masa is generated by projections, and its projections form a complete lattice (\([27, \text{III.1.8.1}]\)). We shall consider a modification of this property in §15.3. Proposition 13.3.3 has the following corollary.

**Corollary 13.3.5.** If \( A \) is a stabilization of a unital C*-algebra then \( \mathcal{D}(A) \) is not an AW*-algebra. \( \square \)

### 13.4 Exercises

The following exercises show that the multiplier algebra construction is the non-commutative analog of the Čech–Stone compactification.

**Exercise 13.4.1.** Suppose \( X \) is a locally compact Hausdorff space. Prove that the multiplier algebra of \( C_0(X) \) is isomorphic to the algebra \( C_b(X) \) of all bounded, continuous, complex valued functions on \( X \).

A topological space \( X \) is completely regular if real-valued functions separate points in \( X \). Tietze’s extension theorem implies that every locally compact Hausdorff space is completely regular.

**Exercise 13.4.2.** Suppose \( X \) is a completely regular topological space. Prove that there exists a topological space \( \beta X \) that has a dense subspace homeomorphic to \( X \) and it is such that every bounded, continuous, real-valued function on \( X \) continuously extends to \( \beta X \).
The space $\beta X$ is the Čech–Stone compactification of $X$.

**Exercise 13.4.3.** Prove that $\beta \mathbb{N}$ is homeomorphic to the space of all ultrafilters on $\mathbb{N}$. The basic open sets are of the form $U_X := \{ \mathcal{U} : X \in \mathcal{U} \}$, for $X \subseteq \mathbb{N}$. Verify that $\beta \mathbb{N}$ is naturally identified with the space of pure states on $\ell_\infty(\mathbb{N})$.

**Exercise 13.4.4.** Suppose that $M$ is a C*-algebra, $A$ is a C*-subalgebra of $M$, and $A$ has a strictly positive element $h$ for $M$. Prove that the $A$-strict topology on $M$ is induced by the single seminorm, $\rho_h + \lambda h$.

**Exercise 13.4.5.** Suppose that $M$ is a II$_\infty$ factor and $A$ is the Breuer ideal of $M$ (Example 4.1.5). Prove that $A$ is not $\sigma$-unital, and it therefore has no strictly positive element (Example 13.2.4 (4)).

**Exercise 13.4.6.** 1. Suppose $A$ is a $\sigma$-unital and nonunital C*-algebra. Prove that $M(A)/A$ is nonseparable. More precisely, prove that the density character of $M(A)/A$ is at least $\aleph_1$.

2. Consider $\mathbb{R}_1$ with respect to the ordinal topology in which the closure of $X \subseteq \mathbb{R}_1$ is the set of suprema of all bounded subsets of $X$. Prove that $J(C_0(\mathbb{R}_1)) \cong \mathbb{C}$.

**Exercise 13.4.7.** Prove that there exists a separable C*-algebra $A$ such that Proj($A$) is not a lattice.

Hint: If for some reason you really don’t feel like using Proposition 13.3.3 and the Löwenheim–Skolem Theorem, you may want to analyze

$$A = \{ f \in C([0, 1], M_2(\mathbb{C})) : f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \text{ for } \lambda \in \mathbb{C} \}.$$

**Exercise 13.4.8.** Prove that Proj($\prod_n M_n(\mathbb{C})/\bigoplus_n M_n(\mathbb{C})$) is not a lattice.

**Exercise 13.4.9.** Suppose $A$ is a C*-algebra and $B$ is a C*-subalgebra of $A$. Also suppose that both $A$ and $B$ are nonunital and that $B$ contains an approximate unit for $A$. Prove that there exists an injective *-homomorphism from $\mathcal{M}(B)$ into $\mathcal{M}(A)$ that extends the identity map on $B$, and that therefore $\mathcal{M}(B)$ can be identified with a subalgebra of $\mathcal{M}(A)$.

**Exercise 13.4.10.** Find a nonunital C*-algebra $A$ and a nonunital C*-subalgebra $B$ of $A$ such that the identity map on $B$ extends to an injective *-homomorphism of $\mathcal{M}(B)$ into $\mathcal{M}(A)$, although $B$ does not contain an approximate unit for $A$.

**Exercise 13.4.11.** Suppose that $A$ and $B$ are C*-algebras and $A$ is nonunital. Prove that the identity map on $A \otimes B$ extends to an injective *-homomorphism from $\mathcal{M}(A) \otimes \mathcal{M}(B)$ into $\mathcal{M}(A \otimes B)$. Then prove that this *-homomorphism is surjective if and only if $B$ is finite-dimensional.

Hint: For the second part, use the fact that each one of $A$ and $B$ contains an infinite-dimensional masa, and therefore an infinite sequence of orthogonal elements of norm 1.

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5 Used in the proof of Lemma 13.1.4.
**Exercise 13.4.12.** Suppose $A$ is a $\sigma$-unital, nonunital, and non-separable $C^*$-algebra.

1. Use Exercise 13.4.9 to represent $\mathcal{M}(A)$ as an inductive limit of the multiplier algebras of a net of separable $C^*$-subalgebras of $A$.
2. Prove that $\mathcal{Q}(A)$ is isomorphic to an inductive limit of a net of coronas of separable $C^*$-algebras.

**Exercise 13.4.13.** Suppose $A$ is a nonunital $C^*$-algebra with a tracial state (see e.g., Exercise 2.8.37). Prove that $\mathcal{Q}(A)$ has a tracial state.

**Exercise 13.4.14.** Suppose $A$ is the stabilization of a UHF algebra. Prove that $\mathcal{Q}(A)$ is not simple.

**Exercise 13.4.15.** Prove that every surjective $\ast$-homomorphism between $C^*$-algebras extends to a surjective $\ast$-homomorphism between their multiplier algebras.

We outline an alternative construction of the multiplier algebra in one (long) exercise.

**Exercise 13.4.16.** Suppose $A$ is a $C^*$-algebra. A double centralizer of $A$ is a pair $(L, R)$ of linear operators on $A$ such that $aL(b) = R(a)b$ for all $a$ and $b$ in $A$. A left centralizer of $A$ is a linear operator $L$ on $A$ such that $L(ab) = L(a)b$ for all $a$ and $b$ in $A$. A right centralizer of $A$ is a linear operator $R$ on $A$ such that $R(ab) = aR(b)$ for all $a$ and $b$ in $A$. Prove the following.

1. Every $b \in \mathcal{M}(A)$ defines a double centralizer by $L_b(a) := ab$ and $R_b(b) := ba$.
2. If $(L, R)$ is a double centralizer then $L$ is a left centralizer and $R$ is a right centralizer.
3. Every left centralizer is bounded, and every right centralizer is bounded.
4. If $(L, R)$ is a double centralizer, then $\|L\| = \|R\|$.
5. Equip the space of double centralizers with the pointwise addition, multiplication defined by $(L_1, R_1)(L_2, R_2) := (L_1 \circ L_2, R_2 \circ R_1)$, adjoint defined by $(L, R)^* := (R^*, L^*)$, and the operator norm (well-defined by (4)). Prove that it is a $C^*$-algebra.
6. The map defined in (1) is an isomorphism from $\mathcal{M}(A)$ onto the algebra defined in (5).

**Notes for Chapter 13**

§13.2 The multiplier algebra $\mathcal{M}(A)$ can be defined in a number of equivalent ways (e.g., Corollary 13.2.2, the discussion preceding it, and Exercise 13.4.16). For a description of an important alternative construction of $\mathcal{M}(A)$ via Hilbert modules, see [27, II.7.3]. An abelian example of a $C^*$-algebra $A$ such that $\mathcal{M}(A)$ is equal to $\hat{A}$ is given in Exercise 13.4.6 (2). There are also a nontrivial noncommutative type I example ([117]) and even a simple example ([216]).
§13.3 Proposition 13.3.3 is an extension of a result of Weaver ([99]) who proved that the poset of projections in the Calkin algebra is not a lattice.

Another property of projections in coronas which is at least as curious as that in Proposition 13.3.3 was proved in [168]: The corona of the stabilization of a certain unital and projectionless $C^*$-algebra (the notorious Jiang–Su algebra) has real rank zero.

A moment of thought reveals that Exercise 13.4.15 deserves to be called a ‘non-commutative Tietze extension theorem.’ It was proved in [194, Proposition 3.12.10].
Chapter 14
Gaps and Incompactness

The tension between compactness and separability of $\mathcal{P}(\mathbb{N})$ (identified with the Cantor space) and sheer monstrosity of the quotient $\mathcal{P}(\mathbb{N})/\text{Fin}$ was successfully utilized since the early 20th century. Hausdorff’s and Luzin’s constructions of gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$ are early examples of this phenomenon (§9.3). Gaps of this kind provide examples of failure of compactness (or, in the language of §7.3, failure of reflection) in quotient structures. They are structures of uncountable cardinality $\kappa$ (typically, $\kappa = \aleph_1$) with a second-order property that all of their smaller substructures lack.

Quotient C*-algebras such as coronas of $\sigma$-unital, nonunital, C*-algebras provide fertile ground for gaps. In this chapter we show that gaps are imported by the canonical embedding of $\mathcal{P}(\mathbb{N})/\text{Fin}$ into the poset of projections of the Calkin algebra and coronas of other $\sigma$-unital, nonunital, C*-algebras. Using their images as parameters, we will construct interesting nonseparable subalgebras of the Calkin algebra. A Hausdorff gap is used to construct a pair of C*-algebras that are nonisomorphic, but arbitrarily near in the Kadison–Kastler distance (this is a theorem of Choi and Christensen). One can choose these algebras to be nuclear, and even AF. A Luzin family is used in §14.5 and §15.5 to construct a non-unitarizable representation of an abelian group into a C*-algebra and an amenable norm-closed algebra of operators on a Hilbert space not isomorphic to a C*-algebra, respectively.

14.1 Gaps in C*-Algebras

In this section we consider three kinds of gaps in C*-algebras. These are (i) gaps in the poset of positive elements of a C*-algebra, (ii) gaps in the poset of its projections, and (iii) gaps whose sides are C*-subalgebras. In addition, we establish the correspondence between hereditary subalgebras, norm-closed left ideals, and order ideals in every C*-algebra.
This section is a ‘noncommutative’ continuation of §9.3. Arbitrary subsets of a C*-algebra will be denoted by script letters $\mathcal{A}, \mathcal{B}$, etc. The following is an analog of Definition 9.3.1.

**Definition 14.1.1.** Suppose $C$ is a C*-algebra and $(\mathcal{A}, \mathcal{B})$ is a pair of subsets of $C$. The pair $(\mathcal{A}, \mathcal{B})$ is a pregap if $ab = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. (Such $\mathcal{A}$ and $\mathcal{B}$ are also said to be orthogonal.) The pair $(\mathcal{A}, \mathcal{B})$ is separated if there exists $c \in C$ such that $1 \cdot ac = a$ for all $a \in \mathcal{A}$ and $cb = 0$ for all $b \in \mathcal{B}$. If such $c$ exists, we say that $c$ separates $\mathcal{A}$ from $\mathcal{B}$. A pregap is a gap if it is not separated. The sets $\mathcal{A}$ and $\mathcal{B}$ are the sides of the (pre)gap $(\mathcal{A}, \mathcal{B})$. If both sides of a gap are separable then the gap is said to be countable.\(^2\)

Depending on the structure of $\mathcal{A}$ and $\mathcal{B}$, we distinguish several types of gaps in C*-algebras. If both $\mathcal{A}$ and $\mathcal{B}$ are subsets of $C_+$ (Proj$(C)$, respectively) the pair $(\mathcal{A}, \mathcal{B})$ is called a gap in $C_+$ (a gap in Proj$(C)$, respectively). If $\mathcal{A}$ and $\mathcal{B}$ are C*-subalgebras of $C$, the pair $(\mathcal{A}, \mathcal{B})$ is called a gap of C*-subalgebras of $C$. The analogous terminology is used for pregaps.

The following lemma simultaneously treats gaps in $C_+$ and gaps in Proj$(C)$.

**Lemma 14.1.2.** Suppose $C$ is a C*-algebra and $(\mathcal{A}, \mathcal{B})$ is a pair of subsets of $C_+$.

1. The pair $(\mathcal{A}, \mathcal{B})$ is a pregap if and only if the pair $(\mathcal{B}, \mathcal{A})$ is a pregap.
2. If the pair $(\mathcal{A}, \mathcal{B})$ is separated, then it is a pregap.
3. The pair $(\mathcal{A}, \mathcal{B})$ is separated by some $c \in C$ if and only if the pair $(\mathcal{B}, \mathcal{A})$ is separated by $1 - c^*$.
4. The pair $(\mathcal{A}, \mathcal{B})$ is a gap if and only if the pair $(\mathcal{B}, \mathcal{A})$ is a gap.

**Proof.** Only (2) requires a proof; proofs of the other equivalences are left to the reader.

Suppose that $c$ separates $\mathcal{A}$ from $\mathcal{B}$. Then $ab = (ac)b = a(cb) = 0$ for all $a \in A$ and all $b \in B$. \(\square\)

The analog of Lemma 14.1.2 holds for every pair $(A, B)$ of C*-subalgebras of $C$.

**Lemma 14.1.3.** Suppose $C$ is a C*-algebra and $A$ and $B$ are C*-subalgebras of $C$.

1. Then $A$ and $B$ are orthogonal if and only if $A_+$ and $B_+$ are orthogonal.
2. Also, $A$ and $B$ form a gap if and only if $A_+$ and $B_+$ form a gap. \(\square\)

As in §9.3, when $C$ is a quotient C*-algebra $M/J$, we frequently identify a gap $(\mathcal{A}, \mathcal{B})$ in $M/J$ with its preimage under the quotient map, $(\pi^{-1}[\mathcal{A}], \pi^{-1}[\mathcal{B}])$. More precisely, define a quasi-ordering $\leq_l$ on $M_\omega$ by

$$ a \leq_l b \text{ if and only if } (\exists c \in J_\omega) a + c \leq b. $$

Clearly, $a \leq_l b$ is equivalent to $\pi(a) \leq \pi(b)$.

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1 We emphasize that $c$ is not required to be self-adjoint.
2 ‘Separable’ would be more accurate but this word is too similar to ‘separated.’
Lemma 14.1.4. If $M = \mathcal{M}(J)$ and $J$ is separable, then the relation $\leq_J$ is analytic in the strict topology of $\mathcal{M}(J)$.

Proof. Lemma 13.1.8 implies that $J$ is a Borel subset of $\mathcal{M}(J)$ and therefore $\leq_J$ is a continuous image of the Borel set $\{(a, b, c) \in (\mathcal{M}(J)_w)^2 \times \mathcal{M}(J)_w : a + c \leq b\}$. □

Every pregap in $\mathcal{P}(\mathbb{N})/\text{Fin}$ can be enlarged to a pregap in $\mathcal{P}(\mathbb{N})/\text{Fin}$ both of whose sides are ideals in such a way that one of those pregaps is a gap if and only if the other one is (Exercise 9.10.3). The analogous statement is true in the context of gaps in $C^*$-algebras, with the right analog of ideals in the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$ being provided by the following definition.

Definition 14.1.5. An order ideal in a $C^*$-algebra $C$ is a norm closed subset $\mathcal{I}$ of $C_+$ which satisfies the following.

1. $\mathcal{I}$ is a cone, i.e., it is closed under addition and multiplication by positive scalars.
2. $\mathcal{I}$ is hereditary, i.e., $a \in \mathcal{I}$ and $0 \leq b \leq a$ implies $b \in \mathcal{I}$.

A word on the protagonists of Proposition 14.1.6 below is in order. If $C$ is a $C^*$-algebra and $\mathcal{I} \subseteq C$ then $\text{her}(\mathcal{I})$ is the smallest hereditary $C^*$-subalgebra ($\S 2.1.6$) of $C$ including $\mathcal{I}$. Norm-closed left ideals already appeared as left kernels of states $L_\omega$ in both the GNS construction and our study of pure states in $\S 3.6$.

Proposition 14.1.6. Suppose $C$ is a $C^*$-algebra.

1. The function $B \mapsto B_+$ is an order-preserving bijection between hereditary $C^*$-subalgebras and order ideals of $C$.
2. The function $\mathcal{B} \mapsto \{a \in C : a^*a \in \mathcal{B}\}$ is an order-preserving bijection between order-ideals and norm-closed left ideals of $C$.
3. The function $J \mapsto J \cap J^*$ is an order-preserving bijection between norm-closed left ideals and hereditary subalgebras of $C$.

Proof. Since neither this proposition nor Corollary 14.1.7 will be used explicitly, we only prove that the function defined in (2) has the codomain as stated. The analogous assertion is evident in the case of (1) and (3), and the monotonicity is evident for each of these three functions. A proof that each of the functions is a bijection can be found in e.g., [194, Theorem 1.5.2].

If $\mathcal{B}$ is an order-ideal in $C$, then by the continuity of operations

$$J_\mathcal{B} := \{a \in C : a^*a \in \mathcal{B}\}$$

is a norm-closed subset of $C$. If $b \in J_\mathcal{B}$ and $a \in C$ then

$$(ab)^*ab = b^*a^*ab \leq b^*b||a||^2$$

and therefore $ab \in J_\mathcal{B}$. To prove that $J_\mathcal{B}$ is an additive subgroup, fix $b$ and $c$ in $J_\mathcal{B}$. By Exercise 1.11.41 we have $(b+c)^*(b+c) \leq 2(b^*b + c^*c)$ and therefore $b + c \in J_\mathcal{B}$. This proves that $J_\mathcal{B}$ is a left ideal. □
Corollary 14.1.7. If $\mathcal{A} \subseteq C_+$ then the smallest order ideal that contains $\mathcal{A}$ is equal to $\text{her}(\mathcal{A})_+$. \hfill \Box

The following proposition will not be used and its (easy) proof is omitted.

Proposition 14.1.8. Suppose $C$ is a $C^*$-algebra. For a pair $\mathcal{A}, \mathcal{B}$ the following are equivalent.

1. The pair $(\mathcal{A}, \mathcal{B})$ is a pregap (gap, respectively) in $C_+$.
2. The pair $(\text{her}(\mathcal{A}), \text{her}(\mathcal{B}))$ is a pregap (a gap, respectively) of $C^*$-subalgebras of $C$. \hfill \Box

14.2 A Gap-Preserving Order-Embedding

In this section we construct an order-preserving and gap-preserving embedding of the Boolean algebra $\mathcal{P}(\mathbb{N})/\text{Fin}$ into the poset of all projections of the corona of any $C^*$-algebra that has a sequential approximate unit consisting of projections. Unlike gaps, maximal almost disjoint families in $\mathcal{P}(\mathbb{N})/\text{Fin}$ are not preserved by this embedding.

This section relies on §9.1, §9.3, and §14.1. An order-embedding between two quasi-ordered sets $\Phi: \mathcal{Q} \to \mathcal{P}$ is gap-preserving if $(\Phi[\mathcal{A}], \Phi[\mathcal{B}])$ is a gap in $\mathcal{P}$ whenever $(\mathcal{A}, \mathcal{B})$ is a gap in $\mathcal{Q}$.

Not every injective $^*$-homomorphism is gap-preserving (Exercise 14.6.2). In particular, being a gap of projections in a $C^*$-subalgebra $C$ of $\mathcal{Q}(\mathcal{A})$ is in general weaker than being a gap in $\text{Proj}(\mathcal{Q}(\mathcal{A}))$ whose sides are included in $\text{Proj}(\mathcal{C})$, as in the conclusion of Theorem 14.2.1.

Gaps can be imported into $\mathcal{Q}(\mathcal{A})$ from $\mathcal{P}(\mathbb{N})/\text{Fin}$ in bulk.

Theorem 14.2.1. If $A$ is a nonunital $C^*$-algebra with a sequential approximate unit consisting of projections then there are an abelian $C^*$-subalgebra $C$ of $\mathcal{M}(\mathcal{A})$ and an order-isomorphism $\Phi: \mathcal{P}(\mathbb{N}) \to \text{Proj}(\mathcal{C})$ that sends gaps in $\mathcal{P}(\mathbb{N})/\text{Fin}$ to gaps in $\text{Proj}(\mathcal{Q}(\mathcal{A}))$.

Proof. Let $e_n$, for $n \in \mathbb{N}$, be an approximate unit of $A$ consisting of projections and let (with $e_1 = 0$) $f_n := e_{n+1} - e_n$ for all $n$. Since $e_m \leq e_n$ implies $e_me_n = e_m = e_ne_m$, the algebra $C_0 := C'(\{e_n: n \in \mathbb{N}\})$ is abelian. Let $C$ be the strict closure of $C_0$ in $\mathcal{M}(A)$. For $X \subseteq \mathbb{N}$ the sequence of partial sums $s_m := \sum_{n \in X \cap m} f_n$, for $m \in \mathbb{N}$, converges to an element in $\mathcal{M}(\mathcal{A})$ by Lemma 13.1.11. We let $p_X$ be the strict limit of this sequence of partial sums and write $p_X := \sum_{n \in X} f_n$.

Consider $\mathcal{P}(\mathbb{N})$ with its Cantor-set topology and $\mathcal{M}(\mathcal{A})_1$ with its strict topology. The former space is compact and metrizable, the latter is Polish (Lemma 13.1.8), and the map from $\mathcal{P}(\mathbb{N})$ to $\mathcal{M}(\mathcal{A})_1$ defined by $\Phi(X) := p_X$ is continuous. It is clearly order-preserving and orthogonality-preserving. By $[X]_{\text{Fin}}$ we denote the Fin-equivalence class of a set $X$. 
Corollary 14.2.3. The standard embedding \( \Psi : \mathcal{P}(\mathbb{N}) \to \text{Proj}(\ell_\infty) \) sends gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) to gaps in \( \mathcal{L}(H) \).

The embedding \( \Psi : \mathcal{P}(\mathbb{N}) \to \mathcal{S}(H) \) as in Corollary 14.2.3 does not preserve all ‘gap-like’ structure of \( \mathcal{P}(\mathbb{N})/\text{Fin} \). The following is a noncommutative analog of the almost disjoint family in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) (Definition 9.2.1).
Definition 14.2.4. A family \( \mathcal{A} \) of projections in \( \mathcal{B}(H) \) is almost orthogonal if \( pq \) is compact for all distinct \( p \) and \( q \) in \( \mathcal{A} \). A family \( \mathcal{A} \) is maximal almost orthogonal if it is almost orthogonal and if \( \mathcal{A}' \supseteq \mathcal{A} \) is almost orthogonal then \( \mathcal{A}' = \mathcal{A} \).

Proposition 14.2.5. There is a maximal almost disjoint family \( \mathcal{A} \subseteq P(N) \) whose image under \( \Psi \) as in Corollary 14.2.3 is not a maximal almost orthogonal family.

Proof. Let \( p_X := \Psi(X) \) for \( X \subseteq N \). Given an orthonormal basis \( \xi_n \), for \( n \in N \), of \( H \) the vectors
\[
\eta_n := 2^{-n/2} \sum_{j=2^n}^{2^{n+1}-1} \xi_j
\]
for \( n \in N \), are orthonormal. Let \( q := \text{proj}_{\{p \in \mathcal{P}(N) : p \leq \xi_n \}} \). As in the proof of Theorem 14.2.1, \( \phi_q(X) := \|qX\| \) defines a lower semicontinuous submeasure on \( N \). Since \( \lim n \|q\xi_n\| = 0 \), Lemma 9.1.10 implies that \( \text{Exh}(\phi_q) \) is a dense \( P \)-ideal on \( N \). Since \( \text{Exh}(\phi_q) \) is dense, by Zorn’s lemma we can find a maximal almost disjoint family \( \mathcal{A} \) contained in \( \text{Exh}(\phi_q) \). Then \( q \) is almost orthogonal to \( p_X \) for all \( X \in \mathcal{A} \), and \( \{p_X : X \in \mathcal{A} \} \) is not a maximal orthogonal family in \( \mathcal{B}(H) \).

14.3 Twists in Massive C*-algebras

In this section we construct non-commutative analogs of Luzin families (§9.3), called twists, in the poset of projections of the Calkin algebra. Lemma 12.4.4 implies that every countable family of commuting projections in \( \mathcal{Q}(H) \) lifts to a family of commuting projections. Theorem 12.3.7 provides a family of \( c \) commuting projections that cannot be lifted to commuting projections. We can do better (Corollary 14.3.4).

Definition 14.3.1. A twist of projections is an uncountable orthogonal family \( \mathcal{F} \) of projections in a quotient \( M/J \) such that the following two conditions hold.

1. A subfamily \( \mathcal{F}_0 \) of \( \mathcal{F} \) can be lifted to a family of commuting projections in \( M \) if and only if it is countable.
2. For every projection \( q \) in \( M/J \) at most one of the sets \( \{p \in \mathcal{F} : pq = 0\} \) and \( \{p \in \mathcal{F} : (1-q)p = 0\} \) is uncountable.

Theorem 14.3.2. There exists a twist of \( \mathcal{R}_1 \) projections in the Calkin algebra.

A proof of Theorem 14.3.2 will be given after a preliminary lemma modeled on Luzin’s condition (see the proof of Theorem 9.3.6). Fix an increasing sequence \( r_n \) of finite-rank projections in \( \mathcal{B}(H) \) such that \( \text{SOT-lim}_n r_n = 1 \). Then \( \|s(a)\| = \inf_n \|(1-r_n)a\| \) for every \( a \in \mathcal{B}(H) \).

Lemma 14.3.3. Suppose that \( \varepsilon > 0 \) and \( \{p_\alpha : \alpha < \mathcal{R}_1\} \) is a family of projections in \( \mathcal{B}(H) \) such that \( p_\beta p_\alpha \) is compact but the set \( \{\alpha < \beta : \|(1-r)p_\alpha, p_\beta\| \leq \varepsilon\} \) is finite for every \( \beta < \mathcal{R}_1 \) and every finite-rank projection \( r \). Then the following hold.
1. If $X \subseteq \mathbb{R}_1$ then $\{ \pi(p_\alpha) : \alpha \in X \}$ have commuting lifts if and only if $X$ is countable.

2. There is no projection $q$ in $\mathcal{B}(H)$ such that both sets $\{ \alpha : p_\alpha q $ is compact$ \}$ and $\{ \alpha : p_\alpha(1-q) $ is compact$ \}$ are uncountable.

**Proof.** (1) Since $\mathcal{C}^*(\pi(p_\alpha) : \alpha \in X)$ is abelian it has real rank zero, hence if $X$ is countable then $\pi(p_\alpha)$ for $\alpha \in X$ can be lifted to commuting projections by Lemma 12.4.4.

Fix an uncountable $X \subseteq \mathbb{R}_1$ and assume there are commuting projections $q_\alpha$, for $\alpha < \mathbb{R}_1$, in $\mathcal{B}(H)$ such that $p_\alpha - q_\alpha$ is compact for all $\alpha$. For each $\alpha$ let $n(\alpha)$ be large enough so that $\|(1-r_n(\alpha))(p_\alpha - q_\alpha)\| < \varepsilon/4$. Since the assumption on $\mathcal{F}$ holds for all of its uncountable subsets, by passing to an uncountable subset of $X$ we may assume that there exists $n$ such that $n(\alpha) = n$ for all $\alpha$.

Since $[q_\beta, q_\alpha] = 0$, we have

$$\|(1-r_n)p_\alpha p_\beta\| < \|(1-r_n)(p_\alpha p_\beta - q_\alpha q_\beta)\| + \|(1-r_n)(q_\beta q_\alpha - p_\beta p_\alpha)\| < \varepsilon$$

for all $\alpha$ and $\beta$. If $\beta \in X$ then $X \cap \beta \subseteq \{ \alpha < \beta : \|(1-r_n)p_\alpha p_\beta\| \leq \varepsilon \}$. Since $X$ is uncountable, there exists $\beta \in X$ such that $X \cap \beta$ is infinite; contradiction.

(2) Assume the contrary and fix $q$ such that both $X_0 := \{ \alpha : p_\alpha q $ is compact$ \}$ and $X_1 := \{ \alpha : p_\alpha(1-q) $ is compact$ \}$ are uncountable. Since $p_\alpha$ and $1-q$ are self-adjoint, $(1-q)p_\alpha$ is compact for all $\alpha \in X_1$ and $q_\alpha p_\beta$ is compact for all $\alpha \in X_0$. For $\alpha \in X_0 \cup X_1$ fix $n(\alpha)$ such that $\max(\|(1-r_n(\alpha)p_\alpha q\|, \|(1-r_n(\alpha))q_\alpha p_\alpha\|) < \varepsilon/4$ if $\alpha \in X_0$ and $\max(\|(1-r_n(\alpha))(1-q)p_\alpha\|, \|(1-r_n(\alpha))p_\alpha(1-q)\|) < \varepsilon/4$ if $\alpha \in X_1$. By a counting argument and shrinking $X_0$ and $X_1$ if necessary we may find $n$ such that $n = n(\alpha)$ for all $\alpha$. For $\alpha \in X_0$ and $\beta \in X_1$ we have

$$\|(1-r_n)p_\alpha p_\beta\| \leq \|(1-r_n)p_\alpha q p_\beta\| + \|(1-r_n)p_\alpha(1-q)p_\beta\| \leq \|(1-r_n)p_\alpha q\| + \|(1-r_n)(1-q)p_\beta\| < \varepsilon/2$$

and similarly $\|(1-r_n)p_\beta p_\alpha\| < \varepsilon/2$. Therefore $\|(1-r_n)p_\alpha p_\beta\| < \varepsilon$ for $\alpha \in X_0$ and $\beta \in X_1$. If $\beta$ is such that $X_0 \cap \beta$ is infinite, then there are infinitely many $\alpha < \beta$ such that $\|(1-r_n)(p_\alpha p_\beta)\| < \varepsilon$; contradiction. $\square$

**Proof (Theorem 14.3.2).** For a projection $p$ write $p^1 := p$ and $p^\perp := 1-p$. We will recursively construct a family of projections $\mathcal{F} := \{ p_\alpha : \alpha < \mathbb{R}_1 \}$ in $\mathcal{B}(H)$ such that for all $\alpha < \beta$ we have

1. $p_\alpha p_\beta$ is compact, and
2. for every $n$ the set $\{ \gamma < \beta : \|(1-r_n)p_\gamma p_\beta\| \leq 1/8 \}$ is finite.

The recursive construction starts by choosing $p_n$, for $n < \mathbb{N}_0$, simultaneously so that their ranges are infinite-dimensional and orthogonal. Then (1) and (2) are satisfied.

Assume $p_\alpha$, for $\alpha < \beta$, have been chosen to satisfy all relevant instances of (1) and (2). Since $\beta$ is countable, by Lemma 12.4.4 applied to $\mathcal{C}^*(\pi(p_\alpha) : \alpha < \beta)$ we can find an orthonormal basis $(\xi_i)$ of $H$ that diagonalizes all $\pi(p_\alpha)$ for $\alpha < \beta$. Enumerate the lifts of these projections as $q_i$, for $i \in \mathbb{N}$, and fix $X_i \subseteq \mathbb{N}$ such that
\[ q_i = \text{proj}_{\text{span}(\xi_{j(i)}; j \in \mathbb{N})} \] for all \( i \). Recursively find natural numbers \( l(i) \) and \( m(i, j) \), for \( i \in \mathbb{N} \) and \( j < 2 \), such that for all \( i \) and \( j < 2 \) the following conditions hold.

3. \( \|(1 - r_{l(i)})(q_j - p_{X_j})\| < \frac{1}{16} \),
4. \( l(i) < m(i, j) < l(i + 1) \),
5. \( m(i, j) \notin X_k \) for all \( k \leq i \), and
6. \( m(i, 0) \in X_\alpha, m(i, 1) \notin X_i \).

Since \( q_j \) is compact, \( l(j) \) only needs to be chosen sufficiently large in order to satisfy (3). The sets \( X_j \), for \( j \in \mathbb{N} \), are almost disjoint and infinite, and therefore one can find \( m(i, 0) \) and \( m(i, 1) \) as required.

The vectors \( \eta_i := \mathcal{S}_i \xi_{m(i, 0)} + \xi_{m(i, 1)} \) form an orthonormal sequence. Let \( p_\beta \) be the projection to the closed linear span of this sequence. Since \( p_\beta p_{X_j} \) has finite rank for all \( j \), the product \( p_\beta q_j \) is compact for all \( j \). In other words, \( p_\beta p_\alpha \) is compact for all \( \alpha < \beta \) and (1) holds.

To prove (2), fix \( i \in \mathbb{N} \) for a moment. Then \( p_\alpha p_\beta \eta_i = p_\alpha \eta_i \) and \( p_\beta p_\alpha \eta_i \approx 1/16 p_\beta \mathcal{S}_{m(i, 0)} \approx 1/16 \eta_i \).

As \( \|\mathcal{S}_{m(i, 0)} - 1\| \eta_i = \|\mathcal{S}_{m(i, 0)} - \mathcal{S}_{m(i, 1)}\| = 1 \frac{1}{2} \cdot \|p_\alpha \cdot p_\beta\| > \frac{1}{8} \) follows.

Each of the sequences \( \eta_i, \mathcal{S}_{m(i, 0)}, \) and \( \xi_{m(i, 1)} \) is orthogonal and therefore weakly null. By the computations from the previous paragraph, for every fixed \( r_n \) we have \( \|(1 - r_n)\| q_j, p_\beta \| > \frac{1}{8} \) for all but finitely many \( i \). Therefore \( \|(1 - r_n)\| p_\alpha, p_\beta \| > \frac{1}{8} \) for all but finitely many \( \alpha < \beta \).

This describes the recursive construction of projections \( p_\alpha \), for \( \alpha < \mathbb{K} \), satisfying (1) and (2). By Lemma 14.3.3, the family \( \mathcal{F} := \{ \pi(p_\alpha) : \alpha < \mathbb{K} \} \) is a twist of projections in \( L(H) \).

**Corollary 14.3.4.** There exists a family of \( \mathbb{K} \) commuting—even orthogonal—projections \( \mathcal{F} \) in \( L(H) \) such that a subfamily of \( \mathcal{F} \) has a commutative lift if and only if it is countable.

14.4 An Application of Gaps to Kadison–Kastler Stability

In this section we study gaps in the Calkin algebra whose sides are subalgebras and construct abelian gaps as well as gaps without characters. These gaps are used to provide a (nonseparable) counterexample to the Kadison–Kastler stability of \( \mathcal{C}^* \)-subalgebras of \( B(H) \).

The **Kadison–Kastler distance** \( d_{\mathsf{KK}} \) on \( \mathcal{C}^* \)-subalgebras of \( B(H) \) is the Hausdorff distance between their unit balls:

\[ d_{\mathsf{KK}}(A, B) = \max(\sup_{a \in A : \|a\| \leq 1} \inf_{b \in B : \|b\| \leq 1} \|a - b\|, \sup_{b \in B : \|b\| \leq 1} \inf_{a \in A : \|a\| \leq 1} \|a - b\|). \]

This metric was introduced in [147] where it was conjectured (somewhat vaguely) that \( d_{\mathsf{KK}} \)-near operator algebras are isomorphic and even unitarily equivalent by a unitary in the neighbourhood of the identity. This conjecture is true for von
Neumann algebras. It is not known whether its first part is true for separable C*-algebras.\(^3\) The main result of this section, Theorem 14.4.7, provides a non-separable example.

**Definition 14.4.1.** A gap of C*-algebras is said to be abelian if both of its sides are abelian (see also Exercise 14.6.8).

Our convention is that a gap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\) is a pair of C*-algebras of \(\mathcal{M}(A)\) whose images under the quotient map are abelian and not necessarily abelian. Corollary 14.3.4 can be used to find a pair of abelian C*-algebras \((A, B)\) in \(\mathcal{D}(H)\) that are orthogonal and not separated that cannot be lifted to abelian C*-algebras of \(\mathcal{B}(H)\) (Exercise 14.6.7).

Since for every nonzero positive element \(h \in \mathcal{D}(H)\) the hereditary subalgebra \(h\mathcal{D}(H)h\) is nonabelian, there are no hereditary abelian gaps in the Calkin algebra.

**Corollary 14.4.2.** Suppose \(D\) is a nonunital C*-algebra with a countable approximate unit consisting of projections.

1. There exists an abelian gap \((A, B)\) in \(\mathcal{D}(D)\) such that both \(A\) and \(B\) have density character \(\aleph_1\).
2. There exists an abelian gap \((A, B)\) in \(\mathcal{D}(D)\) such that \(A\) is separable.
3. There exists an orthogonal family \(p_\alpha\), for \(\alpha < \aleph_1\), of projections in \(\mathcal{D}(D)\) such that any two of its disjoint uncountable subsets form a gap in \(\mathcal{D}(D)\).

**Proof.** Let \((\mathcal{A}, \mathcal{B})\) be a gap in \(\mathcal{P}(\mathbb{N})/\text{Fin}\). With the embedding \(\Phi\) defined in Theorem 14.2.1 let \(A := C^*(\Phi[\mathcal{A}])\) and \(B := C^*(\Phi[\mathcal{B}])\). Then \(A\) and \(B\) are orthogonal abelian algebras and they form a gap by Theorem 14.2.1.

1. If \((\mathcal{A}, \mathcal{B})\) is a Hausdorff gap (Theorem 9.3.7) then both \(\mathcal{A}\) and \(\mathcal{B}\) have density character \(\aleph_1\).
2. If \((\mathcal{A}, \mathcal{B})\) is as in Example 9.3.9 then \(A\) is separable.
3. Apply \(\Phi\) to the Luzin family constructed in Theorem 9.3.6.\(^4\) \(\square\)

A character of a unital C*-algebra is a unital \(^*\)-homomorphism into \(\mathbb{C}\).

**Lemma 14.4.3.** Suppose that \(A\) is a unital C*-algebra that has a unital copy of \(M_n(\mathbb{C})\) for some \(n \geq 2\). Then \(A\) has no characters.

**Proof.** Suppose \(\varphi: A \to \mathbb{C}\) is a \(^*\)-homomorphism. Since \(M_n(\mathbb{C})\) is simple, the restriction of \(\varphi\) to the unital copy of \(M_n(\mathbb{C})\) vanishes. \(\square\)

Given \(n \geq 1\), a C*-algebra \(A\) is \(n\)-homogeneous if each one of its irreducible representations is on an \(n\)-dimensional Hilbert space. It is \(n\)-subhomogeneous if each one of its irreducible representations is on an \(m\)-dimensional Hilbert space for some \(m \leq n\). The simplest \(n\)-homogeneous C*-algebras are of the form \(M_n(C_0(X))\) for a locally compact Hausdorff space \(X\). Every \(n\)-homogeneous C*-algebra is of type I and therefore nuclear.

\(^3\) Its second part is false for separable C*-algebras (see [139]).

\(^4\) As pointed out after Definition 14.4.1 and Exercise 14.6.7, a twist of projections does not provide a proof of (3).
Lemma 14.4.4. There exists a gap $(A,B)$ of subalgebras in $\mathcal{A}(H)$ such that $B$ is abelian and $A$ does not have characters. It can be chosen so that both $A$ and $B$ have density character equal to $\aleph_1$ and $A$ is 2-homogeneous.

Proof. We will find a gap as required in $M_2(\mathcal{A}(H)) \cong \mathcal{A}(H)$. Let $A_0, B_0$ be an abelian gap in $\mathcal{B}(H)$ such that both $A_0$ and $B_0$ have density character $\aleph_1$ (Corollary 14.4.2 (2)) and define

$$B := \left\{ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} : b \in B_0 \right\}, \quad A := M_2(A_0).$$

Clearly, $B$ is abelian and $A$ is 2-homogeneous, $(A,B)$ is a pregap, and Lemma 14.4.3 implies that $A$ has no characters. To see that $A$ and $B$ form a gap, suppose otherwise and fix $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in M_2(\mathcal{A}(H))$ that separates $A$ and $B$. Then $c_{11}$ separates $A_0$ and $B_0$; contradiction.

Lemma 14.4.5. Suppose $\Psi : C_0 \to C_1$ is an isomorphism between $C^*$-algebras such that $C_j = A_j \oplus B_j$ where $A_j$ has no characters and $B_j$ is abelian for $j < 2$. Then $\Psi[A_0] = A_1$ and $\Psi[B_0] = B_1$.

Proof. Fix $j < 2$. Let $\Omega_j$ be the space of all characters of $C_j$. Then $A_j = \bigcap_{\varphi \in \Omega_j} \ker(\varphi)$. Since $\Psi$ is an isomorphism, $\Omega_0 = \{ \varphi \circ \Psi : \varphi \in \Omega_1 \}$.

Therefore $A_0 = \bigcap_{\varphi \in \Omega_1} \ker(\varphi \circ \Psi) = \Psi^{-1}[\bigcap_{\varphi \in \Omega_1} \ker(\varphi)] = \Psi^{-1}[A_1]$. □

Lemma 14.4.6. Suppose $(A,B)$ is a pair of $C^*$-subalgebras of a unital $C^*$-algebra $C$ that form a gap. Suppose $p$ and $q$ are projections of rank 1 in $M_2(\mathbb{C})$. The $C^*$-subalgebras $A_p := \{ p \} \otimes A$ and $B_q := \{ q \} \otimes B$ of $M_2(C)$ form a pregap which is a gap if and only if $pq \neq 0$.

Proof. Since every elementary tensor in $A_p$ is orthogonal to every elementary tensor in $B_q$, $A_p$ and $B_q$ form a pregap.

If $pq = 0$ then $(A_p,B_q)$ is separated by $p \otimes 1$. To prove the converse, suppose that $pq \neq 0$ and the pair $(A_p,B_q)$ is separated. By replacing $p$ and $q$ with unitarily equivalent projections, we may assume $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $q = \begin{pmatrix} \sin^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \cos^2 \theta \end{pmatrix}$ for some $0 < \theta < 2\pi$. Fix $c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ that separates $(A_p,B_q)$.

For $a \in A$ we have $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $ac_{11} = a$ and $ac_{12} = 0$.

For $b \in B$ we have $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} b \sin^2 \theta & b \cos(\theta) \sin(\theta) \\ b \cos(\theta) \sin(\theta) & b \cos^2(\theta) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore $c_{11}b \sin^2 \theta + c_{12}b \sin \theta \cos \theta = 0$. Let $c := c_{11} + \cot \theta c_{12}$. Then $ac = a$ for all $a \in A$ and $cb = 0$ for all $b \in B$; contradiction. □

And now, for the application promised in the introduction to this section.
14.4 An Application of Gaps to Kadison–Kastler Stability

**Theorem 14.4.7.** There are C*-subalgebras $C_t$ of $\mathcal{B}(H)$ for $0 \leq t < 1$ such that $\lim_{t \to 0} d_{\text{KK}}(C_0, C_t) = 0$ but $C_0 \neq C_t$ for all $t > 0$. These C*-algebras can be chosen to be 2-subhomogeneous and of density character $\aleph_1$.

**Proof.** By Lemma 14.4.4, there are C*-subalgebras $A$ and $B$ of $\mathcal{B}(H)$ such that $B$ is abelian, $A$ is 2-homogeneous, $A$ does not have characters, the density characters of $A$ and $B$ are both equal to $\aleph_1$, and $(A, B)$ is a gap in $\mathcal{D}(H)$.

Consider the following path of projections in $M_2(\mathbb{C})$

$$p_t := \begin{pmatrix} \cos^2(t\pi/2) & \cos(t\pi/2)\sin(t\pi/2) \\ \cos(t\pi/2)\sin(t\pi/2) & \sin^2(t\pi/2) \end{pmatrix},$$

for $0 \leq t \leq 1$. As $p_t$ is a scalar projection in $M_2(\mathbb{C})$, both $A \otimes \{p_t\}$ and $B \otimes \{p_t\}$ are subalgebras of $M_2(\mathcal{B}(H)) \cong \mathcal{B}(H) \otimes M_2(\mathbb{C})$ for all $t$.

For $0 \leq t \leq 1$ let $C_t := C^*(A \otimes \{p_t\}, B \otimes \{p_t\}, M_2(\mathcal{K}(H)))$. This is a 2-subhomogeneous C*-subalgebra of $M_2(\mathcal{B}(H))$ of density character $\aleph_1$. Since both $A$ and $B$ are C*-algebras and $ab \in \mathcal{K}(H)$ for all $a \in A$ and $b \in B$, we have

$$C_t = \{a \otimes p_t + b \otimes p_t + c : a \in A, b \in B, c \in M_2(\mathcal{K}(H))\}.$$ 

This implies that every $t$ satisfies

$$d_{\text{KK}}(C_0, C_t) \leq \sup\{\|b \otimes p_t - b \otimes p_t\| : b \in B\} \leq \|p_0 - p_t\|.$$ 

Since $\lim_{t \to 0} \|p_t - p_0\| = 0$, we have $\lim_{t \to 0} d_{\text{KK}}(C_0, C_t) = 0$.

We have yet to prove that $C_0 \neq C_t$ for all $t > 0$. Fix $0 < t < 1$ and, towards obtaining a contradiction, suppose that $\Phi: C_0 \to C_t$ is an isomorphism. As $\mathcal{K}(H)$ is the unique essential ideal in both $C_t$ and $C_0$, we have $\Phi(\mathcal{K}(H)) = \mathcal{K}(H)$. By Exercise 2.8.30 there exists a unitary $u$ such that $\Phi(d) = Adu(d)$ for $d \in \mathcal{K}(H)$. Since $\mathcal{K}(H)$ is an essential ideal in both $C_0$ and $\mathcal{B}(H)$, Lemma 2.5.6 implies $\Phi(a) = Adu(a)$ for all $a \in A$. In particular $\Phi$ extends to an inner automorphism $Adu$ of $\mathcal{B}(H)$ and $\Phi: \mathcal{D}(H) \to \mathcal{D}(H)$ defined by

$$\Phi(a + \mathcal{K}(H)) = \Phi(a) + \mathcal{K}(H)$$

extends to an automorphism of $\mathcal{D}(H)$.

Let $A_1 := A \otimes \{p_t\}$ and $B_t := B \otimes \{p_t\}$. Since $p_t$ and $p_1$ are orthogonal if and only if $t = 0$, Lemma 14.4.6 implies that $(A_1, B)$ forms a gap if and only if $t > 0$. By Lemma 14.4.5, $\Phi([A_1]) = [A_1]$ and $\Phi([B]) = [B_t]$. If $c \in \mathcal{B}(H)_+$ separates $A_1$ from $B_0$ then $\Phi(c)$ separates $A_1$ from $B_t$; contradiction. Therefore $C_0$ and $C_t$ are not isomorphic. \qed
14.5 Uniformly Bounded Group Representations, I

In this section we discuss unitarizability of uniformly bounded representations of a group into a C*-algebra and construct a uniformly bounded representation of an uncountable discrete abelian group which is not unitarizable, although its restriction to any countable subgroup is unitarizable.\(^5\)

We consider algebras of operators on a complex Hilbert space that are not necessarily self-adjoint. Therefore:

Until the end of this section a subalgebra of a C*-algebra is norm-closed, but not necessarily a C*-subalgebra, and a homomorphism between algebras of operators (even if they are C*-algebras) is not necessarily a *-homomorphism.

In Example 14.5.2 below we will need the following obvious estimate for the operator norm in \(M_n(\mathbb{C})\).

Lemma 14.5.1. If \(s = (s_{ij}) \in M_n(\mathbb{C})\) then \(\|s\| \leq n \max_{i,j} |s_{ij}|\).

Example 14.5.2. The simplest example of a finite-dimensional, non-self-adjoint, operator algebra is in order.

1. For \(t \in \mathbb{R}\) let \(s_t := \begin{pmatrix} 1 & 0 \\ t & -1 \end{pmatrix}\). Then \(s_t^2 = 1\), and the algebra generated by \(s_t\) is

\[
A_t := \{xs_t + y : x, y \in \mathbb{C}\} = \left\{ \begin{pmatrix} x+y & 0 \\ tx & -x+y \end{pmatrix} : x, y \in \mathbb{C} \right\}.
\]

Since \(s_t\) has distinct eigenvalues 1 and \(-1\), it is diagonalizable. If \(b_t := \begin{pmatrix} 2 & 0 \\ t & 1 \end{pmatrix}\) then \(b_t s_t b_t^{-1}\) is diagonal and \(b_t A_t b_t^{-1}\) is the C*-algebra \(D_2\) of diagonal matrices in \(M_2(\mathbb{C})\). By Lemma 14.5.1 we have

\[
L := \sup_{0 \leq t \leq 1} \|b_t\| \|b_t^{-1}\| < 4.
\]

2. Suppose \(t\) and \(r\) are distinct nonzero complex numbers. Then \(s_t\) and \(s_r\) as in (1) generate the algebra of all lower-triangular matrices in \(M_2(\mathbb{C})\). Being 3-dimensional and noncommutative, it is not isomorphic to a C*-algebra. This implies that \(s_t\) and \(s_r\) cannot be diagonalized simultaneously.

Recall that \(U(M_2(\mathbb{C}))\) denotes the unitary group of \(M_2(\mathbb{C})\).

Lemma 14.5.3. If \(U\) and \(V\) are disjoint closed sets in \([0, 1]\) and \(K < \infty\) then

\[
L := \inf \{ \text{dist}(as^a^{-1}, U(M_2(\mathbb{C}))) + \text{dist}(as^a^{-1}, U(M_2(\mathbb{C}))) : a \in \text{GL}(M), \|a\| \|a^{-1}\| \leq K, r \in U, t \in V \} > 0.
\]

\(^5\) This result will be refined in §15.5, to which this section serves as a warmup.
Proof. Since \(|a||a^{-1}| \leq K\), the range set \(\mathcal{Z}\) for the parameters \((a, r, t)\) in the definition of \(L\) is compact. As the function \(\text{dist}(\cdot, U(M_2(\mathbb{C})))\) is continuous, it attains its infimum on \(\mathcal{Z}\) at some \((a, r, t)\). If this infimum is zero then \(r \neq t\) and \(a\) simultaneously diagonalizes \(s_{r}\) and \(s_{t}\), contradicting Example 14.5.2 (2).

\[\square\]

**Definition 14.5.4.** A representation \(\pi: \Gamma \to A\) of a discrete group \(\Gamma\) into a unital C*-algebra \(A\) is a homomorphism of \(\Gamma\) into \(\text{GL}(A)\). It is unitary if its range is a subset of the unitary group of \(A\). A representation \(\pi: \Gamma \to A\) is unitarizable if there exists \(\pi \in \text{GL}(A)\) such that \(g \mapsto a^{-1} \pi(g) a\) is a unitary representation. It is uniformly bounded if \(|\pi| := \sup_{g \in \Gamma} \|\pi(g)\|\) is finite.

**Example 14.5.5.** Suppose \(\pi: \Gamma \to A\) is a representation of a discrete group into a unital C*-algebra. Then the linear span of \(\pi[\Gamma]\) is a linear space closed under multiplication, and therefore its norm-closure is a subalgebra of \(A\). If \(\pi\) is in addition a unitary representation then the closure of the linear span of \(\pi[\Gamma]\) is self-adjoint and therefore its norm closure is a C*-subalgebra of \(A\).

Every unitary group representation has norm 1 and is therefore uniformly bounded. Every uniformly bounded representation of an amenable group into a von Neumann algebra is unitarizable (Exercise 15.6.25, but we will prove a more difficult related result in Proposition 15.4.1). The free group \(F_{\infty}\) has a uniformly bounded non-unitarizable representation, and conjecturally the property of having all uniformly bounded representations into \(\mathcal{B}(H)\) unitarizable characterizes amenable groups (see [203]).

**Proposition 14.5.6.** There exists a uniformly bounded and non-unitarizable representation of \(\bigoplus_{\mathbb{R}} \mathbb{Z}/2\mathbb{Z}\) into \(M_2(\ell_\infty(\mathbb{N}))/M_2(c_0(\mathbb{N})).\)

**Proof.** Let \(\{A_\alpha: \alpha < \mathbb{R}_+\}\) be the Luzin almost disjoint family constructed in Theorem 9.3.6. Therefore \(A_\alpha \cap A_\beta\) is finite for all distinct \(\alpha\) and \(\beta\) and for any \(X \subseteq \mathbb{N}\) at most one of the sets \(\{\alpha: A_\alpha \setminus X\text{ is finite}\}\) and \(\{\alpha: A_\alpha \cap X\text{ is finite}\}\) is uncountable. Let \(f: \mathcal{P}(\mathbb{N}) \to [0, 1]\) be a continuous injection, e.g., \(f(X) := \sum_{n \in X} 3^{-n-1}\).

For \(\alpha < \mathbb{R}_+\) let

\[t_\alpha := \begin{pmatrix} 1 & 0 \\ f(A_\alpha) & -1 \end{pmatrix} \otimes 1.\]

Define \(p_\alpha \in M_2(\ell_\infty(\mathbb{N}))\) by \(p_\alpha(n) = I_2\) if \(n \in A_\alpha\) and \(p_\alpha(n) = 0\) if \(n \notin A_\alpha\). This is a central projection. Let \(b_\alpha \in M_2(\ell_\infty(\mathbb{N}))\) be defined by

\[b_\alpha := t_\alpha p_\alpha + (1 - p_\alpha).\]

Lemma 14.5.1 implies \(|b_\alpha| = \max(||t_\alpha||, ||1 - p_\alpha||) \leq 2\).

Since \(t_\alpha^2 = 1\), we have \(b_\alpha^2 = 1\). Therefore there exists \(a \in M_2(\ell_\infty(\mathbb{N}))\) such that \(ab_\alpha a^{-1}\) is a diagonal unitary. For \(\alpha \neq \beta\) the elements \(b_\alpha\) and \(b_\beta\) can be simultaneously diagonalized in \(M_2(\ell_\infty(\mathbb{N}))\) if and only if \(b_\alpha(n)\) and \(b_\beta(n)\) can be simultaneously diagonalized for all \(n\). By the second part of Example 14.5.2, \(b_\alpha\) and \(b_\beta\) can be simultaneously diagonalized if and only if \(A_\alpha\) and \(A_\beta\) are disjoint.

If \(\alpha \neq \beta\) then \(A_\alpha \cap A_\beta\) is finite and therefore (writing \([x, y] = xy - yx\)
Let \( \pi : M_2(\ell_\infty(\mathbb{N})) \to M_2(\ell_\infty(\mathbb{N}))/M_2(c_0(\mathbb{N})) \) denote the quotient map. Let \( g_\alpha \) be the generator of the \( \alpha \)th copy of \( \mathbb{Z}/2\mathbb{Z} \) in \( \bigoplus_{\mathbb{K}_1} \mathbb{Z}/2\mathbb{Z} \). Since \( \pi(b_\alpha) \), for \( \alpha < \mathbb{K}_1 \), are commuting involutions, letting \( \sigma \) the generator of \( \alpha \)\( F \subseteq b \) uncountable.

\[ \bigoplus_{\mathbb{K}_1} \mathbb{Z}/2\mathbb{Z} \to GL(M_2(\ell_\infty(\mathbb{N}))/M_2(c_0(\mathbb{N}))). \]

Each \( \alpha \) satisfies \( \|\pi(b_\alpha)\| \leq \|b_\alpha\| \leq 2 \). Since the sets \( A_\alpha \) are almost disjoint, for \( F \subseteq \mathbb{K}_1 \) we have \( \|\sigma(\prod_{\alpha \in F} g_\alpha)\| = \|\prod_{\alpha \in F} \pi(b_\alpha)\| = \max_{\alpha \in F} \|b_\alpha\| \leq 2 \), and \( \sigma \) is a uniformly bounded representation of \( \bigoplus_{\mathbb{K}_1} \mathbb{Z}/2\mathbb{Z} \).

Suppose that the set \( S_a := \{ \alpha : \pi(ab_\alpha a^{-1}) \) is a unitary \} is uncountable for some \( a \in M_2(\ell_\infty(\mathbb{N})) \). By Lemma 6.6.5 there are disjoint closed intervals \( U \) and \( V \) in \([0,1] \) for which both \( \mathcal{X} := \{ \alpha \in S : f(A_\alpha) \in U \} \) and \( \mathcal{Y} := \{ \alpha \in S : f(A_\alpha) \in V \} \) are uncountable.

Let \( K := \|a\|\|a^{-1}\| \). Lemma 14.5.3 implies that there exists \( \varepsilon > 0 \) such that if \( b \in GL(M_2(\mathbb{C})) \) satisfies \( \|b\|\|b^{-1}\| \leq K \) then \( \text{dist}(bs, b^{-1}) + \text{dist}(bs, b^{-1}) \geq \varepsilon \) for all \( r \in U \) and \( t \in V \). Let
\[ X := \{ n \in \mathbb{N} : \text{dist}(a(n)s, a(n)^{-1}, U(M_2(\mathbb{C}))) < \varepsilon / 2 \text{ for some } r \in U \}. \]

Since \( \pi(ab_\alpha a^{-1}) \) is a unitary for all \( \alpha \), \( A_\alpha \setminus X \) is finite for every \( \alpha \in \mathcal{X} \).

On the other hand, if \( \alpha \in \mathcal{Y} \) then \( \text{dist}(a(n)b_\alpha a(n)^{-1}, U(M_2(\mathbb{C}))) > \varepsilon / 2 \) for at most finitely many \( n \in A_\alpha \), and therefore \( A_\alpha \cap X \) is finite. Therefore \( X \) separates two uncountable subsets, \( \mathcal{X} \) and \( \mathcal{Y} \), of a Luzin family; contradiction.

The discussion of the present section continues in §15.5, after we introduce a machinery that will, among other things, show that the conclusion of Proposition 14.5.6 does not hold for any countable amenable group.

### 14.6 Exercises

**Exercise 14.6.1.** Suppose \( A \) and \( B \) are hereditary, \( \sigma \)-unital, nonunital, \( C^* \)-subalgebras of the Calkin algebra \( \mathcal{Q}(H) \). Prove that there exists an inner automorphism \( \Phi \) of \( \mathcal{Q}(H) \) such that \( \Phi[A] = B \).

**Exercise 14.6.2.** Suppose \( A, B \) is a gap in a \( C^* \)-algebra \( C \). Prove that there exist a \( C^* \)-algebra \( D \) and an injective \( * \)-homomorphism \( \Phi : C \to D \) such that the pair \( (\Phi[A], \Phi[B]) \) is not a gap in \( D \).

Together with Theorem 14.2.1, Corollary 14.4.2, and Theorem 14.3.2, the following exercise shows that the gap spectrum of the Calkin algebra is considerably richer than that of \( \mathcal{Q}(\mathbb{N})/\text{Fin.} \)
Exercise 14.6.3. Prove that for every \( \varepsilon > 0 \) there is \( m \in \mathbb{N} \) such that \( \ell_2(m) \) has orthonormal bases \( e_i^0 \), for \( i < m \), and \( e_i^1 \), for \( i < m \), satisfying \( |\langle e_i^0 | e_j^1 \rangle| < \varepsilon \) for all \( i \) and \( j \).

Exercise 14.6.4. 1. Use Exercise 14.6.3 to prove that there is a gap in \( \text{Proj}(\mathcal{B}(H)) \) modulo \( \mathcal{K}(H) \) which has Borel, \( \sigma \)-directed sides (\( \text{Proj}(\mathcal{B}(H)) \) is taken with the strong operator topology).

2. Prove that there are Borel, \( \sigma \)-directed, order-ideals \( A_r \), for \( r \in \{0,1\}^{\mathbb{N}} \), in \( \text{Proj}(\mathcal{B}(H)) \) such that the pair \( (A_s, A_r) \) is a gap modulo \( \mathcal{K}(H) \) whenever \( r \neq s \).

Exercise 14.6.5. Prove the following analog of Proposition 9.6.11 (see also Corollary 14.3.2 and let \( A \neq 0 \)).

\[
b^* = \min \{|\mathcal{A}| : \text{there is a separable } \mathcal{B} \subseteq \mathcal{L}(H)_+ \text{ such that } (\mathcal{A}, \mathcal{B}) \text{ is a gap}.\]

Exercise 14.6.6. Prove a strengthening of Theorem 14.2.1: Suppose \( D \) is a nonunital, \( \sigma \)-unital, \( C^* \)-algebra. Then there are an abelian \( C^* \)-subalgebra \( C \) of \( \mathcal{A}(D) \) and an order-embedding \( \Phi: \mathcal{P}(\mathbb{N}) \to \text{Proj}(C)_+ \) that sends gaps in \( \mathcal{P}(\mathbb{N})/\text{Fin} \) to gaps in \( \text{Proj}(\mathcal{L}(D)) \).

Exercise 14.6.7. Let \( \mathcal{F} \) be a twist of \( \mathcal{R}_1 \) projections in \( \mathcal{L}(H) \) constructed in Theorem 14.3.2 and let \( A := C^*(\mathcal{F}) \).

1. Prove that a \( C^* \)-subalgebra of this abelian \( C^* \)-algebra has an abelian lift to \( \mathcal{L}(H) \) if and only if it is separable.

2. Use this to find a pair of abelian, orthogonal \( C^* \)-subalgebras of \( \mathcal{L}(H) \) that cannot be separated but are not the image of an abelian gap under the quotient map.

Exercise 14.6.8. Suppose \( D \) is a nonunital, \( \sigma \)-unital \( C^* \)-algebra. Prove that the analog of Corollary 14.4.2 (1)–(3) holds in \( \mathcal{L}(D) \):

1. There exists an abelian gap \( (A, B) \) in \( \mathcal{L}(D) \) such that both \( A \) and \( B \) have density character \( \mathcal{R}_1 \).

2. There exists an abelian gap \( (A, B) \) in \( \mathcal{L}(D) \) such that \( A \) is separable.

3. There exists an orthogonal family \( p_\alpha \), for \( \alpha < \mathcal{R}_1 \), of projections in \( \mathcal{L}(D) \) such that every pair of its disjoint uncountable subsets forms a gap in \( \mathcal{L}(D) \).

Exercise 14.6.9. Suppose \( A \) and \( B \) are \( C^* \)-subalgebras of \( \mathcal{B}(H) \) and \( d_{KK}(A, B) < 1 \). Prove that \( A \) and \( B \) have the same density character.

Compare the following exercise with Definition 8.2.8.

Exercise 14.6.10. Let \( C_t \), for \( 0 \leq t \leq 1 \), be subalgebras of \( \mathcal{B}(H) \) as in the proof of Theorem 14.4.7. Prove that for any \( 0 \leq t, s \leq 1 \) there exist clubs \( D_t \subseteq \text{Sep}(C_t) \) and \( D_s \subseteq \text{Sep}(C_s) \) and an order-isomorphism \( \Phi: D_s \to D_t \) such that \( A \) and \( \Phi(A) \) are isomorphic for all \( A \in D_s \).

Exercise 14.6.11. Use Theorem 14.3.2 to give an alternate proof (as well as a strengthening) of Theorem 11.1.2: Prove that if \( \mathcal{I} \subseteq \text{Proj}(\mathcal{L}(H)) \) is a twist of projections, then the \( C^* \)-subalgebra of \( \mathcal{B}(H) \) generated by \( \pi^{-1}[\mathcal{I}] \) does not have an abelian approximate unit.
§14.1 In the terminology of [42], $C^*$-subalgebras $A, B$ of $B(H)$ are essentially orthogonal if $\pi[A]$ and $\pi[B]$ are orthogonal in the Calkin algebra. $C^*$-subalgebras $A, B$ of $B(H)$ are essentially interconnected if they form a gap.

For a $C^*$-algebra $C$ and a pair $(\mathcal{A}, \mathcal{B})$ in $C_{+1}$, being a gap in $C$ (Definition 14.1.1) is apparently stronger than being a gap in the poset $(C_{+1}, \leq)$ (Definition 9.3.1) as in the latter case we consider only self-adjoint separators. I don’t know whether the two notions are equivalent.

Order ideals in $C^*$-algebras were introduced in [70], where Proposition 14.1.6 was proved. See also [194, §1.5].

§14.2 Gaps in $\text{Proj}(B(H))$ were first considered in [266], where Theorem 14.2.1 was proved in the case of the Calkin algebra. Exercise 14.6.4 is also taken from [266].

Maximal almost orthogonal families were introduced in [262], where Proposition 14.2.5 and several related relative consistency results were proved.

§14.3 Theorem 14.3.2 was inspired by Luzin’s Theorem 9.3.6. We give a version from [24] that improves the original statement from [99] and answers a question posed there. For a further generalization to other coronas see [245].

§14.4 Theorem 14.4.7 was proved in [42].

See e.g., [44] for the recent progress on the Kadison–Kastler stability question. Exercise 14.6.10 shows that the nonseparable example from Theorem 14.4.7 does not reflect (in the terminology of §7.3) to produce a separable example.

In the category of Banach spaces the question analogous to Kadison–Kastler’s has a negative answer. There is a Banach space with separable subspaces that are arbitrarily near in the analog of the Kadison–Kastler distance but not isomorphic ([146]).

§14.5 The results of this section are special cases of the main results of [43] and [249], given in full generality in §15.5.
Degree-1 Saturation

Deadlines are one of the most common ways of inducing stress that provoke an increase in performance. If the latter chapters of this book suddenly improve in quality, you now know why.

Dean Burnett, The Idiot Brain

Proceeding in the direction opposite from that of Chapter 14, we introduce the notion of countable degree-1 saturation. This is a weakening of the quantifier-free saturation (discussed in §16.1). Every corona of a $\sigma$-unital, nonunital, $C^*$-algebra is countably degree-1 saturated, and so is every ultraproduct of a $C^*$-algebra associated with a countably incomplete ultrafilter and this property, among other things, explains why there are no countable gaps in these massive $C^*$-algebras. Countable degree-1 saturation is also used to give a uniform treatment of several distinguishing properties of coronas of $\sigma$-unital $C^*$-algebras.

15.1 Degree-1 Types and Saturation

In this section we introduce countable degree-1 saturation of $C^*$-algebras and prove that the corona of every $\sigma$-unital, nonunital, $C^*$-algebra has this property.

The most common massive $C^*$-algebras are countably degree-1 saturated: coronas of $\sigma$-unital $C^*$-algebras, ultraproducts associated with ultrafilters on $\mathbb{N}$, reduced products associated with the Fréchet filter (i.e., the asymptotic sequence algebras), and quotients of $II_\infty$ factors with respect to Breuer ideals.

All massive quotient $C^*$-algebras have closure properties resembling those of von Neumann algebras, with two important differences. First, ‘limits’ can be found only for countable sets. Second, these ‘limits’ are neither unique nor canonical. These closure properties are unified by the model-theoretic notion of saturation.

The following is a variant of the model-theoretic definition of a type (see Definition 16.1.4 and the subsequent discussion of types as sets of conditions).
**Definition 15.1.1.** Fix a C$^*$-algebra $C$ and $n \in \mathbb{N}$. A *degree-1 condition* over $C$ in variables $\bar{x} = (x_j : j < n)$ is an expression of the form

$$\| \sum_{j<n} a_0, x_j^*, a_1, x_j + \sum_{j<n} a_2, x_j^*, d_j + a \| = r$$  \hspace{1cm} (15.1)$$

with all coefficients in $C$ and $r \in \mathbb{R}^\ast$. The condition $\| P(\bar{x}) \| = r$ is *satisfied* in $C$ by a tuple $\bar{b}$ of the appropriate length if $\| P(\bar{b}) \| = r$.

**Definition 15.1.2.** A *degree-1 $n$-type* over $C$ is a set of degree-1 conditions over $C$ in variables $\bar{x} = (x_j : j < n)$. We will write ‘type $t$’ instead of ‘degree-1 $n$-type’ whenever there is no danger of confusion. A type $t(\bar{x})$ is *realized* in $C$ if there exists $\bar{b}$ in the unit ball of $C$ such that every condition in $t(\bar{x})$ is satisfied by $\bar{b}$. A type $t(\bar{x})$ is *approximately realized* in $C$ (or *satisfiable*) if for every finite subset $t_0(\bar{x})$ of $t(\bar{x})$ and every $\epsilon > 0$ there exists $\bar{b}$ in the unit ball of $C$ such that for every condition $\| P(\bar{x}) \| = r$ in $t_0(\bar{x})$ we have $\| P(\bar{b}) \| - r < \epsilon$. Such $\bar{b}$ is a *partial realization* of $t(\bar{x})$.

Every realized type is satisfiable, but the converse is not necessarily true.

**Example 15.1.3.** Suppose $C$ is a C$^*$-algebra and $A$ is a C$^*$-subalgebra of $C$.

1. The *relative commutant* type $t^A_\text{rel}(x) := \{ \| [x,a] \| = 0 : a \in A \}$ is realized by $c \in C$ if and only if $c \in A' \cap C$.
2. If $A$ and $B$ are C$^*$-algebras then a *derivation* is a linear map $\delta : A \to B$ satisfying

$$\delta(ab) = \delta(a)b + a\delta(b).$$

for all $a$ and $b$ in $A$. A derivation from $A$ to itself of the form $\delta_\text{ad}(a) := ad - da$ is *inner*. If $A$ is a subalgebra of $C$ and $\delta : A \to C$ is a derivation then the type

$$t_\delta(x) := \{ \| \delta(a) - ax + xa \| = 0 : a \in A \}$$

is realized by $d \in C$ if and only if $\delta$ and $\delta_\text{ad}$ agree on $A$.

Each one of the types $t^A_\text{rel}$ and $t^\delta_\text{ad}$ has uncountably many conditions. However, these conditions are of the form $\| P(x,a) \| = 0$, for a fixed $*$-polynomial and a parameter $a \in A$. For a type $t$ of this form we can fix a dense subset $D$ of $A$ and consider the countable type $t'$ consisting only of conditions in $t$ with parameters in $D$. The types $t$ and $t'$ are *equivalent* in the sense that every (partial) realization of one is a (partial) realization of the other.

**Definition 15.1.4.** A C$^*$-algebra $C$ is *countably degree-1 saturated* if every satisfiable countable degree-1 type over $C$ is realized in $C$.

The following is the main result of this section (but see Theorem 15.2.4 below).

**Theorem 15.1.5.** The corona of every $\sigma$-unital, nonunital, C$^*$-algebra is countably degree-1 saturated.

---

1 This applies only to the present chapter; in Chapter 16.1 we will introduce the standard model-theoretic definition of a type.
Proof. Suppose $A$ is a $\sigma$-unital, nonunital, $C^*$-algebra and let $M := \mathcal{M}(A)$. Fix $n_0 \geq 1$ and a satisfiable countable degree-1 $n_0$-type

$$t(\bar{x}) = \{\|P_j(\bar{x})\| = r_j : j \in \mathbb{N}\}$$

over $M/A$. A lift of $a \in M/A$ is $\bar{a} \in M$ such that $\pi(\bar{a}) = a$ (§2.5). A lift of a *-polynomial $P(\bar{x})$ with coefficients in $M/A$ is a *-polynomial whose coefficients are lifts of the corresponding coefficients of $P(\bar{x})$. A lift of a condition $\|P(\bar{x})\| = r_j$ is a condition of the form $\|\bar{P}(\bar{x})\| = r_j$ where $\bar{P}(\bar{x})$ is a lift of $P(\bar{x})$.

Fix a lift $\|\bar{P}_m(\bar{x})\| = r_m$ of every condition $\|P_m(\bar{x})\| = r_m$ in $t$. Since $t$ is satisfiable, for every $n$ there exists $\bar{b}_n$ in the unit ball of $M$ such that

$$\max_{j < n}\|\bar{P}_j(\pi(\bar{b}_n))\| - r_j < 1/n.$$ 

Let $X$ be the union of the sets of coefficients of the polynomials $\bar{P}_n$ and the entries of all $\bar{b}_n$, for $n \in \mathbb{N}$. Let $B$ be the $C^*$-subalgebra of $M$ generated by $X$ and a countable approximate unit for $A$. Then $B$ is separable, and by Proposition 1.9.3 there exists a sequence $X$-quasi-central sequential approximate unit $e_n$, for $n \in \mathbb{N}$, in $B \cap A$ which moreover satisfies $e_ne_{n+1} = e_n$ for all $n$.

Fix an enumeration $X = \{x_j : j \in \mathbb{N}\}$. We shall tacitly use the fact that the quasi-central approximate units can be ‘sped up’ as described in the following claim.

Claim. Suppose $e_n$, for $n \in \mathbb{N}$, is an $X$-quasi-central approximate unit in some $C^*$-algebra. If $\varepsilon_n$, for $n \in \mathbb{N}$, is any strictly decreasing sequence in $(0, 1]$ with $\lim_n \varepsilon_n = 0$ then there is an increasing sequence $m(j)$, for $j \in \mathbb{N}$, in $\mathbb{N}$ such that $\max_{k < j} \|e_{m(j)}x_k - x_k e_{m(j)}\| < \varepsilon_j$ for all $j \in \mathbb{N}$.

Proof. The assumptions imply $\lim_n \max_{k < j} \|e_{m(j)}x_k - x_k e_{m(j)}\| < \varepsilon_j$ for every $j$. Fix $m(j)$ sufficiently large to have $\max_{k < j} \|e_{m(j)}x_k - x_k e_{m(j)}\| < \varepsilon_j$. \qed

Lemma 1.4.8 implies that for every *-polynomial $P(\bar{x})$ there is a universal constant $K < \infty$ depending only on $P(\bar{x})$ such that

$$\|P(\bar{b}), a\| \leq K \max_{j \leq n_0} \|b_j\| \max_{\varepsilon} \|c, a\|,$$

where $c$ ranges over the coefficients of $P$ and the entries of $\bar{b}$. Lemma 1.4.8 also implies that there is a continuous function $g : (0, 1] \to (0, 1]$ such that $\lim_{t \to 0} g(t) = 0$ and for all $0 \leq a \leq 1$ and $b$ with $\|b\| \leq 1$ and $\|[a, b]\| < g(\delta)$ we have

$$\|[a^{1/2}, b]\| < \delta.$$  

By going to a subsequence $(e_{m(j)})$ of $(e_j)$ if necessary, and replacing $e_j$ with $e_{m(j)}$ for all $j$, we may assume that $\sum_j \|[x_j, e_n]\| < \infty$ for all $x_j \in X$ and that

$$g(2\|[a, e_n]\|) < g(2^{-n})$$  

for all $j \leq m \leq n$ and every coefficient $a$ of $\bar{P}_j$. Let (with $e_{-1} := 0$)
so that \( \sum_{n \leq m} f_n^2 = \epsilon_{m+1}^{-1} \) for all \( m \). Since \( \tilde{P}_j(\tilde{x}) \) is a degree-1 polynomial, each of its non-constant terms is of the form \( a_0 x_j a_1 \) or \( a_0 x_j^* a_1 \) for some coefficients \( a_0 \) and \( a_1 \). Since (15.2) and (15.3) together imply that \( \| [f_n, a_j] \| < 2^{-n} \) for \( j \in \{0, 1\} \), with \( K_j \) denoting the number of terms of \( \tilde{P}_j \) we have

\[
\| \tilde{P}_j(f_m \tilde{b}_n f_n) - f_m \tilde{P}_j(\tilde{b}_n f_n) \| < 2^{-n} K_j \quad \text{and} \quad \| \tilde{P}_j(f_m \tilde{b}_n f_n) - \tilde{P}_j(\tilde{b}_n) f_m^2 \| < 2^{-n} K_j.
\]

On every bounded subset of \( M \), the weak topology induced by \( \rho_{\epsilon_n} \) and \( \lambda_n \) for \( n \in \mathbb{N} \) is equivalent to the strict topology (Lemma 13.1.4).

From now on, until the end of the proof of Theorem 15.1.5, the strict topology corresponding to \( \{ \epsilon_n : n \in \mathbb{N} \} \) will be referred to as the strict topology. In particular, ‘strictly Cauchy’ will stand for ‘Cauchy in the \( \{ \epsilon_n : n \in \mathbb{N} \} \)-strict topology.’

**Claim.** 1. For every \( \tilde{a} \in \ell_\omega(A) \) the sequence of partial sums \( s_m := \sum_{n \leq m} f_n a_n f_n \) is strictly Cauchy and it converges to an element of \( M \), denoted \( \sum f_n a_n f_n \).

2. The map \( \Theta : \ell_\omega(A) \to M \) defined by \( \Theta(\tilde{a}) := \sum f_n a_n f_n \) is unital and completely positive (see §3.2).

3. If \( \tilde{a} \in \ell_\omega(A) \) is such that \( \lim_n \| a_n \| - \| a_n f_n^2 \| = 0 \) then

\[
\| \pi(\sum_{n \leq m} a_n f_n^2) \| \geq \limsup_m \| a_n f_n^2 \|.
\]

**Proof.** (1) Fix \( n \in \mathbb{N} \). If \( m > n + 1 \) then \( e_n f_m = 0 \), hence \( e_n (s_m - s_k) e_n = 0 \) for all large enough \( m \) and \( k \). This implies the first part of the claim. Since \( M \) is strictly complete, the second part follows.

(2) Since \( M \) is strictly complete, the first claim implies that \( \Theta \) maps \( \ell_\omega(A) \) into \( M \). For every \( m \) the map \( \tilde{a} \mapsto \sum_{n < m} f_n a_n f_n \) is completely positive as a sum of completely positive maps. Therefore \( \Theta \) is completely positive. It is unital since \( \sum f_n a_n f_n \) strictly converges to \( 1_M \) by the choice of \( f_n \).

(3) If \( \limsup_m \| a_n f_n^2 \| = 0 \) there is nothing to prove. We may therefore assume \( r := \limsup_m \| a_n f_n^2 \| > 0 \) is nonzero. By replacing each \( a_n \) with \( a_n/r \), we may assume \( \limsup_m \| a_m \| = \limsup_m \| a_m f_m^2 \| = 1 \). Fix \( \epsilon > 0 \). Let \( n \) be large enough and such that \( 1 - \epsilon < \| a_n f_n^2 \| < 1 + \epsilon \) and \( \| a_n f_n^2 \| - \| a_n \| < \epsilon \). Fix a unit vector \( \xi \) that satisfies \( \| a_n f_n^2 \xi_n \| > \| a_n f_n^2 \| - \epsilon \). This implies \( \| \xi_n - a_n f_n^2 \xi_n \| < 2\epsilon \) and therefore \( \| \sum_{n \geq m} a_n f_n^2 \xi_n \| \geq \| a_n f_n^2 \xi_n \| - \| \sum_{n > m} a_n f_n^2 \xi_n \| > 1 - 4\epsilon \).

Since this holds for arbitrarily large \( n \) and the sequence \( \sum_{m \leq k} f_m^2 \), for \( k \in \mathbb{N} \), is an approximate unit for \( A \), we have \( \| \pi(\sum a_n f_n^2) \| \geq 1 - 4\epsilon \).

Since \( \epsilon > 0 \) was arbitrary, \( \| \pi(\sum a_n f_n^2) \| \geq 1 \), as required.

With \( \tilde{b} := \Theta((\tilde{b}_n : n \in \mathbb{N})) \), we claim that \( \pi(\tilde{b}) \) realizes \( t(\tilde{x}) \) in \( M/A \).

Fix \( j \). By (15.4) there is \( K_j < \infty \) such that \( c_n := \tilde{P}_j(f_m \tilde{b}_n f_n) - \tilde{P}_j(\tilde{b}_n) f_m^2 \) satisfies \( \| c_n \| \leq K_j 2^{-n} \) for every \( n \). Since \( c_n \in A \), the sum of the absolutely convergent series \( \sum_n c_n = \tilde{P}_j(\tilde{b}) - \sum_n \tilde{P}_j(\tilde{b}_n) f_m^2 \) belongs to \( A \). Part (3) of the Claim, together with

\footnote{Notation alert: \( \tilde{a} \) is an infinite sequence, while each \( \tilde{b}_n \) is an \( n \)-tuple.}
\[ \|\pi(x)\| = \lim_n \| x(1 - e_n) \|, \text{ implies} \]
\[ \|\pi(\tilde{P}_j(\tilde{b}))\| = \|\sum_n f_n \tilde{P}_j(\tilde{b}_n) f_n\| \geq \limsup_n \|\tilde{P}_j(\tilde{b}_n)\| = r_j. \]

Part (2) of the Claim implies
\[ \|\pi(\tilde{P}_j(\tilde{b}))\| = \|\sum_n f_n \tilde{P}_j(\tilde{b}_n) f_n\| \leq \limsup_n \|\tilde{P}_j(\tilde{b}_n)\| = r_j. \]

Since \( \pi(\tilde{P}_j(\tilde{b})) = P_j(\pi(\tilde{b})) \), the condition \( \|P_j(\tilde{b})\| = r_j \) is satisfied in \( M/A \) by \( \tilde{b} \).
Since \( \|P_j(\tilde{x})\| = r_j \) was an arbitrary condition in \( t \), this completes the proof. \( \square \)

**Corollary 15.1.6.** If \( A \) is a separable \( C^* \)-subalgebra of a countably degree-1 saturated algebra \( C \) then the relative commutant \( A' \cap C \) is countably degree-1 saturated.

**Proof.** Let \( t(\bar{x}) \) be a satisfiable degree-1 \( n \)-type over \( A' \cap C \). Let \( t'(\bar{x}) \) denote the union of \( t(\bar{x}) \) and a countable dense subset of the relative commutant type \( t^n_\alpha(x_j) \) (Example 15.1.3 (1)), for all \( j < n \). The set of realizations of \( t_c \) in \( C \) is \( A' \cap C \).

The type \( t(\bar{x}) \) is satisfiable (realized, respectively) over \( A' \cap C \) and if only if \( t'(\bar{x}) \) is satisfiable (realized, respectively) over \( C \). Since \( t \) was arbitrary and \( t' \) is countable if \( t \) is, countable degree-1 saturation of \( C \) implies countable degree-1 saturation of \( A' \cap C \). \( \square \)

## 15.2 Variations on the Theme of Saturation

In this section we introduce quantifier-free types and saturation and a self-strengthening of countable degree-1 saturation.

Degree-1 saturation is a restricted form of the full model-theoretic saturation that will be discussed at length in Chapter 16. For now, we define an intermediate notion, quantifier-free saturation, but only in the context of \( C^* \)-algebras.

**Definition 15.2.1.** A quantifier-free condition in variables \( \bar{x} = (x_j : j < n) \) is an expression of the form \( \|P(\bar{x})\| = r \) where \( P(\bar{x}) \) is a \( * \)-polynomial in noncommuting variables.\(^3\) If the polynomial has degree \( n \), we say that this is a degree-\( n \) condition. A quantifier-free type in variables \( \bar{x} \) is a set of quantifier-free conditions in \( \bar{x} \).

A degree-\( n \) type is defined analogously for all \( n \geq 2 \).\(^4\) Consistency (i.e., satisfiability) and realizations of quantifier-free types are defined as in Definition 15.1.2. A \( C^* \)-algebra \( C \) is countably quantifier-free saturated if every satisfiable countable quantifier-free type over \( C \) is realized in \( C \).

**Example 15.2.2.** If \( \Phi \) is an automorphism of a separable \( C^* \)-subalgebra \( B \) of a \( C^* \)-algebra \( C \), let \( t_\Phi(x) := \{\|xx^* - 1\|, \|x^*x - 1\|\} \cup \{\|xx^* - \Phi(a)\| = 0 : a \in A\} \). This degree-2 type is satisfiable if and only if \( \Phi \) is approximately inner. It is realized in \( C \) if and only if \( \Phi \) is implemented by a unitary in \( C \).

\(^3\) The standard definition of a quantifier-free type involves conditions associated with arbitrary quantifier-free formulas and appears to be more general, but the two definitions are equivalent.

\(^4\) If \( n = 1 \) then the definition agrees with the one given in §15.1.
Example 15.2.3. The Calkin algebra is not countably degree-2 saturated. There exists a unital isomorphic copy $B$ of the CAR algebra in $\mathcal{D}(H)$ and an automorphism $\Phi$ of $B$ which is approximately inner,\(^5\) but not implemented by a unitary in $\mathcal{D}(H)$. (See e.g., [86, Proposition 4.2], also Exercise 10.6.2). By Example 15.2.2, this implies that the Calkin algebra is not countably quantifier-free saturated.

Theorem 15.2.4 below is a strengthening of Theorem 15.1.5. The proof of the latter requires only one modification in order to give a proof of the former, and it is left as an exercise (Exercise 15.6.22). See also Exercise 15.6.20, Exercise 15.6.21 and Notes to this chapter for some limiting examples and discussion.

**Theorem 15.2.4.** Suppose $A$ is a nonunital C$^*$-algebra such that in $A_+$ there exists an increasing sequence which strictly converges to $1_{\mathcal{M}(A)}$. Then the corona $\mathcal{Q}(A)$ is countably degree-1 saturated. \(\square\)

Clearly, every $\sigma$-unital, nonunital, C$^*$-algebra satisfies the assumptions of Theorem 15.2.4.

Example 15.2.5. The Breuer ideal of a $\mathrm{II}_\infty$ factor $M$ with separable predual (Example 4.1.5 (2)) satisfies the assumptions of Theorem 15.2.4. Such $\mathrm{II}_\infty$ factor is isomorphic to the von Neumann algebra tensor product $N \otimes \mathcal{B}(H)$, where $N$ is a $\mathrm{II}_1$ factor and $H$ is an infinite-dimensional separable Hilbert space. If $r_n$, for $n \in \mathbb{N}$, is a sequence of finite-rank projections in $\mathcal{B}(H)$ that strongly converges to $1_{\mathcal{B}(H)}$, then the sequence $1_N \otimes r_n$, for $n \in \mathbb{N}$, strictly converges to $1_M$. However, the Breuer ideal of $M$ is not $\sigma$-unital (Exercise 4.5.9).

**Corollary 15.2.6.** If $M$ is a $\mathrm{II}_\infty$ factor with a separable predual and $J$ is its Breuer ideal, then the C$^*$-algebra $M/J$ is countably degree-1 saturated. \(\square\)

A minor, but useful, generalization of degree-1 conditions follows.

**Definition 15.2.7.** A generalized degree-1 condition (or shortly a generalized condition) over a C$^*$-algebra $M$ is an expression of the form $\|P(\bar{x})\| \in K$ where $P$ is a degree-1 polynomial with coefficients in $M$ and $K \subseteq \mathbb{R}$ is a nonempty compact set. A generalized degree-1 type is a set of generalized degree-1 conditions and degree-1 properties. Consistency (i.e., satisfiability) and realizations of generalized types are defined as in Definition 15.1.2.

**Definition 15.2.8.** A property $Q(\bar{x})$ of a tuple of elements of a C$^*$-algebra is a degree-1 property if there is a countable generalized degree-1 type $t_Q(\bar{x})$ such that $Q(\bar{a})$ holds if and only if $\bar{a}$ realizes $t_Q(\bar{x})$.

**Lemma 15.2.9.** All of the following are degree-1 properties:

1. $x$ is self-adjoint,
2. $x \geq 0$,

\(^5\) All automorphisms of the CAR algebra are approximately inner, and even asymptotically inner; this follows from the classification results, see [208].
3. $x$ is self-adjoint and $\text{sp}(x) \subseteq [s, t]$ for fixed $s \leq t$ in $\mathbb{R}$.
4. $x$ and $y$ are self-adjoint and $x \leq y$.
5. $x = y$.

**Proof.** In each instance it suffices to add finitely many conditions to a given type. We state what these conditions are and omit proofs whenever they are obvious.

1. The required condition is $\|x - x^*\| = 0$.
2. If $\|x\| \leq 1$ and $x = x^*$ then $x \geq 0$ if and only if $\text{sp}(x) \subseteq [0, 1]$. This is equivalent to $\|1 - x\| \leq 1$, $\|x\| \leq 1$, and $x = x^*$. Therefore the required generalized conditions are $\|x - x^*\| = 0$, $\|x\| \in [0, 1]$, and $\|1 - x\| \in [0, 1]$.
3. If $x = x^*$ then $\max(\text{sp}(x)) \leq t$ if and only if $\|x\| \leq t$ and $\min(\text{sp}(x)) \geq s$ if and only if $x - s \geq 0$, and therefore the conclusion follows from (2).

If $x$ and $y$ are self-adjoint then $x \leq y$ if and only if $y - x$ is positive. Therefore (4) can be proved by combining (1) and (2).

In (5), add the condition $\|x - y\| = 0$.

\[ \Box \]

**Proposition 15.2.10.** A $C^\ast$-algebra $C$ is countably degree-1 saturated if and only if every satisfiable generalized degree-1 type over $C$ is realized in $C$.

**Proof.** Only the direct implication requires a proof. Suppose that $C$ is countably degree-1 saturated. We will prove that every satisfiable generalized countable degree-1 type over $C$ is realized in $C$. Let $t(\bar{x}) := \{\|P_n(\bar{x})\| \in K_n : n \in \mathbb{N}\}$ be a satisfiable countable generalized type over $C$. We will define an ‘honest’ satisfiable countable degree-1 type $t'(\bar{x})$ with the same set of realizations as $t(\bar{x})$. Since $t$ is satisfiable we can find $\bar{b}_n$ for $n \in \mathbb{N}$, such that $\text{dist}(\|P_j(\bar{b}_n)\|, K_j) < 1/n$ for all $j < n$. Fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. For $j \in \mathbb{N}$ let $r_j := \lim_{n \to \mathcal{U}} \|P_j(\bar{b}_n)\|$ (the limit exists and belongs to $K_j$ by the compactness of $K_j$). Then $\{\|P_j(\bar{x})\| = r_j : j \in \mathbb{N}\}$ is a satisfiable type whose realizations are realizations of $t(\bar{x})$. Therefore $t$ is realized in $C$. \[ \Box \]

### 15.3 Applications of Countable Degree-1 Saturation

There are only 17,000 three-letter acronyms.

Paul Boultin, answering the question “What do you think will be the biggest problem of computing in the 90s?”

In this section we show that countable degree-1 saturation implies a large number of well-studied properties of massive $C^\ast$-algebras including coronas, ultrapowers, asymptotic sequence algebras, central sequence algebras, as well as the relative commutants of separable $C^\ast$-subalgebras of these algebras. All of these algebras are $\text{SAW}^\ast$, $\text{CRISP}$, $\text{AA-CRISP}$, and even $\text{KTT}$.

Via the Gelfand–Naimark duality (Theorem 1.3.2), the category of abelian $C^\ast$-algebras is equivalent to the category of locally compact Hausdorff spaces. The study of $C^\ast$-algebras can therefore be considered as ‘noncommutative topology.’ Some instances of countable degree-1 saturation in $C^\ast$-algebras were studied by...
operator algebraists for decades. In the course of his seminal study of ultrapowers of C*-algebras, Kirchberg abstracted several properties of ultrapowers that resemble those of coronas ([156]).

G.K. Pedersen introduced noncommutative variants of topological properties of massive spaces such as coronas $\beta X \setminus X$ ([195], [197]). These properties include the absence of countable gaps and limits (cf. §14.1), which resemble properties of von Neumann algebras such as the existence of the supremum for every increasing net of positive elements (Lemma 3.1.3). Pedersen observed that the coronas of $\sigma$-unital C*-algebras satisfy a weak separable analog of this property, and introduced the class of SAW*-algebras. SAW* stands for ‘Separable AW’ (AW*-algebras are C*-algebras that share some of the properties of von Neumann algebras; see Corollary 13.3.5 and the paragraph preceding it). We will see that many of the properties considered by Pedersen and Kirchberg are subsumed by the concept of countable degree-1 saturation.

**Definition 15.3.1.** 1. A C*-algebra $C$ is an SAW*-algebra if any two orthogonal $\sigma$-unital C*-subalgebras of $C$ are separated.

2. A C*-algebra $C$ has CRISP (countable Riesz separation property) if for all positive elements $a_n, b_n$, for $n \in \mathbb{N}$, in $C$ satisfying $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n$ there exists a positive $c \in C$ such that $a_n \leq c \leq b_n$ for all $n$.

The following is proved by parsing Definition 15.3.1 and Definition 9.3.1.

**Lemma 15.3.2.** 1. A C*-algebra $C$ has CRISP if and only if the poset $(C_{+,1}, \leq)$ does not contain countable asymmetric gaps.

2. A C*-algebra $C$ is an SAW*-algebra if and only if the poset $(C_{+,1}, \leq)$ does not contain countable symmetric gaps. $\Box$

Lemma 3.1.3 implies that every von Neumann algebra has CRISP and is an SAW*-algebra. For separable C*-algebras each of these two properties is equivalent to being finite-dimensional (Exercise 15.6.1 and Exercise 15.6.3). Similarly, the only countably degree-1 saturated C*-algebras that are von Neumann algebras are finite-dimensional (Exercise 15.6.4).

In the classical (discrete) model theory a type with only finitely many conditions is satisfiable if and only if it is realized. In logic of metric structures this is not necessarily the case—the type used in the proof of Lemma 15.3.3 contains only three conditions!

**Lemma 15.3.3.** Every countably degree-1 saturated C*-algebra $C$ is an SAW*-algebra.

**Proof.** Suppose $A$ and $B$ are orthogonal $\sigma$-unital C*-subalgebras of $C$. Every $\sigma$-unital C*-algebra has a strictly positive element by Lemma 1.8.3. Let $h$ be a strictly positive element of $A$ and let $g$ be a strictly positive element of $B$. Consider the generalized degree-1 type $t(x) := \{xh = h, xg = 0, 0 \leq x\}$.

Since $\lim_n \|h^{1/n}h - h\| = 0$ and $h^{1/n}g = 0$ for all $n$, this type is approximately realized by $h^{1/n}$ for a large enough $n$. By Proposition 15.2.10, it is realized in $C$, and its realization is a positive contraction that separates $A$ and $B$. $\Box$
Lemma 15.3.4. Every countably degree-1 saturated C*-algebra C has CRISP.

Proof. Fix $a_n$ and $b_n$ for $n \in \mathbb{N}$ as in the definition of CRISP. Then $b_n \neq 0$ for all $n$ and by rescaling each $b_n$ by $\|b_1\|^{-1}$ we may assume $\|b_1\| = 1$. Consider the generalized degree-1 type $t(x) := \{x = x^*, a_n \leq x, x \leq b_n : n \in \mathbb{N}\}$.

Every finite subset of $t(x)$ is realized by $a_n$ for a large enough $n$. By Proposition 15.2.10 some $c \in C$ realizes $t$. Therefore $c$ is self-adjoint and it satisfies $a_n \leq c \leq b_n$ for all $n$; the CRISP of $C$ follows.

A property related to both SAW* and CRISP is the absence of decreasing sequences of nonzero positive elements with no nonzero lower bound. This is the noncommutative analog of the topological property that every nonempty $G_δ$ set has a nonempty interior. This property will be used in the forthcoming discussion of excision of states on countably degree-1 saturated C*-algebras. Its proof is similar to the proofs of the last two lemmas and it is left as Exercise 15.6.13.

Lemma 15.3.5. Suppose $C$ is a countably degree-1 saturated C*-algebra and $a_n$, for $n \in \mathbb{N}$, is a sequence in $C_{+1}$ such that $a_n a_{n+1} = a_{n+1}$ for all $n$. Then there exists $a \in C_{+1}$ such that $a_n a = a$ for all $n$.

Saturation is relevant to the excision of states (§5.2) and quantum filters (§5.3).

Definition 15.3.6. If $\varphi$ is a state on a C*-algebra $A$ and $X \subseteq A$ excises $\varphi$ on $X$ if for every $x \in X$ we have $\inf_{a \in X} \|axa - \varphi(x)a^2\| = 0$.

Example 15.3.7. Suppose $\varphi$ is a pure state on a C*-algebra $A$.

1. The maximal quantum filter $\mathcal{F}_\varphi$ excises $\varphi$ by Theorem 5.2.1. In addition, for every $X \subseteq A$ a Löwenheim–Skolem/Blackadar closing-off argument produces $\mathcal{F} \subseteq \mathcal{F}_\varphi$ of cardinality $|X| + \aleph_0$ which excises $\varphi$ on $X$.

2. Even if $X$ is finite there need not be a finite $\mathcal{F}$ that excises $\varphi$ on $X$. Take for example $C([0,1])$, the evaluation functional $\varphi(f) := f(1)$, and $X := \{\text{id}_{[0,1]}\}$.

Example 15.3.7 (2) is not an issue in countably degree-1 saturated C*-algebras.

Proposition 15.3.8. Suppose $C$ is a countably degree-1 saturated C*-algebra, $\varphi$ is a pure state on $C$, and $B$ is a separable C*-subalgebra of $C$. Then a single element of $C_{+1}$ excises $\varphi$ on $B$.

Proof. Let $b_n$, for $n \in \mathbb{N}$, be an enumeration of a dense subset of $B$. An $a \in C_{+1}$ such that $ab_n a^* = \varphi(b_n)a^2$ for all $n$ is required. The “obvious” type of such $a$ is satisfiable (and, as we will see, realized) but it is not a degree-1 type since it includes the condition $xb_n x - \varphi(b_n)x^2 = 0$ for every $n \in \mathbb{N}$. Back to the drawing table: By Proposition 5.3.4 there is a norm-dense, directed $\mathcal{D} \subseteq \mathcal{F}_\varphi$ that excises $\varphi$ on $B$. Fix a countable directed $\mathcal{D} \subseteq \mathcal{D}$ that excises $\varphi$ on $B$. Let

$$t(x) := \{|x| = 1, 0 \leq x, xc = x : c \in \mathcal{F}\}.$$

Since $\mathcal{F}$ is directed, every finite subset of the generalized type $t(x)$ is realized by some $d \in \mathcal{F}$. Since $C$ is countably degree-1 saturated, by Lemma 15.3.5 some $a \in C$
realizes \( t(x) \). Then \( a \in C_{+1} \) and \( ac = ca \) for all \( c \in \mathcal{F} \), hence \( a \leq c \) for all \( c \in \mathcal{F} \).

For \( b \in B \) and \( c \in \mathcal{F} \) we have

\[
aba - \varphi(b)a^2 = a(b - \varphi(b))a = ac(b - \varphi(b))ca \leq a^2\|c(b - \varphi(b))c\|.
\]

Therefore \( aba - \varphi(b)a^2 \leq a^2 \inf_{c \in \mathcal{F}} \|cbc - \varphi(b)c^2\| = 0 \). Since \( b \in B \) was arbitrary, \( a \) is as required. \( \square \)

### 15.4 Further Applications of Countable Degree-1 Saturation

This section is divided into three independent subsections, each one containing a proof that every countably degree-1 saturated \( C^* \)-algebra \( C \) has a particular property: (i) Every uniformly bounded representation of a countable amenable group into \( C \) is unitarizable, (ii) \( C \) allows for a poor man’s version of Borel function calculus (‘discontinuous’ functional calculus) on normal elements, and (iii) \( C \) is essentially non-factorizable (the latter result is proved for SAW*-algebras).

#### 15.4.1 Uniformly Bounded Group Representations II: Countable Groups

We continue the discussion of \( \S 14.5 \), after recalling Definition 14.5.4. A representation \( \pi : \Gamma \to \text{GL}(A) \) of a discrete group \( \Gamma \) into a unital \( C^* \)-algebra \( A \) is uniformly bounded if its norm \( \|\pi\| := \sup_{g \in \Gamma} \|\pi(g)\| \) is finite. It is unitary if its range is included in the unitary group of \( A \) and unitarizable if there is \( a \in \text{GL}(A) \) such that \( g \mapsto a^{-1}\pi(g)a \) is a unitary representation.

Since an invertible element \( a \) is a unitary if and only if \( \|a\| = \|a^{-1}\| = 1 \), a representation \( \pi \) is unitary if and only if \( \|\pi\| = 1 \).

One of the equivalent criteria for amenability of a discrete group \( \Gamma \) is the existence of a Følner net (e.g., [121]): a net \( \{F_\lambda\} \) of finite subsets of \( \Gamma \) such that for all \( g \in G \) we have

\[
\lim_{\lambda} \frac{|gF_\lambda \Delta F_\lambda|}{|F_\lambda|} = 0.
\]

The following proposition will be reused in the proof of parts of Theorem 15.5.1.

**Proposition 15.4.1.** Suppose \( C \) is a unital countably degree-1 saturated \( C^* \)-algebra. Then every uniformly bounded representation \( \pi \) of a countable amenable group \( \Gamma \) into \( C \) is unitarizable.

**Proof.** The ‘obvious’ type of an \( a \in C \) that unitarizes \( \pi \) contains conditions of the form \( \|x\pi(g)x^{-1}\| = 1 \) that are not degree-1. Let

\[
t(x) := \{ x = x^*, x \leq \|\pi\|^2, \|\pi\|^{-2} \leq x \} \cup \{ \pi(g)x \pi(g)^* = x : g \in \Gamma \}.
\]
This defines a representation of $a \in F$ such that it also satisfies 
\[ \| \pi(h_\pi \pi(g)^\dagger - h_\pi \| \leq \| \pi \| |gF\Delta F| \leq \varepsilon. \]

Also, $h := h_\pi$ satisfies $h = h^\dagger$, and since $\| \pi \|^{-2} \leq \pi(g)\pi(g)^\dagger \leq \| \pi \|^2$ for all $g \in G$, it also satisfies $\| \pi \|^{-2} \leq h \leq \| \pi \|^2$.

Because $\varepsilon > 0$ and $t_0(x)$ were arbitrary, the type $t(x)$ is satisfiable. A realization $a \in C$ of $t$ is self-adjoint and $\| \pi \|^{-2} \leq a \leq \| \pi \|^2$, in particular $a$ is invertible. Let 
\[ \pi'(g) := a^{1/2} \pi(g)a^{-1/2}. \]

This defines a representation of $\Gamma$ into $A$, and for $g \in \Gamma$ we have 
\[ \pi'(g)\pi'(g)^\dagger = a^{-1/2} \pi(g)a\pi(g)^\dagger a^{-1/2} = a^{-1/2}aa^{-1/2} = 1. \]
Therefore $\pi'$ is a unitary representation and $a^{1/2}$ unitarizes $\pi$. \hfill \Box

### 15.4.2 ‘Discontinuous’ Functional Calculus

Countably degree-1 saturated $C^*$-algebras admit a limited (and non-canonical) analog of the Borel functional calculus, Theorem 15.4.3 below. Its roots can be traced back to the so-called Second Splitting Lemma from [33]. A proof of the following lemma is left to the reader.

**Lemma 15.4.2.** Suppose $U$ is an open subset of a compact Hausdorff space $K$, $g : U \to \mathbb{C}$ is continuous and bounded, and $f \in C(K)$ satisfies $\text{supp}(f) \subseteq U$. The function $h$ on $K$ that agrees with $gf$ on $U$ and vanishes on $K \setminus U$ is continuous. \hfill \Box

In Theorem 15.4.3 and its proof we abuse the notation and identify $gf$ with $h$ as in Lemma 15.4.2.

**Theorem 15.4.3.** Suppose $C$ is a countably degree-1 saturated $C^*$-algebra, $a \in C$ is normal, $B \subseteq (a)^\prime \cap C$ is separable,\(^6\) $U \subseteq \text{sp}(a)$ is open, and $g : U \to \mathbb{C}$ is a bounded continuous function. Then there is $c \in C \cap C^*(B, a)^\prime$ such that for every $f \in C(\text{sp}(a))$ satisfying $\text{supp}(f) \subseteq U$ we have $cf(a) = (gf)(a)$.

If $g$ is real-valued, then $c$ can be chosen to be self-adjoint.

---

\(^6\) Fuglede’s Theorem implies that $(a)^\prime \cap C$ is a $C^*$-subalgebra of $C$ if $a$ is normal (Corollary C.6.13).
Proof. Let \( b_n \), for \( n \in \mathbb{N} \), enumerate a countable dense subset of \( B \) and let \( f_n \), for \( n \in \mathbb{N} \), enumerate a countable dense subset of the unit ball of the hereditary \( C^* \)-subalgebra \( \{ f \in C(\text{sp}(a)) : \text{supp}(f) \subseteq U \} \) of \( C(\text{sp}(A)) \). Let \( M := \sup_{z \in U} |g(z)| \) and
\[
t(x) := \{ ||x|| \leq M, ||[a,x]|| = 0 \} \cup \{ ||[b_n,x]|| = 0, ||xf_n(a) - (gf_n)(a)|| = 0 : n \in \mathbb{N} \}.
\]

Any realization \( c \) of \( t(x) \) in \( C \) is in the commutant of \( B \cup \{ a \} \) and it satisfies \( cf(a) = (gf)(a) \) for all \( f \in C(\text{sp}(a)) \) with \( \text{supp}(f) \subseteq U \). The type \( t(x) \) is a countable generalized degree-1 type and by Proposition 15.2.10 it is realized in \( C \) if and only if it is satisfiable in \( C \).

We proceed to show that \( t(x) \) is satisfiable in \( C \). Fix \( n_0 \in \mathbb{N} \) and \( \epsilon > 0 \). We will find \( d \) of norm \( \leq 1 \) such that \( [a,d] = 0, [b_n,d] = 0 \) and \( ||df_n(a) - (gf_n)(a)|| \leq \epsilon \) for all \( n \leq n_0 \).

The set \( K := \{ x \in \text{sp}(a) : \max_{n \leq n_0} |f_n(x)| \geq \epsilon/2 \} \) is a compact subset of \( U \) and there is a continuous \( h : \text{sp}(a) \to [0,1] \) such that \( h(z) = 0 \) for \( z \notin U \) and \( h(z) = 1 \) for \( z \in K \). Then \( gh \in C(\text{sp}(a)) \) and \( d := (gh)(a) \) belongs to \( C^*(a) \). (If \( g \) is real-valued then so is \( gh \), and \( d \) is self-adjoint.) Since \( d \in C^*(a) \), it commutes with all \( b_n \) and with \( a \). By the continuous functional calculus,
\[
||d|| = \sup_{z \in \text{sp}(a)} |gh(z)| \leq \sup_{z \in U} |g(z)|.
\]

To verify \( ||df_n(a) - (gf_n)(a)|| \leq \epsilon \), we need to check \( ||(ghf_n)(z) - (gf_n)(z)|| \leq \epsilon \) for all \( z \in \text{sp}(a) \). If \( z \notin U \) or if \( z \notin K \) then the two expressions are equal, and if \( z \in U \setminus K \) then \( \max(|(ghf_n)(z)|,|(gf_n)(z)|) \leq \epsilon/2 \), and the conclusion follows.

It remains to prove that \( c \) can be chosen to be self-adjoint if \( g \) is real-valued, recall that in this case \( d = (gh)(a) \) was self-adjoint and use Proposition 15.2.10. \( \Box \)

Unlike the ‘honest’ functional calculus, Theorem 15.4.3 does not give a unique element.

Lemma 15.4.4. In the notation of Theorem 15.4.3, if \( U \) is not a closed subset of \( \text{sp}(a) \) and \( g \neq 0 \), then \( c \) is not unique.

Proof. Consider the type \( t'(x_1,x_2) \) which contains \( t(x_1) \) and \( t(x_2) \) as well as the condition \( ||x_1 - x_2|| = \sup_{z \in U} |g(z)| \). It is approximately finitely satisfiable and its realization gives distinct elements satisfying the conclusion of Theorem 15.4.3. \( \Box \)

Lemma 15.4.4 can be improved to show that the set of realizations \( c \) as in Theorem 15.4.3 is infinite, and even nonseparable. See Exercise 15.6.29.

15.4.3 Essential Non-Factorizability

A \( C^* \)-algebra \( A \) is essentially non-factorizable if it is not isomorphic to a tensor product of two infinite-dimensional \( C^* \)-algebras. For example, if \( H \) is an infinite-dimensional Hilbert space then \( \mathcal{B}(H) \cong M_n(\mathcal{C}) \otimes \mathcal{B}(H) \) for all \( n \geq 1 \), but \( \mathcal{B}(H) \) is essentially non-factorizable by Exercise 15.6.33.
Theorem 15.4.5. Every SAW*-algebra is essentially non-factorizable. In particular, coronas of σ-unital, nonunital C*-algebras and ultrapowers associated with nonprincipal ultrafilters on $\mathbb{N}$ are essentially non-factorizable.

We require a lemma from topology and one from combinatorics.

Lemma 15.4.6. Suppose $C$ is an infinite-dimensional SAW*-algebra and $a_n \in C_{+1}$, for $n \in \mathbb{N}$, are orthogonal. If a state $\varphi_m$ of $C$ satisfies $\varphi_m(a_n) = 1$ for all $m \in \mathbb{N}$, then the weak* closure of $\{ \varphi_m : m \in \mathbb{N} \}$ in $\mathcal{S}(C)$ is homeomorphic to $\beta \mathbb{N}$.

Proof. If $m \neq n$ then Lemma 1.7.5 implies $\varphi_m(a_n) = \varphi_m(a_n a_n) = 0$. The space $\beta \mathbb{N}$ is the unique compact Hausdorff space $Z$ that has $\mathbb{N}$ as a dense subspace and every $f \in \ell_\omega(\mathbb{N})$ extends uniquely to $Z$. Therefore we have only to prove that for every $X \subseteq \mathbb{N}$ the sets $\{ \varphi_n : n \in X \}$ and $\{ \varphi_n : n \in \mathbb{N} \setminus X \}$ have disjoint closures. Since $C$ is an SAW*-algebra there exists a positive contraction $e \in C$ such that $ea_n = a_n$ for $n \in X$ and $ea_n = 0$ for $a \notin X$.

Then $\{ \psi \in \mathcal{S}(C) : \psi(e) > 1/2 \}$ and $\{ \psi \in \mathcal{S}(C) : \psi(e) < 1/2 \}$ are disjoint weak*-open neighbourhoods of $\{ \varphi_n : n \in X \}$ and $\{ \varphi_n : n \in \mathbb{N} \setminus X \}$, respectively. \hfill $\Box$

Lemma 15.4.7. If $f : \mathbb{N}^2 \to \mathbb{N}$ is such that $f(m,n) \neq f(k,k)$ for all $(m,n,k) \subseteq \mathbb{N}$ satisfying $m < n$ then there is no continuous map $\bar{f} : (\beta \mathbb{N})^2 \to \beta \mathbb{N}$ extending $f$.

Proof. Suppose otherwise. If $\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}$ then every open neighbourhood of $(\mathcal{U}, \mathcal{U})$ in $(\beta \mathbb{N} \setminus \mathbb{N})^2$ contains a pair $(m,n) \in \mathbb{N}^2$ such that $m < n$. Therefore the closures of $\mathcal{X} := \{ (m,n) \in \mathbb{N}^2 : m < n \}$ and $\mathcal{Y} := \{ (k,k) : k \in \mathbb{N} \}$ in $(\beta \mathbb{N})^2$ have a nonempty intersection. However, the images of $\mathcal{X}$ and $\mathcal{Y}$ are disjoint subsets of $\mathbb{N}$, and therefore their closures in $\beta \mathbb{N}$ are disjoint; contradiction. \hfill $\Box$

Proof (Theorem 15.4.5). Suppose the contrary, and find infinite-dimensional C*-algebras $A$ and $B$ and an isomorphism $\Phi : C \to A \otimes_\alpha B$ for some tensor product of $A$ and $B$. Identify $C$ with $A \otimes_\alpha B$. By Exercise 2.8.9 there are orthogonal $a_n$, for $n \in \mathbb{N}$, in $A_{+1}$. For $m \in \mathbb{N}$ let $\varphi_m$ be a pure state on $A$ such that $\varphi_m(a_m) = 1$. Then $\varphi_m(a_n) = \delta_{mn}$ for all $m$ and all $n$. Similarly, we can find orthogonal $b_n \in B_{+1}$ and pure states $\psi_n \in \mathcal{S}(B)$ such that $\psi_n(b_n) = 1$ for all $n \in \mathbb{N}$. Then $\psi_m(b_n) = \delta_{mn}$ for all $m$ and all $n$. By Exercise 5.7.10 (1), $\theta_{mn} := \varphi_m \otimes \psi_n$ is a uniquely defined state on $C$ such that

$$\theta_{mn}(a_j \otimes b_k) = \delta_{mj}\delta_{nk}$$

for all $m,n,j,$ and $k$. Since $C$ is an SAW*-algebra, by Lemma 15.4.6 the weak* closure of $Z := \{ \theta_{mn} : m \in \mathbb{N}, n \in \mathbb{N} \}$ is homeomorphic to $\beta \mathbb{N}$. Fix a bijection from $Z$ onto $\mathbb{N}$ and let $g$ be the unique homeomorphism from the weak* closure of $Z$ to $\beta \mathbb{N}$ extending this bijection. Let $f_0 : (\beta \mathbb{N})^2 \to \mathcal{S}(A) \times \mathcal{S}(B)$ be the unique continuous extension of the function $(m,n) \mapsto (\varphi_m, \psi_n)$. The map $f : (\beta \mathbb{N})^2 \to \mathcal{S}(C)$ defined by $f(\varphi, \psi) \mapsto \varphi \otimes \psi$ is weak* continuous. Therefore $g \circ f : (\beta \mathbb{N})^2 \to \beta \mathbb{N}$ is a continuous map whose restriction to $\mathbb{N}^2$ is an injection into $\mathbb{N}$. This contradicts Lemma 15.4.7. \hfill $\Box$
15.5 An Amenable Operator Algebra not Isomorphic to a C*-algebra

In this section we pick up the thread of §14.5 and construct an amenable subalgebra of \( \mathcal{B}(H) \) that is not isomorphic to a C*-algebra. This algebra is nonseparable and it is an inductive limit of club many separable subalgebras each of which is isomorphic to a separable C*-algebra.

Until the end of this section a subalgebra of a C*-algebra is norm-closed, but not necessarily a C*-subalgebra and a homomorphism between algebras of operators (even if they are C*-algebras) is not necessarily a *-homomorphism.

Two subalgebras \( A \) and \( B \) of \( \mathcal{B}(H) \) are said to be similar if there exists an invertible \( c \in \mathcal{B}(H) \) such that \( a \mapsto cac^{-1} \) is an isomorphism between \( A \) and \( B \). Such an isomorphism is not necessarily a *-isomorphism (see Example 1.3.5).

**Theorem 15.5.1.** There is a subalgebra \( B \) of \( M_2(\ell_\infty(\mathbb{N})) \) with the following properties.

1. It has \( M_2(c_0(\mathbb{N})) \) as an essential ideal.
2. The quotient \( B/M_2(c_0(\mathbb{N})) \) is an abelian C*-algebra of real rank zero.
3. The algebra \( B \) is an inductive limit of a directed and \( \sigma \)-complete (§7.3) family of separable subalgebras, each of which is similar to a C*-algebra.
4. No nonseparable subalgebra of \( B \) is similar to a C*-algebra.

Together with some standard results, Theorem 15.5.1 implies that there exists an amenable norm-closed subalgebra of \( \mathcal{B}(H) \) (and even \( M_2(\ell_\infty(\mathbb{N})) \)) that is not isomorphic to a C*-algebra. Background and further references are provided in the Notes for this chapter.

The remainder of this section is devoted to proving Theorem 15.5.1. We import and review the notation used in the proof of Proposition 14.5.6.

**Definition 15.5.2.** Fix a Luzin almost disjoint family \( A_\alpha \), for \( \alpha < \aleph_1 \), and a continuous injection \( f : \mathcal{P}(\mathbb{N}) \to [0,1] \). Let \( t_\alpha := \begin{pmatrix} 1 & 0 \\ f(A_\alpha) & -1 \end{pmatrix} \otimes 1 \), and let \( p_\alpha \) be the central projection in \( M_2(\ell_\infty(\mathbb{N})) \) corresponding to \( A_\alpha \) so that \( p_\alpha(n) = 1 \) if \( n \in A_\alpha \) and \( p_\alpha(n) = 0 \) otherwise. Also define \( b_\alpha \in M_2(\ell_\infty(\mathbb{N})) \) by

\[
 b_\alpha := t_\alpha p_\alpha + (1 - p_\alpha).
\]

Define a representation \( \sigma : \bigoplus_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to GL(M_2(\ell_\infty(\mathbb{N}))/M_2(c_0(\mathbb{N}))) \) by \( (g_\alpha \text{ is the generator of the } \alpha\text{-th copy of } \mathbb{Z}/2\mathbb{Z}) \sigma(g_\alpha) := \pi(b_\alpha) \).

In order to relax the notation, for \( x \in M_2(\ell_\infty(\mathbb{N})) \) we write \( (\pi \text{ is the quotient map}) \)

\[
 \tilde{x} := \pi(x).
\]

**Lemma 15.5.3.** If \( \alpha \neq \beta \) then \( b_\alpha^2 = 1 \), \( \hat{p}_\alpha b_\alpha = i_\alpha \hat{p}_\alpha \), \( \hat{p}_\alpha b_\beta = \hat{p}_\alpha \), and

\[
 b_\alpha b_\beta = b_\alpha + b_\beta - 1 = b_\beta b_\alpha.
\]
Proof. Since $t_a^2 = 1$, we have $b_a^2 = 1$. Since a Luzin family is almost disjoint, we have $\hat{p}_\alpha \hat{p}_\beta = 0$ if $\alpha \neq \beta$. Therefore $\hat{p}_\alpha b_\alpha = i_\alpha \hat{p}_\alpha$ and $\hat{p}_\beta b_\beta = \hat{p}_\beta$ if $\alpha \neq \beta$, and

$$\begin{align*}
\hat{b}_\alpha \hat{b}_\beta &= \hat{p}_\alpha b_\alpha \hat{b}_\beta + \hat{p}_\beta b_\beta \hat{b}_\beta + (1 - \hat{p}_\alpha - \hat{p}_\beta) b_\alpha b_\beta \\
&= i_\alpha \hat{p}_\alpha + i_\beta \hat{p}_\beta + (1 - \hat{p}_\alpha - \hat{p}_\beta) b_\alpha b_\beta \\
&= b_\alpha + b_\beta - 1.
\end{align*}$$

An analogous computation gives $\hat{b}_\beta b_\alpha = b_\alpha + b_\beta - 1$. \hfill \Box

**Lemma 15.5.4.** The algebra $B := \overline{\text{span}}(\sigma[\bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}])$ satisfies

$$\pi[B] = \overline{\text{span}}(\{1\} \cup \{b_\alpha : \alpha < \mathbb{R}_1\}).$$

*Proof.* Clearly $\overline{\text{span}}(\{1\} \cup \{b_\alpha : \alpha < \mathbb{R}_1\})$ is a norm-closed subspace of $\pi(B)$. Lemma 15.5.3 implies that this set is closed under multiplication, and therefore equal to $\pi[B]$. \hfill \Box

**Lemma 15.5.5.** The C$^*$-algebra $B$ as in Lemma 15.5.4 satisfies the following.

1. It is an inductive limit of a directed and $\sigma$-complete family of separable subalgebras, each of which is similar to a C$^*$-algebra.
2. The quotient $\pi[B]$ is isomorphic to $C(\mathbb{R}_1 \cup \{\infty\})$, where $\mathbb{R}_1 \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}_1$ taken with the discrete topology.

*Proof.* (1) For a countable ordinal $\alpha$ the group $\Gamma_\alpha := \bigoplus_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$ is identified with a subgroup of $\bigoplus_{\mathbb{R}_1} \mathbb{Z}/2\mathbb{Z}$. The function $\alpha \mapsto \Gamma_\alpha$ is an order-isomorphism between $\mathbb{R}_1$ and a club in $[\bigoplus_{\mathbb{R}_1} \mathbb{Z}/2\mathbb{Z}]^\mathbb{R}$. The algebras

$$B_\alpha := \overline{\text{span}}(\sigma[\Gamma_\alpha]),$$

for $\alpha < \mathbb{R}_1$, form a club in $\text{Sep}(B)$. Proposition 15.4.1 implies that every uniformly bounded representation of a countable amenable group in a countably degree-1 saturated algebra is unitarizable. Since $M_2(\ell_\infty(\mathbb{N}))/M_2(c_0(\mathbb{N}))$ is countably degree-1 saturated, for every $\alpha < \mathbb{R}_1$ there is a positive $a_\alpha \in M_2(\ell_\infty(\mathbb{N}))$ such that $\pi(a_\alpha)$ unitarizes $\sigma[\Gamma_\alpha]$. Therefore $a_\beta B_\alpha a_\alpha^{-1}$ is self-adjoint and hence a C$^*$-subalgebra.

(2) The plan is to prove that $\pi[B]$ is an inductive limit of separable abelian C$^*$-subalgebras. Let $a_\alpha$ and $B_\alpha$ be as in the proof of (1). If $\beta < \alpha$ then $\hat{a}_\alpha$ commutes with $\hat{p}_\beta$ and $\hat{a}_\alpha \hat{b}_\beta \hat{a}_\alpha^{-1}$ is a self-adjoint unitary unitarily equivalent to

$$c_\beta := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \hat{p}_\beta + (1 - \hat{p}_\beta).$$

The projections $q_\beta := \frac{1}{2}(1 - c_\beta)$ are orthogonal since $q_\beta \leq \hat{p}_\beta$ for all $\beta$. Since $\hat{p}_\beta \hat{p}_\gamma = 0$ for all $\gamma < \beta < \alpha$ and $\alpha$ is countable, we can find disjoint sets $A'_\beta$, for $\beta < \alpha$ such that $A_\beta A'_\beta$ is finite for all $\beta$ and $\bigcup_{\beta < \alpha} A'_\beta = \mathbb{N}$. Working on these disjoint sets of indices, replace $a_\alpha$ with $a_\alpha u$ for a unitary $u$ and obtain $\hat{a}_\alpha \hat{b}_\beta \hat{a}_\alpha^{-1} = \hat{c}_\beta$ for all $\beta < \alpha$. Then $D_\alpha := a_\alpha B_\alpha a_\alpha^{-1}$ is a C$^*$-algebra isomorphic to $B_\alpha$. Its quotient
Lemma 15.5.6. This matrix is equal to
\[
\begin{pmatrix}
\lambda + \theta & 0 \\
\lambda f(A_\alpha) - \lambda + \theta
\end{pmatrix}
\]
by \( \Phi_\alpha(b) := \pi(a_1 b a_1^{-1}) \). If \( \alpha < \alpha' \) then the \( * \)-homomorphisms \( \Phi_\alpha \) and \( \Phi_{\alpha'} \) agree on \( B_\alpha/(B_\alpha \cap M_2(c_0(\mathbb{N}))) \), and therefore \( \| \Phi_\alpha \| \leq \| \Phi_{\alpha'} \| \). Since the cofinality of \( \mathbb{R}_1 \) is uncountable, \( M := \sup_{\alpha < \mathbb{R}_1} \| \Phi_\alpha \| < \infty \), and the set-theoretic union of graphs of all \( \Phi_\alpha \) is an isomorphism from \( B/M_2(c_0(\mathbb{N})) \) onto \( E \).

Here is an opportunity to catch a breath before we plunge into computations.

**Lemma 15.5.6.** A \( 2 \times 2 \) matrix \( \lambda t_1 + \theta 1_2 \) is unitarizable if and only if the scalars \( \lambda \) and \( \theta \) satisfy \( |\lambda + \theta| = | -\lambda + \theta | = 1 \).

**Proof.** This matrix is equal to \( \begin{pmatrix} \lambda + \theta & 0 \\ \lambda f(A_\alpha) - \lambda + \theta \end{pmatrix} \). It is unitarizable if and only if its eigenvalues lie on the unit circle, and its eigenvalues are \( \lambda + \theta \) and \( -\lambda + \theta \).

Given \( \varepsilon \geq 0 \), consider the following compact subset of \( \mathbb{C}^2 \):
\[
Y_\varepsilon := \{ (\lambda, \theta) \in \mathbb{C}^2 : |\lambda + \theta| = | -\lambda + \theta | = 1 \text{ and } |\lambda| \geq \varepsilon \}\.
\]
The following extension of Lemma 14.5.3 is by no means an eye-pleaser, but it is suggestive of the direction that the proof of Lemma 15.5.8 below will take.

**Lemma 15.5.7.** If \( U_0 \) and \( U_1 \) are disjoint closed subsets of \([0, 1]\) and \( K < \infty \) then
\[
L := \inf \{ \max_{j < 2} \text{dist}(a(\lambda_j s_{r(j)} + \theta_j 1_2) a^{-1}, U(M_2(\mathbb{C}))) : a \in \text{GL}(M_2(\mathbb{C})), |a||a^{-1}| \leq K, r(j) \in U_j, (\lambda_j, \theta_j) \in Y_{1/k} \text{ for } j < 2 \} > 0.
\]

**Proof.** The range of the parameters \( (a, r_j, \lambda_j, \theta_j : j < 2) \) in the definition of \( L \) is compact. As the function \( \max_{j < 2} \text{dist}(a(\lambda_j s_{r(j)} + \theta_j 1_2) a^{-1}, U(M_2(\mathbb{C}))) \) is continuous, it attains its infimum at some \( (a, r_j, \lambda_j, \theta_j : j < 2) \). Suppose that this infimum is equal to zero. Since both \( \lambda_0 \) and \( \lambda_1 \) are nonzero and \( r_0 \neq r_1 \), \( a \) simultaneously diagonalizes \( s_{r(0)} \) and \( s_{r(1)} \). But \( r(0) \neq r(1) \) and the matrices \( s_{r(0)} \) and \( s_{r(1)} \) cannot be simultaneously diagonalized (Example 14.5.2); contradiction.

**Lemma 15.5.8.** The algebra \( B \) as in Definition 15.5.2 has no nonseparable subalgebra similar to a \( \mathbb{C}^* \)-algebra.
Proof. Suppose the contrary, that some nonseparable subalgebra $C$ of $B$ is similar to a C$^*$-algebra. Let $a \in M_2(ℓ_∞(N))$ be such that $aCa^{-1}$ is a C$^*$-algebra. Lemma 15.5.5 implies that $aCa^{-1}/M_2(c_0(N))$ is an abelian C$^*$-subalgebra of $B/M_2(c_0(N))$.

We may assume that $C$ is unital by replacing it with $C^*(C \cup \{1_B\})$. By Exercise 1.11.16 (6), $aCa^{-1}$ is spanned by its unitaries. Therefore the unitary group of $aCa^{-1}$ is nonseparable. Fix $ε > 0$ and $d_α$, for $α < R_1$, such that $αd_αa^{-1}$ is a unitary for all $α$ and

$$
\|d_α - d_β\| > ε
$$

for $α ≠ β$. Fix $α < R_1$. Lemma 15.5.4 implies $d_α \in \overline{\text{span}}\{1\} \cup \{b_α : α < R_1\}$, hence there exists a countable $Y_α \subseteq R_1$ such that

$$
d_α \in \overline{\text{span}}\{1\} \cup \{b_α : α ∈ Y_α\}.
$$

By Lemma 15.5.3, $ρ_αb_γ = ρ_α$ if $α ≠ γ$, so the set $Z_α := \{γ ∈ R_1 : ρ_γd_α ∉ \overline{ρ_γ}\}$ is included in $Y_α$, and therefore countable. Fix an enumeration

$$
Z_α = \{μ(α, j) : j ∈ N\}.
$$

By Lemma 15.5.4, $ρ_μ(μ(α,j))d_α \in \overline{\text{span}}\{μ(α,j)ρ_μ(μ(α,j)) : j ∈ N\}$ for each $j ∈ N$. We can therefore fix scalars $λ(α, j)$ and $θ(α, j)$ such that

$$
ρ_μ(μ(α,j))d_α = λ(α, j)μ(α,j)ρ_μ(μ(α,j)) + θ(α, j)ρ_μ(μ(α,j)).
$$

The projection $ρ_μ(μ(α,j))$ is central, hence the matrix $λ(α, j)s_μ(μ(α,j)) + θ(α, j)1_2$ is unitarizable and Lemma 15.5.6 implies $(λ(α, j), θ(α, j)) ∈ Y_0$.

Since $d_α \in \overline{\text{span}}\{1_μ, b_γ : γ ∈ Z_α\}$, we have $\lim_j λ(α, j) = 0$. Therefore there exists $n(α)$ large enough so that $|λ(α, j)| < ε/5$ for $j ≥ n(α)$. Let

$$
F_α := \{μ(α, j) : j < n(α)\}.
$$

As this set is finite, by the Δ-System Lemma there exist an uncountable $S ⊆ R_1$, a finite $R ⊆ R_1$, and $n$ such that $n = n(α)$ for all $α ∈ S$ and the family $F_α$, for $α ∈ S$, is a Δ-system with root $R$. Let $m := |R|$. By re-enumerating, we may assume that $R = \{λ(α, j) : j < m\}$ for all $α ∈ S$. By Lemma 6.6.5 and passing to an uncountable subset of $S$ if necessary we may assume that

$$
|λ(α, j) − λ(γ, j)| < ε/5
$$

for all $j < n$ and all $α$ and $γ$ in $S$.

Suppose for a moment that $\max_j |λ(α, j)| < ε/5$ for some $α ∈ S$. By (15.6) this implies $\max_j |λ(β, j)| < 2ε/5$ for all $β ∈ S$ and therefore $\|d_α - d_β\| < 3ε/5$ for all $α$ and $β$ in $S$; contradiction.

Therefore for every $α ∈ S$ there exists $j(α) ≥ m$ such that $|λ(α, j(α))| ≥ ε/5$. Since the sets $F_α \setminus R$, for $α ∈ S$, are disjoint, the ordinals

$$
ξ(α) := μ(α, j(α))
$$


are distinct for $\alpha \in S$. Write $\lambda(\alpha) := \lambda(\alpha, j(\alpha))$ and $\theta(\alpha) := \theta(\alpha, j(\alpha))$. By replacing $S$ with an uncountable subset, we may assume that $M := \sup_{\alpha \in S} |\theta(\alpha)|$ is finite. Fix $\alpha \in S$ and let $\hat{p} := \hat{p}_{\xi(\alpha)}$. Now (15.5) reads as follows:

$$\hat{p} \hat{a} \xi(\alpha) = \hat{p}(\lambda(\alpha) \hat{a} \xi(\alpha) \hat{a}^{-1} + \theta(\alpha)).$$

Since $\hat{p}$ belongs to the center of $M_2(\ell_\infty(\mathbb{N}))/M_2(c_0(\mathbb{N}))$, $\hat{p} \hat{a} \xi(\alpha)$ is a unitary in $\hat{p}M_2(\ell_\infty(\mathbb{N}))/M_2(c_0(\mathbb{N}))$. With $A_\alpha := \{k(i) : i \in \mathbb{N}\}$, this implies a statement that takes us into the ballpark of Lemma 15.5.7:

$$\lim_{i} \text{dist}(a(k(i)) (\lambda(\alpha) \hat{a} \xi(\alpha) + \theta(\alpha) 1_2) a(k(i))^{-1}, U(M_2(\mathbb{C}))) = 0.$$

The remaining part of this proof is very similar to the final paragraph of the proof of Proposition 14.5.6. By Lemma 6.6.5 there are disjoint closed intervals $U$ and $V$ in $[0, 1]$ such that both $\mathcal{X} := \{\alpha \in S : t_\alpha \in U\}$ and $\mathcal{Y} := \{\alpha \in S : t_\alpha \in V\}$ are uncountable.

Let $K := \|a\| \|a^{-1}\|$. Lemma 15.5.7 implies that there exists $\delta > 0$ such that if $b \in \text{GL}(M_2(\mathbb{C}))$ satisfies $\|b\| \|b^{-1}\| \leq K$ then for all $(\lambda_0, \theta_0) \in Y_{r/5}$ we have

$$\max_{j < 2} \text{dist}(b(\lambda_j s_{r(j)} + \theta_j 1_2) b^{-1}, U(M_2(\mathbb{C}))) \geq \delta$$

for all $r \in U$ and all $t \in V$. Let

$$\mathcal{X} := \{n \in \mathbb{N} : \text{dist}(a(n)(\lambda_0 s_r + \theta_0 1_2) a(n)^{-1}, U(M_2(\mathbb{C}))) < \delta/2$$

for some $r \in U$ and $(\lambda_0, \theta_0) \in Y_0\}.$

The set $\mathcal{X}$ separates $\{\xi(\alpha) : \alpha \in \mathcal{X}\}$ and $\{\xi(\alpha) : \alpha \in \mathcal{Y}\}$, by an argument analogous to that in the proof of Proposition 14.5.6. But these are two uncountable subsets of a Luzin family; contradiction. \hfill $\Box$

**Proof (Theorem 15.5.1).** It remains to check that the algebra $B$ defined above satisfies the requirements of Theorem 15.5.1. (1) is clear from the construction. Lemma 15.5.5 implies (2) and (3), and Lemma 15.5.8 implies (4). \hfill $\Box$

### 15.6 Exercises

**Exercise 15.6.1.** Prove that every $C^*$-algebra that has CRISP is an SAW*-algebra.

**Exercise 15.6.2.** Suppose that $C$ is countably degree-1 saturated $C^*$-algebra and $(A, B)$ are orthogonal, $\sigma$-unital, and nonunital $C^*$-subalgebras of $C$. Prove that there is $c \in C_{+, 1}$ orthogonal to both $A$ and $B$.

**Exercise 15.6.3.** Prove that every separable SAW*-algebra is finite-dimensional. Conclude that all separable algebra with CRISP are finite-dimensional.
Exercise 15.6.4. Suppose that a C*-algebra C is countably degree-1 saturated and infinite-dimensional. Prove the following.

1. C is nonseparable.
2. Every maximal abelian C*-subalgebra of C is nonseparable.
3. C has density character $\geq c$.\(^8\)
4. C is not a von Neumann algebra.

Exercise 15.6.5. Prove that for C*-subalgebras A and B of a C*-algebra C the following are equivalent.

1. A and B are orthogonal, i.e., $ab = 0$ for all $a \in A$ and $b \in B$.
2. $ab = ba = 0$ for all $a \in A$ and $b \in B$.
3. $ab = 0$ for all $a \in A_+$ and all $b \in B_+$.

Exercise 15.6.6. A C*-algebra C has AA-CRISP (asymptotically abelian, countable Riesz separation property) if the following holds. Suppose $a_n, b_n$, for $n \in \mathbb{N}$, are positive elements of C such that $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n$. Furthermore suppose $D \subseteq C$ is separable and $\lim_n \| [a_n, d] \| = 0$ for every $d \in D$. Then there exists $c \in D' \cap C_+$ such that $a_n \leq c \leq b_n$ for all $n$.

Prove that every countably degree-1 saturated algebra has AA-CRISP.

Exercise 15.6.7. Suppose B is a separable C*-subalgebra of a countably degree-1 saturated C*-algebra C. Prove that every derivation of B into C is inner.

Hint: See Example 15.1.3 (2). You will need [194, 8.6.12].

The following is a variant of Higson’s and Skandalis’s formulation of Kasparov’s Technical Theorem ([128], also [197, Theorem 8.1]).

Definition 15.6.8. Assume A, B and D are C*-subalgebras of a C*-algebra C. It is said that D derives A if for every $d \in D$ the inner derivation $\delta_d(x) := dx - xd$ maps A into itself.

A C*-algebra C is a KTT-algebra if the following holds: Assume A, B, and D are C*-subalgebras of M such that A and B are orthogonal and D derives A. Furthermore assume A and B are $\sigma$-unital and D is separable. Then there is $c \in C_+$ such that $c \in D' \cap C_+$ and $c$ separates A and B.

Exercise 15.6.9. Prove that every countably degree-1 saturated C*-algebra is a KTT-algebra.

Exercise 15.6.10. A C*-algebra C is sub-Stonean if for all $a$ and $b$ in C such that $ab = 0$ there are positive contractions $f$ and $g$ such that $af = a$, $gb = b$ and $fg = 0$.

1. Prove that every sub-Stonean C*-algebra is SAW$^\ast$.
2. Prove that every countably degree-1 saturated C*-algebra is sub-Stonean.

\(^8\) Used in the proof of Theorem 16.7.5.
Exercise 15.6.11. Suppose that $B$ is a separable $C^*$-subalgebra of a countably degree-1 saturated $C^*$-algebra $C$. If $J$ is an ideal of $B$ then there is a positive contraction $f \in C \cap B^*$ such that $af = a$ for all $a \in J$.

If $c \in C$ satisfies $Jc = \{0\}$ then $f$ can be chosen to satisfy $fc = 0$ and $fJc = \{0\}$.

Exercise 15.6.12. A $C^*$-algebra $C$ is $\sigma$-sub-Stonean ([156]) if for every separable $C^*$-subalgebra $A$ of $C$ and all positive $c$ and $d$ in $C$ such that $cAd = \{0\}$ there are contractions $f$ and $g$ in $A' \cap C$ such that $fg = 0$, $fc = c$ and $gd = d$. Prove that every countably degree-1 saturated $C^*$-algebra is $\sigma$-sub-Stonean.

Exercise 15.6.13. Prove Lemma 15.3.5: Suppose $C$ is a countably degree-1 saturated $C^*$-algebra and $a_n$, for $n \in \mathbb{N}$, is a sequence of positive elements of $C$ of norm 1 such that $a_n a_{n+1} = a_{n+1}$ for all $n$. Prove that there exists a positive element $a \in C$ of norm 1 such that $a_n a = a$ for all $n$.

Exercise 15.6.14. Suppose that a $C^*$-algebra $C$ is countably degree-1 saturated. Is every masa in $C$ necessarily countably degree-1 saturated?

Exercise 15.6.15. Recall that $\mathcal{Z}_0$ denotes the ideal of asymptotic density zero subsets of $\mathbb{N}$ (Exercise 9.10.4) and let $J$ be the ideal in $\ell_\infty(\mathbb{N})$ generated by $\{\text{proj}_X : X \in \mathcal{Z}_0\}$. Prove that $\ell_\infty(\mathbb{N})/J$ is not countably degree-1 saturated.

Exercise 15.6.16. Suppose $A$ is $\sigma$-unital and $B \subseteq \mathcal{Q}(A)$ is a separable and unital $C^*$-subalgebra. Prove that every asymptotically inner automorphism $\Phi$ of $B$ is implemented by a unitary $u$ in $\mathcal{Q}(A)$, so that $\Phi(b) = ubu^*$ for all $b \in B$.

Hint: Don’t try to apply countable degree-1 saturation of $\mathcal{Q}(A).$ Instead, use a quasi-central approximate unit directly.

Exercise 15.6.17. Suppose that $B$ is a separable, infinite-dimensional, $C^*$-subalgebra of $C(\beta\mathbb{N} \setminus \mathbb{N})$. Prove that there is no conditional expectation of $C$ onto $B$.

Exercise 15.6.18. Suppose that $B$ is a separable, infinite-dimensional, $C^*$-subalgebra of a countably degree-1 saturated $C^*$-algebra $C$. Prove that there is no conditional expectation of $C(\beta\mathbb{N} \setminus \mathbb{N})$ onto $B$.

For the notation used in the following two exercises see §12.3.1.

Exercise 15.6.19. Suppose that $\lambda \leq \kappa$ are infinite cardinals. Prove that $\mathcal{H}_{\lambda}(\ell_2(\kappa))$ is not strictly $\mathcal{H}(\ell_2(\kappa))$-complete, but that it is sequentially strictly $\mathcal{H}(\ell_2(\kappa))$-complete. (I.e., every strictly Cauchy sequence converges.)

Exercise 15.6.20. Prove that $\mathcal{B}(\ell_2(\mathbb{R}_1))/\mathcal{H}_{\kappa_1}(\ell_2(\mathbb{R}_1))$ is not countably degree-1 saturated.

While you are at it, prove that for any pair of cardinals $\lambda \leq \kappa$ the quotient $\mathcal{H}_{\lambda}(\ell_2(\kappa))/\mathcal{H}_{\kappa}(\ell_2(\kappa))$ is countably degree-1 saturated if and only if $\lambda = \mathfrak{R}_0$.

Exercise 15.6.21. Use the result of Exercise 15.6.20 to prove that there exists a unital $C^*$-algebra $M$ with an essential ideal $A$ with the following three properties.

\[\ldots\] and if you do and succeed, let me know how.
1. There is an increasing sequence $g_n$, for $n \in \mathbb{N}$, in $A_{+,1}$ whose supremum is $1_M$.
2. Every increasing uniformly bounded sequence in $A_+$ strictly converges to an element of $M$.
3. The quotient $M/A$ is not countably degree-1 saturated.

**Exercise 15.6.22.** Prove Theorem 15.2.4: Suppose $A$ is a nonunital C*-algebra such that in $A_{+,1}$ there exists an increasing sequence which strictly converges to $1_{\mathcal{M}(A)}$. Prove that the corona $\mathcal{Q}(A)$ is countably degree-1 saturated.

Degree-$n$ conditions and saturation were defined in Definition 15.2.1.

**Exercise 15.6.23.** Prove that the following properties of a C*-algebra are equivalent.
1. Countable degree-2 saturation.
2. Countable degree-$n$ saturation for any $n \geq 2$.
3. Quantifier-free saturation.

**Exercise 15.6.24.** The proof of Proposition 15.4.1 shows that the representation $\pi$ can be unitarized by a positive element. Prove that this is always the case: If a representation of a group can be unitarized, then it can be unitarized by a positive element.

**Exercise 15.6.25.** Prove that every uniformly bounded representation of a discrete amenable group into a von Neumann algebra is unitarizable.

*Hint:* Use the proof of Proposition 15.4.1, replacing saturation by weak compactness of bounded balls.

**Exercise 15.6.26.** Suppose that $C$ countably degree-1 saturated, $B \subseteq C$ is separable and $a \in B' \cap C$ is self-adjoint. Prove that there exists a self-adjoint $c \in B' \cap C$ such that $cf(a) = 0$ for all $f \in C(\text{sp}(a))$ such that $\text{supp}(f) \subseteq [0, 1/2]$ and $cf(a) = f(a)$ for all $f \in C(\text{sp}(a))$ such that $\text{supp}(f) \subseteq [1/2, 1]$.

**Exercise 15.6.27.** Suppose $A$ is a a countably quantifier-free saturated C*-algebra. Prove that the metric structure obtained by adding the distance predicate on $A$ as a von Neumann algebra is unitarizable.

**Exercise 15.6.28.** Suppose $A$ is a countably quantifier-free saturated C*-algebra that in addition has real rank zero and a unique trace $\tau$. Suppose $a \in C_{+,1}$ is such that every $f \in C([0, 1])$ satisfies $\tau(f(a)) = \int_0^1 f(t) \, dt$. Prove that there are projections $p_I \in C^*(a)' \cap A$, for every open interval $I \subseteq [0, 1]$, such that the following holds for all $I$ and $J$.

1. For $f \in C([0, 1])$ such that $\text{supp}(f) \subseteq I$ we have $p_I f(a) = f(a)$.
2. If $I \cap J = \emptyset$ then $p_I p_J = 0$.
3. The length of $I$ is equal to $\tau(p_I)$.

The following exercise generalizes Lemma 15.4.4. It is also related to the argument used in the proof of Lemma 16.6.2.
Exercise 15.6.29. Suppose $C$ is a countably degree-1 saturated $C^*$-algebra, $n \geq 1$, and $t(\bar{x})$ is a countable degree-1 $n$-type over $C$. Prove that exactly one of the following possibilities holds.

1. The type $t(\bar{x})$ is not satisfiable in $C$.
2. The set of realizations of $t(\bar{x})$ in $C$ is nonempty and compact.
3. The set of realizations of $t(\bar{x})$ in $C$ is nonseparable and norm-closed.

Exercise 15.6.30. Prove that all abelian $C^*$-subalgebras of the $C^*$-algebra $B$ defined in Lemma 15.5.4 are separable.

Exercise 15.6.31. Find a $C^*$-subalgebra $C$ of $M_2(\ell_\infty(\mathbb{N}))$ such that for every $\varepsilon > 0$ there is an amenable Banach algebra $B_\varepsilon$ with the following properties.

1. $d_{KK}(C, B_\varepsilon) < \varepsilon$.
2. $B_\varepsilon$ is an inductive limit of a directed and $\sigma$-complete chain of separable, amenable, subalgebras each of which is isomorphic to a $C^*$-algebra.
3. $B_\varepsilon$ has no nonseparable subalgebra isomorphic to a $C^*$-algebra.
4. All abelian $C^*$-subalgebras of $B_\varepsilon$ as are separable.

Exercise 15.6.32. Prove that the property of a Banach algebra $A$, ‘$A$ is isomorphic to a $C^*$-algebra’ reflects to separable subalgebras.

Exercise 15.6.33. Prove that a $(C^*$-algebraic)$^{10}$ tensor product of two infinite-dimensional $C^*$-algebras cannot be a von Neumann algebra. In particular, $\mathcal{B}(H) \otimes \mathcal{B}(K)$ is not isomorphic to $\mathcal{B}(H \otimes K)$ unless at least one of the Hilbert spaces $H$ and $K$ is finite-dimensional.

Notes for Chapter 15

§15.1 Countable degree-1 saturation was introduced in [86], on which §15.1, 15.2, and §15.3 are largely based. Theorem 15.1.5 is [86, Theorem 1] and its proof is an adaptation of [128].

Theorem 15.2.4 is [68, Theorem 3.8]). Its proof in [68] contains a minor problem (see the argument using vectors $\xi_n$ in the proof of clause (10) of Lemma 3.7 on p. 2644). The corollary of Theorem 15.2.4 asserting that the quotient $M/A$ where $M$ is a II$_\infty$ factor with separable predual and $A$ is its Breuer ideal is countably degree-1 saturated has been taken from [68].

In [254] Voiculescu constructed countably degree-1 saturated $C^*$-algebras as quotients of Banach $C^*$-subalgebras of $\mathcal{B}(H)$ that are not $C^*$-algebras themselves. The proof of the countable degree-1 saturation of these $C^*$-algebras uses the tridiagonal method, a variant of which was used in the proof of Lemma 1.9.4. This method can be used to provide a proof of Theorem 15.1.5.

---

10 There is a natural tensor product in the category of von Neumann algebras (§3.1.5), and it is much better behaved than its $C^*$-algebraic counterpart.
All known C*-algebras that are countably quantifier-free saturated are countably saturated, but it is not known whether the converse is true (see [68]).

§15.4 Theorem 15.4.3 was announced in [84]. It is a generalization of Lemma 7.3 (the Second Splitting Lemma) from [33] (this is the case C = ℋ(H) of Exercise 15.6.26). A close relative is [240, Lemma 1.6] (this is the case when A is the universal UHF C*-algebra ⊗_{n≥1}M_n(ℂ) of Exercise 15.6.28).

Simon Wassermann conjectured that the Calkin algebra is essentially non-factorizable. This is confirmed by Theorem 15.4.5, first proved in [115]. There is a simple nonseparable AF C*-algebra that cannot be presented as a nontrivial tensor product ([234]). Such C*-algebras are called tensorially prime. All separable, tensorially prime AF algebras are finite-dimensional, but infinite-dimensional, separable tensorially prime C*-algebras exist ([207]).

The analog of Theorem 15.4.5 for the ultrapowers of II_1 factors was proved earlier in [75]. Theorem 15.4.5 is a noncommutative descendant of [60] and [79]. In the former, van Douwen introduced the ‘βN-spaces’—compact Hausdorff spaces in which the closure of every countable discrete subset is homeomorphic to βN. He observed that Čech–Stone compactifications and Čech–Stone remainders of separable, locally compact spaces have this property and initiated the study of the ‘directions of coordinate axes’ in βN-spaces. Closely related results appear in [206, §10].

A function f : X × Y → Z depends on at most one coordinate if one of the following applies

1. there exists g : Y → Z such that f(x, y) = g(y) for all (x, y) ∈ X × Y, or
2. there exists g : X → Z such that f(x, y) = g(x) for all (x, y) ∈ X × Y.

A function f : X × Y → Z is elementary if there is m ∈ ℤ and partitions X = \( \bigcup_{j<i} X_j \) and Y = \( \bigcup_{j<i} Y_j \) such that the restriction of f to \( X_i \times Y_j \) depends on at most one coordinate for all \( i < m \) and all \( j < m \).

It is not known whether the following theorem (taken from [79, Theorem 4]) has a reasonable noncommutative analog.

**Theorem 15.6.34.** Suppose \( X_j, \forall j \in J \), are compact Hausdorff spaces, Y is a locally compact, σ-compact Hausdorff space, and f : \( \prod_{j \in J} X_j \to βY \) \( \setminus Y \) is continuous. Then \( \prod_{j \in J} X_j \) can be partitioned into finitely many clopen subsets \( U_k, \forall k < n \), such that the restriction of f to each \( U_k \) depends on at most one coordinate. \( \square \)

§15.5 As promised, we provide some background information on amenability for Banach algebras, introduced by Johnson (see [138], [214], [202]).

Suppose A is a Banach algebra and X is a Banach A-bimodule (i.e., both left and right Banach A-module). A derivation from A into X is a continuous linear map \( δ : A \to X \) that satisfies \( δ(ab) = aδ(b) + δ(a)b \). A derivation is inner if it is of the form \( δ_φ(a) := ax − xa \) for some \( x ∈ X \). If X is an A-bimodule, then the dual space \( X^* \) is an A-bimodule with the naturally defined operations, \( aφ(x) := φ(ax) \) and \( φ(a)x := φ(ax) \) for \( a ∈ A, φ ∈ X^* \), and \( x ∈ X \). An A-bimodule of the form \( X^* \) is called a dual Banach A-bimodule.

A Banach algebra A is amenable if every derivation of A into a dual Banach A-bimodule is inner. For example, a locally compact group G is amenable if and only
if the group Banach algebra $L^1(G)$ is amenable. By a result of Johnson ([137]), a Banach algebra is amenable if and only if it has a so-called *virtual diagonal*. In the case of $C^*$-algebras, this is equivalent to every representation having an approximate diagonal in the sense of Definition 5.1.1.

Amenability of Banach algebras is an isomorphism invariant, and a $C^*$-algebra is amenable if and only if it is nuclear. (This profound result is the culmination of the work of many hands; see e.g., [27, §IV.3].) Two amenable $C^*$-subalgebras of $B(H)$ are isomorphic if and only if they are similar (with the similarity implemented by an element of $M$); this is a consequence of Pisier’s partial solution to Kadison’s similarity problem ([202, Theorem 7.16]). This gives the following consequence of Theorem 15.5.1.

**Corollary 15.6.35.** There exists a nonseparable amenable Banach subalgebra $B$ of $M_2(l_\infty(N))$ which is an inductive limit of a directed, $\sigma$-complete, chain of separable, amenable, subalgebras each of which is isomorphic to a $C^*$-algebra, but it has no nonseparable subalgebra isomorphic to a $C^*$-algebra.

Compare the following with Definition 7.3.1 and Exercise 15.6.32).

**Corollary 15.6.36.** For Banach algebras, not being isomorphic to a $C^*$-algebra does not reflect to separable subalgebras.

This shows that the methods of this section cannot be used to produce a separable amenable Banach algebra not isomorphic to a $C^*$-algebra.

Theorem 15.5.1 was proved in [249], generalizing the main result of [43]. The result of Exercise 15.6.31 was also taken from [249]. It is an open problem whether every separable amenable norm-closed algebra of operators on a Hilbert space is isomorphic to a $C^*$-algebra (see e.g., [176]).

Exercise 15.6.30 and Exercise 15.6.31 provide examples of nonseparable $C^*$-algebras all of whose abelian $C^*$-subalgebras are separable. Such algebra was first constructed in [8] using the Continuum Hypothesis. This assumption was removed in [204] where it was proved that $C^*_r(F_\kappa)$ has this property (see Theorem 11.3.2) and, by a construction related to the one in [8], in [24].

The properties of the ultrapowers appearing in Exercise 15.6.10 and Exercise 15.6.12 were isolated in [156]. This paper is a goldmine.
Chapter 16

Full Saturation

Your theorem is not as good as you first believed it to be, nor is it as bad as you thought five days later.

Attributed to G.K. Pedersen

In this chapter we introduce the ‘full’ model-theoretic saturation, and some familiarity with basic model theory of metric structures is assumed.

It is a common knowledge that both set-theoretic and model-theoretic methods are relevant to any subject in which ultrapowers play a role.\(^1\) All ultrapowers, along with all relative commutants of a fixed separable C*-algebra associated to a non-principal ultrafilter on \(\mathbb{N}\) are so similar that the question whether they are isomorphic or not becomes irrelevant. This similarity is characterized by the existence of a ‘\(\sigma\)-complete back-and-forth system’ of partial isomorphisms between any two such ultrapowers. The question whether such system gives rise to an isomorphism between two ultrapowers is trivialized by the Continuum Hypothesis and convoluted by the negation of the Continuum Hypothesis, but its resolution has no bearing on the relation of the underlying algebra with separable C*-algebras.\(^2\)

The exposition of this chapter is, to some extent, parallel to that of Chapter 15. We prove the countable saturation of ultraproducts associated with countably incomplete ultrafilters (Theorem 16.4.1) and the countable saturation of the reduced products associated with the Fréchet ideal (i.e., the asymptotic sequence algebras in Theorem 16.5.1. Combined with the \(\sigma\)-complete back-and-forth systems introduced in §8.2 and with the aid of the Continuum Hypothesis, countable saturation is used to construct many automorphisms of ultrapowers, asymptotic sequence algebras, and the associated relative commutants (§16.7). We also prove two results showing how the theory of a reduced product depends on theories of the original structures. In the case of the ultraproducts this is Łoś’s Theorem, Theorem 16.2.8 (also known as the Fundamental Theorem of Ultraproducts) and in the case of general reduced

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\(^1\) This is certainly the case in the field of operator algebras, as can be seen from the seminal papers such as [46], [180], [157], [156], [123], [179], [159].

\(^2\) This can be construed as a consequence of Shoenfield’s Absoluteness Theorem (Theorem B.2.12).
products this is the Feferman–Vaught Theorem, or rather its continuous variant due to Ghasemi (Theorem 16.3.1).

### 16.1 Full Types and Saturation

In this section we give the model-theoretic definition of types and saturation. This is closely related to the notions of degree-1 types and saturation (§15.1).

Definitions of basic model-theoretic notions and notation can be found in §D.1. The degree-1 type of an element \( a \) of a \( C^* \)-algebra \( A \), and even its quantifier-free type (Definition 15.2.1), cannot provide complete information about the ‘location’ of an element \( a \) within a \( C^* \)-algebra \( A \). The following builds on Example 2.6.6 (3).

**Example 16.1.1.** In the Calkin algebra \( \mathcal{Q}(H) \), consider the images \( \pi(s) \) and \( \pi(b) \) of the unilateral shift and the bilateral shift on the orthonormal basis of \( H \). Both of these elements are unitaries with the spectrum equal to \( T \). By the continuous functional calculus, there exists an isomorphism \( \Phi: C^*(\pi(s)) \rightarrow C^*(\pi(b)) \) that sends \( \pi(s) \) to \( \pi(b) \). This implies that for every *-polynomial \( p(x) \) we have \( \| p(\pi(s)) \| = \| p(\pi(b)) \| \), and therefore \( \pi(s) \) and \( \pi(b) \) have the same quantifier-free type. However, \( \pi(b) \) has a square root and \( \pi(s) \) does not, and therefore no automorphism of \( \mathcal{Q}(H) \) extends \( \Phi \).

This deficiency can be remedied by requiring the partial isomorphism \( \Phi \) to preserve all additional information (of some prescribed sort) about the elements of its domain. Miraculously, in some situations it suffices to interpret ‘additional information’ as ‘the information coded by first-order continuous logic’ (see §D.2).

Given a metric structure \( C \) in a language \( \mathcal{L} \) and a subset \( X \) of its domain, \( \mathcal{L}_X \) denotes the language \( \mathcal{L} \) expanded by constants for element of \( X \). Definition 16.1.2 is the analog of Definition 15.1.1 and Definition 15.2.1. By \( F^i_{\mathcal{L}_X} \) we denote the set of all \( \mathcal{L}_X \)-formulas whose free variables are included in \( \bar{x} = (x_0, \ldots, x_{n-1}) \) (Definition D.2.2). Each variable in logic of metric structures has a sort associated to it (§D.2.2). In the case of \( C^* \)-algebras, sorts correspond to \( n \)-balls for \( n \geq 1 \). Therefore each variable \( x \) is equipped with a natural number \( n \geq 1 \) such that \( \| x \| \leq n \) in every interpretation of \( x \). Some authors adorn the variables with superscripts indicating their sorts, but this will not be necessary for our purposes.

By \( \varphi^M(\bar{b}) \) we denote the interpretation of a formula \( \varphi(\bar{x}) \) at \( \bar{b} \) in a structure \( M \).

**Definition 16.1.2.** Fix a metric structure \( C \), \( X \subseteq C \), and \( n \in \mathbb{N} \). A **condition** over \( X \) in variables \( \bar{x} = (x_j : j < n) \) is an expression of the form \( \varphi(\bar{x}) = r \) with \( r \in \mathbb{R} \) and \( \varphi \in F^i_{\mathcal{L}_X} \). It is **satisfied** in \( C \) by a tuple \( \bar{b} \) of the appropriate sort if \( \varphi^C(\bar{b}) = r \).

A type over \( C \) is a set of conditions over \( C \). If all conditions in a type \( t \) have all of their free variables among \( \bar{x} \), we write \( t(\bar{x}) \). A type \( t(\bar{x}) \) is **realized** in \( C \) if there exists \( \bar{b} \) of the appropriate sort in \( C \) such that every condition in \( t(\bar{x}) \) is satisfied by \( \bar{b} \).

A type \( t(\bar{x}) \) is **approximately realized** (or **satisfiable**) in \( C \) if for every finite subset \( t_0(\bar{x}) \) of \( t(\bar{x}) \) and every \( \varepsilon > 0 \) there exists \( \bar{b} \) of the appropriate sort in \( C \) such that
for every condition \( \varphi(\bar{x}) = r \in t_0(\bar{x}) \) we have \( |\varphi^C(\bar{b}) - r| < \varepsilon \). Such \( \bar{b} \) is a partial realization of \( t(\bar{x}) \). A type \( t \) is omitted in \( C \) if no tuple in \( C \) realizes \( t \).

In classical logic, a finite type is consistent if and only if it is realized. This is in general not true in logic of metric structures (see Exercise 16.8.5 and Example 15.3.3).

**Example 16.1.3.**

1. Every degree-1 type is a type. In Definition 15.1.2 it was postulated that all variables in a degree-1 type range over the 1-ball. By homogeneity, every degree-1 type in variables belonging to an \( n \)-ball is equivalent to a degree-1 type in variables belonging to the unit ball.

2. A set of relations in some finite set of generators (§2.3) is a type. All conditions occurring in a type of this sort are quantifier-free.

3. Let \( t_\mathbb{N}(x) \) be the type with a single condition, \( \|xx^*-x^*1\| = 0 \). Hence for a fixed \( n \geq 1 \) the set of normal elements in the \( n \)-ball of a given \( \mathbb{C}^* \)-algebra \( A \) is the realization of \( t_\mathbb{N}(x) \) (with the variable \( x \) of an appropriate sort). Similarly, projectionss in \( A \) are realization of the type \( \{ \|xx^*-x\| = 0, \|x^*-1\| = 0 \} \). If we consider the language with a constant for the unit, then the unitaries in a unital \( \mathbb{C}^* \)-algebra \( A \) are realizations of the type

\[
\{ \|xx^*-x^*1\| = 0, \|xx^* - 1\| = 0, \|x^*1\| = 0 \}.
\]

Analogous remarks apply to all distinguished classes of elements appearing in Definition 1.4.1.

4. Example 2.3.3 shows that the set of units of type\(^3 M \) is the realization of the type in the variables \( x_{ij} \), for \( i < n \) and \( j < n \), with conditions \( \|x_{ij}x_{kl} - \delta_{ijkl}\| = 0 \) and \( \|x_{ij} - x_{ij}^*\| = 0 \) for all \( i,j,k,l \).

**Definition 16.1.4.** If \( C \) is a metric structure, \( X \subseteq C \), and \( \bar{a} \) is an \( n \)-tuple in \( C \), then the type of \( \bar{a} \) over \( X \), \( \text{type}(\bar{a}/X) \), is the following type in the expanded language \( \mathcal{L}_X \):

\[
\text{type}(\bar{a}/X) := \{ \varphi^C(\bar{x}, \bar{b}) = r : \bar{b} \in C, \varphi \in F_{\mathcal{L}_X}^\bar{x}, \text{ and } \varphi^C(\bar{a}, \bar{b}) = r \}
\]

where \( \bar{y} \) is a tuple of variables of the same sort as \( \bar{b} \).

There is some ambiguity in the choice of the sorts of the variables occurring in the definition of \( \text{type}(\bar{a}/X) \). For definiteness, we always choose the sort associated with the smallest possible \( n \) such that a given variable or constant belongs to the \( n \)-ball.

**Definition 16.1.5.** Given an infinite cardinal \( \kappa \), a metric structure is said to be \( \kappa \)-saturated\(^4 \) if every satisfiable type in \( \mathcal{L}_M \) of cardinality less than \( \kappa \) is realized in \( M \). An \( \mathbb{R}_1 \)-saturated metric structure is also called countably saturated. A metric structure \( M \) is saturated if it is \( \chi(M) \)-saturated. We will say that \( M \) is fully (countably)

\(^3\) Just another innocuous clash of terminology...

\(^4\) Some authors (notably, [39, §5.1]) define \( \kappa \)-saturation in a slightly different manner. The two definitions agree when the language is separable.
saturated} whenever it is (countably) saturated and there is a possibility of confusion with degree-1 saturation or with quantifier-free saturation.

Example 16.1.6. The classical ‘discrete’ logic is a special case of logic of metric structures in which all structures are taken with the discrete metric. An \( \eta_1 \) set (Definition 8.2.2) is, by definition, a countably quantifier-free saturated dense linear ordering without endpoints. Since the theory of dense linear orderings without endpoints admits elimination of quantifiers (see [39, §1.5]), every \( \eta_1 \) set is fully countably saturated (see [125]).

The space of formulas \( F_{\mathcal{L}} \) is equipped with the norm (Definition D.2.4)

\[
\| \phi \| := \sup \{ |\phi^M(\bar{a})| : M \text{ is an } \mathcal{L} \text{-structure, } \bar{a} \in M \},
\]

and it is an algebra over \( \mathbb{R} \) with respect to this norm. A type \( t(\bar{x}) \) is separable if it is a separable in this norm as subset of \( F_{\mathcal{L}} \).

Lemma 16.1.7. An \( \mathcal{L} \)-structure \( M \) is countably saturated if and only if every consistent separable type over \( M \) is realized in \( M \).

Proof. We need to prove that if every consistent countable type is realized then every consistent separable type is realized. Clearly a subset of a consistent type is consistent. Since the interpretation of a formula is a continuous function, a type is realized if and only if its dense subset is realized.

Definition 16.1.8. Suppose \( \kappa \) is a cardinal. An \( \mathcal{L} \)-structure \( M \) is said to be \( \kappa \)-universal if for every \( \mathcal{L} \)-structure \( A \) of density character smaller than \( \kappa \) elementarily equivalent to \( M \) there exists an elementary embedding \( \Phi: A \to M \).

An \( \mathcal{L} \)-structure \( M \) is said to be \( \kappa \)-homogeneous if for all \( \mathcal{L} \)-structures \( A \) and \( B \) of density character smaller than \( \kappa \) and elementary embeddings \( \Phi: B \to A \) and \( \Psi: B \to M \) there exists an elementary embedding \( \Theta: A \to M \) such that the following diagram commutes.

\[
\begin{array}{c}
A \xrightarrow{\Phi} B \\
\downarrow \Theta \downarrow \Psi \\
M
\end{array}
\]

A structure is said to be separably universal (separably homogeneous) if it is \( \mathcal{R}_1 \)-universal (\( \mathcal{R}_1 \)-homogeneous).

A proof of the following standard fact is left as Exercise 16.8.17.

Theorem 16.1.9. Given an infinite cardinal \( \kappa \), a metric structure is \( \kappa \)-saturated if and only if it is \( \kappa \)-universal and \( \kappa \)-homogeneous.
16.2 Reduced Products and Ultraproducts

In this section we define ultraproducts and reduced products of C*-algebras and prove Łoś's Theorem and countable saturation of ultraproducts associated with countably incomplete ultrafilters.

Given a family of C*-algebras $B_j$, for $j \in \mathbb{J}$, and an ideal $\mathcal{J}$ on $\mathbb{J}$, we let $\bigoplus \mathcal{J} B_j := \{ \hat{b} \in \prod_{j \in \mathbb{J}} B_j : \limsup_{j \to \mathcal{J}} \| b_j \| = 0 \}$ (Definition 2.5.8).

**Definition 16.2.1.** The reduced product of an indexed family $B_j$, for $j \in \mathbb{J}$, of C*-algebras associated with an ideal $\mathcal{J}$ on $\mathbb{J}$ is the quotient $\prod_{j \in \mathbb{J}} B_j / \bigoplus \mathcal{J} B_j$. We will sometimes denote it $\prod_{j \in \mathbb{J}} B_j / \mathcal{J}$.

The general definition of a reduced product of metric structures of the same language is given in Definition D.2.13.

The dual filter of an ideal $\mathcal{J}$ on a set $\mathbb{J}$ is $\mathcal{J}_s := \{ \mathbb{J} \setminus X : X \in \mathcal{J} \}$ (Definition 9.1.2). Some authors refer to the reduced product $\prod_{j \in \mathbb{J}} B_j / \bigoplus \mathcal{J} B_j$ as the reduced product with respect to the dual filter.

Two ‘extreme’ cases of reduced products are also the most important ones.

**Example 16.2.2.** The algebra $\bigoplus_{\text{Fin}} B_j$ is the direct sum, $\bigoplus_{j \in \mathbb{J}} B_j$. An important example of a reduced product associated to Fin is the algebra $\prod_{j \in \mathbb{J}} M_j(\mathbb{C}) / \bigoplus_{j \in \mathbb{J}} M_j(\mathbb{C})$. If $B_n = B$ for all $n$, then $\bigoplus_{j \in \mathbb{J}} B = c_0(B)$, $\prod_{j \in \mathbb{J}} B = \ell_\infty(B)$, and the algebra $\ell_\infty(B)/c_0(B)$ is called the asymptotic sequence algebra.

**Definition 16.2.3.** Suppose that $\mathcal{J}$ is the dual ideal of an ultrafilter $\mathcal{U}$. The reduced product $\prod_{j \in \mathbb{J}} B_j / \bigoplus \mathcal{J} B_j$ is denoted $\prod_{j \in \mathbb{J}} B_j / \mathcal{U}$ and called ultraproduct. If $B_j = B$ for all $j$, then $c_\mathcal{U}(B) := \bigoplus \mathcal{J} B$ hence $c_\mathcal{U} = \{ \hat{b} \in \prod_{j \in \mathbb{J}} : \lim_{j \to \mathcal{U}} \| b_j \| = 0 \}$. In this case the ultraproduct $\prod_{j \in \mathbb{J}} B_j / \mathcal{U}$ is denoted $B_\mathcal{U}$ and called ultrapower.

**Remark 16.2.4.** In the C*-algebra literature, the norm ultrapower as in Definition 16.2.3 is usually denoted $A_\otimes$ (where $\otimes$ stands for whatever one might use to denote an ultrafilter). The notation $A_\otimes$ is reserved for the tracial ultrapower of a C*-algebra $A$ such that $\mathcal{T}(A) \neq \emptyset$. The tracial ultrapower is the ultrapower in the language of C*-algebras in which the metric is given by the uniform 2-norm $\|a\|_2,a := \sup_{\tau \in \mathcal{T}(A)} \tau(a^*a)^{1/2}$. As this is just another special instance of the general definition of an ultrapower, and there are more languages than possible locations for subscripts or superscripts, from a logician’s point of view the best way to indicate which ultrapower is being used is to specify the language. All this said, the tracial ultrapower construction is very important and it deserves a special notation. It will appear implicitly in Example 16.4.5, Proposition 16.4.6, and Exercise 16.8.23.

**Definition 16.2.5.** An element $b$ of $\prod_{j \in \mathbb{J}} B_j / \bigoplus \mathcal{J} B_j$ is routinely identified with its representing sequence $(b_j)$. This formally incorrect practice, akin to treating the elements of an $L_p$-space as functions and not equivalence classes of functions, is mostly harmless. A C*-algebra $B$ is identified with its image under the diagonal embedding $b \mapsto (b, b, \ldots)$ inside $\ell_\infty(B)/c_\mathcal{J}(B)$ and inside an ultrapower $B_\mathcal{U}$. The
relative commutant of $B$ in the asymptotic sequence algebra, $B' \cap \ell_\infty(B)/c_0(B)$, is known as the central sequence algebra. The relative commutant $B' \cap B^\mathcal{U}$ is the relative commutant.

Most known instances of the ultraproduct/ultrapower constructions are instances of the general model-theoretic Definition D.2.14. The analog of the following lemma is true in virtually every category of metric structures equipped with the ultraproduct functor.

**Lemma 16.2.6.** If $A_j$, for $j \in J$, are $C^*$-algebras, then the definitions of the ultraproduct $\prod \mathcal{U} A_j$ given in definitions 16.2.3, C.7.1, and D.2.14 agree.

**Proof.** It suffices to see that the $n$-ball of $\prod \mathcal{U} A_j$ is the ultraproduct with respect to $\mathcal{U}$ of the $n$-balls of $A_i$ for $i \in I$ for all $n \geq 1$. \hfill $\Box$

Here is the precise statement for the $C^*$-algebras (see [87]).

**Theorem 16.2.7.** The functor that associates the metric structure $\mathcal{M}(A)$ to a $C^*$-algebra $A$ (Definition D.2.1) commutes with taking ultraproducts. \hfill $\Box$

In the ongoing discussion, $\mathcal{L}$ will denote an arbitrary metric language. The distinguishing feature of the ultrapower functor is the fact that it preserves the theory of a structure.

**Theorem 16.2.8 (Łoś’s Theorem).** If $(M_j)_{j \in J}$ are $\mathcal{L}$-structures, $\mathcal{U}$ is an ultrafilter on $J$, $\varphi(\bar{a})$ is a formula, and $M := \prod \mathcal{U} M_j$, then $\varphi^M(\bar{a}) = \lim_{j \to \mathcal{U}} \varphi^{M_j}(\bar{a}_j)$ for all $\bar{a}$ in $\prod \mathcal{U} M_j$ of the appropriate sort.

**Proof.** The proof proceeds by induction on the complexity of $\varphi$ (Definition D.2.2). In this proof we suppress all redundant parameters of $\varphi$.

If $\varphi$ is atomic, then $\varphi^M = \lim_{j \to \mathcal{U}} \varphi^{M_j}$ by the definition.

Suppose that $n \geq 1$, $f : \mathbb{R}^n \to \mathbb{R}$ is continuous, $\psi = f(\varphi_0, \ldots, \varphi_{n-1})$, and the assertion is true for $\varphi_j$, for $j < n$. Then

$$\lim_{j \to \mathcal{U}} \psi^{M_j} = \lim_{j \to \mathcal{U}} f(\varphi_0, \ldots, \varphi_{n-1})^{M_j} = f(\lim_{j \to \mathcal{U}} \varphi_0^{M_j}, \ldots, \lim_{j \to \mathcal{U}} \varphi_{n-1}^{M_j}) = \psi^M$$

by the continuity of $f$.

Suppose the assertion is true for $\varphi(x)$. Then for every $b \in M$ and its representing sequence $(b_j)$ we have $\lim_{j \to \mathcal{U}} \varphi(b_j)^{M_j} = \varphi^M(b)$. Since $\mathcal{U}$ is an ultrafilter $\sup_j$ and $\lim_{j \to \mathcal{U}}$ commute, and the conclusion follows. The proof for $\inf_j \varphi(x)$ is analogous.

We proved that the set of formulas for which the conclusion of Theorem 16.2.8 holds contains all atomic formulas and that it is closed under both (4) and (5) as in Definition D.2.2. Therefore the assertion is true for all formulas. \hfill $\Box$

\footnote{In some situations this is not quite obvious, as finding the correct model-theoretic language for a given category may take some effort.}
Łoś’s Theorem is also known as the Fundamental Theorem of Ultraproducts. If a theory is identified with a character on the space of formulas (Definition D.2.8), then Łoś’s Theorem states that

$$\text{Th}(\prod_{\mathcal{U}} B_j) = \text{weak}^*\text{-lim}_{j\to\mathcal{U}} \text{Th}(B_j),$$

and by expanding the language the analogous formula holds for the type of a tuple $\vec{a}$ in $\prod_{\mathcal{U}} B_j$. This is not the case for reduced products of metric structures in general, even if the sequence $\text{Th}(B_j)$, for $j \in \mathbb{N}$, converges pointwise. Nevertheless, a useful analog of Łoś’s Theorem for reduced products exists, but it deserves a section of its own.

### 16.3 The Metric Feferman–Vaught Theorem

In this section we prove Ghasemi’s metric Feferman–Vaught Theorem.

We fix an arbitrary metric language $\mathcal{L}$. The main result of this section is a mouthful (Definition 16.3.2 and Theorem 16.3.3) but only the following corollary will be needed later on, and only in Section 16.5 and some of the exercises.

**Theorem 16.3.1.** For every $\mathcal{L}$-formula $\varphi(\vec{x})$ there is a countable set $\mathbb{F}$ of $\mathcal{L}$-formulas all of whose free variables are included in those of $\varphi(\vec{x})$ with the following property. If $(M_j)_{j \in \mathbb{J}}$ and $(N_j)_{j \in \mathbb{J}}$ are $\mathcal{L}$-structures and $\mathcal{J}$ is an ideal on $\mathbb{J}$, then

$$\sup_{\theta \in \mathbb{F}} \limsup_{j \to \mathbb{J}} |\varphi(\vec{a}_j)^{M_j} - \varphi(\vec{b}_j)^{N_j}| = 0$$

implies

$$\varphi(\vec{a})^{\prod_{\mathcal{J}} M_j / \mathcal{J}} = \varphi(\vec{b})^{\prod_{\mathcal{J}} N_j / \mathcal{J}}$$

for all $\vec{a}_j \in M_j$ and $\vec{b}_j \in N_j$ of the appropriate sort.

Let $\mathcal{L}_{BA}^+$ be the language of Boolean algebras with the standard symbols $\land, \lor, \neg, 0,$ and $1$, also equipped with constants $Z^\zeta_t$ for every $\mathcal{L}$-formula $\zeta$ and every $t \in \mathbb{Q}$.

Definition 16.3.2 and Theorem 16.3.3 are stated for $\mathcal{L}$-formulas whose range is included in $[0, 1]$. Since the range of every $\mathcal{L}$-formula $\varphi(\vec{x})$ is a bounded interval, the range of $r(\varphi(\vec{x}) - t)$ is $[0, 1]$ for appropriately chosen real numbers $r$ and $t$ and this will not be a loss of generality.

**Definition 16.3.2.** For $k \geq 2$, an $\mathcal{L}$-formula $\varphi(\vec{x})$ is $k$-determined if objects with the following properties exist.

1. A finite set $\mathbb{F}[\varphi, k]$ of $\mathcal{L}$-formulas whose free variables are included in the free variables of $\varphi(\vec{x})$,
2. $\mathcal{L}_{BA}^+$-formulas $\theta^\varphi_{l,k}$ for $0 \leq l \leq k$, such that
   a. all variables of $\theta^\varphi_{l,k}$ are among $Z^\zeta_t$ for $\zeta \in \mathbb{F}[\varphi, k]$ and $t \in \mathbb{Q} \cap [0, 1]$, and
   b. the formula $\theta^\varphi_{l,k}$ is increasing, in the sense that if $X_i \leq X'_i$ for all $i$ then $\theta^\varphi_{l,k}(\vec{X})$ implies $\theta^\varphi_{l,k}(\vec{X}')$. 

For $\mathcal{L}$-formulas whose range is included in $[0, 1]$, the following properties hold.

1. A finite set $\mathbb{F}[\varphi, k]$ of $\mathcal{L}$-formulas whose free variables are included in the free variables of $\varphi(\vec{x})$,
2. $\mathcal{L}_{BA}^+$-formulas $\theta^\varphi_{l,k}$ for $0 \leq l \leq k$, such that
   a. all variables of $\theta^\varphi_{l,k}$ are among $Z^\zeta_t$ for $\zeta \in \mathbb{F}[\varphi, k]$ and $t \in \mathbb{Q} \cap [0, 1]$, and
   b. the formula $\theta^\varphi_{l,k}$ is increasing, in the sense that if $X_i \leq X'_i$ for all $i$ then $\theta^\varphi_{l,k}(\vec{X})$ implies $\theta^\varphi_{l,k}(\vec{X}')$. 

These properties are used to construct the metric Feferman–Vaught Theorem.
Given \(\mathcal{L}\)-structures \((M_i)_{i \in \mathcal{I}}\) and an ideal \(\mathcal{I}\) on \(\mathcal{I}\), for \(\zeta(\bar{x}) \in \mathbb{F}[\varphi,k]\) and \(0 \leq t \leq 1\) we write \(\theta_t^{\varphi,k}[\bar{a}]\) for the value of \(\theta_t^{\varphi,k}\) in \(\mathcal{P}(\mathcal{I})/\mathcal{I}\) with\(^6\)

\[
Z_t^{\zeta}[\bar{a}] := \{j : (\zeta(\bar{a}_j))^{M_j} > t\} \mathcal{J}.
\]

Then the following holds (writing \(M := \prod_j M_j/\mathcal{J}\))

3. \(\varphi(\bar{a})^{M} > (l + 1)/k\) implies \(\theta_t^{\varphi,k}[\bar{a}]\) and
4. \(\theta_t^{\varphi,k}[\bar{a}]\) implies \(\varphi(\bar{a})^{M} > (l - 1)/k\).

In particular, Definition 16.3.2 asserts that the value of \(\varphi(\bar{a})\) is determined up to \(2/k\) by a finite set of formulas \(\theta_t^{\varphi,k}\) for \(0 \leq l \leq k\), which are in turn determined by the evaluations of formulas in the finite set \(\mathbb{F}[\varphi,k]\) in every \(M_j\).

The proof of the following theorem of Ghasemi is only notationally different from the original ([116]).

**Theorem 16.3.3.** Every \(\mathcal{L}\)-formula is \(k\)-determined for all \(k \geq 2\).

**Proof.** The proof proceeds by induction on complexity of \(\varphi\), simultaneously for all \(k \geq 2\). By using Proposition D.2.6, it suffices to prove that the set of all \(k\)-determined formulas satisfies the following closure properties:

1. All atomic formulas are \(k\)-determined.
2. If \(\varphi\) is \(k\)-determined, so is \(\frac{1}{2}\varphi\).
3. If \(\varphi\) and \(\psi\) are \(2k\)-determined, then \(\varphi - \psi\) is \(k\)-determined.
4. If \(\varphi\) is \(k\)-determined, so are \(\sup_x \varphi\) and \(\inf_x \varphi\) for every variable \(x\).

For readability of the ongoing proof, we will describe the required Boolean formulas \(\theta_t^{\varphi,k}\) informally. We will also combine the recursive construction of these objects with a proof that they have the desired properties for arbitrary \((M_i)_{i \in \mathcal{I}}\) and \(\mathcal{I}\), with \(M = \prod_j M_j/\mathcal{J}\) and \(\bar{a} \in M\) of the appropriate type. (Needless to say, the constructed objects will not depend on the choices of \((M_j), \mathcal{I}, \) and \(\bar{a}\).) This will make this subtle proof proceed smoother.

Let \(\theta_t^{\varphi,k} := \text{False}\) if \(l > k\) and \(\theta_t^{\varphi,k} := \text{True}\) if \(l \leq 0\). This convention will be used tacitly.

If \(\varphi\) is atomic or constant (i.e., a scalar), then let \(\mathbb{F}[\varphi,k] := \{\varphi\}\) and let \(\theta_t^{\varphi,k}\) be the formula \(Z_t^{\varphi,k} \neq 0\). Clearly \(\theta_t^{\varphi,k}\) is increasing. Since

\[
\varphi(\bar{a})^{M} = \limsup_{j \to \mathcal{J}} \varphi(\bar{a}_j)^{M_j},
\]

we have that \(\varphi(\bar{a})^{M} > (l + 1)/k\) implies \(\theta_t^{\varphi,k}[\bar{a}]\).

Similarly, \(\theta_t^{\varphi,k}[\bar{a}]\) implies \(\varphi(\bar{a})^{M} > (l - 1)/k\), as required.

Suppose that \(\varphi(\bar{x}) = \frac{1}{2}\psi(\bar{x})\) and the inductive hypothesis holds for \(\psi\). Let \(\mathbb{F}[\varphi,k] := \mathbb{F}[\psi,k]\) and let \(\theta_t^{\varphi,k} := \theta_t^{2\psi,k}\). These objects satisfy the requirements by the definitions.

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\(^6\) Here [\(\mathcal{X}\)] \(\mathcal{I}\) denotes the equivalence class of \(\mathcal{X}\) modulo \(\mathcal{I}\).
Suppose that $\varphi = \psi \cdot \eta$ and the inductive hypothesis holds for $\psi$ and $\eta$. Let

$$F[\varphi, k] := F[\psi, 2k] \cup \{1 - \xi : \xi \in F[\eta, 2k]\}.$$ 

In order to define $\theta^{\psi, k}$, we need an additional bit of notation. Given a Boolean formula $\theta(X_0, \ldots, X_{n-1})$, where $X_0, \ldots, X_{n-1}$ is the list of all variables occurring in $\theta$, let (if $X = Z^\xi_y$ then $\bar{X}$ denotes $(Z^\eta_1 - \xi^\eta_0)^\eta$):

$$\bar{\theta}(X_0, \ldots, X_{n-1}) := -\theta(\bar{X}_0, \ldots, \bar{X}_{n-1}).$$

The following claims are proved by an easy computation.

**Claim.** If $\theta$ is increasing then so is $\bar{\theta}$. □

**Claim.** Suppose that $\theta_i^{\eta, k}$ satisfies (3) and (4) of Definition 16.3.2. Then for every $M = \prod M_j / \mathcal{J}$ and $\bar{a} \in M$ we have $\bar{\theta}_i^{\eta, k}[\bar{a}]$ implies $\eta(\bar{a})^M < (l + 1)/k$ and $\eta(\bar{a})^M \geq (l - 1)/k$ implies $\bar{\theta}_i^{\eta, k}[\bar{a}]$. □

We will prove that the formula

$$\theta_m^{\psi, k} := \bigvee_{l=0}^{2(k-m)} (\theta_{2m+l}^{\psi, 2k} \land \bar{\theta}_{2m+l}^{\eta, 2k})$$

satisfies the requirements. By the first Claim, it is increasing.

Suppose that $\theta_m^{\psi, k}[\bar{a}]$ holds. Fix $l$ such that both $\theta_{2m+l}^{\psi, 2k}[\bar{a}]$ and $\bar{\theta}_{2m+l}^{\eta, 2k}[\bar{a}]$ hold. By the inductive assumption and Claim, $(2m + l - 1)/2k < \psi(\bar{a})^M$ and $(l + 1)/2k \geq \eta(\bar{a})^M$. Therefore if $\psi(\bar{a})^M > 0$ then $\eta(\bar{a})^M = \psi(\bar{a})^M - \eta(\bar{a})^M > (m - 1)/k$ as required.

Now suppose that $\psi(\bar{a})^M > (m + 1)/k$. This implies that $\psi(\bar{a})^M > 0$ and therefore $\eta(\bar{a})^M = \psi(\bar{a})^M - \eta(\bar{a})^M$. Fix $l$ such that $(2m + l)/2k < \psi(\bar{a})^M < (2m + l + 1)/2k$. Then $\eta(\bar{a})^M < (2m + l + 1)/2k - (2m + 2)/2k < (l - 1)/2k$, hence $\theta_{2m+l}^{\psi, 2k}[\bar{a}] \land \bar{\theta}_{2m+l}^{\eta, 2k}[\bar{a}]$ holds. By monotonicity both $\theta_{2m+l-1}^{\psi, 2k}[\bar{a}] \land \bar{\theta}_{2m+l-1}^{\eta, 2k}[\bar{a}]$ and $\theta_m^{\psi, k}[\bar{a}]$ hold, as required.

Now suppose that $\varphi(\bar{x}) = \sup_y \psi(\bar{x}, y)$ and the inductive hypothesis holds for $\psi$. Let

$$\mathcal{B} := \{t : Z^\xi \text{ occurs in } \theta_i^{\psi, k} \text{ for some } 0 \leq l \leq k\},$$

$$\mathcal{C} := \{\alpha : \text{there is } F \subseteq F[\psi, k] \text{ such that } \alpha : F \to \mathcal{B}\}.$$ 

Since $\mathcal{B}$ is finite, so is $\mathcal{C}$. We’ll need the set $\mathcal{C}_0 := \{\alpha \in \mathcal{C} : \text{dom}(\alpha) = F[\psi, m]\}$ of ‘maximal’ elements of $\mathcal{C}$. For $\alpha \in \mathcal{C}$ consider the $\mathcal{L}$-formula

$$\xi_\alpha(\bar{x}) := \sup_y \min_{\zeta \in \text{dom}(\alpha)} (\zeta(\bar{x}, y) - \alpha(\zeta)).$$

The salient property of this formula is that for every $\bar{a} \in \prod_j M_j / \mathcal{J}$, with the interpretations of the Boolean constants as in Definition 16.3.2, we have
\[
Z_0^{\vec{\alpha}}[\vec{a}] = \bigwedge_{\zeta \in \dom(\alpha)} Z_{\alpha(\zeta)}^\ast[\vec{a}].
\]

Let \( F[\varphi, k] := F[\psi, k] \cup \{ \xi_{\alpha} : \alpha \in \mathcal{C} \} \).

For \( 0 \leq m \leq k \) let \( \theta_m^{\varphi, \psi} \) be the Boolean formula asserting that there exist elements \( Y[\alpha] \), for \( \alpha \in \mathcal{C} \), such that

1. \( Y[\alpha] \leq Z_0^{\vec{\alpha}} \) for all \( \alpha \).
2. If \( \alpha \in \mathcal{C} \) and \( \dom(\alpha) = \mathbb{G} \cup \mathbb{H} \), with both \( \mathbb{G} \) and \( \mathbb{H} \) nonempty, then

\[
Y[\alpha] \cap \mathbb{G} \cup Y[\alpha] \cap \mathbb{H} = Y[\alpha].
\]

3. The elements \( Y[\gamma] \), for \( \gamma \in \mathcal{C}_0 \), form a partition of unit, i.e., \( \bigvee_{\gamma \in \mathcal{C}_0} Y[\gamma] = 1 \) and \( Y[\gamma] \cap Y[\beta] = 0 \) if \( \gamma \) and \( \beta \) are distinct elements of \( \mathcal{C}_0 \).
4. The formula obtained from \( \theta_m^{\varphi, \psi} \) by replacing \( Z_{\alpha(\zeta)}^\ast \) with \( Y[\alpha] \) for all \( \zeta \in F[\psi, k] \) and \( \alpha \in \mathcal{C} \) with \( \dom(\alpha) = \{ \xi \} \) holds.

We claim that \( \theta_m^{\varphi, \psi} \) for \( 0 \leq m \leq k \) are as required.

Suppose that \( \varphi(\vec{a})^M > (m + 1)/k \). Fix \( b \) such that \( \psi(\vec{a}, b)^M > (m + 1)/k \) and identify both \( \vec{a} \) and \( b \) with their representing sequences, \( (\vec{a}_j) \) and \( (b_j) \) for \( j \in J \), respectively. For each \( j \in J \) define \( \gamma_j : F[\psi, k] \to \mathbb{R} \) by

\[
\gamma_j(\zeta) := \max\{ t \in \mathbb{R} : t < \zeta(\vec{a}_j, b_j)^M \}.
\]

Then \( \gamma_j \in \mathcal{C}_0 \) for every \( j \) and the sets \( Y[\gamma] := \{ j : \gamma = \gamma_j \} \), for \( \gamma \in \mathcal{C}_0 \), form a partition of \( J \).

Therefore \( Y[\gamma] := \bar{Y}[\gamma] \), for \( \gamma \in \mathcal{C}_0 \), form a partition of unit as required in (3). For \( \alpha \in \mathcal{C} \) let

\[
\bar{Y}[\alpha] := \{ j : \alpha(i) = \max\{ t \in \mathbb{R} : t < \zeta_i(\vec{a}_j, b_j)^M \} \text{ for all } i \in \dom(\alpha) \}
\]

and let \( Y[\alpha] := [\bar{Y}[\alpha]]_\mathcal{J} \). This definition is compatible with the above definitions in the case when \( \gamma \in \mathcal{C}_0 \), and these elements satisfy (1) and (2) from the definition of \( \theta_m^{\varphi, \psi} \). Moreover, by the inductive hypothesis on \( \psi \), the formula obtained from \( \theta_m^{\varphi, \psi} \) by replacing \( Z_{\alpha(\zeta)}^\ast \) with \( Y[\alpha] \) for all \( i \in m \) and \( \alpha \in \mathcal{C} \) such that \( \dom(\alpha) = \{ \xi \} \) holds. This is (4) of Definition 16.3.2, and therefore \( \theta_m^{\varphi, \psi}[\vec{a}] \) holds as required.

Now suppose that \( \theta_m^{\varphi, \psi}[\vec{a}] \) holds. Fix \( Y[\alpha] \), for \( \alpha \in \mathcal{C} \), that satisfy (1)-(4) as asserted by \( \theta_m^{\varphi, \psi}[\vec{a}] \). Fix \( \bar{Y}[\alpha] \subset J \) that lifts \( Y[\alpha] \) for every \( \alpha \in \mathcal{C} \). Since there are only finitely many requirements on \( Y[\alpha] \), for \( \alpha \in \mathcal{C} \), after removing a set in \( \mathcal{J} \) we may assume that these lifts satisfy the analogs of conditions (2) and (3). Then removing a set in \( \mathcal{J} \) from \( J \) does not affect \( \prod_j M_j / \mathcal{J} \), for simplicity of notation we may assume that the lifts \( Y[\alpha] \), for \( \alpha \in \mathcal{C} \), satisfy the exact analogs of conditions (2) and (3). Thus \( \bar{Y}[\alpha] \cup \bar{Y}[\alpha] \cap \mathbb{H} = Y[\varphi, \alpha] \) whenever and \( \alpha \in \mathcal{C} \), \( \dom(\alpha) = \mathbb{G} \cup \mathbb{H} \), \( \mathbb{G} \) and \( \mathbb{H} \) are nonempty, and in addition \( \bar{Y}[\gamma] \), for \( \gamma \in \mathcal{C}_0 \), form a partition of \( J \).

Fix \( j \in J \). Then \( j \in \bar{Y}[\gamma] \) for a unique \( \gamma_j \in \mathcal{C}_0 \). Since \( [\bar{Y}[\gamma_j]]_\mathcal{J} \leq Z_0^{\xi_{\gamma_j}} \) (this requirement is embedded in \( \theta_m^{\varphi, \psi} \)), we have
Therefore we can choose \( b_j \) such that \( \zeta(\bar{a}, b_j)^{M_j} > \gamma_j(\zeta) \) for all \( \zeta \in \mathcal{F}[\psi,k] \). This defines a representing sequence of \( b \in \prod_j M_j \) which satisfies \( Z^\zeta_{\gamma(\zeta)}[\bar{a}, b] \geq Y[\gamma] \). Since \( \theta_{\psi,k} \) is increasing, the inductive hypothesis implies

\[
\psi(\bar{a}, b) \geq \sup_y \psi(\bar{a}, y)^{\Pi_y / \mathscr{F}} > (m - 1)/k,
\]

as required.

The proof in the case when \( \varphi(\bar{x}) = \inf_y \psi(\bar{x}, y) \) for some \( \psi \) that satisfies the inductive assumption is analogous to the proof in the case when \( \varphi(\bar{x}) = \sup_y \psi(\bar{x}, y) \). This completes the proof. \( \square \)

**Proof (Proof of Theorem 16.3.1).** By Theorem 16.3.3, for every sentence \( \varphi(\bar{x}) \) and every \( k \geq 2 \) there is a finite set \( \mathcal{F}[\varphi,k] \) of formulas such that the value \( \varphi(\bar{a})|_{\Pi_y M_y / \mathscr{F}} \) is determined up to \( 2/k \) by the sets

\[
Z^\zeta_{\psi}[\bar{a}, M] := \{ \{ j \in \mathcal{J} : \zeta(\bar{a})^{M_j} > t \} \}_{\mathscr{F}},
\]

for \( \zeta \in \mathcal{F}[\varphi,k] \) and \( t \) in some finite set. Similarly, the sets

\[
Z^\zeta_{\psi}[\bar{b}, N] := \{ \{ j \in \mathcal{J} : \zeta(\bar{b})^{N_j} > t \} \}_{\mathscr{F}},
\]

determine the value of \( \varphi(\bar{b})|_{\Pi_y N_y / \mathscr{F}} \) up to \( 2/k \). Let \( \mathcal{F} := \bigcup_k \mathcal{F}[\varphi,k] \). Suppose that \( \Pi_j M_j / \mathscr{F} \) and \( \Pi_j N_j / \mathscr{F} \) satisfy \( \lim_{j \to \mathscr{F}} |\theta(\bar{a})^{M_j} - \theta(\bar{b})^{N_j}| = 0 \) for all \( \theta \in \mathcal{F} \).

Hence \( Z^\zeta_{\psi}[\bar{a}, M] = Z^\zeta_{\psi}[\bar{b}, N] \) for all \( \zeta \in \bigcup_k \mathcal{F}[\varphi,k] \) and all \( t \). Since \( k \) was arbitrary, \( \varphi(\bar{a})|_{\Pi_y M_y / \mathscr{F}} = \varphi(\bar{a})|_{\Pi_y N_y / \mathscr{F}} \) follows as required. \( \square \)

### 16.4 Saturation of Ultraproducts

In this section we prove that the ultraproducts associated with countably incomplete ultrafilters are countably saturated and briefly consider the trace-kernel ideal and the tracial ultrapower of a C*-algebra.

An ultrafilter \( \mathcal{U} \) on a set \( X \) is **countably incomplete** if there exists a partition of \( X \) into countably many sets \( X_j \), for \( j \in \mathbb{N} \), neither of which belongs to \( \mathcal{U} \). The most common examples of countably incomplete ultrafilters are the nonprincipal ultrafilters on \( \mathbb{N} \).

**Theorem 16.4.1.** If \( \mathcal{U} \) is a countably incomplete ultrafilter on \( \mathcal{J} \) and \( M_j \), for \( j \in \mathcal{J} \), are metric structures of the same language, then the ultraproduct \( \prod_{\mathcal{U}} M_j \) is countably saturated.

Proof. Let \( t(\bar{x}) \) be a countable type over \( M := \prod_{\mathcal{U}} M_j \), and enumerate its conditions as \( \varphi_n(\bar{x}) = r_n \) for \( n \in \mathbb{N} \). Since \( t(\bar{x}) \) is finitely satisfiable, for every \( n \geq 1 \) there exists \( \bar{b}(n) \in M \) such that \( \max_{j < n} |\varphi_j^M(\bar{b}(n)) - r_j| \leq \frac{1}{n} \). Fix a representing sequence \( (\bar{b}(n)_j) \) for \( \bar{b}(n) \). By Łoś’s Theorem for every \( n \) there exists \( Y_n \in \mathcal{U} \) such that for all \( j < n \) and all \( i \in Y_j \) we have \( \max_{j < n} |\varphi_j^M(\bar{b}(n)_j)^M - r_j| \leq \frac{1}{n} \). The key set-theoretic component of the proof follows. The sets \( Z_j := \bigcap_{k < j} \bigcup_{l \leq j} X_k \) form a decreasing sequence with an empty intersection and each \( Z_j \) belongs to \( \mathcal{U} \). For \( i \in Z_{n+1} \setminus Z_n \) let \( \bar{c}_i := \bar{b}(n)_i \). The choice of \( \bar{b}(n)_j \in M_{i_j} \) for \( i_j \notin Z_0 \) is of no consequence. Let \( \bar{c} \) be the element of \( M \) with the representing sequence \( (\bar{c}_i) \). Łoś’s Theorem implies \( \varphi_j^M(\bar{c}) = r_j \) for all \( j \), and therefore \( \bar{c} \) realizes \( t \) in \( M \).

Since \( t \) was an arbitrary satisfiable countable type, this concludes the proof. \( \Box \)

Higher levels of saturation can be obtained by using the so-called regular ultrafilters (Definition 16.8.18).

Corollary 16.4.2. If \( \mathcal{U} \) is a nonprincipal ultrafilter on \( \mathbb{N} \) and \( M_j \), for \( j \in \mathbb{N} \), is a sequence of \( L \)-structures, then \( \prod_{\mathcal{U}} M_j \) is countably saturated. \( \Box \)

Corollary 16.4.3. Every countable consistent type over a \( C^* \)-algebra \( A \) is realized in an elementary extension of \( A \). \( \Box \)

The assumption that the type is countable is not necessary in Corollary 16.4.3 (Exercise 16.8.31). Given an expansion (Definition D.2.10) of a \( C^* \)-algebra \( A \) by a predicate or a function (e.g., a state, an automorphism, the distance function to a \( C^* \)-subalgebra, . . . ) its ultrapower is a countably saturated expansion of \( A^\mathcal{U} \). The following may be the most important example.

Proposition 16.4.4. Suppose \( A \) is a \( C^* \)-algebra and \( \mu \) is a seminorm on \( A \) dominated by the operator norm. For an ultrafilter \( \mathcal{U} \), \( \mu^\mathcal{U}(a) := \lim_{j \to \mathcal{U}} \mu(a_j) \) defines a seminorm on \( A^\mathcal{U} \). The expansion \((A, \mu) \) is an elementary submodel of \((A^\mathcal{U}, \mu^\mathcal{U})\), and the latter structure is countably saturated.

Proof. Since \( \mu \) is dominated by \( \| \cdot \| \), it is uniformly continuous and expanding the language by adding a function symbol for \( \mu \) is justified. By Łoś Theorem \((A, \mu) \) is an elementary submodel of \((A^\mathcal{U}, \mu^\mathcal{U})\), and countable saturation follows by Corollary 16.4.2. Finally, being a seminorm is clearly axiomatizable. \( \Box \)

Example 16.4.5. Suppose \( B \) is a unital \( C^* \)-algebra with a nonempty space \( T(B) \) of tracial states. Consider the uniform 2-seminorm \( \|a\|_{2,\mathcal{U}} := \sup_{\tau \in T(B)} \tau(a^*a)^{1/2} \) on \( B \). By Proposition 16.4.4 it extends to a seminorm on \( B^\mathcal{U} \), still denoted \( \| \cdot \|_{2,\mathcal{U}} \). If \( \mathcal{U} \) is a nonprincipal ultrafilter on \( \mathbb{N} \), then the countable saturation of \((B^\mathcal{U}, \| \cdot \|_{2,\mathcal{U}})\) implies that

\[ J_\mu := \{ a \in B^\mathcal{U} : \|a\|_{2,\mathcal{U}} = 0 \} \]

is a nontrivial, even nonseparable, ideal in \( B^\mathcal{U} \). This is the trace-kernel ideal. One considers the metric structure \((B, \| \cdot \|_{2,\mathcal{U}})\) which has the same algebraic operations.

\[ ^7 \] A joke.
as $B$, but its metric is defined by $\| \cdot \|_{2,\mu}$ and the operator norm is not a part of the language. A routine verification (left to the reader) shows that the so-called \textit{tracial ultrapower} $(B, \| \cdot \|_{2,\mu})^\mathcal{U}$ is isomorphic to $B^\mathcal{U}/J_\mu$.

The analysis of the interplay between the tracial and norm ultrapowers of a tracial $C^*$-algebra $B$ plays an important role in the classification programme of $C^*$-algebras (see e.g., [179], [159]). Also see Exercise 16.8.23.

An ideal $J$ in a $C^*$-algebra with the property that for every countable $X \subseteq J$ there exists $c \in J_{+,1}$ such that $cx = x$ for all $x$ is called a $\sigma$-\textit{ideal}. The following analog of CRISP (Definition 15.3.1) implies that the trace-kernel ideal is a $\sigma$-ideal.\footnote{Unlike the definition of CRISP, we do not require $a_n$ to be positive.}

\textbf{Proposition 16.4.6.} Suppose $B$ is a tracial $C^*$-algebra, $\mathcal{U}$ is a countably incomplete ultrafilter, $J_B$ is the trace-kernel ideal (Example 16.4.5), $a_n \in J_\mathcal{U}$, and $b_n \in B^\mathcal{U}$ for all $n$. Then there is $c \in (J_B)_{+,1}$ such that $ca_n = a_n$ and $[c, b_n] = 0$ for all $n$.

\textit{Proof.} Consider the type

$$t(x) := \{ \| xa_n - a_n \| = 0, \| [x, b_n] \| = 0, \| x \|_{2,\mu} = 0, \| x \|_{2,\mu} = 1, x = x^*, x \geq 0 \}. $$

Take a separable elementary submodel $(C, \| \cdot \|_{2,\mu})$ of $(B^\mathcal{U}, \| \cdot \|_{2,\mu})$ containing all $b_n$. Then $J_B \cap C$ is $\sigma$-unital, and by Proposition 1.9.3 it has a $\{b_n : n \in \mathbb{N}\}$-quasi-central approximate unit. Thus for every $n \geq 1$ there is $c \in C_{+,1}$ such that $\| ca_n - a_n \| < 1/n$, $\| [c, b_n] \| < 1/n$, and $\| c \|_{2,\mu} = 0$. Therefore $t(x)$ is consistent. Since $\mathcal{U}$ is countably incomplete, $t(x)$ is realized by some $c \in (B^\mathcal{U})_{+,1}$, and this $c$ is as required. \hfill $\square$

See also Exercise 16.8.23.

\section{16.5 Saturation of Reduced Products}

In this section we prove that the asymptotic sequence algebras are countably saturated and discuss saturation of relative commutants.

All results in this section are stated and proved for general metric structures in some language $\mathcal{L}$. The following is the analog of Theorem 16.4.1.

\textbf{Theorem 16.5.1.} If $M_j$, for $j \in \mathbb{N}$, are $\mathcal{L}$-structures then the reduced product $\prod_j M_j/\text{Fin}$ is countably saturated.

\textit{Proof.} Fix a countable consistent type $t(\bar{x})$ over $M$, and enumerate it as a sequence of conditions $\varphi_i(\bar{x}) = s_i$, for $i \in \mathbb{N}$. By composing each $\varphi_i$ with a linear function we may assume that its range is included in $[0,1]$. Each $\varphi_i$ may have parameters in $M$, but we can expand the language by constants for these parameters. Applying Theorem 16.3.1 to each $\varphi_i(\bar{x})$ and $k \geq i$, we obtain $m(i) \in \mathbb{N}$, formulas $\xi^i_k(\bar{x})$, for $j < m(i)$, and Boolean formulas $\theta^i_{jk}$, for $0 \leq l \leq k$, such that for every $\bar{a}$ in $M$ of the
appropriate sort, the value of $\Phi^M_i(\bar{a})$ is $1/k$-determined by $\theta^{i,h}_{j,k}[\bar{a}]$, for $0 \leq l \leq k$. The value of this Boolean formula is determined by the isomorphism type of the finite Boolean algebra generated by the sets $Z^i_{j,l/k}(\bar{a}) := \{(n : \zeta^i_{j,l}(\bar{a})_n \geq 1/k)\}_{\text{Fin}}$.

This isomorphism type is uniquely determined by (let $\bar{m}(k) = \max_{i \leq k} m(i)$ and declare hitherto undefined sets to be empty):

$$Y^*_k(\bar{a}) := \{S \subseteq k^2 \times \bar{m}(k) : \cap_{l \leq i \leq k} (Z^i_{j,l/k}(\bar{a}) \setminus Z^i_{j,l(t+1)/k}(\bar{a})) \text{ is finite}\}.$$ 

Since the type $t(\bar{s})$ is consistent, for every $k$ there exists $\bar{b}(k)$ in $M$ such that $\max_{i \leq k} |\Phi^i(\bar{b}(k))| - s_i < 1/k$. Let $\mathcal{U}$ be a nonprincipal ultrafilter on $\mathbb{N}$. For a fixed $k$ there are only finitely many possibilities and there is $Y^*_k \subseteq \mathcal{P}(k^2 \times \bar{m}(k))$ such that $\{n \in \mathbb{N} : Y^*_k(\bar{b}(n)) = Y^*_k\} \in \mathcal{U}$. For every $k$ fix $n(k)$ so that $Y^*_k(\bar{b}(n(k))) = Y^*_k$. Then

1. $Y^*_k = \{S \cap (k^2 \times \bar{m}(k)) : S \in Y^*_{k'}\}$ for all $k < k'$,

because each one of these sets is equal to $Y^*_k(b(n(k')))$. For $\bar{a}$ in $M$, $k \in \mathbb{N}$, and $S \subseteq k^2 \times \bar{m}(k)$, let

$$\Phi^i_{S,k}(\bar{a}) := \cap_{l \leq i \leq k} (Z^i_{j,l/k}(\bar{a}) \setminus Z^i_{j,l(t+1)/k}(\bar{a})).$$

If $Y^*_k(\bar{a}) = Y^*_k$, then $\Phi^i_{S,k}(\bar{a})$ is finite if and only if $S \subseteq Y^*_k$, for every $S \subseteq k^2 \times \bar{m}(k)$. Therefore, if $Y^*_k(\bar{a}) = Y^*_k$ for all $k$ then $\bar{a}$ satisfies $t(\bar{s})$.

We will choose disjoint finite $Y_k \subseteq \mathbb{N}$, for $2 \leq k$, such that for all $k$ the following conditions hold:

2. $j \in Y^*_j$,

3. $|\Phi^i_{S,k}(\bar{b}(n(k))) \cap Y_k| \geq n$ for all $S \subseteq k^2 \times \bar{m}(k)$ such that $S \notin Y^*_k$, and

4. $\Phi^i_{S,k}(\bar{b}(n(k+1))) \subseteq \cup_{j \leq k} Y_j$ for all $S \subseteq k^2 \times \bar{m}(k)$ such that $S \in Y^*_k$.

We proceed to describe the recursive construction of the sequence $(Y_k)$. For $i \leq 2$ and $S \notin Y^*_S$ the set $\Phi^i_{S,2}(\bar{b}(n(2)))$ is infinite and for a large enough $m(S)$ we have $|\Phi^i_{S,2}(\bar{b}(n(2))) \cap m(S)| \geq 2$. With $m = \max\{m(S) : S \notin Y^*_S\}$ the set

$$Y_2 := \{2\} \cup \cup\{\Phi^i_{S,3}(\bar{b}(n(3)) : S \subseteq 3^2 \times m(3), S \in Y^*_3\}$$

satisfies (2), (3), and (4).

Suppose that the sets $Y_2, \ldots, Y_k$ have been chosen to satisfy (2), (3), and (4). The set $\Phi^i_{S,k+1}(\bar{b}(n(k+1)))$ is infinite for every $S \notin Y^*_k$, and therefore a large enough $m$ satisfies $|\Phi^i_{S,k+1}(\bar{b}(n(k+1))) \setminus \cup_{j \leq k} Y_j| \geq k + 1$ for all $S \notin Y^*_k$. With

$$Y_{k+1} := (m \setminus \cup_{j \leq k} Y_j)$$

$$\cup\{\Phi^i_{S,k+2}(\bar{b}(n(k+2)) : S \subseteq (k+2)^2 \times \bar{m}(k+2), S \in Y^*_k\}$$

the sets $Y_2, \ldots, Y_{k+1}$ satisfy (2), (3), and (4) with $k + 1$ replacing $k$.

This describes the recursive construction. The sets $Y_k$, for $k \in \mathbb{N}$, form a partition of $\mathbb{N}$ into finite sets. Define $\bar{d} \in M$ by its representing sequence $\bar{a}_j := \bar{b}(n(k))$, if
C is one of the following forms: A, U, \ell \geq 16.6 The Back-and-Forth Method II. Saturation

If M, Corollary 16.5.2. together imply \Phi is infinite, hence S and S. Suppose that A is a separable Corollary 16.5.4. out a few corollaries of this fact.

In this section we continue the study of \sigma-complete back-and-forth systems started in §8.2, and apply the Continuum Hypothesis to massive C*-algebras such as ultra-products or asymptotic sequence algebras.

Proposition 16.6.1. Suppose C and D are metric structures of density character \aleph_1. The following are equivalent.

1. There exists a \sigma-complete back-and-forth system between C and D.
2. The structures C and D are isomorphic.

\( j \in Y_k \). To prove that \( \bar{a} \) realizes \( t(\bar{x}) \), it suffices to prove \( \Upsilon_k(\bar{a}) = \Upsilon_k^* \) for all \( k \). Fix \( k \) and \( S \subseteq k^2 \times \bar{m}(k) \).

If \( S \notin \Upsilon_k^* \), then (3) implies that \( |\Phi_{\Sigma,k}(\bar{a})| \geq j \) for all \( j \geq k \). Therefore \( \Phi_{\Sigma,k}(\bar{a}) \) is infinite, hence \( S \) is an infinite set in \( \Upsilon_k(\bar{a}) \).

Now suppose \( S \in \Upsilon_k^* \). Then \( S \in \Upsilon_k^* \) for all \( j \). Fix any \( l \geq k \). Then (4) and (1) together imply \( \Phi_{\Sigma,l}(\bar{a}) \cap \Upsilon_l = \emptyset \) for all \( j \geq l + 1 \). Hence \( \Phi_{\Sigma,l}(\bar{a}) \) is finite, and since \( l \geq k \) was arbitrary, \( S \notin \Upsilon_k(\bar{a}) \).

Corollary 16.5.2. If \( M_j \), for \( j \in \mathbb{N} \), are C*-algebras then the asymptotic sequence algebra \( \prod_j M_j / \bigoplus_j M_j \) is countably saturated.

For quantifier-free types and saturation see Definition 15.2.1.

Proposition 16.5.3. If C is a countably saturated C*-algebra and A is a separable C*-subalgebra of C, then \( A' \cap C \) is countably quantifier-free saturated.

Proof. Let \( t(\bar{x}) \) in \( \bar{x} = (x_0, \ldots, x_{k-1}) \) be a quantifier-free type over \( A' \cap C \). Fix a dense subset \( \{a_n : n \in \mathbb{N} \} \) of \( A \) and let \( t'(\bar{x}) := t(\bar{x}) \cup \{|[a_n, x_j]| = 0 : n \in \mathbb{N}, j < k \} \). Then \( t'(\bar{x}) \) is a consistent type over \( C \). Its realization \( \bar{a} \) belongs to \( A' \cap C \) and since all formulas in \( t' \) are quantifier-free, \( \bar{a} \) realizes \( t(\bar{x}) \) in \( A' \cap C \).

Every countably saturated C*-algebra is countably degree-1 saturated. We single out a few corollaries of this fact.

Corollary 16.5.4. Suppose that A is a separable C*-algebra and a C*-algebra C is of one of the following forms: \( A^U \) for a nonprincipal ultrafilter \( U \) on \( \mathbb{N} \), \( \ell_\infty(A)/c_0(A) \), \( B' \cap A^U \), or \( B' \cap \ell_\infty(A)/c_0(A) \) for a separable C*-subalgebra B of \( A^U \) or of \( \ell_\infty(A)/c_0(A) \), respectively.

Then C is an SAW*-algebra, every uniformly bounded representation \( \pi \) of a countable amenable group \( \Gamma \) into C is unitarizable, admits the ‘discontinuous functional calculus’ of §15.4.2, and it is essentially non-factorizable.

Proof. Combine Theorem 16.4.1, Corollary 16.5.2, and Proposition 16.5.3 with Lemma 15.3.3, Proposition 15.4.1, Theorem 15.4.3, and Theorem 15.4.5.

16.6 The Back-and-Forth Method II. Saturation

In this section we continue the study of \sigma-complete back-and-forth systems started in §8.2, and apply the Continuum Hypothesis to massive C*-algebras such as ultra-products or asymptotic sequence algebras.

Corollary 16.5.2. If M_j, for j \in \mathbb{N}, are C*-algebras then the asymptotic sequence algebra \( \prod_j M_j / \bigoplus_j M_j \) is countably saturated.
Proof. If $\Phi : C \to D$ is an isomorphism and $E$ is a club in $\text{Sep}(C)$, then

$$F := \{(A, \Phi[A], \Phi \upharpoonright A) : A \in C\}$$

is a $\sigma$-complete back-and-forth system between $C$ and $D$.

Conversely, suppose $F$ is a $\sigma$-complete back-and-forth system between $C$ and $D$. By using the fact that $C$ and $D$ both have density character $\aleph_1$, we can choose an increasing sequence $p(\gamma)$, for $\gamma < \aleph_1$, in $F$ such that $A^{p(\gamma)}$, for $\gamma < \aleph_1$ is a club in $\text{Sep}(C)$ and $B^{p(\gamma)}$, for $\gamma < \aleph_1$ is a club in $\text{Sep}(D)$. Then $\bigcup \Phi^{p(\gamma)}$ is an isomorphism between $C$ and $D$. \qed

The following lemma will be used to construct many automorphisms of a given countably saturated structure of density character $\aleph_1$. For separable types see Lemma 16.1.7 and the paragraph preceding it. Recall that we write $B \preceq A$ if $B$ is an elementary submodel of $A$.

Lemma 16.6.2. Suppose $A$ is a nonseparable countably saturated metric structure, $B \preceq A$ is separable, and $\Psi \in \text{Aut}(B)$. For every club $C \subseteq \text{Sep}(A)$ there is $E \in C$ such that $B \subseteq E$ and there are distinct extensions $\Phi^0$ and $\Phi^1$ of $\Psi$ to $\text{Aut}(E)$.

Proof. Fix $a \subseteq A \setminus B$. The type of $a$ over $B$, $\text{type}(a/B)$, (Definition 16.1.4) is separable. With $\varepsilon := \text{dist}(a, B)$, define a 2-type $t(x, y)$ such that a pair $c, d$ realizes $t(x, y)$ if and only if $\text{type}(c/B) = \text{type}(d/B) = \text{type}(a/B)$ and $d(c, d) \geq \varepsilon$. In symbols:

$$t(x, y) := \{P(x), P(y) : P(x) \text{ is a condition in } \text{type}(a/B)\} \cup \{d(x, y) \geq \varepsilon\}.$$

This type is satisfiable in $A$: Since every finite fragment of $\text{type}(a/B)$ is approximately realized in $B$ by elementarity, for every finite fragment $t_0(x, y)$ of $t(x, y)$ and every $\delta > 0$ there is $b \in B$ which $\delta$-approximately realizes the fragment of $\text{type}(a/B)$ that belongs to $t_0(x, y)$. Since $d(a, b) \geq \text{dist}(a, B) = \varepsilon$, the pair $(a, b)$ $\delta$-approximately realizes $t_0(x, y)$.

By the countable saturation of $A$, some pair $c, d$ in $A$ realizes $t(x, y)$. In order to construct $C$ as required, we use Proposition 7.2.7 to find a dense subset $D$ of $A$ such that the set $\{p \in [D]^\varepsilon_0 : D \cap p = p\}$ includes a club. Then

$$F := \{p \in [D]^\varepsilon_0 : p \in C \land D \cap p = p\}$$

includes a club. By Theorem 6.4.1, fix $f : [D]^\varepsilon_0 \to D$ such that every countable subset of $D$ closed under $f$ belongs to

$$F_1 := \{p \in F : \exists \sigma \in C\}.$$

Using the Löwenheim–Skolem Theorem, find a separable $E_0 \preceq A$ which includes $B \cup \{c, d\}$. We have $c \neq d$ because $d(c, d) \geq \varepsilon$. By the separable universality of $A$ (Theorem 16.1.9) there exist elementary embeddings $\Phi_j^0 : E_0 \to A$, for $j < 2$, which both extend $\Psi$ and satisfy $\Phi_0^0(c) = e$ and $\Phi_1^0(c) = d$.

We proceed to build separable elementary submodels $E_n$ of $A$, elementary embeddings $\Phi_n : E_n \to A$, and $p_n \in F_1$ such that the following conditions hold for all $n$:
1. $E_n \preceq A$.
2. $C^*(E_n, \Phi^0_n[E_n], \Phi^1_n[E_n], p_n) \subseteq E_{n+1}$, and $E_{n+1} \preceq A$.
3. $\Phi^0_{n+1}$ extends $(\Phi^0_n)^{-1}$, $\Phi^1_{n+1}$ extends $(\Phi^1_n)^{-1}$, and
4. $\overline{p_n} \supseteq E_n$.

Suppose that $E_j$, $\Phi^0_j$, and $\Phi^1_j$ have been chosen for $j \leq n$. By the Downward Löwenheim–Skolem Theorem, some separable elementary submodel $E_{n+1}$ of $A$ includes $E_n \cup \Phi^0_n[E_n] \cup \Phi^1_n[E_n] \cup p_n$. Since $F^j := \Phi^j_n[E_n]$ is a separable elementary submodel of $A$ for $j < 2$, by the separable universality of $A$ the elementary embedding $(\Phi^j_n)^{-1} : F^j \to A$ can be extended to an elementary embedding $\Phi^j_{n+1} : E_{n+1} \to A$. Finally choose $p_{n+1}$ so that $E_{n+1} \subseteq \overline{p_{n+1}}$.

Once $E_n$, $p_n$, $\Phi^0_n$, and $\Phi^1_n$ have been chosen for all $n$, $E := \bigcup_n E_n$ is a separable elementary substructure of $A$. Since $\Phi^0_{n+2}$ extends $(\Phi^0_{n+1})^{-1}$, which in turn extends $\Phi^0_n$, $\Phi^0 := \bigcup_n \Phi^0_n$ is an automorphism of $E$. Similarly, $\Phi^1 := \bigcup_n \Phi^1_n$ is an automorphism of $E$. By construction, $\Phi^0(c) = e$ and $\Phi^1(c) = d$ and therefore $\Phi^0 \neq \Phi^1$.

In addition, the set $p := \bigcup_n p_n$ is dense in $E$ and, being equal to the union of an increasing sequence in $F_1$, it belongs to $F_1$. The latter fact has two consequences. First, $\overline{p} \cap D = p$. Second, $\overline{p} \in C$. Therefore $E \in C$, as required. \hfill \Box

The proof of the following theorem uses $\{0, 1\}^{<\mathfrak{r}_1}$, the complete binary tree of height $\mathfrak{r}_1\,$(§8.4), as a bookkeeping device.

**Theorem 16.6.3.** Every countably saturated metric structure $A$ of density character $\mathfrak{r}_1$ has $2^{\mathfrak{r}_1}$ automorphisms. Given any separable $X \subseteq A$, each one of these automorphisms can be chosen to be equal to the identity on $X$.

**Proof.** We define a $\sigma$-complete back-and-forth system $F$ between $A$ and $A$ with the property that each one of its conditions has two incompatible extensions in $F$. (Two elements of $F$ are said to be *incompatible* if they do not have a common extension in $F$.)

Let

$$F := \{ (B, B, \Phi) : B \text{ is separable}, X \subseteq B \preceq A, \Phi \in \text{Aut}(B), \text{ and } \Phi \upharpoonright X = \text{id}_X \}.$$ 

The assumed separable universality of $A$ implies that every condition of $F$ has a proper extension in $F$.

With everything in place, one can now recursively embed $\{0, 1\}^{<\mathfrak{r}_1}$ into $F$ so that distinct branches of $\{0, 1\}^{<\mathfrak{r}_1}$ correspond to different automorphisms as follows. Fix an enumeration $A = \{ a_\xi : \xi < \mathfrak{r}_1 \}$. We will define $p_s := (B_s, B_s, \Phi_s) \in F$ for $s \in \{0, 1\}^{<\mathfrak{r}_1}$ so that the following conditions hold for all $s$ and $t$ in $\{0, 1\}^{<\mathfrak{r}_1}$.

1. $s \subseteq t$ implies $p_s \preceq p_t$.
2. $p_{00}$ and $p_{10}$ are incompatible.
3. $p_{01}$ and $p_{11}$ are incompatible.

The conditions $p_s$, for $s \in \{0, 1\}^{<\mathfrak{r}_1}$, are chosen by transfinite recursion on the well-founded set $\{0, 1\}^{<\mathfrak{r}_1}$ (Proposition A.2.2) as follows.

\footnote{If the definition of $F$ seems like a typo, it may be a good moment to work out Exercise 8.7.12.}
Suppose that \( t \in \{0, 1\}^{<\mathbb{R}^+} \) is such that \( p_s \) has been constructed for all \( s \subseteq t \). We first consider the case when \( \alpha := \text{dom}(t) \) is a limit ordinal. Since \( \alpha \) is countable, we let \( \beta(n) \), for \( n \in \mathbb{N} \), be an increasing sequence of ordinals such that \( \sup_n \beta(n) = \alpha \). Since \( F \) is a \( \sigma \)-complete back-and-forth system, \( p_s = \sup_n p_{s|\beta(n)} \) is well-defined. If \( t \subseteq s \) then \( t \subseteq s|_{\beta(n)} \) for a large enough \( n \), and therefore \( p_s \) is as required.

Now suppose that \( \alpha := \text{dom}(t) \) is a successor ordinal. Fix \( s \) such that \( t \) is an immediate successor of \( s \). It does not matter whether \( t = s_0 \) or \( t = s_1 \), since we shall find \( p_{s_0} \) and \( p_{s_1} \) simultaneously. Since \( F \) is a \( \sigma \)-complete back-and-forth system, we can find \( r \geq p_s \) in \( F \) such that \( a_\alpha \in B_r \). By Lemma 16.6.2, \( r \) has incompatible extensions in \( F \); these conditions are \( p_{s_0} \) and \( p_{s_1} \) as required. \( \square \)

**Theorem 16.6.4.** Suppose \( C \) and \( D \) are countably saturated metric structures. The following are equivalent.

1. The metric structures \( C \) and \( D \) are elementarily equivalent.
2. There exists a \( \sigma \)-complete back-and-forth system between \( C \) and \( D \).

**Proof.** Suppose that there exists a \( \sigma \)-complete back-and-forth system between \( C \) and \( D \). Since club many separable substructures of any nonseparable structure are elementary submodels, Lemma 8.2.9 (1) implies that \( C \) and \( D \) are elementarily equivalent. Now suppose that \( C \) and \( D \) are elementarily equivalent and let

\[
F := \{(A, B, \Phi) : (A, B) \in \text{Sep}(C) \times \text{Sep}(D) : A \preceq C, B \preceq D, \\
\Phi : A \rightarrow B \text{ is an isomorphism}\},
\]

Then \( F \) is trivially \( \sigma \)-complete, but we need to prove that for every \( (A, B, \Phi) \in F \), every \( c \in C \), and every \( d \in D \) there exists \( (E, F, \Psi) \in F \) that extends \( (A, B, \Phi) \) and is such that \( c \in E \) and \( d \in F \).

Choose a separable \( A_1 \preceq C \) such that \( A \subseteq A_1 \) and \( c \in A_1 \). Theorem 16.1.9 implies that some elementary embedding \( \Phi_1 : A_1 \rightarrow D \) extends \( \Phi \). Let \( B_1 := \Phi_1[A_1] \). Now choose a separable \( B_2 \preceq D \) such that \( B_1 \cup \{d\} \subseteq B_2 \). Theorem 16.1.9 implies that an elementary embedding \( \Phi_2 : B_2 \rightarrow C \) extends \( \Phi_1^{-1} \).

The triple \( (E, F, \Psi) := (\Phi_2[B_2], B_2, \Phi_2^{-1}) \) belongs to \( F \) and it is as required. \( \square \)

It should be noted that Theorem 16.6.4 has been proved in ZFC. Although Corollary 16.6.5 has a more appealing conclusion, Theorem 16.6.4 suffices for all practical purposes.\(^\text{10}\)

**Corollary 16.6.5.** The Continuum Hypothesis implies that countably saturated structures \( C \) and \( D \) of density character \( c \) are elementarily equivalent if and only if they are isomorphic.

Moreover, if \( A \preceq C \), \( B \preceq D \), \( A \) and \( B \) are separable, and \( (A, B, \Phi) \) is a partial isomorphism, then an isomorphism between \( C \) and \( D \) can be chosen to extend \( \Phi \).

**Proof.** This is a consequence of Theorem 16.6.4 and Proposition 16.6.1. \( \square \)

\(^{10}\) At least as long as the ‘practical purposes’ concern only separable structures.
16.7 Isomorphisms and Automorphisms

The assumption that $C$ and $D$ are elementary substructures of $A$ cannot be dropped from Corollary 16.6.5 even when $C = D$; see Exercise 16.8.12.

We conclude this section with an off-tangent discussion directed at the readers left wondering whether a countably saturated structure of density character $\aleph_1$ can exist if the Continuum Hypothesis fails. The case when $\kappa = \aleph_1$ of the following example gives a positive answer.

**Example 16.6.6.** Suppose $\kappa$ is an uncountable cardinal. Then each of the following metric structures is (in ZFC) $\kappa$-saturated.

1. The Hilbert space $\ell_2(\kappa)$, considered in the language of Banach spaces. This can be proved in more than one way, and it is left as Exercise 16.8.8.
2. The measure algebra associated with the Haar measure $\mu$ on $\{0,1\}^\kappa$. The language is the language of Boolean algebras and the metric is defined by $d(X,Y) := \mu(X \Delta Y)$.

The existence of a countably saturated $C^*$-algebra of density character $\aleph_1$ is equivalent to the Continuum Hypothesis (Exercise 15.6.4). The dividing line between the theories that have countably saturated models of density character $\aleph_1$ provably in ZFC and those that do not (usually not expressed in this way) was isolated by Saharon Shelah and it has a prominent place in the modern model theory ([220]).

16.7 Isomorphisms and Automorphisms

Theorem a day, doctor away.

Neil Hindman

In this section we study ultraproducts and other reduced products of $C^*$-algebras. By using full countable saturation (§16.1) and $\sigma$-complete back-and-forth systems (§8.2, §16.6) we prove that the Continuum Hypothesis implies that many of these quotients are isomorphic.

As famously said by Rufus Willett (and others), to a man with a hammer everything looks like a nail. Now is the time to roll out the sledgehammer which was developed in the earlier sections of the present Chapter and put it to good use: The present short section has the highest theorem to page ratio in this entire book.

**Proposition 16.7.1.** Suppose the Continuum Hypothesis. If $A$ is a separable and infinite-dimensional $C^*$-algebra or a non-compact metric structure, and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then the following holds.

---

11 This is not to be confused with the even more famous A.H. Maslow’s quip “I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.” Ours is a good hammer.
1. Each one of $\prod_{N} A / \text{Fin}$ (or $\ell_{\infty} (A) / c_{0} (A)$) and $A^{\mathcal{U}}$ has $2^{c}$ automorphisms that fix $A$ pointwise.

2. If $A$ is a C*-algebra, then each one of $A' \cap \ell_{\infty} (A) / c_{0} (A)$ and $A' \cap A^{\mathcal{U}}$ has $2^{c}$ automorphisms.

Proof. Since both $\ell_{\infty} (A) / c_{0} (A)$ and $A^{\mathcal{U}}$ are countably saturated (Corollary 16.5.2), (1) is a consequence of Theorem 16.6.3 applied to $X = A$ and $\ell_{\infty} (A) / c_{0} (A)$. All of these automorphisms fix the relative commutant setwise, and (2) follows. \[\square\]

What, if anything, can be said about the structure of the automorphisms of an ultrapower $A^{\mathcal{U}}$ of a separable C*-algebra? A desirable answer to this question would consist of isolating ‘obvious’ automorphisms, followed by a proof that all automorphisms are of this kind. We are still far from having a good description of automorphisms of $A^{\mathcal{U}}$, even assuming additional set-theoretic axioms, but we can show that the most obvious guess is too simple-minded, at least assuming the Continuum Hypothesis. All of §17 is devoted to the analogous problem for the coronas, and the Calkin algebra in particular.

Assume that $A$ is a separable metric structure and $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$. An automorphism $\Phi$ of $A^{\mathcal{U}}$ is of product type if there exist $E$ and $F$ in Part $\mathbb{N}$ (§9.7) and an isomorphism $\Phi_{n}: A^{E_{n}} \to A^{F_{n}}$ for every $n$ such that $a \mapsto \langle \Phi_{n} (a \mid E_{n}) \rangle_{n}$ is a lifting of $\Phi$. Clearly, every map of this form lifts an automorphism of $A^{\mathcal{U}}$.

Corollary 16.7.2. Suppose $A$ is a separable, infinite-dimensional, C*-algebra and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. If the Continuum Hypothesis holds, then $A^{\mathcal{U}}$ has an automorphism $\Phi$ which cannot be lifted by an automorphism, or even an endomorphism, of $\ell_{\infty} (A)$.

Proof. Proposition 16.7.1 implies that $A^{\mathcal{U}}$ has $2^{\aleph_{1}}$ automorphisms. It will therefore suffice to show that there are only $\aleph_{1}$ automorphisms of $A^{\mathcal{U}}$ of product type. Since $\mathfrak{c} < 2^{\mathfrak{c}}$, this will complete the proof.

There are $\mathfrak{c}$ elements of Part $\mathbb{N}$. Since $A^{E_{n}}$ and $A^{F_{n}}$ are separable and every *-homomorphism $\Phi_{n}: A^{E_{n}} \to A^{F_{n}}$ is continuous, there are only $\mathfrak{c}$ of them. Therefore for a fixed pair $E, F$ there are $\mathfrak{c}^{\mathfrak{c}} = \mathfrak{c}$ product type automorphisms of $A^{\mathcal{U}}$, and the conclusion follows. \[\square\]

By using the Gelfand–Naimark duality and the Stone duality (§1.3) we obtain the following 1956 theorem of W. Rudin.

Corollary 16.7.3. The Continuum Hypothesis implies that the Čech–Stone remainder $\beta \mathbb{N} \setminus \mathbb{N}$ has $2^{\mathfrak{c}}$ autohomeomorphisms and, via the Stone duality, that the Boolean algebra $\mathcal{P} (\mathbb{N}) / \text{Fin}$ has $2^{\mathfrak{c}}$ automorphisms. \[\square\]

Theorem 16.7.4. The Continuum Hypothesis implies that every separable C*-algebra $A$ satisfies the following for all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$.

1. The ultrapowers $A^{\mathcal{U}}$ and $A^{\mathcal{V}}$ are isomorphic.

2. The relative commutants $A' \cap A^{\mathcal{U}}$ and $A' \cap A^{\mathcal{V}}$ are isomorphic.
3. If in addition $A$ is infinite-dimensional, then each one of $A^U$ and $A' \cap A^U$ has $2^c$ automorphisms.

Proof. This is an application of $\sigma$-complete back-and-forth systems (§8.2, §16.6) and model-theoretic saturation (§16.1). We prove (1) and (2) simultaneously.

If $A$ is finite-dimensional then $A^U \cong A^V \cong A$ and both statements are trivial. We may therefore assume $A$ is infinite-dimensional. Łoś’s Theorem, Theorem 16.2.8, implies $A \preceq A^U$. Example 8.1.3 (11) and the Continuum Hypothesis together imply that $A^U$ has cardinality $\aleph_1$.

Fix nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$. By Corollary 16.6.5, the identity on $A$ extends to an isomorphism $\Phi: A^U \to A^V$. This implies both $A^U \cong A^V$ and $A' \cap A^U \cong A' \cap A^V$, as required. Clause (3) is a consequence of Theorem 16.6.3. \[\square\]

The proof of Theorem 16.7.4 gives the following characterization of ultrapowers.

**Theorem 16.7.5.** Suppose that $A$ is a separable C*-algebra. If $B$ is a countably saturated elementary extension of $A$ of density character $\aleph_1$, then for every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ there exists an isomorphism $\Phi: A^U \to B$ that extends the identity map on $A$.

Proof. If there exists a countably saturated C*-algebra of density character $\aleph_1$, then the Continuum Hypothesis holds (Exercise 15.6.4). Therefore our assumption implies that $A^U$ has density character $\aleph_1$. By Łoś’s Theorem, $A^U$ and $B$ are elementarily equivalent. Since the Continuum Hypothesis holds, they are also saturated, and therefore isomorphic. \[\square\]

The following is a special case of the Keisler–Shelah Theorem; for more details see Notes to this Chapter.

**Theorem 16.7.6 (Keisler).** Assume the Continuum Hypothesis. Two separable metric structures $A$ and $B$ are elementarily equivalent if and only if $A^U \cong B^U$ for some (any) nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Proof. By Łoś’s Theorem, if $A^U \cong B^U$ then $A$ and $B$ are elementarily equivalent. The converse follows by Łoś’s Theorem and Corollary 16.6.5. \[\square\]

We conclude the present section with a converse to Theorem 16.7.4.

**Theorem 16.7.7.** For an infinite-dimensional separable unital C*-algebra $A$ the following are equivalent.

1. For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ the ultrapowers $A^U$ and $A^V$ are isomorphic via an isomorphism that is equal to the identity on $A$.

2. There are fewer than $2^c$ isomorphism classes of ultrapowers $A^U$, for a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

\[\text{If } A \text{ is infinite-dimensional, then such } B \text{ exists if and only if the Continuum Hypothesis holds (Exercise 16.8.8). Compare with Example 16.6.6.}\]
3. For all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ the relative commutants of $A' \cap A^\mathcal{U}$ and $A' \cap A^\mathcal{V}$ are isomorphic.
4. There are fewer than $2^\mathfrak{c}$ isomorphism classes of relative commutants $A' \cap A^\mathcal{U}$, for a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$.
5. The Continuum Hypothesis holds.\(^{13}\)

Proof. Clearly (1) implies (2) and (3). By Theorem 16.7.4, (5) implies (1) and (4). Proofs that (2) implies (5) and that (4) implies (5) are beyond the scope of this book. A few hints are given in Notes to this Chapter. \(\square\)

### 16.8 Exercises

**Exercise 16.8.1.** Suppose $\mathcal{U}$ is a principal ultrafilter on $\mathbb{J}$. Verify that $\prod_j B_j / \mathcal{U}$ is isomorphic to $B_j$ for some $j \in \mathbb{J}$.

**Exercise 16.8.2.** Suppose that a simple and infinite-dimensional C*-algebra $A$ has a tracial state. Prove that the ultrapower of $A$ associated with a nonprincipal ultrafilter on $\mathbb{N}$ is not simple.

For Exercises 16.8.3–16.8.6 we recommend using Łoś’s Theorem.

**Exercise 16.8.3.** Prove that simplicity of C*-algebras is not axiomatizable.

**Exercise 16.8.4.** Prove that being AF is not axiomatizable. Repeat for AM.

**Exercise 16.8.5.** Prove that there exists a type over $\mathcal{O}_2$ which has finitely many conditions, it is consistent, and it is not realized in $\mathcal{O}_2$.

**Hint:** Prove a more general statement: If $A$ is a finitely generated C*-algebra such that $A \otimes M_2(\mathbb{C}) \cong A$ (this is true for $\mathcal{O}_2$) then there exists a type over $A$ which has finitely many conditions, it is consistent, and it is not realized in $A$.

**Exercise 16.8.6.** Prove that a projection in an ultraproduct $\prod_\mathcal{U} A_j$ can be lifted to a projection in $\prod_j A_j$. Prove the analogous statements for unitaries and partial isometries.

The following is a C*-twin of Exercise 15.6.15.

**Exercise 16.8.7.** Prove that for any unital C*-algebra $A$ the quotient $\ell_\infty(A) / \bigoplus \mathcal{Z}_0 A$ is not countably saturated ($\mathcal{Z}_0$ denotes the ideal of asymptotic density zero subsets of $\mathbb{N}$, see Exercise 9.10.4).

**Exercise 16.8.8.**
1. Verify that the theory of Hilbert space is axiomatizable (§D.2.4), and that $\ell_2(\mathcal{K})$ is $\kappa$-saturated for every uncountable cardinal $\kappa$.
2. Suppose that there exists a saturated C*-algebra of density character $\aleph_1$. Prove that the Continuum Hypothesis holds.

\(^{13}\) Clearly (5) does not depend on $A$, but this is the correct statement.
Exercise 16.8.9. Prove that an infinite-dimensional and simple AF algebra cannot be countably saturated, regardless of its density character. Then prove a stronger statement, that every countably saturated AF algebra is subhomogeneous.

Exercise 16.8.10. Suppose that $A$ is a counterexample to the Naimark’s problem of density character $\aleph_1$. Prove that $A$ cannot be countably saturated.

The following exercise is analogous to Exercise 13.4.11.

Exercise 16.8.11. Prove that $C^*$-algebras $A$ and $B$ and a countably incomplete ultrafilter $\mathcal{U}$ satisfy $A^\mathcal{U} \otimes B^\mathcal{U} \cong (A \otimes B)^\mathcal{U}$ if and only if at least one of $A$ and $B$ is finite-dimensional. Now prove this without using Corollary 16.5.4.

Exercise 16.8.12. Find a separable $C^*$-subalgebra $A$ of $(M_2^\infty)^\mathcal{U}$ and an automorphism $\Phi$ of $A$ that cannot be extended to an automorphism of $(M_2^\infty)^\mathcal{U}$.

Hint: The tracial state on $(M_2^\infty)^\mathcal{U}$ is unique.

Exercise 16.8.13. Prove the following

1. Every saturated $C^*$-algebra of density character $\kappa$ has $2^\kappa$ automorphisms.
2. If metric structures $A$ and $B$ are elementarily equivalent, saturated, and have the same density character $\kappa$, then they are isomorphic. Moreover, there are $2^\kappa$ distinct isomorphisms between them.

Exercise 16.8.14. Assume the Continuum Hypothesis. Prove that there is a separable $C^*$-algebra $A$ such that $A^\mathcal{U} \cong (M_2^\infty)^\mathcal{U}$, but $A$ is not isomorphic to $M_2^\infty$.

Exercise 16.8.15. Assume the Continuum Hypothesis. Using the fact that there is no universal separable $C^*$-algebra ([143]), prove that there exists a family $A_\alpha$, for $\alpha < c$, of nonisomorphic separable $C^*$-algebras such that $A_\alpha^\mathcal{U} \cong A_\beta^\mathcal{V}$ for all nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$.

Exercise 16.8.16. 1. Prove the Upwards Löwenheim–Skolem Theorem: If a first-order, single-sorted, theory $T$ has an infinite model then it has a model of an arbitrarily large cardinality.
2. Prove the metric Upward Löwenheim–Skolem Theorem: If a first-order metric theory $T$ has a model in which the interpretation of one of the sorts of $T$ is not compact, then it has a model of an arbitrarily large density character.\(^{14}\)

Exercise 16.8.17. Prove Theorem 16.1.9: Given an infinite cardinal $\kappa$, a metric structure is $\kappa$-saturated if and only if it is $\kappa$-universal and $\kappa$-homogeneous.

Definition 16.8.18. Given an infinite cardinal $\kappa$, an ultrafilter $\mathcal{U}$ on a set $X$ is $\kappa$-regular if there exists a family $X_\alpha$, for $\alpha < \kappa$, of sets in $\mathcal{U}$ such that every $x \in X$ belongs to at most finitely many of the $X_\alpha$’s.

Exercise 16.8.19. Prove that an ultrafilter is $\aleph_0$-regular if and only if it is countably incomplete.

\(^{14}\) We considered only the single-sorted first-order logic.
Exercise 16.8.20. Prove that if \( \mathcal{U} \) is a \( \kappa \)-regular ultrafilter then the ultraproduct \( \prod_{\mathcal{U}} A_i \) is \( \kappa^+ \)-saturated for any family \( \langle A_i \rangle \) of \( \mathcal{L} \)-structures.

Conclude that for every \( \mathcal{L} \)-structure \( A \) and every consistent type \( t(\bar{x}) \) over \( A \) there exists an elementary extension of \( A \) that realizes \( t(\bar{x}) \).

Exercise 16.8.21. Suppose that \( \kappa \) is an infinite cardinal and \( T \) is a complete theory in a metric language \( \mathcal{L} \). Use Corollary 16.4.3 and bookkeeping to prove that \( T \) has a \( \kappa \)-saturated model. If in addition \( |\mathcal{L}| < \kappa \), \( \kappa^{<\kappa} = \kappa \), then \( T \) has a saturated model of density character at most \( \kappa \).

Exercise 16.8.22. Suppose \( \mathcal{I} \) and \( \mathcal{J} \) are ideals on sets \( \mathbb{I} \) and \( \mathbb{J} \), respectively, and \( (A_i) \) is an indexed family of metric structures. Let \( A_i := \prod_{\mathcal{I}} A_i / \bigoplus_{\mathcal{J}} A_i \) for \( i \in \mathbb{I} \). Prove that \( \prod_{i \in \mathbb{I}} A_i / \bigoplus_{i \in \mathcal{J}} A_i \) is isomorphic to a reduced product of \( (A_i) \) associated with an ideal on \( \mathbb{I} \times \mathbb{J} \) and describe this ideal.

Exercise 16.8.23. Suppose that \( B \) is a \( C^* \)-algebra, \( \mu \) is a seminorm on \( B \) such that \( \mu(bb^*) = \mu(b^*b) \) for all \( b \in B \), and \( \mathcal{U} \) is a countably incomplete ultrafilter. As in Proposition 16.4.4 and Example 16.4.5, let \( J_{\mathcal{U}} := \{ b \in B^\mathcal{U} : \mu^\mathcal{U}(b^*b) = 0 \} \). Also suppose that \( A \) is a separable \( C^* \)-algebra and that \( \Phi: A \to B^\mathcal{U} / J_{\mathcal{U}} \) is a \( * \)-homomorphism.

1. Prove that there exists a c.p.c. map \( \Psi: A \to B^\mathcal{U} \) that lifts \( \Phi \), in the sense that \( \Phi = \pi_{\mathcal{U}} \circ \Psi \), where \( \pi_{\mathcal{U}}: B^\mathcal{U} \to B^\mathcal{U} / J_{\mathcal{U}} \) is the quotient map.
2. Prove that \( \pi_{\mathcal{U}}[\Psi(A) \cap B^\mathcal{U}] = \Phi[A] \cap B^\mathcal{U} / J_{\mathcal{U}} \).

Exercise 16.8.24. Suppose that \( A \) is a non-type I \( C^* \)-algebra and \( \mathcal{U} \) is a countably incomplete ultrafilter. Prove that some \( C^* \)-subalgebra \( B \) of \( A^\mathcal{U} \) has a quotient isomorphic to \( \prod_{\mathcal{U}} M_n(\mathbb{C}) \).

Hint: Tinker with the proof of Glimm’s Theorem.

Exercise 16.8.25. Assume the Continuum Hypothesis and fix a separable \( C^* \)-algebra \( A \). Let \( \mathcal{U} \) be a nonprincipal ultrafilter on \( \mathbb{N} \). Prove that \( \ell_\infty(A^\mathcal{U}) / c_0(A^\mathcal{U}) \) and \( (\ell_\infty(A) / c_0(A))^\mathcal{U} \) are isomorphic.

Exercise 16.8.26. Suppose that \( A_n \), for \( n \in \mathbb{N} \), is a sequence of unital \( C^* \)-algebras. Prove that there exists an infinite \( X \subseteq \mathbb{N} \) such that the theories of \( A_n \), for \( n \in X \), (considered as characters of \( \text{Sent}_{\mathcal{L}} \), see Definition D.2.8) converge in the logic topology.

Exercise 16.8.27. Assume the Continuum Hypothesis. Use Exercise 16.8.26 to prove that there exists an infinite \( X \subseteq \mathbb{N} \) such that for any infinite \( Y \subseteq X \) the algebras \( \prod_{x \in X} M_1(\mathbb{C}) / \bigoplus_{y \in Y} M_1(\mathbb{C}) \) and \( \prod_{x \in X} M_1(\mathbb{C}) / \bigoplus_{y \in Y} M_1(\mathbb{C}) \) are isomorphic.

Exercise 16.8.28. Prove that there are \( \epsilon \) many different theories of \( C^* \)-algebras of the form \( \prod_{x \in X} M_1(\mathbb{C}) / \bigoplus_{y \in Y} M_1(\mathbb{C}) \) for \( X \subseteq \mathbb{N} \). Conclude that there are \( \epsilon \) nonisomorphic \( C^* \)-algebras of this form.

Exercise 16.8.29. Assume the Continuum Hypothesis. For a separable \( C^* \)-algebra \( A \) let \( A_1 := \ell_\infty(A) / c_0(A) \) and let \( A_{n+1} := \ell_\infty(A_n) / c_0(A_n) \) for \( n \geq 1 \). Prove that \( A_m \) is isomorphic to \( A_1 \) for all \( m \geq 1 \).
Exercise 16.8.30. Prove that the Calkin algebra is not countably saturated, and that it is therefore not isomorphic to any nontrivial ultrapower.

Exercise 16.8.31. Suppose $M$ is a metric structure and $t$ is a consistent type over $M$. Prove that $M$ has an elementary extension in which $t$ is realized.

An ultrafilter is countably complete if it is not countably incomplete.

Exercise 16.8.32. Use Ulam’s matrices to prove that the minimal cardinal $\kappa$ that carries a nonprincipal, countably incomplete, ultrafilter is (if it exists) a regular limit cardinal.

Exercise 16.8.33. Suppose that $\kappa$ is the least cardinal that carries a nonprincipal, countably incomplete, ultrafilter $\mathcal{U}$.

1. Prove that the ultrapower $V^\mathcal{U}_\lambda$ (see §A.5) associated with a nonprincipal, countably incomplete, ultrafilter is well-founded for all ordinals $\lambda$.
2. By combining Mostowski’s Collapsing Theorem and Łoś’s Theorem, prove that for every $\lambda > \kappa$ there exists a transitive set $M(\lambda)$ and an elementary embedding $j: V^\mathcal{U}_\lambda \rightarrow M(\lambda)$ such that $j(\kappa > \kappa$ and $j(\alpha) = \alpha$ for all $\alpha < \kappa$.
3. Use this to improve the conclusion of Exercise 16.8.32 as much as you can.

We conclude this section with the definition of a class of large quotient $C^*$-algebras that plays an important role in the $E$-theory of Connes and Higson ([26, §25]) and in the Phillips–Weaver construction of an outer automorphism of the Calkin algebra ([200], see also §17.1). Fix a $C^*$-algebra $A$. Let $C_b([0,1],A)$ denote the algebra of all $A$-valued, bounded continuous functions on $[0,1]$. The algebra $C_0([0,1],A)$ is an ideal in $C_b([0,1],A)$, and $A$ is identified with the subalgebra consisting of the equivalence classes of constant functions in

$$A_\infty := C_b([0,1],A)/C_0([0,1],A).$$

Exercise 16.8.34. Prove that the density character of $A_\infty$ is at least $c$ for every $C^*$-algebra $A$.

Exercise 16.8.35. For every $C^*$-algebra $A$ prove that every projection $p$ in $A_\infty$ is Murray–von Neumann equivalent to a projection in $A$.

Exercise 16.8.36. Prove that $A_\infty$ is not countably degree-1 saturated for some separable $C^*$-algebra $A$.

Notes for Chapter 16

The results of §16.1, §16.2, §16.4, and §16.6 belong to the folklore of Model Theory. Model theory of metric structures was introduced in [22] and adapted to $C^*$-algebras in [90] and [87].
$\S$ 16.3 Theorem 16.3.3 is the metric version of the classical Feferman–Vaught Theorem ([100]) proved in [116].

$\S$ 16.5 Theorem 16.5.1 is [98, Theorem 1.5]. The simplified proofs of Theorem 16.5.1 and Theorem 16.3.3 given here first appeared in the Appendix to [85]. The conclusion of Proposition 16.5.3 can be improved under some additional assumptions to prove that the relative commutant of a separable C*-subalgebra of an ultrapower is countably saturated ([88]).

$\S$ 16.7 The full Keisler–Shelah Theorem asserts that two structures of the same language are elementarily equivalent if and only if they have isomorphic ultrapowers (see e.g., [39, §6] and [22, Theorem 5.7] for the continuous logic). Theorem 16.7.6 asserts that if the structures are separable and the Continuum Hypothesis holds then their ultrapowers associated to any nonprincipal ultrafilter on $\mathbb{N}$ are isomorphic. In general, the ultrafilters have to be chosen to be on a set of cardinality $\max(2^{\chi(A)}, 2^{\chi(B)})$. It is relatively consistent with ZFC that two countable, elementarily equivalent, structures do not have isomorphic ultrapowers associated to ultrafilters on $\mathbb{N}$ ([222]).

Theorem 16.7.4 and parts of Theorem 16.7.7 predate the logic of metric structures. They were proved in [113] by using classical first-order logic. In [113] it was also proved that the conclusion of Theorem 16.7.4 is equivalent to the Continuum Hypothesis.

We omitted the proof that the negation of the Continuum Hypothesis implies that a separable, infinite-dimensional C*-algebra $A$ has $2^\kappa$ nonisomorphic ultrapowers and $2^\kappa$ nonisomorphic relative commutants $A' \cap A^\mathbb{U}$. This is Theorem 16.7.7. (2) implies (5) and (4) implies (5). Two ultrafilters giving rise to nonisomorphic ultrapowers were constructed in [113] and two ultrafilters giving rise to nonisomorphic relative commutants were constructed in [89, Theorem 5.1]. Using Shelah’s non-structure theory ([226]) these estimates were improved to $2^\kappa$ in [96]. Bits and pieces of the construction of these ultrafilters are scattered through chapters of this book. A family of $2^\kappa$ ultrafilters is constructed by a generalization of the proof of Theorem 9.4.6. While the latter proof is propelled by chipping off finite fragments of an independent family in $\mathcal{P}(\mathbb{N})$ (Definition 9.2.3), the proof of Theorem 16.7.7 uses an independent family in $\mathbb{N}^\mathbb{N}$ as defined in Exercise 9.10.14. The remainder of the proof has three distinct stages. (i) The construction of $2^\kappa$ sufficiently different linear orderings (more precisely, $\eta_1$-sets) for every $\kappa \geq \mathfrak{r}_2$. (ii) For each one of these linear orderings $L$, the construction of an ultrafilter $\mathcal{U}(L)$ such that $L$ embeds as a maximal linear subordering into a relation definable in the ultrapower $A^\mathcal{U}(L)$. (iii) A proof that at most $\kappa$ of the constructed orderings embed into any given $A^\mathbb{U}$. For details see [96].

Ideals of the form $\bigoplus A_n$ appear in [114], and model-theoretic analysis of the quotients of the form $\prod A_n / \bigoplus A_n$ using the metric analog of the Feferman–Vaught theorem (Theorem 16.3.1) was given in [116].

In [114] it was proved that the conclusion of Exercise 16.8.27 (which is a result from [116]) is false in some models of ZFC.

15 More precisely, it was proved that $\kappa = \mathfrak{r}_\alpha$ implies there are $|\alpha|$ such ultrafilters.
By Corollary 16.7.2, the Continuum Hypothesis implies that every nontrivial ultrapower of an infinite-dimensional separable C*-algebra has ‘nontrivial’ automorphisms. It is not known whether it is consistent with ZFC that for some separable C*-algebra $A$ and some nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ all automorphisms of $A^\mathcal{U}$ have a lifting of product type. The latter statement is relatively consistent with ZFC in the case of ultraproducts of finite fields and certain graphs (see [222, 223, 225]).

Not all reduced products of the form $\prod_n A_n / \bigoplus J A_n$ for separable and unital C*-algebras $A_n$ and a nonprincipal ideal $J$ on $\mathbb{N}$ are countably saturated, or even countably degree-1 saturated. See Exercise 16.8.7, taken from [88, Example 3.2]. This has to be taken with a grain of salt.\(^\text{16}\) Let $\mathcal{Z}_0$ denote the ideal of the sets of asymptotic density zero, see Exercise 9.10.4. The Boolean algebra $\mathcal{P}(\mathbb{N}) / \mathcal{Z}_0$ is a metric structure with respect to the metric $d_{\mathcal{Z}_0}(X, Y) := \limsup_n |(X \Delta Y) \cap n|/n.$ As a metric Boolean algebra, it is countably saturated. A proof of this, as well as a proof that many other similar-looking quotients $\mathcal{P}(\mathbb{N}) / \mathcal{Z}_f$ are elementarily equivalent to $\mathcal{P}(\mathbb{N}) / \mathcal{Z}_0,$ and therefore isomorphic under the Continuum Hypothesis, can be extracted from [145].

Exercise 16.8.2 was first proved in the case of UHF algebras in [113]. The tracial ideal of an ultrapower, has played an important role in the recent progress in Elliott’s programme on classification of separable and nuclear C*-algebras (see e.g.,[157], [156], [179], [159]).

Instances of Exercise 16.8.23 appear in [217] and [159]. Exercise 16.8.24 is a result of Kirchberg, reproduced in [87].

By a result of Kirchberg ([156]), for every separable infinite-dimensional C*-algebra $A$ and every nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ the relative commutant $A' \cap A^\mathcal{U}$ is nontrivial. In [234] Suzuki constructed a simple nonseparable AF algebra $A$ such that $A' \cap A^\mathcal{U}$ is trivial.

A cardinal $\kappa$ that carries a $\kappa$-complete nonprincipal ultrafilter (if such a cardinal exists) is called a measurable cardinal. Exercise 16.8.32 and Exercise 16.8.33 provide a glimpse into the extent of the largeness of these cardinals. Although ZFC does not handle proper classes well, the conclusion of the latter exercise can be extended to show that if a measurable cardinal exists, then there exists a nontrivial elementary embedding of the universe $V$ into a transitive class. The Axiom of Choice implies that such embedding necessarily moves some cardinal, and the leasts such cardinal is the critical point of $j.$ It is not difficult to prove that this critical point is necessarily a measurable cardinal. The existence of such elementary embedding has nontrivial consequences to the structure of $V,$ and even to the regularity properties of sets of real numbers; see [149].

\(^{16}\) The following few lines have little to do with operator algebras but are, in the author’s opinion, an important part of the bigger picture.
Chapter 17
Automorphisms of Massive Quotient C*-Algebras

But there is always supposed to be an invisible asterisk when scientists say “this is true”—because nothing is certain. (And, yes, that is true of my work too, including everything in this book. Sorry about that.)

P.E. Tetlock and D. Gardner, Superforecasting: The Art and Science of Prediction

My bubble, my rules.

Bart Simpson

Mathematics is not a science. We the mathematicians posit axioms and, lead by the infallible\(^1\) rules of logic, establish truths about any world that these axioms apply to. This would surely be a rather futile exercise if it wasn’t for the “unreasonable effectiveness of mathematics in the natural sciences,” to quote E. Wigner. Every now and then, a natural statement about basic mathematical objects transpires to be neither provable nor refutable from the established axioms. This chapter is devoted to an instance of this phenomenon.

The Calkin algebra provides the setting for the Weyl–von Neumann theorem and its extension to normal operators by Berg and Sikonia (Corollary 12.4.6). It also provides the setting for the Brown–Douglas–Fillmore theory of extensions of C*-algebras—a seminal work that introduced methods of homological algebra and algebraic topology to operator algebras and for the analytic K-homology ([34], see also [129]). It is a question of Brown, Douglas, and Fillmore about outer automorphisms of the Calkin algebra that precipitated some of the most stimulating applications of set theory to C*-algebras. This question will be discussed in the present chapter. We first prove that the Continuum Hypothesis implies that the Calkin al-

\(^1\) This is at least what we like to think. However, Gödel’s theorem implies that any sufficiently strong theory recursively presented in Hilbert-style predicate logic can prove its consistency if and only if it is inconsistent. The large cardinal hierarchy ([149]) provides very strong heuristic evidence that this is not a problem, as long as one has no issue with having turtles all the way down—or rather, all the way up. In any case, the set of provable statements in a consistent and recursively axiomatizable theory cannot be recursive. In this book we follow the established practice and ignore this inconvenience while we can.
automorphism. Second, forcing axioms imply that all automorphisms of the Calkin algebra are inner.

In §17.1 we prove that the Continuum Hypothesis implies that the Calkin algebra has $2^\mathbb{R}_1$ outer automorphisms. This is a special case of the theorem that if $A$ is $\sigma$-unital and stable $C^*$-algebra then the equality $\mathfrak{d} = \mathfrak{r}_1$ implies that $\mathcal{Q}(A)$ has $2^\mathbb{R}_1$ automorphisms. These coronas are frequently not countably saturated and the proof uses an oblique strategy. In §17.2 we prove that the approximate $^*$-homomorphisms between finite-dimensional $C^*$-algebras are Ulam-stable. The remaining part of this chapter contains a complete proof that forcing axioms (more precisely, OCA) imply all automorphisms of the Calkin algebra are inner.

17.1 The Calkin Algebra Has Outer Automorphisms

In this section we prove that a weakening of the Continuum Hypothesis ($\mathfrak{d} = \mathfrak{r}_1$ plus $\mathfrak{c} < 2^{\aleph_1}$) implies that the Calkin algebra has $2^{\aleph_1}$ outer automorphisms. Outer automorphisms of $\mathcal{Q}(H)$ will be associated to automorphisms of an inverse limit of a certain system of quotient groups of $\mathcal{T}\mathbb{N}$ indexed by elements of the poset Part$_\mathbb{N}$.

The present section relies on §9.5 and §9.7. A unital $^*$-homomorphism $\Phi$ of a unital $C^*$-algebra $A$ into $\mathcal{Q}(H)$ is called an extension. An extension is split if there exists a $^*$-homomorphism $\Phi : A \to \mathcal{B}(H)$ such that $\pi \circ \Phi = \text{id}_A$. Corollary 12.4.8 implies that every two split extensions of a unital, abelian, singly generated $C^*$-algebra are unitarily equivalent. Even singly generated $C^*$-algebras may have non-equivalent extensions (e.g., $C(\mathbb{T})$ has both nontrivial and trivial extensions, see Proposition 12.4.3). However, the Brown–Douglas–Fillmore theory ([34]) provides a characterization of the unitary equivalence of extensions of unital, abelian, $C^*$-subalgebras of $\mathcal{Q}(H)$. For example, two unital copies of $C(\mathbb{T})$ are unitarily equivalent if and only if the lifts of their generators have the same Fredholm index.

Recall that Part$_\mathbb{N}$ is the poset of partitions of $\mathbb{N}$ into finite intervals (§9.7). Fix a basis $\xi_n$, for $n \in \mathbb{N}$, of $H$. The corresponding atomic masa in $\mathcal{B}(H)$ (Example 3.1.20) is identified with $\ell_\infty(\mathbb{N})$. An element of $\ell_\infty(\mathbb{N})$ is a unitary if and only if its range is included in $\mathcal{T}\mathbb{N}$. We therefore identify the unitary group of the atomic masa with $\mathcal{T}\mathbb{N}$.

As in §9.7.1, for $E = \langle E_n : n \in \mathbb{N} \rangle$ in Part$_\mathbb{N}$ define $E^{\text{even}}$ and $E^{\text{odd}}$ in Part$_\mathbb{N}$ by

$$E^{\text{even}}_n := E_{2n} \cup E_{2n+1},$$
$$E^{\text{odd}}_n := E_{2n-1} \cup E_{2n}.$$

To $E \in \text{Part}_\mathbb{N}$ associate the von Neumann algebra $\mathcal{D}[E]$ of block-diagonal operators respecting $E$ (Definition 9.7.5) and let

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2 More difficult to state, but arguably more plausible than the Continuum Hypothesis; see §8.6.

3 This is a (standard) abuse of the language, justified by the fact that such $\Phi$ corresponds to an extension $0 \to \mathcal{K}(H) \to E \to A \to 0$ of $A$ by $\mathcal{K}(H)$ (see [179] or [26]).
\[ \mathcal{F}[E] := \{ a_0 + a_1 : a_0 \in \mathcal{D}[E^{\text{even}}], a_1 \in \mathcal{D}[E^{\text{odd}}] \}. \] (17.1)

This is a Banach subspace, but not a subalgebra, of \( \mathcal{B}(H) \).

**Lemma 17.1.1.** For \( E \in \text{Part}_\mathbb{N} \) we have

\[ \mathcal{F}[E] = \{ a \in \mathcal{B}(H) : (\forall m \in \mathbb{N}) (\forall n \in \mathbb{N}) (a_{mn}|_{\mathbb{N}^n} \neq 0 \implies (\exists j)(m, n) \subseteq E_j \cup E_{j+1}) \}. \]

**Proof.** Denoting the right-hand side of the displayed equation by \( \mathcal{F}' \), we clearly have \( \mathcal{F}[E] \subseteq \mathcal{F}' \). For the converse, fix \( a \in \mathcal{F}' \). With \( q_j := \text{proj}_{\mathbb{N}^n \cap (E_j \cup E_{j+1})} \), \( a_0 := \sum q_j a q_j \) is in \( \mathcal{D}[E^{\text{even}}] \) and \( a_1 := a - a_0 \) is in \( \mathcal{D}[E^{\text{odd}}] \).

**Proposition 17.1.2.** For every separable subalgebra \( A \) of \( \mathcal{B}(H) \) there is \( E \in \text{Part}_\mathbb{N} \) such that \( \pi[A] \subseteq \pi[\mathcal{F}[E]] \).

**Proof.** Fix an enumeration \( a_n \), for \( n \in \mathbb{N} \), of a countable dense subset of \( A \). By Lemma 9.7.6 there are \( E \in \text{Part}_\mathbb{N}, a_n^{\text{even}} \in \mathcal{D}[E^{\text{even}}], \) and \( a_n^{\text{odd}} \in \mathcal{D}[E^{\text{odd}}] \), for all \( n \in \mathbb{N} \) such that \( a_n - a_n^{\text{even}} - a_n^{\text{odd}} \) is compact for all \( n \). Therefore \( \pi(a_n) \in \pi[\mathcal{F}[E]] \) for all \( n \). Since \( \mathcal{F}[E] \) is norm-closed, the conclusion follows.

**Definition 17.1.3.** For \( E \in \text{Part}_\mathbb{N} \) and \( u, v \in T^\mathbb{N} \) (identified with \( U(\ell_\infty(\mathbb{N})) \)) we write \( u \sim_E v \) if \( u a_n^* - v a_n^* \in \mathcal{F}(H) \) for all \( a \in \mathcal{F}[E] \).

Evidently, \( u \sim_E v \) if and only if \( \pi(u v^*) \in \pi[\mathcal{F}[E]]' \cap \mathcal{L}(H) \) and \( \sim_E \) is an equivalence relation (see also Lemma 17.1.9 below).

**Lemma 17.1.4.** Suppose \( \mathcal{E} \) is a \( \leq' \)-cofinal subset of \( \text{Part}_\mathbb{N} \) and \( u_E \in T^\mathbb{N} \), for \( E \in \mathcal{E} \), satisfy \( u_E \sim_E u_F \) whenever \( E \leq' F \) for \( E \) and \( F \) in \( \mathcal{E} \). Then there exists a unique automorphism of \( \mathcal{L}(H) \) which agrees with \( \text{Ad} \pi(u_E) \) on \( \pi[\mathcal{F}[E]] \) for all \( E \in \mathcal{E} \).

**Proof.** Proposition 17.1.2 implies that \( \mathcal{L}(H) = \bigcup_{E \in \mathcal{E}} \pi[\mathcal{F}[E]] \). Therefore

\[ \Phi(a) := \text{Ad}(\pi(u_E))(a), \] for \( E \in \mathcal{E} \) such that \( a \in \pi[\mathcal{F}[E]] \),

defines a self-adjoint linear isometry from \( \mathcal{L}(H) \) to \( \mathcal{L}(H) \). It is a \( ^*\)-homomorphism because Proposition 17.1.2 implies that for every pair \( a, b \) in \( \mathcal{L}(H) \) there exists a single \( E \in \mathcal{E} \) such that \( a, b, \) and \( ab \) all belong to \( \pi[\mathcal{F}[E]] \).

By an analogous argument, \( \Psi : \mathcal{L}(H) \to \mathcal{L}(H) \) defined by \( \Psi(a) := \text{Ad}(\pi(u_E))(a), \) for \( E \in \mathcal{E} \) such that \( a \in \pi[\mathcal{F}[E]] \), is an endomorphism of \( \mathcal{L}(H) \). Since \( \Phi \circ \Psi = \Phi \circ \Phi = \text{id}_{\mathcal{L}(H)} \), \( \Phi \) is an automorphism of \( \mathcal{L}(H) \). It clearly agrees with \( \text{Ad} \pi(u_E) \) on \( \pi[\mathcal{F}[E]] \) for every \( E \in \mathcal{E} \).

Our next task is to find a practical equivalent reformulation of the equivalence relation \( \sim_E \). It will be expressed in terms of a ‘bi-pseudo-metric’ on \( \mathbb{N} \times T^\mathbb{N} \). For \( i \) and \( j \) in \( \mathbb{N} \), \( x \) and \( y \) in \( T^\mathbb{N} \), and \( F \in \mathbb{N} \) let
\[ \Delta_{[i,j]}(x,y) := |x(i)\bar{y}(j) - y(i)\bar{y}(j)|, \] and
\[ \Delta_F(x,y) := \max_{i,j \in F} \Delta_{[i,j]}(x,y). \] (17.2) (17.3)

By \( \text{diam}(X) \) we denote the diameter of a set \( X \) with respect to a metric clear from the context.

**Lemma 17.1.5.** If \( F \subseteq \mathbb{N} \) is nonempty, \( i, j \) are in \( \mathbb{N} \), and \( x, y, z \) are in \( T^\mathbb{N} \) then the following hold.

1. \( \Delta_{[i,j]}(x,y) = |x(i)\bar{y}(j) - x(j)\bar{y}(j)|. \)
2. \( \Delta_F(x,1) = \text{diam}\{x(i) : i \in F\}. \)
3. \( \Delta_{[i,k]}(x,y) \leq \Delta_{[i,j]}(x,y) + \Delta_{[j,k]}(x,y), \) hence \( \Delta_{\{x,y\}}(x,y) \) is a pseudometric on \( \mathbb{N} \).
4. \( \Delta_F(x,z) \leq \Delta_F(x,y) + \Delta_F(y,z), \) hence \( \Delta_F \) is a pseudometric on \( T^\mathbb{N} \).
5. \( \Delta_F(x,y) = \Delta_F(xz,yz). \)
6. \( \min_{\lambda \in T} \sup_{i \in F} |x(i) - \lambda y(i)| \leq \Delta_F(x,y) \leq 2 \min_{\lambda \in T} \sup_{i \in F} |x(i) - \lambda y(i)|. \)

**Proof.** (1), (2), and (3) comprise a proof of (4) broken into small steps, and (5) is obvious.

(6) Fix \( j \in F \) and let \( \lambda := x(j)\bar{y}(j) \). Then
\[ \sup_{i \in F} |x(i) - \lambda y(i)| = \sup_{i \in F} \Delta_{[i,j]}(x,y) \leq \Delta_F(x,y). \]
The other inequality follows (with \( j \) still fixed) from (3). \( \square \)

**Lemma 17.1.6.** Let \( F \) and \( E \) be finite subsets of \( \mathbb{N} \). Then for all \( i \in F, j \in E, \) and all \( x \) and \( y \) we have \( \Delta_{F \cup E}(x,y) \leq \Delta_F(x,y) + \Delta_E(x,y) + \Delta_{[i,j]}(x,y). \)

**Proof.** Lemma 17.1.5 (3) for \( k \in E \) and \( l \in F \) implies that
\[ \Delta_{[k,l]}(x,y) \leq \Delta_{[k,j]}(x,y) + \Delta_{[i,j]}(x,y) + \Delta_{[j,l]}(x,y), \]
and the desired inequality follows. \( \square \)

For \( E \in \text{Part}_\mathbb{N} \) and \( x \) and \( y \) in \( T^\mathbb{N} \) Lemma 17.1.5 (4) implies that
\[ \Delta_E(x,y) := \limsup_{j \to \infty} \Delta_{E_j \cup E_{j+1}}(x,y) \]
defines a pseudometric on \( T^\mathbb{N} \).

**Lemma 17.1.7.** If \( E \subseteq F \) and \( x, y, z \) are in \( T^\mathbb{N} \) then \( \Delta_E(x,z) \leq \Delta_E(x,y) + \Delta_E(y,z) \) and \( \Delta_E(x,y) \leq \Delta_F(x,y). \)

**Proof.** The second inequality follows from the definitions, and the first one follows from Lemma 17.1.5 (4). \( \square \)

**Definition 17.1.8.** Let \( F_E := \{x \in T^\mathbb{N} : \Delta_E(x,1) = 0\}, G_E := T^\mathbb{N}/F_E, \) for \( E \in \text{Part}_\mathbb{N}, \)

\( ^4 \) This bit will be needed only in the proof of Theorem 17.8.5.
Lemma 17.1.7 implies that $F_E$ is a subgroup of $\mathbb{T}^N$. Also $E \preceq^* F$ implies $F_E \supseteq F_F$ and therefore $G_F = G_E / (F_E / F_F)$. We moreover have a commutative diagram whose rows are exact sequences ($\pi_{FE}$ is the inclusion map and $\pi_{EF}$ is the quotient map):

$$
\begin{array}{ccc}
0 & \longrightarrow & F_F \\
\downarrow \iota_{EF} & \quad & \downarrow \pi_{FE} \\
0 & \longrightarrow & \mathbb{T}^N \\
\downarrow \text{id} & \quad & \downarrow \pi_{EF} \\
0 & \longrightarrow & G_F \\
\downarrow & \quad & \downarrow \\
0 & \longrightarrow & \mathbb{T}^N \\
\downarrow & \quad & \downarrow \\
0 & \longrightarrow & G_E \\
\end{array}
$$

Lemma 17.1.9. Suppose $E \in \text{Part}_N$ and $u$ and $v$ belong to $\mathbb{T}^N$. Then $u \sim_E v$ if and only if $uv^* \in F_E$.

Proof. Lemma 17.1.5 (2) implies that $uv^* \in F_F$ if and only if

$$
\lim_{n} \text{diam}(\{uv^*(j) : j \in E_n \cup E_{n+1}\}) = 0.
$$

With $q_n := \text{Proj}_{\text{sp}}(\mathbb{Z}/\mathbb{Z}[E_n \cup E_{n+1}])$, the set $\{uv^*(j) : j \in E_n \cup E_{n+1}\}$ is equal to the spectrum of $w_n := q_n uv^*$, considered as a unitary in $\mathcal{B}(q_n[H])$. Thus $uv^* \in F_F$ if and only if $\lim_n \text{diam}(\text{sp}(w_n)) = 0$. By considering $\|w_n - \lambda \cdot q_n\|$ for $\lambda \in \text{sp}(w_n)$, one sees that $\frac{1}{2} \text{diam}(\text{sp}(w_n)) \leq \text{dist}(w_n, \mathbb{C}q_n) \leq \text{diam}(\text{sp}(w_n))$. Corollary 3.3.8 implies $\text{diam}(\text{sp}(w_n)) \approx \sup_{|a| \leq 1} \|a - (\text{Ad}(uv^*))(a)\|$. Therefore $\lim_n \text{diam}(\text{sp}(w_n)) = 0$ if and only if $a - (\text{Ad}(uv^*))(a)$ is compact for all $a \in \mathcal{F}[E]$; but the latter condition is equivalent to $u \sim_E v$.

To create some elbow room, we introduce a speedup of the relation $\preceq^*$ and write $E \ll^* F$ if $E \preceq^* F$ and for every $m$ there exist $n$ and $k$ such that $\bigcup_{j=n}^{n+m-1} E_j \subseteq F_k$. It is not difficult to see that $\ll^*$ is a partial ordering on $\text{Part}_N$.

Lemma 17.1.10. If $E \ll^* F$, then $F_F$ is a proper subgroup of $F_E$.

Proof. We already know that $E \preceq^* F$ implies $F_F \subseteq F_E$. Using $E \ll^* F$, we find $n(m)$ and $k(m)$ such that $n(m) + m < n(m+1)$ and $\bigcup_{j=n}^{n+m-1} E_j \subseteq F_k$ for all $m \in \mathbb{N}$. Define $x \in \mathbb{T}^N$ by $x(k) := \exp(i2\pi j/m)$ if $k \in E_{n(m)+j}$ for some $j$ and $x(k) := 1$ if $k \notin \bigcup_{j=n}^{n+m-1} E_j$. Then $\lim_{j} \Delta_{E_{j} \cup E_{j+1}}(x, 1) = 0$, and therefore $x \in F_F$. Finally, if $\bigcup_{j=n}^{n+m-1} E_j \subseteq F_k(m)$, then $\Delta_{E_{k}(m)}(x, 1) = \max_{j < m} |1 - \exp(i2\pi j/m)|$. This implies $\Delta_{F_k}(x, 1) = 2$ and $x \notin F_F$.

The following is the last ZFC result of the present section.

Proposition 17.1.11. For every $\ll^*$-increasing sequence $E(\alpha)$, for $\alpha < \mathfrak{r}_1$, the inverse limit $\lim_{\alpha} G_{E(\alpha)}$ has cardinality $2^{\aleph_1}$.

Proof. Let $F(\alpha) := F_E(\alpha)$ and $G(\alpha) := G_E(\alpha)$.

Claim. If $\alpha$ is a limit ordinal then $x \mapsto (\pi_{E(\beta)} E(\alpha))(x) : \beta < \alpha$ is a surjection from $G(\alpha)$ onto $\lim_{\beta < \alpha} G(\beta)$. 
Proof. Let \( \alpha(n) \), for \( n \in \mathbb{N} \), be an increasing sequence of ordinals with the supremum equal to \( \alpha \). Then \( \lim_{\alpha \beta < \alpha} G(\beta) = \lim_{\alpha} G(\alpha(n)) \).

Fix \((x_n : n \in \mathbb{N}) \in (\mathbb{T}^\mathbb{N})^\mathbb{N}\) so that \( x_n x_n^{-1} \in F(\alpha(m)) \) whenever \( m \leq n \). Such a sequence represents a thread in \( \lim_{\alpha} G(\alpha(n)) \), and every thread in \( \lim_{\alpha} G(\alpha(n)) \) has a representing sequence of this sort. We write

\[
\Delta_n(x, y) := \sup \{ \Delta_{E(\alpha(n)) \cup E(\alpha(n))_{i+1}}(x, y) : \min E(\alpha(n))_i \geq m \}.
\]

For \( m \in \mathbb{N} \) and \( E \) and \( F \) in \( \mathcal{P}_\mathbb{N} \) we write \( E \leq_m F \) if for all \( j \) satisfying \( m \leq \min(F_j) \) there exists \( i \) such that \( E_i \subseteq F_j \). Then \( E \leq^* F \) if and only if \( E \leq^*_m F \) for some \( m \in \mathbb{N} \).

Let \( m(k) \), for \( k \in \mathbb{N} \), be an increasing sequence such that for all \( i < j \leq k \) we have \( E(\alpha(i)) \leq^{m(k)} E(\alpha(j)) \) and \( \Delta^{m(k)}_{\alpha(i)}(x(t_j), x(t_j)) < \frac{1}{i} \). Define \( x(j) := x_i(j) \), if \( m(i) \leq j < m(i + 1) \). Then \( xx^{-1} \in F(\alpha(i)) \) for all \( i \) and \( x \) is as required. \( \square \)

Again we use \( \{0, 1\}^{<\mathbb{R}_1} \) (§8.4) as a bookkeeping device. Define \( x(s) \in G(\alpha) \) for \( s \in \{0, 1\}^{\mathbb{R}_1} \) so that for all \( s \sqsubseteq t \in \{0, 1\}^{<\mathbb{R}_1} \) the following conditions hold:

1. \( \pi_{E(\alpha \cup \beta)}(x(t)) = x(s) \) (where \( \alpha \) and \( \beta \) are the levels of \( s \) and \( t \), respectively),
2. \( x(s0) \neq x(s1) \).

Suppose that \( x(s) \) has been defined for \( s \in \{0, 1\}^{<\alpha} \). If \( \alpha \) is a successor, fix \( \beta \) such that \( \alpha = \beta + 1 \) and fix \( x(t) \in \{0, 1\}^\beta \). By Lemma 17.1.10 the quotient map from \( G(\beta + 1) \) onto \( G(\beta) \) is not injective and we can therefore choose distinct \( x(t0) \) and \( x(t1) \) as required. If \( \alpha \) is a limit ordinal, then by the Claim for \( s \in \{0, 1\}^\alpha \) we can find \( x(s) \) as required.

By Proposition A.2.3, we can define \( x(s) \) for all \( s \in \{0, 1\}^{<\mathbb{R}_1} \). Therefore the group \( \lim_{\alpha < \mathbb{R}_1} G(\alpha) \) contains all branches of the complete binary tree of height \( \mathbb{R}_1 \) and its cardinality is at least \( 2^{\mathbb{R}_1} \). \( \square \)

**Theorem 17.1.12.** The equality \( \mathcal{D} = \mathbb{R}_1 \) implies that the Calkin algebra has \( 2^{\mathbb{R}_1} \) automorphisms.

**Proof.** Theorem 9.7.8 implies that \( (\mathbb{P}_{\mathbb{N}}, \leq^*) \) is cofinally equivalent to \( (\mathbb{N}^\mathbb{N}, \leq^*) \). Therefore \( \mathcal{D} = \mathbb{R}_1 \) implies that there exists a \( \leq^* \)-increasing and cofinal \( \mathbb{R}_1 \)-sequence, \( E(\alpha) \), for \( \alpha < \mathbb{R}_1 \). Since \( \mathbb{P}_{\mathbb{N}} \) does not have a \( \leq^* \)-maximal element, we may assume that this sequence is \( \leq^* \)-increasing. Since this sequence is cofinal, Proposition 17.1.2 implies \( \lim_{\alpha} E(\alpha) = \mathcal{D}(H) \).

By Proposition 17.1.11, \( \Gamma := \lim_{\alpha} G(\alpha) \) has cardinality \( 2^{\mathbb{R}_1} \). Lemma 17.1.4 implies that every element of \( \Gamma \) defines an automorphism of \( \mathcal{D}(H) \). If two elements of \( \Gamma \) differ on \( G(\alpha) \), then the corresponding inner automorphisms differ on \( \mathcal{D}[E(\alpha)] \) by Lemma 17.1.9. \( \square \)

**Corollary 17.1.13.** If \( \mathcal{D} = \mathbb{R}_1 \) and \( \varepsilon < 2^{\mathbb{R}_1} \) then the Calkin algebra has \( 2^{\mathbb{R}_1} \) outer automorphisms. In particular, the Continuum Hypothesis implies that the Calkin algebra has \( 2^{\mathbb{R}_1} \) outer automorphisms.
17.2 Ulam-stability of Approximate ∗-homomorphisms

Proof. Since every unitary in $\mathcal{D}(H)$ lifts to a partial isometry in $\mathcal{B}(H)$, $\mathcal{D}(H)$ has only $c$ inner automorphisms. The first assertion now follows from Theorem 17.1.12. Since the Continuum Hypothesis implies both $\mathcal{d} = \aleph_1$ and $c = \aleph_1 < 2^{\aleph_1}$, this completes the proof. □

Example 17.1.14. For every presently known outer automorphism of $\mathcal{D}(H)$ (Theorem 17.1.12, [200], [82, §1]), its restriction to any separable C∗-subalgebra is implemented by a unitary (Exercise 17.9.2; cf. Proposition 7.3.9).

Some results about the Calkin algebra readily generalize to coronas of stable, separable, C∗-algebras. (Recall that $A$ is stable $A \otimes \mathcal{K}(H) \cong A$.) Here is one of them; its proof is, being similar to that of Theorem 17.1.12, left as Exercise 17.9.4.

Theorem 17.1.15. Suppose that $A$ is a stable and σ-unital C∗-algebra. The equality $\mathcal{d} = \aleph_1$ implies that $\mathcal{Q}(A)$ has $2^{\aleph_1}$ automorphisms. □

By Theorem 17.1.15 and a counting argument, the Continuum Hypothesis implies that the corona of every stable and σ-unital C∗-algebra has an outer automorphism. However, in many cases ‘obvious’ outer automorphisms exist (Exercise 17.9.3). In §17.3 we will isolate the notion of a ‘topologically trivial’ automorphism that includes all ‘obviously defined’ automorphisms of coronas.

17.2 Ulam-stability of Approximate ∗-homomorphisms

To say that something is almost rigorous makes as much sense as saying a woman is almost pregnant.

Ed Nelson (see [230])

In this section we prove that for a small enough $\epsilon > 0$ a Borel-measurable unitary $\epsilon$-representation of a compact group into a unital C∗-algebra is $2\epsilon$-approximated by a representation, and that a Borel-measurable unital $\epsilon$-∗-homomorphism from a full matrix algebra into a von Neumann algebra with a faithful tracial state can be $16\epsilon$-approximated by a ∗-homomorphism.

A reader who is wondering what is a section on almost ∗-homomorphisms doing here may want to peek ahead at Proposition 17.5.4.

Definition 17.2.1. An $\epsilon$-representation of a group $G$ in a unital C∗-algebra $A$ is a function $\Theta: G \to U(A)$ such that $\sup_{x,y \in G} \|\Theta(xy) - \Theta(x)\Theta(y)\| \leq \epsilon$ and $\Theta(1) = 1$.

The following is used as a template for Theorem 17.2.3. Its proof can be extracted from the proof of the latter so easily that this is not even given as an exercise.

Theorem 17.2.2. Assume $G$ is a compact group and $A$ is a von Neumann algebra. If $\epsilon < 1/10$ then for every Borel-measurable $\epsilon$-representation $\Theta: G \to U(A)$ there exists a unitary representation $\Lambda: G \to U(A)$ such that $\|\Lambda - \Theta\| \leq 2\epsilon$. □
By $A_m$ we denote the $m$-ball of a $C^*$-algebra $A$.

Theorem 17.2.3. Assume $A$ and $B$ are von Neumann algebras, $A$ is finite-dimensional, $\varepsilon < 1/28$, and $\Theta : A_1 \to B_2$ is a Borel-measurable function and satisfies $\Theta[U(A)] \subseteq U(B)$ and $\|\Theta(ga) - \Theta(g)\Theta(a)\| \leq \varepsilon$ for all $g \in U(A)$ and all $a \in A_1$, and $\Theta(1) = 1$. Then there exists a Borel-measurable $\Lambda : A_1 \to B_2$ which satisfies $\|\Lambda - \Theta\| \leq 4\varepsilon$ and $\Lambda(ga) - \Lambda(g)\Lambda(a) = 0$ for all $g \in U(A)$ and all $a \in A_1$.

Proof. Let $\mu$ denote the Haar measure on $U(A)$. Throughout this proof $g$ and $x$ stand for elements of $U(A)$ and $a$ stands for an element of $A_1$. As in §3.3, define $\Theta' : A_1 \to B$ by

$$\Theta'(a) := \int \Theta(x)^*\Theta(xa)d\mu(x).$$

Since $\Theta(x) \in U(A)$, we have $\|\Theta(x)^*\Theta(xa) - \Theta(xa)\| = \|\Theta(xa) - \Theta(x)\Theta(a)\| \leq \varepsilon$, and therefore $\|\Theta' - \Theta\| \leq \varepsilon$. The invariance of $\mu$ implies

$$\int \Theta(xg^{-1})^*\Theta(xa)d\mu(x) = \int \Theta(x)^*\Theta(xga)d\mu(x) = \Theta'(ga).$$

With $a = 1$ and replacing $g$ with $g^{-1}$ this gives $\Theta'(g^{-1}) = \Theta'(g)^*$. Using these observations, $\Theta(x)^*\Theta(x) = 1$, and linearity of the integral,

$$\mathcal{I} := \int (\Theta(xg^{-1}) - \Theta(x)\Theta(g^{-1}))^*(\Theta(xa) - \Theta(x)\Theta(a))d\mu(x)$$

$$= \Theta'(ga) - \Theta'(g)\Theta(a) - \Theta(g^{-1})^*\Theta'(a) + \Theta(g^{-1})^*1 \cdot \Theta(a)$$

$$= \Theta'(ga) - \Theta'(g)\Theta'(a) + \Theta'(g)(\Theta'(a) - \Theta(a)) - \Theta(g^{-1})^*(\Theta'(a) - \Theta(a))$$

$$= \Theta'(ga) - \Theta'(g)\Theta'(a) + (\Theta'(g)^* - \Theta(g^{-1})^*)(\Theta'(a) - \Theta(a)).$$

Since $\|\mathcal{I}\| \leq \varepsilon^2$ and $\|(\Theta'(g)^* - \Theta(g^{-1})^*)(\Theta'(a) - \Theta(a))\| \leq \varepsilon^2$, we have

$$\|\Theta'(ga) - \Theta'(g)\Theta'(a)\| \leq 2\varepsilon^2. \tag{17.4}$$

There is no reason to expect $\Theta'(g)$ to belong to $U(B)$, but we can estimate how far it lands. Clearly $\|\Theta'(g)\| \leq \sup_{h \in U(A)} \|\Theta(h)^*\Theta(hg)\| \leq 1$, and $\Theta'(g)^*\Theta'(g) \leq 1$.

By (17.4), with $a = g$ and using $\Theta(1) = 1$, we have

$$\|1 - \Theta'(g)^{-1}\Theta'(g)\| = \|1 - \Theta'(g)^*\Theta'(g)\| \leq 2\varepsilon^2.$$  

By Lemma 1.2.6, $\Theta'(g)$ is invertible and (using $\varepsilon < 1/28$, which is an overkill here)

$$\|\Theta'(g)^{-1} - 1\| \leq \frac{1}{1 - 2\varepsilon^2} - 1 \leq 3\varepsilon^2.$$

Define $\Theta'' : A_1 \to B_2$ by $\Theta''(g) := \Theta'(g)|\Theta'(g)|^{-1}$ and $\Theta''(a) := \Theta'(a)$. Then we have $\Theta''[U(A)] \subseteq U(B)$, $\|\Theta'' - \Theta\| \leq \sup_{g \in U(A)} \|\Theta'(g)|\Theta'(g)|^{-1}| - 1\| \leq 3\varepsilon^2$, and $\|\Theta'' - \Theta\| \leq \varepsilon + 2\varepsilon^2 < 2\varepsilon.$\footnote{By now it should be evident that no attempt has been made to determine the optimal constants.}


\[ \| \Theta''(ga) - \Theta''(g)\Theta''(a) \| \leq \| \Theta'(ga) - \Theta'(g)\Theta'(a) \| + (2 + \| \Theta'(a) \|)\| \Theta'' - \Theta \| \leq 14\varepsilon^2. \]

Therefore \( \Theta'' \) satisfies the assumptions of this theorem with \( \varepsilon \) replaced by \( 14\varepsilon^2 \).

Let \( \Theta_n : A_1 \to B_2 \) be defined by \( \Theta_0 := \Theta \) and \( \Theta_{n+1} := \Theta_n'' \) if \( n \geq 0 \). Let \( \varepsilon_0 := \varepsilon \) and \( \varepsilon_{n+1} := 14\varepsilon_n^2 \) for \( n \geq 0 \). Then \( \Theta_n \) is an \( \varepsilon_n^* \)-homomorphism and \( \| \Theta_{n+1} - \Theta_n \| \leq 2\varepsilon_n \) for all \( n \). Therefore \( \Lambda(a) := \lim_n \Theta_n(a) \) is, being a limit of a Cauchy sequence, well defined. Since \( \varepsilon_{n+1} \leq \varepsilon_n/2 \), we have \( \| \Lambda - \Theta \| \leq 4\varepsilon \) and \( \| \Lambda(ab) - \Lambda(a)\Lambda(b) \| = \lim_n \| \Theta_n(ab) - \Theta_n(a)\Theta_n(b) \| = 0 \).

When does a homomorphism between unitary groups of two \( C^* \)-algebras extend to a *-homomorphism between the underlying \( C^* \)-algebras? The following, rather specific, sufficient condition will do for our purposes.

**Lemma 17.2.4.** Suppose \( A \) and \( B \) are unital \( C^* \)-algebras and \( \Lambda : U(A) \to U(B) \) is a group homomorphism. If \( A \) has a faithful tracial state \( \tau \), \( B \) has a faithful tracial state \( \sigma \), and \( \sigma(\Lambda(u)) = \tau(u) \), then \( \Lambda \) has a unique extension to a *-homomorphism.

**Proof.** Every \( a \in A \) can be written uniquely as a linear combination of four unitaries (Exercise 1.11.16). We claim that \( \Phi : \sum_{\lambda \in \Lambda} \lambda_j u_j \mapsto \sum_{\lambda \in \Lambda} \lambda_j \Lambda(u_j) \)

is a well-defined, unital, *-homomorphism from \( A \) into \( B \). To see that \( \Phi \) is well-defined, it will suffice to prove that for an arbitrary linear combination of unitaries \( \sum_{\lambda \in \Lambda} \lambda_j u_j \), \( \sum_{\lambda \in \Lambda} \lambda_j u_j = 0 \) implies \( \sum_{\lambda \in \Lambda} \lambda_j \Lambda(u_j) = 0 \).

But \( \sum_{\lambda \in \Lambda} \lambda_j u_j = 0 \) if and only if \( (\sum_{\lambda \in \Lambda} \lambda_j u_j)^*(\sum_{\lambda \in \Lambda} \lambda_j u_j) = 0 \), and since \( \tau \) is faithful and \( \tau((\sum_{\lambda \in \Lambda} \lambda_j u_j)^*(\sum_{\lambda \in \Lambda} \lambda_j u_j)) = \sum_{\lambda \in \Lambda} \lambda_j \Lambda(u_j)^* \Lambda(u_j) \), this is equivalent to \( \sum_{\lambda \in \Lambda} \lambda_j \Lambda(u_j) = 0 \). Since \( \sigma \) is faithful, an analogous computation implies that \( \sum_{\lambda \in \Lambda} \lambda_j \Lambda(u_j) = 0 \) if and only if \( \sum_{\lambda \in \Lambda} \lambda_j \Lambda(u_j)^* \Lambda(u_j) = 0 \) for all \( j \) and \( k \), and therefore \( \sum_{\lambda \in \Lambda} \lambda_j u_j = 0 \) if and only if \( \sum_{\lambda \in \Lambda} \lambda_j \Lambda(u_j) = 0 \). Since this was an arbitrary linear combination of the unitaries in \( A \), this completes the proof that \( \Phi \) is well-defined. It is linear by the definition.

The verifications that \( \Phi \) is self-adjoint and multiplicative are straightforward. It is therefore a *-homomorphism. The uniqueness of \( \Phi \) follows from the fact that every element of \( A \) is a linear combination of four unitaries used in the beginning of this proof.

Since a *-homomorphism is uniquely determined by its restriction to the unit ball, we require an \( \varepsilon^* \)-homomorphism between \( C^* \)-algebras to be defined only on a unit ball (in some literature the domain of an \( \varepsilon^* \)-homomorphism is a \( C^* \)-algebra).

**Definition 17.2.5.** Given \( \varepsilon > 0 \) and \( C^* \)-algebras \( A \) and \( B \), some \( \Theta : A_1 \to B_1 \) is an \( \varepsilon \)-*homomorphism if for all \( x, y \in A_1 \) and \( \lambda \in \mathbb{C} \), \( |\lambda| \leq 1 \), each one of \( \Theta(x^*) - \Theta(x)^* \), \( \Theta(x + y) - \Theta(x) - \Theta(y) \), \( \Theta(xy) - \Theta(x)\Theta(y) \), and \( \Theta(\lambda x) - \lambda \Theta(x) \) has norm not greater than \( \varepsilon \). It is unital if in addition \( \Theta(U(A)) \subseteq U(B) \) and \( \Theta(1) = 1 \).
Theorem 17.2.6. Suppose $\varepsilon < 1/28$, $m \geq 1$, $A$ is a von Neumann algebra with a faithful tracial state $\sigma$, and $\Theta : M_n(\mathbb{C}) \to A$ is a unital $\varepsilon$-homomorphism. Then there exists a $*$-homomorphism $\Phi : M_n(\mathbb{C}) \to A$ such that $\|\Theta - \Phi\| \leq 16\varepsilon$.

Proof. By Theorem 17.2.3, there is $\Lambda : \mathcal{U}(M_n(\mathbb{C})) \to A$ such that $\Lambda(uv) = \Lambda(u)\Lambda(v)$ for all $u$ and $v$, and $|\Lambda - \Theta| \leq 4\varepsilon$ For a projection $p$, the unitary $u_p := 1 - 2p$ is self-adjoint and satisfies $\Lambda(u_p)^2 = \Lambda(u_p) = 1$. Therefore $\Lambda(u_p)$ is self-adjoint and $\Lambda(p) := \frac{1}{2}(1 - \Lambda(1 - 2p))$ is a projection.

Claim. Suppose $p$ and $q$ are projections.

1. If $p$ and $q$ are Murray–von Neumann equivalent, then so are $\Lambda(p)$ and $\Lambda(q)$.
2. If $p$ and $q$ commute, then so do $\Lambda(p)$ and $\Lambda(q)$.
3. If $pq = 0$, then $\Lambda(p)\Lambda(q) = 0$ and $\Lambda(p + q) = \Lambda(p) + \Lambda(q)$.
4. We have $\Lambda((1)) = 1$.
5. If $\sum_{j=m}^{p_j} = 1$ for projections $p_j$, then $\sum_{j=m}^{p_j} \Lambda(p_j) = 1$.

Proof. (1) Since $p$ and $q$ belong to $M_n(\mathbb{C})$, Murray–von Neumann equivalence coincides with the unitary equivalence. If $w$ is a unitary and $wpw^* = q$ then $uwv = u_q$, and $\Lambda(w)\Lambda(u_p)\Lambda(w)^* = \Lambda(u_q)$. Therefore

$$\Lambda(p) = \Lambda(w)\frac{1}{2}(1 - \Lambda(u_p))\Lambda(w)^* = \frac{1}{2}(1 - \Lambda(u_q)) = \Lambda(q).$$

(2) Since $[p, q] = 0$ if and only if $[u_p, u_q] = 0$, apply $\Lambda$ as in the proof of (1).

(3) We have $pq = 0$ if and only if $p + q$ is a projection, if and only if $u_pu_q = u_{p+q}$. We can now apply $\Lambda$ to this equality and obtain both conclusions.

(4) For $p := 1$ we have $u_p = -1$. Since $\Theta$ is unital,

$$\|\Lambda(u_p) + 1\| \approx 2\varepsilon \|\Theta(u_p) + 1\| \approx \varepsilon \|\Theta(-u_p) - 1\| = 0.$$

Since $\Lambda(u_p) = 1 - 2\Lambda(p)$, we have $\|\Lambda(p) - 1\| = \frac{1}{2}\|\Lambda(u_p) + 1\| \leq \frac{3}{2} \varepsilon < 1$. By Lemma 1.5.7, $\Lambda(p)$ and $1$ are unitarily equivalent in $A$, thus $\Lambda(p) = 1$.

(5) This follows immediately from (4) and the additivity claim in (3). \qed

Let $\tau$ be the unique tracial state on $M_n(\mathbb{C})$ Our next task is to prove that $\sigma(A) = \sigma(A(u))$ for every unitary $u$ and every tracial state $\sigma$ of $A$. (Since $A$ is finite-dimensional, it has a tracial state.) By the spectral theorem, $u = \sum_{j=m}^{\lambda_j} \exp(i\lambda_j p_j)$, where $p_j$ are rank-1 projections and $\exp(i\lambda_j)$ are the eigenvalues of $u$ (eigenvalues of multiplicity $n$ are repeated $n$ times). Fix a tracial state $\sigma$ of $A$. Since all $p_j$ are Murray–von Neumann equivalent, all $\Lambda(p_j)$ are Murray–von Neumann equivalent and therefore $\sigma(\Lambda(p_j)) = \sigma(\Lambda(p_{\lambda})).$ Also $\sum_{j<m}^{p_j} \Lambda(p_j) = 1$. Therefore $\sigma(\Lambda(p_j)) = 1/m$ for all $j$. Then (using $\sum_{j<m}^{\Lambda(p_j)} = 1$ in the last equality)

$$\Lambda(u) = \prod_{j=m}^{\exp(i\lambda_j \Lambda(p_j))} = \prod_{j=m}^{\exp(i\lambda_j) \Lambda(p_j)} + 1 = \sum_{j=m}^{\exp(i\lambda_j) \Lambda(p_j)}.$$

Since $\sigma(\Lambda(p_j)) = 1/m = \tau(p_j)$ for all $j$, we have

$$\tau(u) = \frac{1}{m} \sum_{j=m}^{\exp(i\lambda_j) = \sigma(\Lambda(A)).}$$
Since $A$ is finite-dimensional, $\sigma$ can be chosen to be a faithful trace. By Lemma 17.2.4, $\Phi(\sum_{j<4} \lambda_j u_j) := \sum_{j<4} \lambda_j \Lambda(u_j)$ is a well-defined $*$-homomorphism.

Also, $\|\Phi(a) - \Theta(a)\| \leq 4 \sup_{x \in U(M_m(C))} \|\Phi(u) - \Lambda(u)\| \leq 8 \varepsilon$, as required. □

17.3 Liftings of $*$-homomorphisms Between Coronas

In this section we study liftings of $*$-homomorphisms between coronas and outline how the machinery of §17.1 applies to construct many automorphisms of coronas other than the Calkin algebra. Liftings will be used in the proof that OCA$_T$ implies all automorphisms of $\mathcal{D}(H)$ are inner.

**Definition 17.3.1.** A lifting of a $*$-homomorphism $\Phi: \mathcal{D}(A) \to \mathcal{D}(B)$ is a function $\Phi_*: \mathcal{M}(A) \to \mathcal{M}(B)$ such that the following diagram commutes ($\pi_A$ and $\pi_B$ denote the quotient maps).

$$
\begin{array}{ccc}
\mathcal{M}(A) & \xrightarrow{\Phi_*} & \mathcal{M}(B) \\
\pi_A \downarrow & & \downarrow \pi_B \\
\mathcal{D}(A) & \xrightarrow{\Phi} & \mathcal{D}(B)
\end{array}
$$

If this diagram commutes on some $\mathcal{X} \subseteq \mathcal{M}(A)$, then $\Phi_*$ is called a lifting of $\Phi$ on $\mathcal{X}$. When convenient, instead we say that $\Phi$ is a lifting on $\pi[\mathcal{X}]$.

The Axiom of Choice implies that every $*$-homomorphism between coronas has a lifting.\footnote{We are not assuming that $\Phi_*$ has any additional properties—it is not necessarily a $*$-homomorphism, and it is not necessarily measurable with respect to any reasonable $\sigma$-algebra.} Our proof that OCA$_T$ implies all automorphisms of the Calkin algebra are inner (Theorem 17.8.5) can be summarized as a struggle to obtain liftings with better and better properties. For every $\Phi$ we will fix a lifting $\Phi_*$ with the properties guaranteed by the following Lemma.

**Lemma 17.3.2.** Every $*$-homomorphism $\Phi: \mathcal{D}(A) \to \mathcal{D}(B)$ has a lifting $\Phi_*$ such that $\|\Phi_*(a)\| \leq \|a\|$ for all $a$. If $\mathcal{M}(B)$ has real rank zero then we can assure that $\Phi_*(p)$ is a projection for every projection $p$.

**Proof.** This is the question of the existence of lifts. The first property can be secured by Lemma 2.5.4 (2). For the second, use Lemma 3.1.13. □

**Definition 17.3.3.** A $*$-homomorphism $\Phi$ between coronas of separable C$^*$-algebras is said to be topologically trivial if it has a lifting which is Borel-measurable with respect to the strict topology (this is a Polish topology, see Lemma 13.1.8).

This definition is more natural than it may appear (see Exercise 17.9.7). We have the following.
Proposition 17.3.4. If a \( \ast \)-homomorphism between coronas can be lifted by a \( \ast \)-homomorphism between the corresponding multiplier algebras, then it is topologically trivial. In particular, all inner automorphisms are topologically trivial.

Example 17.3.5. There is a separable abelian \( \mathrm{C}^\ast \)-algebra \( A \) such that \( \mathcal{D}(A) \) has a topologically trivial automorphism that cannot be lifted by a \( \ast \)-homomorphism. Let \( X \) be a connected space obtained by adding a compact space to \( \mathbb{R} \) so that \( X \) does not have an orientation-reversing homeomorphism. (One can for example attach two nontrivial and non-homeomorphic connected compact spaces to \( \mathbb{R} \) at 0 and at 1.) Then \( \mathcal{D}(C_0(X)) \) is isomorphic to \( \mathcal{D}(C_0(\mathbb{R})) \), but the orientation-reversing automorphism of \( \mathcal{D}(C_0(X)) \) cannot be lifted by a \( \ast \)-homomorphism. It is not difficult to see that this automorphism is nevertheless topologically trivial.

The reader may want to verify that the inner automorphism of \( \mathcal{D}(H) \) implemented by the unilateral shift cannot be lifted by a \( \ast \)-homomorphism, but its inverse can.

Lemma 17.3.6. If \( A \) is a separable, nonunital \( \mathrm{C}^\ast \)-algebra then \( \mathcal{D}(A) \) has at most \( \mathfrak{c} \) topologically trivial automorphisms.

Proof. Since \( \mathcal{M}(A) \) is a Polish space (Lemma 13.1.8), there are only \( \mathfrak{c} \) Borel-measurable functions from \( \mathcal{M}(A) \) to itself (Example 8.1.3 (6)).

Together with Theorem 17.1.15, this gives the following.

Corollary 17.3.7. If the Continuum Hypothesis holds and \( A \) is a separable, stable, nonunital \( \mathrm{C}^\ast \)-algebra then \( \mathcal{D}(A) \) has topologically nontrivial automorphisms.

17.4 Aaçai, I. Discretizations and Liftings of Product Type

In sections \( \S 17.4\)–\( \S 17.8 \) we prove that OCA\( _T \) (see \( \S 8.6 \)) implies all automorphisms of the Calkin algebra are inner (Theorem 17.8.5). These sections heavily rely on the material from \( \S 8.6, \S 9.5, \S 9.7, \S 9.9 \), and the earlier parts of Chapter 17. The proof is presented in the order that I believe makes it easiest to digest. Aaçai stands for ‘All automorphisms of the Calkin algebra are inner.’ Forgive me, Gert Pedersen, wherever you are.

In this section we introduce ‘discretizations’ of \( \mathcal{D}[E] \), pass from continuous liftings to liftings of product type, and make connection with Ulam-stability.

Definition 17.4.1. Throughout \( \S 17.4\)–\( \S 17.8 \) we fix a separable Hilbert space \( H \) with an orthonormal basis \( (\zeta_n) \). If \( E \in \text{Part}_\mathbb{N} \) and \( X \subseteq \mathbb{N} \) then \( \rho^E_X := \text{proj}_{\text{span}\{\xi_\ell: \ell \in \bigcup_{n \in X} E_n\}} \).

If \( \Phi \) is an endomorphism of \( \mathcal{D}(H) \) we write

\[\text{proj}_{\text{span}\{\xi_\ell: \ell \in \bigcup_{n \in X} E_n\}} \]
Lemma 17.4.3.

Proof. If \( \varepsilon \geq 0 \), defined by \( x \mapsto D_X \) for concreteness, we fix a discretization of \( H \) topology, then the subspace topology on \( D \) and \( D \) and \( D \) of \( H \) contains both 0 and 1 and that \( D \) and \( D \) is \( \varepsilon \)-dense in \( U(A(n)) \). For \( a \in D \) let \( \text{supp}(a) := \{ n : a(n) \neq 0 \} \) and identify \( D_X[E] \) with \( \{ a \in D[E] : \text{supp}(a) \subseteq X \} \).

We omit \( E \) whenever it is clear from the context and write \( p_X \) and \( q_X \) in place of \( p_X^E \) and \( q_X^E \).

Definition 17.4.2. Let \( A(n) := D_{\{n\}}[E] \). Then \( A(n) \cong M_m(\mathbb{C}) \) with \( m = |E_n| \). Let \( D(n) \) be a finite, \( 2^{-n} \)-dense,\(^8\) subset of the unit ball of \( A(n) \). Fix an infinite \( X \subseteq \mathbb{N} \) and let \( D := \prod_n D(n) \) and \( D_X[E] := \prod_{n \in X} D(m) \). Then \( D[E] \) is a discretization of \( D[E] \) and \( D_X[E] \) is a discretization of \( D_X[E] \). It will be convenient to assume that \( D(n) \) contains both 0 and 1 and that \( D(n) \cap U(A(n)) \) is \( 2^{-n} \)-dense in \( U(A(n)) \). For \( a \in D \) let \( \text{supp}(a) := \{ n : a(n) \neq 0 \} \) and identify \( D_X[E] \) with \( \{ a \in D[E] : \text{supp}(a) \subseteq X \} \).

We omit \( E \) whenever it is clear from the context and write \( D \) and \( D_X \) in place of \( D[E] \) and \( D_X[E] \).

When \( \prod_{n \in \mathbb{N}} A(n) \) is identified with \( D[E] \) and considered with the weak operator topology, then the subspace topology on \( D[E] \) is compact and it agrees with the product topology, compatible with the metric \( d(a, b) := 1 / (1 + \min\{ n : a_n \neq b_n \}) \). Also, \( D[E] \) inherits the algebraic operations from \( D[E] \). While \( D[E] \) is not closed under these operations, if \( a \) and \( b \) in \( D[E] \) are disjointly supported, then \( a + b \in D[E] \).

For concreteness, we fix a discretization \( D[E] \) of each \( E \in \text{Part}_\mathbb{N} \). By Lemma 17.4.7, the actual choice of \( \mathcal{D}[E] \) is inconsequential.

Lemma 17.4.3. If \( H \) is a separable Hilbert space then the relation \( \approx_{\varepsilon}^{\mathcal{H}} \) on \( \mathcal{B}(H)_{\leq 1} \) defined by \( x \approx_{\varepsilon}^{\mathcal{H}} y \) if \( \| \pi(x - y) \| \leq \varepsilon \) is Borel in the weak operator topology for \( \varepsilon \geq 0 \).

Proof. If \( \{ e_n \} \) is an approximate unit of \( \mathcal{K}(H) \) and \( \{ \eta_m \} \) is dense in the unit sphere of \( H \), then \( x \approx_{\varepsilon}^{\mathcal{H}} y \) if and only if \( \inf_n \sup_m \| (1 - e_n)(x - y)(\eta_m) \| \leq \varepsilon \).

Definition 17.4.4. A function \( \Theta : D_X[E] \to \mathcal{B}(H)_{\leq 1} \) is an \( \varepsilon \)-approximation of \( \Phi \) on \( D_X \) if \( \Theta(a) \approx_{\varepsilon}^{\mathcal{H}} \Phi(a) \) for all \( a \in D_X \).

A function between Polish spaces is \( C \)-measurable if it is measurable with respect to the \( \sigma \)-algebra generated by analytic sets (see §B.2.2).

Lemma 17.4.5. If \( E \in \text{Part}_\mathbb{N} \) and an endomorphism of \( \mathcal{D}(H) \) has a \( C \)-measurable \( \varepsilon \)-approximation on \( D[E] \) for every \( \varepsilon > 0 \), then it has a continuous lifting on \( D_Y[E] \) for some infinite \( Y \subseteq \mathbb{N} \).

Proof. Let \( \Phi \) be an endomorphism of \( \mathcal{D}(H) \) and fix a \( C \)-measurable \( 1/d \)-approximation \( \Theta_d \) on \( D[E] \) for every \( d \geq 1 \). Since \( C \)-measurable functions are Baire-measurable (see §B.2.1), Corollary 9.9.2 implies that there are an infinite \( X \subseteq \mathbb{N} \)

\(^8\) A subset of a metric space is \( \varepsilon \)-dense if the open \( \varepsilon \)-balls around its elements cover the space.
and \( b \in D_{N}X[E] \) such that for every \( d \geq 1 \) the function \( a \mapsto \Theta_{d}(a + b) \) is continuous on \( D_{X}[E] \). Then the function on \( D_{X}[E] \) defined by \( \Theta_{d}^{1}(a) := \Theta_{d}(a + b)q_{X}^{E} \) is continuous. Also,

\[
\Theta_{d}^{1}(a) = \Theta_{d}(a + b)q_{X}^{E} \approx_{1/d} \Phi_{x}(a + b)q_{X}^{E} \approx_{X} \Phi_{x}(a)
\]

hence \( \Theta_{d}^{1} \) is a 1/d-approximation of \( \Phi_{x} \) on \( D_{X}[E] \). Lemma 17.4.3 implies that the set

\[
\mathcal{W}_{d} := \{(a, b) \in D_{X}[E] \times \mathcal{B}(H)_{\leq 1} : \Theta_{d}^{1}(a) \approx_{1/d} b \}
\]

is Borel for every \( d \geq 1 \), and therefore \( \mathcal{W} := \bigcap_{d \geq 1} \mathcal{W}_{d} \) is Borel as well. By Theorem B.2.13, \( \mathcal{W} \) has a C-measurable uniformization, \( \Theta_{2}^{3} \). For every \((a, b) \in \mathcal{W} \) we have \( \Phi_{x}(a) - b \in \mathcal{X}(H) \), hence this uniformization is a lifting of \( \Phi_{x} \) on \( D_{X}[E] \). By Corollary 9.9.2, there exist an infinite \( Y \subseteq X \) and \( c \in D_{N}[Y][E] \) such that the function on \( D_{Y}[E] \) defined by

\[
\Theta^{3}(a) := \Theta^{2}(a + c)q_{Y}^{E} \approx_{X} \Phi_{x}(a),
\]

is continuous. For every \( d \geq 1 \) we have \( \Theta^{3}(a) \approx_{d} \Phi_{x}(a + c)q_{Y}^{E} \approx_{X} \Phi_{x}(a) \), and therefore \( \Theta^{3} \) is a continuous lifting of \( \Phi_{x} \) on \( D_{Y}[E] \).

In the following, \( (D(n)) \) is any sequence of finite sets and \( D := \prod_{n}D(n) \).

**Definition 17.4.6.** A function \( \Xi : D \to \mathcal{B}(H)_{\leq 1} \) is of a product type if there are orthogonal projections \( r_{n} \in \mathcal{B}(H) \) and \( \Xi_{n} : D(n) \to r_{n}(\mathcal{B}(H)_{\leq 1})r_{n} \) for \( n \in \mathbb{N} \) such that \( \Xi(a) = \sum_{n} \Xi_{n}(a_{n}) \) for all \( a \in D \).

**Lemma 17.4.7.** Suppose \( E \in \text{Part}_{N} \) and that \( D[E] \) and \( D'[E] \) are two discretizations of \( D[E] \).

1. There exists a continuous function of product type \( \Theta : D[E] \to D'[E] \) such that \( x - \Theta(x) \in \mathcal{X}(H) \) for all \( x \).
2. If \( E \in \text{Part}_{N} \) and \( \Phi \) is an endomorphism of \( D[H] \), then \( \Phi \) has a continuous lifting on some discretization of \( D[E] \) if and only if it has a continuous lifting on every discretization of \( D[E] \).

**Proof.** (1) Fix a linear ordering \( \prec_{n} \) of each \( D_{n} \). Define \( \Theta_{n} : D'[E] \to D(n) \) by \( \Theta_{n}(x) := y \) if \( ||x - y|| < 2^{-n} \) and \( ||x - z|| \geq 2^{-n} \) for all \( z \in D(n) \) such that \( z \prec_{n} y \). Since \( D(n) \) is finite and \( 2^{-n} \)-dense in \( A(n) \), such \( y \) exists and \( \Theta_{n} \) is well-defined. Identify \( x \in D'[E] \) with \( (x_{a}) \in \prod_{n}D_{n}' \) and define \( \Theta : D'[E] \to D[E] \) by \( \Theta(x) := \sum_{n} \Theta_{n}(x_{a}) \).

Then \( \Theta \) is continuous and \( x - \Theta(x) \in \mathcal{X}(H) \) for all \( x \).

To see that \( \Phi \) is of product type, for \( n \in \mathbb{N} \) let \( r_{n} \) be the unit of \( A(n) \).

(2) Compose the lifting with \( \Theta \) provided by (1).

**Lemma 17.4.8.** If \( \Phi \) is an endomorphism of the Calkin algebra which has a continuous lifting \( \Theta \) on \( D[E] \) for some \( E \in \text{Part}_{N} \), then it has a lifting of product type on \( D_{X}[E] \) for some infinite \( X \subseteq \mathbb{N} \).

---

9 We have \( ||\Xi_{n}(a_{n})|| \leq 1 \) for all \( n \) and that \( \Xi_{n}(a_{n}) \) are pairwise orthogonal. Therefore the series on the right-hand side of the displayed formula converges strongly to an element of \( \mathcal{B}(H)_{\leq 1} \).
Proof. We recursively find an increasing sequence \((n(j))_j, s(j) \in D_{(m(j),n(j+1))}\) (with \(n(0) := 0\)), and an increasing sequence of finite-rank projections \((r_j)_j\) so that the following holds for all \(j, a \text{ and } b \in D_{(0,m(j))}\), and all \(c \text{ and } d \in D_{(n(j+1),\infty)}\):

1. \(\|\Theta(a + s(j) + c) - \Theta(b + s(j) + c)(1 - r_j)\| < 2^{-j}\),
2. \(\|[(1 - r_j)(\Theta(a + s(j) + c) - \Theta(b + s(j) + c))] < 2^{-j}\),
3. \(\|r_j(\Theta(a + s(j) + c) - \Theta(b + s(j) + d))\| < 2^{-j}\).

Set \(n(0) := 0\). Suppose that \(n(j), s(j), \text{ and } r_j, \text{ for } j < k\), have been chosen to satisfy (5)–(8). Assume that \(n(k) \text{ and } s(k) \text{ that satisfy (5)}\) cannot be found. We will recursively find an increasing sequence \(m(i) \in \mathbb{N}\) and \(t(i) \in D_{(m(i),m(i+1))}\), so that \(m(0) = n(k-1) + 1\), and for every \(i\) there are \(a(i) \text{ and } b(i) \in D_{(0,m(i-1))}\) and \(c \in D_{(m(i+1),\infty)}\) such that \(p_i := \text{proj}_{\Xi_i}([\xi_i < l]}\) (with \(\xi_i \text{ as in Definition 17.4.1})\) satisfies

\[
\|[(\Theta(a(i) + \sum_{l \leq i} t(l)) + c) - \Theta(b(i) + \sum_{l \leq i} t(l)) + c)](1 - p_i)\| \geq 2^{-k+1}.
\]

Our assumption implies that such \(a(i), b(i), \text{ and } c \text{ exist for every choice of } t(l) \text{ and } m(l'), \text{ for } l \leq i \text{ and } l' \leq i + 1\). By the continuity of \(\Theta\), there is a large enough \(m(2)\) such that with \(t(i+1) := c \cap m(i+2)\), for all \(d \in D_{(m(l+1),\infty)}\), we have

\[
\|[(\Theta(a(i) + \sum_{l \leq i+1} t(l)) + d) - \Theta(b(i) + \sum_{l \leq i+1} t(l+1) + d)](1 - p_i)\| \geq 2^{-k+1}.
\]

Since there are only finitely many choices for \(a(i), b(i)\), some pair \(\bar{a}, \bar{b}\) appears as \(a(i), b(i)\) infinitely often. Let \(l := \sum_i t(l)\) (the partial sums converge to an element of \(D\)). Then \(\|[(\Theta(\bar{a} + t) - \Theta(\bar{b} + t))(1 - p_i)\| \geq 2^{-k+1}\) for all \(i\), and \(\Theta(\bar{a} + t) - \Theta(\bar{b} + t)\) is not compact. But \((\bar{a} + t) - (\bar{b} + t)\) has finite rank. This contradicts the assumption that \(\Theta\) lifts \(\Phi\).

We can therefore choose \(\tilde{n}^0(k), \tilde{n}^0(k) \in D_{(n(k),n'(k))}\), and \(r^0_k\) such that (5) holds. An analogous argument shows that we can find \(n^1(k) > n^0(k), s^1(k) \in D_{(n(k),n'(k))}\) extending \(\tilde{n}^0(k)\), and \(r_k\) such that (6) holds. This extension does not affect the condition (5).

Since each one of \(x \mapsto \Theta(x)r_k \text{ and } x \mapsto r_k \Theta(x)\) is a continuous function with a compact range, we can find \(n(k) \geq n^1(k)\) and \(s(k) \geq s^1(k)\) to produce a basic open set \([(n(k-1), n(k)), s(k)]\) such that both (7) and (8) hold for all \(a, c, \text{ and } d\). This describes the recursive construction.

Let \(X := \{n(j) : j \in \mathbb{N}\}\). The sum \(\sum_j s(j)\) converges to an element of \(D_{\mathbb{N}\setminus X}\).

Define \(\tilde{\Xi}_j : D_{n(j)} \rightarrow (r_{j+1} - r_j)\mathcal{B}(H)_{\leq 1}(r_{j+1} - r_j)\) by

\[
\tilde{\Xi}_j(x) := (r_{j+1} - r_j)\Theta(s + x)(r_{j+1} - r_j).
\]

The function of product type determined by \((\tilde{\Xi}_j)\) is not necessarily a lifting on \(D_X\). Letting \(\Xi^0(a) := q_X \tilde{\Xi}(a)q_X\) we obtain a function \(\Xi^0(a) := \sum_j \Xi^0(a(n(j)))\) that is a lifting on \(D_X\), but not obviously of product type.

Since \(r_j\), for \(j \in \mathbb{N}\), form an approximate unit for \(\mathcal{X}(H)\), there exists an infinite set \(Y \subseteq \mathbb{N}\) such that for all \(i < j \in Y\) we have \(\|r_{i+1}q_X(1 - r_j)\| < 2^{-i-j}\). For \(j \in Y\) let
are isomorphic to $M_n$. For $\Theta$ defines an isomorphism from $D_Y$ shows that $\xi$ is a projection and $\Theta(q)$ satisfies a moment fix $m$ we have $\xi(q) = \xi(q(i))$. Then $\xi$ is an isometry. For a moment fix $m \in Y$. Then $q_m := \text{proj}_{(\xi(q(i)) \in E_m)}$

is a projection and $q_m \preceq p_{(m)}^{E | (n,m)}$. Both corners $p_{(m)}^{E | (n,m)}$ and $q_{(m)}^{E | (n,m)}$ are isomorphic to $M_l(C)$, where $l := |F_m|$. A computation of the images of $\xi(q(i))$ shows that $v^*p_{(m)}^{E | (n,m)}v = q_m v = p_{(m)}^{F | (n,m)}$ and $v p_{(m)}^{F | (n,m)}v^* = q_m$. Hence $q := \sum_{i \in Y} q_i$ satisfies $q v^* = q$.

For $a \in R_Y[F]$ we have (with the SOT-convergent sum identified with its limit)

$$v^* a v = \sum_{i \in Y} v p_{(k)}^{F | (n,m)} a v = \sum_{i \in Y} v p_{(k)}^{F | (n,m)} a v^* \in q R_X[E] q,$$

and since $v$ is an isometry, $v^* (v a v^*) v = a$. Therefore $a \mapsto v a v^*$ is an isomorphism from $R_Y[F]$ into a von Neumann subalgebra of $R_X[E] q$. Since this $^*$-homomorphism sends the central projection $p_{(m)}^{F | (n,m)}$ of the domain to the central projection $q_m$ of the range, and since it is an isomorphism between the corresponding corners, it is an isomorphism.

### 17.5 Acai, II. The Isometry Trick

What if hokey pokey is what it’s all about?

Anonymous

In this section we introduce the ‘isometry trick’ that will be used to cut several corners in the proof of Theorem 17.8.5.

We will use the notation from Definition 17.4.1. Thus $\xi(q(i))$ is an orthonormal basis for $H$. If $g : \mathbb{N} \to \mathbb{N}$ is an injection, then $v(\xi(q(i))) := \xi(q(i))$ defines an isometry on $H$. An isometry of this form is, locally to this chapter, called a injection isometry.

**Lemma 17.5.1.** Suppose $E$ and $F$ are in Part $\mathbb{N}$, $X$ and $Y$ are infinite subsets of $\mathbb{N}$, and $\lim_{n \in X} |E_n| = \infty$. Then there exist a permutation isometry $v$ such that $a \mapsto v a v^*$ defines an isomorphism from $R_Y[F]$ onto $v v^* R_X[E] v v^*$.

**Proof.** For $m \in Y$ choose $n(m) \in X$ such that $|E_n(m)| \geq F_m$ and $m \neq m'$ implies $n(m) \neq n(m')$. Let $g : \mathbb{N} \to \mathbb{N}$ be an injection such that such that for all $m \in Y$ and all $i \in F_m$ we have $g(i) \in E_n(m)$. Define $v$ by $v(\xi(q(i))) := \xi(q(i))$. Then $v$ is an isometry. For a moment fix $m \in Y$. Then

$$q_m := \text{proj}_{(\xi(q(i)) \in E_m)}$$

is a projection and $q_m \preceq p_{(m)}^{E | (n,m)}$. Both corners $p_{(m)}^{E | (n,m)}$ and $q_{(m)}^{E | (n,m)}$ are isomorphic to $M_l(C)$, where $l := |F_m|$. A computation of the images of $\xi(q(i))$ shows that $v^* p_{(m)}^{E | (n,m)} v = v^* q_m v = p_{(m)}^{F | (n,m)}$ and $v p_{(m)}^{F | (n,m)} v^* = q_m$. Hence $q := \sum_{i \in Y} q_i$ satisfies $v q v^* = q$.

For $a \in R_Y[F]$ we have (with the SOT-convergent sum identified with its limit)

$$v a v = v \sum_{k \in X} p_{(k)}^{F | (n,m)} a v = \sum_{k \in Y} v p_{(k)}^{F | (n,m)} a v^* \in q R_X[E] q,$$

and since $v$ is an isometry, $v^* (v a v) v = a$. Therefore $a \mapsto v a v^*$ is an isomorphism from $R_Y[F]$ into a von Neumann subalgebra of $q R_X[E] q$. Since this $^*$-homomorphism sends the central projection $p_{(m)}^{F | (n,m)}$ of the domain to the central projection $q_m$ of the range, and since it is an isomorphism between the corresponding corners, it is an isomorphism.
Lemma 17.5.2. Suppose $\Phi : \mathcal{D}(A) \rightarrow \mathcal{D}(B)$ is a $^\ast$-homomorphism between coronas of nonunital $C^\ast$-algebras, $\mathcal{X} \subseteq \mathcal{M}(A)$, $v$ is an isometry in $\mathcal{M}(A)$, and $\Gamma$ is a lifting of $\Phi$ on $\mathcal{V}$. Then $b \mapsto \Phi_s(v)^\ast \Gamma(vbv^\ast) \Phi_s(v)$ is a lifting of $\Phi$ on $\mathcal{X}$.

Proof. If $b \in \mathcal{X}$ then since $v$ is an isometry we have $b = v^\ast bv^\ast v$ and therefore $\Phi(\pi(b)) = \Phi(\pi(v^\ast)) \Phi(\pi(vbv^\ast)) \Phi(\pi(v))$ is lifted by $\Phi_s(v)^\ast \Gamma(vbv^\ast) \Phi_s(v)$.

The isometry trick is a simple idea used in Lemma 17.5.3 and elsewhere. The reader is spared from a laundry list of all of its variants. A lifting of product type on $\mathcal{D}[E]$ is a lifting on $\mathcal{D}[E]$ which happens to be of product type (Definition 17.4.6).

Lemma 17.5.3. Suppose $E$ and $F$ are in Part$_n$, $X \subseteq \mathcal{N}$, $v \in \mathcal{D}(H)$ is an injection isometry such that $a \mapsto vav^\ast$ defines an isomorphism from $\mathcal{D}[F]$ onto $vv^\ast \mathcal{D}[X]vv^\ast$, and $\Phi$ is an endomorphism of $\mathcal{D}(H)$.

1. If the restriction of $\Phi$ to $\mathcal{D}[X]E$ is implemented by $w$, then the restriction of $\Phi$ to $\mathcal{D}[F]$ is implemented by any lift of $\Phi(\pi(v^\ast)) \pi(wv)$.
2. If $\Phi$ has a lifting of product type on $\mathcal{D}[X]E$ and $v$ then it has a lifting of product type on $\mathcal{D}[F]$.
3. If $\Theta$ is a $C$-measurable $\varepsilon$-approximation of $\Phi$ on $\mathcal{D}[X]E$ then
   
   $a \mapsto \Phi_s(v^\ast) \Theta(vav^\ast) \Phi_s(v)$

   is a $C$-measurable $\varepsilon$-approximation of $\Phi$ on $\mathcal{D}[F]$.

Proof. (1) As in the proof of Lemma 17.5.2, if $b \in \mathcal{D}[F]$ then $b = v^\ast bv^\ast v$ and we have

   $\Phi(\pi(b)) = \Phi(\pi(v^\ast)) \Phi(\pi(vbv^\ast)) \Phi(\pi(v))$

   
   

   $= \Phi(\pi(v^\ast)) \pi(wv) \pi(b) \pi(v^\ast w^\ast) \Phi(\pi(v))$

   and any lift of $\Phi(\pi(v^\ast)) \pi(wv)$ implements $\Phi$ on $\mathcal{D}[F]$.

   (2) Let $q := vv^\ast$. Since $a \mapsto vav^\ast$ is an isomorphism of product type from $\mathcal{D}[F]$ onto $q\mathcal{D}[X]E[q]$, $b \mapsto vbv$ is its inverse. Since $v$ is an injection isometry, $v^\ast D_X[E]v$ is a discretization of $\mathcal{D}[F]$. By Lemma 17.4.7 there exists a continuous function $\Theta : \mathcal{D}[F] \rightarrow v^\ast D_X[E]v$ of product type such that $a - \Theta(a) \in \mathcal{N}$ for all $a \in \mathcal{D}[F]$. If $\Sigma$ is a lifting of product type of $\Phi$ on $\mathcal{D}[X]E$, so is its restriction to $v^\ast D_X[E]v$ and $a \mapsto \Phi_s(v^\ast) \Sigma(\Theta(vav^\ast)) \Phi_s(v)$ is a lifting of product type of $\Phi$ on $\mathcal{D}[F]$, as required.

   (3) Since the composition of a continuous function with a $C$-measurable function is $C$-measurable, this is immediate.

We temporarily trade discretizations for $^\ast$-$^\ast$-homomorphisms. Recall that the domain of a (unital) $^\ast$-$^\ast$-homomorphism is the unit ball of a $C^\ast$-algebra (Definition 17.2.5).

Proposition 17.5.4. Suppose that $\Phi$ is an endomorphism of the Calkin algebra which has a continuous lifting $\Theta$ on $\mathcal{D}[E]$ for some $E \in$ Part$_n$ such that $\lim_n |E_n| = \infty$. Then for every $F \in$ Part$_n$, $\Phi$ has a lifting $(\Theta'_E)$ of product type on $\mathcal{D}[F]$ such that each
\( \Theta' \) is a unital, Borel-measurable, \( \varepsilon_n^{*-}\)-homomorphism on \( \mathcal{D}[F] \) for some sequence \( (\varepsilon_n) \) converging to 0.

**Proof.** By Lemma 17.4.8 followed by Lemma 17.5.3 (2), \( \Phi \) has a lifting \( (\Xi_n) \) of product type on \( D[E] \).

Let \( a_n := \Xi_n(p_{n|n}) \) for \( n \in \mathbb{N} \). For every \( X \subseteq \mathbb{N} \), \( \sum_{n \in X} p_{n|n} \) is a projection and \( \sum_{n \in X} a_n \) lifts its \( \Phi \)-image. The latter is therefore equal to a projection modulo the compacts, and since \( a_n \) belong to to orthogonal full matrix subalgebras of \( \mathcal{B}(H) \), we have \( \|a_n - a'_n\| \to 0 \) and \( \|a^2_n - a_n\| \to 0 \) as \( n \to \infty \). By Exercise 2.8.10 (2), there are projections \( q_n \) such that \( \|a_n - q_n\| \to 0 \). The ranges of the \( \Xi_n \)'s belong to orthogonal full matrix subalgebras of \( \mathcal{B}(H) \), and by carrying out this argument in these algebras the projections \( q_n \) can be chosen to be orthogonal.

As in Definition 17.4.2, let \( A(n) := \mathcal{D}[n] \cdot E \). If \( u_n \in A(n) \) is a unitary and \( b_n := \Xi_n(u_n) \), then \( \|b_n b_n^* - 1_{A(n)}\| \to 0 \) and \( \|b_n^* b_n - 1_{A(n)}\| \to 0 \) as \( n \to \infty \). By Exercise 2.8.10 (5), there is a unitary \( v(u_n) \) in \( p_n \mathcal{B}(H) p_n \) such that \( \|b_n - v(u_n)\| \to 0 \) as \( n \to \infty \) for any choice of \( (u_n) \).

We can therefore modify \( (\Xi_n) \) and obtain a lifting \( (\hat{\Xi}_n) \) of \( \Phi \) of product type on \( \mathcal{D}[E] \) such that \( \|\hat{\Xi}_n - \Xi_n\| \to 0 \), \( q_n = \hat{\Xi}_n(p_{n|n}) \) are orthogonal projections, and \( \hat{\Xi}_n \) sends unitaries of \( A(n) \) to unitaries of \( q_n \mathcal{B}(H) q_n \).

Recall that the unitaries in \( D(n) \) are \( 2^{-n} \)-dense in the unitaries of \( A(n) \). Let \( x_0, \ldots, x_{m-1} \) be an enumeration of \( D(n) \) in which \( x_0 = 1 \) and the unitaries in the list are listed first. Define \( \Theta'_n(a) := \Theta'_m(x_j) \), where \( j \) is the smallest index such that \( \|a - x_j\| < 2^{-n} \) and \( x_j \) is a unitary if \( a \) is. Then \( \Theta'_n \) sends the unit to the unit and the unitaries to unitaries, hence it is unital in the sense of Theorem 17.2.6. The range of \( \Theta'_n \) is finite and the preimage of every point is a difference of open balls, hence it is Borel-measurable. Therefore \( \Theta' \) is Borel-measurable and it sends the unit and the unitaries to the right spot.

We now check that for every \( \varepsilon > 0 \), \( \Theta'_n \) is an \( \varepsilon^{*-}\)-homomorphism for all large enough \( n \). If \( \limsup_n \max_{a,b \in A(n)} \|\Theta'_n(ab) - \Theta'_n(a)\Theta'_n(b)\| > 0 \), then there are \( \epsilon > 0 \), and a pair \((a,n),(b,n)\) in \( A(n) \) such that \( \|\Theta'(a(n)b(n)) - \Theta'(a(n))\Theta'(b(n))\| \geq \varepsilon \) for all \( n \) in some infinite \( X \subseteq \mathbb{N} \). With \( a := \sum_{n \in X} a(n) \) and \( b := \sum_{n \in X} b(n) \) (these operators are well-defined, since all elements of every \( A(n) \) have norm at most 1) the difference \( \Theta'(ab) - \Theta'(a)\Theta'(b) \) is not compact; contradiction.

Analogous proofs show that each of \( \max_{a,b \in A(n)} \|\Theta'_n(a + b) - \Theta'_n(a) - \Theta'_n(b)\| \), \( \max_{a \in A(n), \lambda \in \mathbb{C}, |\lambda| \leq 1} \|\Theta'_n(\lambda a) - \lambda \Theta'_n(a)\| \) and \( \max_{a \in A(n)} \|\Theta'_n(a^*) - \Theta'_n(a)^*\| \) converges to 0 as \( n \to \infty \). Therefore for every \( \varepsilon > 0 \) and all large \( n \), \( \Theta'_n \) satisfies the requirements from the definition of a unital \( \varepsilon^{*-}\)-homomorphism.

By Lemma 17.5.3 (2), this lifting can be transferred to \( \mathcal{D}[F] \) for any \( F \in \text{Part}_{\mathbb{N}} \) while preserving its relevant properties. \( \square \)

Here is the reason for the presence of §17.2 in this text.

**Proposition 17.5.5.** Suppose that \( \Phi \) is an endomorphism of the Calkin algebra which has a continuous lifting on \( D[E] \) for some \( E \in \text{Part}_{\mathbb{N}} \) such that \( \lim_n \|E_n\| = \infty \). Then for every \( F \in \text{Part}_{\mathbb{N}} \), \( \Phi \) has a lifting on \( \mathcal{D}[F] \) which is a \( ^*\)-homomorphism.
Proof. Fix $F$. By Proposition 17.5.4, $\Phi$ has a lifting of product type on $\mathcal{D}[F]$ whose $n$th component $\Xi_n$ is a unital $1/n^*$-homomorphism. By Theorem 17.2.6, there exists a unital $^*$-homomorphism

$$\Xi_n' : \mathcal{D}[n] [F] \to \Xi'_n(p_{[n]} F) \mathcal{B}(H) \Xi'_n(p_{[n]} F)$$

such that $\|\Xi_n - \Xi'_n\| \to 0$ and $\Xi'_n(p_{[n]} F)$ are orthogonal projections for $n \in \mathbb{N}$. Therefore $a \mapsto \sum_n \Xi'_n(p_{[n]} a p_{[n]} F)$ is a $^*$-homomorphism and a lifting of $\Phi$ on $\mathcal{D}[F]$. \hfill $\square$

We haven’t used the assumption that $\Phi$ was an automorphism... yet.

Let $r_n := \text{proj}_{C_{b_n}}$, (where $(\xi_j)$ is a fixed basis of $H$ as in Definition 17.4.1).

**Lemma 17.5.6.** Suppose $\Phi : \ell_\infty \to \mathcal{B}(H)$ is a lifting of a $^*$-homomorphism from $\ell_\infty/c_0$ into $\mathcal{D}(H)$ whose range is a masa. Then $\Phi(r_n)$ is a projection of rank at most 1 for all but finitely many $n$, and $1 - \sum_n \Phi(r_n)$ has finite rank.

**Proof.** Assume otherwise. Then for infinitely many $n$ there exists a nonzero projection $q_n \leq \Phi(r_n)$ such that $\Phi(r_n) - q_n$ is nonzero. The sum of all $q_n$ is a projection that commutes with the image of $\ell_\infty/c_0$ but does not belong to it, contradicting the assumption that the image of $\ell_\infty/c_0$ was a masa. Since $\Phi(r_n)$ are orthogonal, $q := \sum_n \Phi(r_n)$ is a projection. If $1 - q$ is not compact, then the annihilator of the image of $\ell_\infty/c_0$ is nontrivial, again contradicting the assumption that the image of $\ell_\infty/c_0$ was a masa. \hfill $\square$

**Proposition 17.5.7.** Suppose that $\Phi$ is an automorphism of the Calkin algebra which has a continuous lifting $\Theta$ on $\mathcal{D}[E]$ for some $E \in \text{Part}_\mathbb{N}$ such that $\lim_n |E_n| = \infty$. Then for every $F \in \text{Part}_\mathbb{N}$, $\Phi$ has a lifting on $\mathcal{D}[F]$ of the form $\text{Ad}_u$ for some partial isometry $u$.

**Proof.** Proposition 17.5.5 implies that $\Phi$ has a lifting ($\Xi_n$) which is a $^*$-homomorphism, and a $^*$-homomorphism is automatically of product type. Since $\Phi$ is an automorphism, it sends masas to masas. By Lemma 17.5.6 each $\Xi_n'$ is an isomorphism between corners of $\mathcal{B}(H)$, and it is therefore implemented by a partial isometry $v_n$. Since the $v_n$’s have orthogonal range projections and orthogonal domain projections, $\sum v_n$ strongly converges to a partial isometry $v$. Clearly Ad$_v$ agrees with $\Xi$ on $\mathcal{D}[E]$, and $\Theta$ is implemented by $v$ on $\mathcal{D}[E]$. \hfill $\square$

### 17.6 Aaćai, III. Open Colourings and $\sigma$-narrow Approximations

The reader may wonder how this proof works.

Barry Johnson ([140])

In this section we (at last) apply OCA$_T$ to produce a $\sigma$-narrow $1/d$-approximation to a given automorphism of $\mathcal{D}(H)$ on a discretization $D_X[E]$. 
Welcome to the middle of the proof that OCA\(_T\) implies all automorphisms of \(\mathcal{D}(H)\) are inner! The story so far: Assuming \(\Phi \in \operatorname{Aut}(\mathcal{D}(H))\) and that for some \(E \in \operatorname{Part}_N\) and \(X \subseteq \mathbb{N}\) we can produce \(C\)-measurable \(\varepsilon\)-approximations of \(\Phi\) on \(D_X[E]\) for all \(\varepsilon > 0\), in §17.4 we produced a lifting of product type on \(D_Y[E]\) for some infinite \(Y \subseteq X\). In §17.5 we saw how to transfer essentially any kind of a lifting of \(\Phi\) on \(D_Y[E]\) to a lifting of the same kind on \(D[F]\) for any \(E\) and \(F\) in \(\operatorname{Part}_N\), as long as \(\sup_{n \in Y} |E_n| = \infty\).

All we need is to produce such \(E, X\), and a reasonable lifting of \(\Phi\) on \(D_X[E]\). Lemma 17.6.3 and Lemma 17.7.1 are the places where the magic happens. The former lemma is one of the two places in the proof of Theorem 17.8.5 where OCA\(_T\) is used. The applications of OCA\(_T\) bookend the proof: Lemma 17.6.3 is really the first step (held back until now), and the second application of OCA\(_T\), Corollary 17.8.4, completes the proof. Each time, OCA\(_T\) is used \(\aleph_0\) times.

Recall that for \(a\) and \(b\) in \(\mathcal{B}(H)\) and \(\varepsilon > 0\) we write \(a \approx_{\varepsilon}^X b\) if \(\|\pi(a - b)\| \leq \varepsilon\) and \(a \approx_{\varepsilon}^X b\) if \(a - b \in \mathcal{X}(H)\).

**Definition 17.6.1.** A subset \(\mathcal{L}\) of \(\mathcal{B}(H)\) is narrow if for every \(a \in \mathcal{B}(H)\), all \((a, b)\) and \((a, c)\) in \(\mathcal{L}\) we have \(b \approx_{\varepsilon}^X c\). It is \(\varepsilon\)-narrow if for every \(a \in \mathcal{B}(H)\), all \((a, b)\) and \((a, c)\) in \(\mathcal{L}\) we have \(b \approx_{\varepsilon}^X c\).

A function \(f: \mathcal{B}(H)_{\leq 1} \to \mathcal{B}(H)_{\leq 1}\) is \(\sigma\)-narrow if its graph can be covered by a countable family of narrow Borel sets. It is \(\sigma\)-\(\varepsilon\)-narrow if its graph can be covered by a countable family of \(\varepsilon\)-narrow Borel sets.

An endomorphism \(\Phi\) of \(\mathcal{D}(H)\) has a \(\sigma\)-narrow lifting if its restriction to the unit ball has a lifting which is \(\sigma\)-narrow. It has a \(\sigma\)-\(\varepsilon\)-narrow approximation if there is a \(\sigma\)-\(\varepsilon\)-narrow function \(\Theta\) such that every \(a \in \mathcal{B}(H)_{\leq 1}\) satisfies \(\Phi_\varepsilon(a) \approx_{\varepsilon}^X \Theta(a)\).

A \(\sigma\)-narrow lifting on \(\mathcal{D}[E]\) is a \(\sigma\)-narrow \(\varepsilon\)-approximation on \(\mathcal{D}[E]\) are defined analogously.

If the notion of a \(\sigma\)-narrow lifting seems a bit awkward, then the following example may be appreciated.

**Example 17.6.2.** There exists a unital endomorphism \(\Psi\) of \(\ell_\infty/c_0\) that has a \(\sigma\)-narrow lifting, but no continuous (or even Baire-measurable) lifting. Let \(\mathcal{U}_n\), for \(n \in \mathbb{N}\), be distinct nonprincipal ultrafilters on \(\mathbb{N}\). Define \(\Upsilon: \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})\) by \(\Upsilon(A) := \{n: A \in \mathcal{U}_n\}\). Then \(\Upsilon\) is an endomorphism of the Boolean algebra \(\mathcal{P}(\mathbb{N})\) which lifts an endomorphism of \(\mathcal{P}(\mathbb{N})/\text{Fin}\). By the Gelfand–Naimark/Stone dualities (§1.3.1) it determines an endomorphism \(\Psi\) of \(\ell_\infty/c_0\). A verification that \(\Psi\) has a \(\sigma\)-narrow lifting, but no Baire-measurable one, is left as Exercise 17.9.11.

**Lemma 17.6.3.** Assume OCA\(_T\). If \(\Phi\) is an endomorphism of \(\mathcal{D}(H)\) and \(\varepsilon > 0\), then \(\Phi\) has a \(\sigma\)-narrow \(\varepsilon\)-approximation on \(D_X[E]\) for some \(E \in \operatorname{Part}_N\) and infinite \(\hat{X}\).

**Proof.** It will be convenient to use the notation \(\mathcal{D}_X[E]\) and \(D_X[E]\) with the intervals in \(E\) indexed by \(\{0, 1\}^\mathbb{N}\) and \(X \subseteq \{0, 1\}^{<\mathbb{N}}\). Fix \(E\) so that \(\lim_n \min_{|E| = n} |E_n| = \infty\). If \(X \subseteq \{0, 1\}^{<\mathbb{N}}\) is infinite and included in a branch of \(\{0, 1\}^{<\mathbb{N}}\), then this branch is denoted \(B(X)\). Fix a discretization \(D[E]\) of \(\mathcal{D}[E]\). The definition of a discretization
depends on the indexing of the intervals of $E$ by $\mathbb{N}$, but all that we need is that for every $n$ and every $s \in \{0, 1\}^n$ the set $D_n$ is $2^{-n}$-dense in the unit ball of $\mathcal{D}_{[n]}[E]$. Also fix $d \geq (2\varepsilon)^{-1}$. Let (omitting $[E]$ when clear from the context)

$$\mathcal{R} := \{(X, a) : B(X) \text{ is defined and } a \in D_X\}.$$ 

and let $\{(X, a), (Y, b)\} \in M_0^d$ if the following conditions are satisfied:

(M$_0^d$ 1) $B(X) \neq B(Y),$ 
(M$_0^d$ 2) $p_X b = p_Y a,$ and 
(M$_0^d$ 3) $\max(||\Phi_s(a)q_Y - q_X\Phi_s(b)||, ||q_Y\Phi_s(a) - \Phi_s(b)q_X||) > 1/d.$

These conditions are symmetric and define a partition $[\mathcal{R}]^2 = M_0^d \cup M_1^d.$ In order to topologize $\mathcal{R},$ identify $(X, a) \in \mathcal{R}$ with

$$(B(X), X, a, q_X, \Phi_s(a)) \in \{0, 1\}^N \times \mathcal{P}(\{0, 1\}^N) \times D \times \mathcal{B}(H)^2_{\leq 1}$$

where $\{0, 1\}^N, \mathcal{P}(\{0, 1\}^N),$ and $D$ are equipped with their standard compact metric topologies and $\mathcal{B}(H)_{\leq 1}$ is equipped with the weak operator topology. Consider $\mathcal{R}$ with respect to the subspace topology.

**Claim.** The partition $[\mathcal{R}]^2 = M_0^d \cup M_1^d$ is open.

**Proof.** The condition (M$_0^d$ 1) is open. Once it is satisfied, $p_X b$ and $p_Y b$ are taking values in the finite set $\prod_{s \in [N]} D(s)$ and therefore (M$_0^d$ 2) is open relative to (M$_0^d$ 1). Since the set $\{b : ||b|| > 1/d\}$ is WOT-open, the condition (M$_0^d$ 3) is open. \(\square\)

**Claim.** There are no uncountable $M_0^d$-homogeneous subsets of $\mathcal{R}$ for any $d \geq 1.$

**Proof.** Assume otherwise and fix $d$ such that some $\mathcal{H} \subseteq \mathcal{R}$ is uncountable and $M_0^d$-homogeneous. Since $ap_Y = bp_X$ for all $(X, a)$ and $(Y, b)$ in $\mathcal{H},$ there exists $c \in D$ such that $c p_X = a$ for all $(X, a) \in \mathcal{H}$ and $\|c\| \leq 1.$ (Let $c(s) = a(s)$ for any $(X, a) \in \mathcal{H}$ such that $X \in [s].$)

Fix $\delta < 1/(6d).$ Since $\Phi_s$ lifts $\Phi, q_X\Phi_s(c) - \Phi_s(a)$ and $\Phi_s(c)q_X - \Phi_s(a)$ are compact for every $(X, a) \in \mathcal{H}.$ Some compact positive contraction $q = q(X, a)$ satisfies $\max(||(1 - q)(\Phi_s(c)q_X - \Phi_s(a))||, ||(1 - q)(q_X\Phi_s(c) - \Phi_s(a))||) < \delta.$ By Lemma 6.6.5 there is a positive, finite rank, operator $q'$ such that $\mathcal{H}_1 := \{(X, a) : ||q'(X, a) - q'|| < \delta\}$ is uncountable. Then every $(X, a) \in \mathcal{H}_1$ satisfies

$$\max(||(1 - q')(\Phi_s(c)q_X - \Phi_s(a))||, ||(1 - q')(q_X\Phi_s(c) - \Phi_s(a))||) < 2\delta.$$ 

Since $q'H_{\leq 1}$ is separable, there are an uncountable $\mathcal{H}_2 \subseteq \mathcal{H}_1$ and $d_0, d_1$ in $\mathcal{B}(H)$ such that $\max(||q'(\Phi_s(a)q_Y - d_0)||, ||q'(q_Y\Phi_s(a) - d_1)||) < \delta$ for all $(X, a)$ and $(Y, b)$ in $\mathcal{H}_2.$ Fix $(X, a)$ and $(Y, b)$ in $\mathcal{H}_2.$ Then

$$(1 - q')\Phi_s(a)q_Y \approx 2\delta (1 - q')q_X\Phi_s(c)q_Y \approx 2\delta (1 - q')q_X\Phi_s(b)$$

and therefore

$$\max(||(1 - q)(\Phi_s(c)q_X - \Phi_s(a))||, ||(1 - q)(q_X\Phi_s(c) - \Phi_s(a))||) < 2\delta.$$
\[ \| \Phi_*(a)q_Y - q_X \Phi_*(b) \| \leq \| q'(\Phi_*(a)q_Y - q_X \Phi_*(b)) \| + 4\delta \leq 6\delta < 1/d. \]

An analogous argument shows that \( \| q_Y \Phi_*(a) - \Phi_*(b)q_X \| < 1/d \). This implies that \( \{(X,a), (Y,b)\} \in M_1^d \); contradiction. \qed

By OCA\(_T\) and the last Claim, \( \mathcal{H} \) can be covered by \( M_1^d \)-homogeneous sets \( \mathcal{H}_n^d \), for \( n \in \mathbb{N} \). Fix a countable \( \mathcal{E}_n^d \subseteq \mathcal{H}_n^d \) dense in the topology described after the definition of \( M_1^d \). Since \( M_1^d \) is open, the closure of any \( M_1^d \)-homogeneous set is \( M_1^d \)-homogeneous. Fix any branch \( \mathcal{B} \) of \( \{0,1\}^{<\mathbb{N}} \) that does not belong to the countable set \( \{ \mathcal{B}(X) : (X,a) \in \bigcup_n \mathcal{E}_n^d \} \).

For distinct \((X,a)\) and \((Y,b)\) in \( \mathcal{H} \) and \( k \in \mathbb{N} \) write

\[ \Delta((X,a),(Y,b)) := \min\{k : (\exists s \in \{0,1\}^k)(s \in XAY \text{ or } s \in X \cap Y \text{ and } a(s) \neq b(s))\}. \]

Fix an approximate unit, \( e_m \), for \( m \in \mathbb{N} \), of \( \mathcal{H}(H) \).

Fix \( n \) and \( k \). Both sets \( \{0,1\}^{\leq n} \) and \( \{0,1\}^{n+1} \) are finite, each \( e_k \) has finite rank, and \( \mathcal{E}_n^d \) is dense in \( \mathcal{H}_n^d \). Hence we can choose \( F_k \in \mathcal{E}_n^d \) so that for every \((X,a) \in \mathcal{H}_n^d \)

there exists \((Y,b) \in F_k \) that satisfies \( \Delta((X,a),(Y,b)) > k \) and

\[ \max(\| \Phi_*(p_X) - \Phi_*(p_Y) \| e_k, \| (\Phi_*(a) - \Phi_*(b)) e_k \| ) < 1/k. \]

Then \( \bigcup_{k \in \mathcal{Y}} F_k \) is dense in \( \mathcal{H}_n^d \) for every infinite \( Y \subseteq \mathbb{N} \). Let \( k(0) := 0 \) and for \( j \in \mathbb{N} \) let \( k(j+1) > k(j) \) be the minimal such that (writing \( \mathcal{B} \) for the node \( s \) in \( \mathcal{B} \) with \( |s| = k \)) \( \mathcal{B} \upharpoonright k(j+1) \neq \mathcal{B}(Y) \upharpoonright k(j+1) \) for all \((Y,b) \in \bigcup_{n \leq k} F_k\).

Let \( \check{X} := \{ \check{b} \upharpoonright k(j) : j \in \mathbb{N} \} \) and consider the compact metrizable space \( D_\check{X} \times \mathcal{B}(H)_{\leq 1} \); it's descriptive set theory time again! For \((Y,b) \in \mathcal{H} \) and \( k \in \mathbb{N} \) the set

\[ \mathcal{W}(Y,b,k) := \{(a,c) \in D_\check{X} \times \mathcal{B}(H)_{\leq 1} : \check{X}, a \in \mathcal{H}, \Delta((\check{X},a),(Y,b)) > k, \max(\| \Phi_*(b) - c e_k \|, \| (q_{\check{X}} - q_Y) e_k \| ) \leq 1/k \} \]

is closed. Therefore \( \mathcal{Z}_n := \bigcap_{j \geq n} \bigcup \{ \mathcal{W}(Y,b,k(j)) : (Y,b) \in F_{k(j),n} \} \) is Borel.

Claim. 1. If \((\check{X},a) \in \mathcal{Z}_n^d \) then \((a,\Phi_*(a)) \in \mathcal{Z}_n \).

2. If \((a,c) \in \mathcal{Z}_n \) then \( q_{\check{X}} c \approx_{1/d} \Phi_*(a) \).

3. The set \( \mathcal{Z}_n^d := \{ (a,q_{\check{X}} c) : (a,c) \in \mathcal{Z}_n \} \) is \( 2/d \)-narrow.

Proof. (1) follows from the density of \( \bigcup_{n \in \mathbb{N}} F_{k(j),n} \) in \( \mathcal{Z}_n^d \).

(2) Suppose that \((a,c) \in \mathcal{Z}_n \). Assume for a moment that

\[ \| \Phi_*(a)q_{\check{X}} - q_{\check{X}} c \| > 2\delta + 1/d. \]

for some \( \delta > 0 \). Choose \( j \) large enough to have \( \| (\Phi_*(a)q_{\check{X}} - q_{\check{X}} c) e_{k(j)} \| > 2\delta + 1/d \) and \( j \geq \max(n,2/\delta) \). If \((Y,b) \in F_{k(j),n} \) is such that \((a,c) \in \mathcal{W}(Y,b,k(j)) \), then

\[ \max(\| (\Phi_*(b) - c) e_{k(j)} \|, \| (q_{\check{X}} - q_Y) e_{k(j)} \| ) \leq 1/k(j) < \delta. \]
Also, \( \{(X, a), (Y, b)\} \in M_d^2 \) holds. By the choice of \( k(j + 1) \) and of \( X \), for every \( x \in X \) we have \( |s| \leq k(j) \) or \( |s| \geq k(j + 1) \), hence \( \Delta((X, a), (Y, b)) > k(j) \) and \( (M_d^2) \) holds as well. Therefore \( (M_d^3) \) fails, and \( \| \Phi_n(a)q_X - q_X\Phi_n(b) \| \leq 1/d \).

Writing \( x \approx_{e, k} y \) if \( \|(x - y)e_k\| < \delta \), we have

\[
\Phi_n(a)q_X \approx_{\delta, k(j)} \Phi_n(a)q_Y \approx_{1/d} q_X\Phi_n(b) \approx_{\delta, k(j)} q_Xc.
\]

Therefore \( \| (\Phi_n(a)q_X - q_Xc)e_{k(j)} \| \leq 2\delta + 1/d \); contradiction. Since \( \delta > 0 \) was arbitrary, \( \| \Phi_n(a)q_X - q_Xc \| \leq 1/d \). Since \( \Phi_n(a) - \Phi_n(a)q_X \) is compact, \( \Phi_n(a) \approx \|_{1/d} q_Xc \) follows.

(3) To prove that \( X_n \) is \( 2/d \)-narrow, note that by (2) for all \( (a, c) \) and \( (a, c') \) in \( X_n \) we have \( q_Xc \approx_{1/d} \Phi_n(a) \approx_{1/d} q_Xc' \).

The sets \( X_n \), for \( n \in \mathbb{N} \), defined in Claim are Borel, each one of them is \( 2/d \)-narrow, and they cover the graph of the restriction of \( \Phi_n \) to \( D_X[\mathcal{E}] \). Since \( 2/d \leq \epsilon \), this restriction is a \( \sigma \)-narrow \( \epsilon \)-approximation of \( \Phi \) on \( D_X[\mathcal{E}] \).

\( \square \)

17.7 Aaçai, IV. From \( \sigma \)-narrow to Continuous Approximations

In this section we start from a \( \sigma \)-narrow \( 1/d \)-approximation of an automorphism \( \Phi \) of \( \mathcal{L}(H) \) obtained in \( \S 17.6 \) and produce a \( C \)-measurable \( 3/d \) approximation of \( \Phi \), both on appropriate subalgebras of \( \mathcal{L}(H) \).

Our immediate goal is to, starting with \( X \subseteq \mathbb{N} \) and a \( \sigma \)-narrow \( 1/d \)-approximation of the automorphism \( \Phi \) on \( D_X[\mathcal{E}] \) as obtained in Lemma 17.6.3, find an infinite \( A \subseteq X \) such that there exists a \( C \)-measurable \( 3/d \)-approximation of \( \Phi \) on \( D_X[\mathcal{E}] \) (Lemma 17.7.2). Recall that \( \Theta : D_X \rightarrow \mathcal{L}(H)_{
abla 1} \) is a \( 1/d \)-approximation of \( \Phi \) on \( D_X \) if \( \Theta(a) \approx_{1/d} \Phi(a) \) for all \( a \in D_X \) (Definition 17.4.4).

**Lemma 17.7.1.** Suppose \( \Phi \) is an endomorphism of \( \mathcal{L}(H) \), \( d \geq 1 \), and \( \mathcal{E} \in \mathcal{P}_{\mathcal{X}} \).

Then for every \( A \subseteq X \) such that both \( A \) and \( X \setminus A \) are infinite at least one of the following applies.

1. There is a \( C \)-measurable \( 3/d \)-approximation of \( \Phi \) on \( D_A \).
2. For every \( 1/d \)-narrow analytic set \( \mathcal{Z} \subseteq D_X \times \mathcal{L}(H)_{
abla 1} \) there are \( B \subseteq X \setminus A \), \( a \in D_A \), and \( b \in D_B \) such that both \( B \) and \( X \setminus (A \cup B) \) are infinite and every uniformization \( \Xi \) of \( \mathcal{Z} \) and \( c \in D_X \setminus (A \cup B) \) such that \( a + b + c \in \text{dom}(\Xi) \) satisfy \( \Xi(a + b + c)q_A \approx_{1/d} \Phi_n(a) \).

**Proof.** Fix a \( 1/d \)-narrow analytic set \( \mathcal{Z} \). The set

\[
\mathcal{Y} := \{(a, b, c) \in D_A \times D_X \setminus A \times \mathcal{L}(H)_{
abla 1} : (\exists c' \in \mathcal{L}(H)_{
abla 1})(a + b, c') \in \mathcal{Z}, c \approx_{1/d} c' q_A \}
\]

10 In this lemma and its proof we will again write \( D \) and \( D_X[\mathcal{E}] \) for \( D[\mathcal{E}] \) and \( D_X[\mathcal{E}] \), respectively.
is analytic as a continuous image of \( \mathcal{X} \times \{ x \in \mathcal{B}(H) : \text{dist}(x, \mathcal{X}(H)) \leq 1/d \} \).

Suppose for a moment that for every \( a \in D_A \) the set
\[
\mathcal{W}(a) := \{ b \in D_{\hat{X}|A} : (a, b, \Phi_+(a)) \in \mathcal{Y} \}
\]
is relatively comeager in \( D_{\hat{X}|A} \). The set
\[
\{(a, c) \in D_A \times \mathcal{B}(H) : \{ b \in D_{\hat{X}|A} : (a, b, c) \in \mathcal{Y} \} \text{ is relatively comeager in } D_{\hat{X}|A} \}
\]
is analytic by Theorem B.2.14. By Theorem B.2.13, it has a \( C \)-measurable uniformization \( \Theta \).

Fix \( a \in D_A \). Since the intersection of two comeager sets is nonempty, there exists \( b \in D_{\hat{X}|A} \) such that both \( (a, b, \Theta(a)) \) and \( (a, b, \Phi_+(a)) \) belong to \( \mathcal{Y} \). Then there are \( c' \) such that \( (a+b, c') \in \mathcal{X} \) and \( \Theta(a) \approx_{1/d} c'q_X \) and \( c'' \) such that \( (a+b, c'') \in \mathcal{X} \) and \( \Phi_+(a) \approx_{1/d} c''q_X \). Since \( \mathcal{X} \) is 1/d-narrow, \( c' \approx_{1/d} c'' \) and therefore \( \Theta(a) \approx_{1/d} \Phi_+(a) \).

Since \( a \in D_A \) was arbitrary, \( a \mapsto \Theta(a) \) is a \( C \)-measurable 3/d-approximation of \( \Phi \) on \( D_A \), and (1) follows.

We can therefore assume that there is \( a \in D_A \) such that \( \mathcal{W}(a) \) is not relatively comeager in \( D_{\hat{X}|A} \). Since \( \mathcal{W}(a) \) is being a continuous image of an analytic set, analytic, it has the Property of Baire (§B.2.1). There is therefore a nonempty open \( U \subseteq D_{\hat{X}|A} \) such that \( \mathcal{W}(a) \cap U \) is relatively meager in \( U \). By Theorem 9.9.1 there are \( I(n) \in \mathbb{N} \) and \( s(n) \in D_{I(n)} \) such that
\[
U \cap \{ b : (\exists n) b \mid I(n) = s(n) \}
\]
is disjoint from \( \mathcal{W}(a) \). Fix a basic open subset \( [J, r] \) of \( U \) and let \( k \) be large enough to have \( I(2n) \cap J = \emptyset \) for all \( n \geq k \). Let \( B := J \cup \bigcup_{n \geq k} I(2n) \) and \( b := r + \sum_{n \geq k} s(2n) \).

Both \( B \) and \( \hat{X} \setminus (A \cup B) \) are infinite and every \( c \in D_{\hat{X}|(A \cup B)} \) satisfies \( b+c \not\in \mathcal{W}(a) \). Therefore \( (a, b+c, \Phi_+(a)) \notin \mathcal{Y} \) and if \( \Xi \) is a uniformization of \( \mathcal{X} \) then
\[
\Xi(a + b + c)q_A \not\approx_{1/d} \Phi_+(a)
\]
and (2) follows.

\[\square\]

**Lemma 17.7.2.** Suppose \( \Phi \) is an endomorphism of \( \mathcal{X}(H) \), \( d \geq 1 \), \( E \in \text{Part}_\mathbb{N} \), and \( \Phi \) has a \( \sigma \)-narrow 1/d-approximation on \( D_{\hat{X}|E} \). Then the following holds.

1. There are an infinite \( A \subseteq \hat{X} \) and a \( C \)-measurable 3/d-approximation to \( \Phi \) on \( D_A[E] \).
2. There is a \( C \)-measurable 3/d-approximation of \( \Phi \) on \( D[F] \) for all \( F \in \text{Part}_\mathbb{N} \).

**Proof.** (1) Assume otherwise. Fix 1/d-narrow analytic sets \( \mathcal{X}_n \), for \( n \in \mathbb{N} \), that cover the graph of a 1/d-approximation of \( \Phi \) on \( D_{\hat{X}} \) (needless to say, in this proof we will omit the parameter \( E \)). For each \( n \) fix a uniformization \( \Xi_n \) of \( \mathcal{X}_n \). We will find a partition \( \hat{X} = \bigsqcup_a A(n) \cup \bigsqcup_b B(n) \) into infinite sets, \( a(n) \in D_{A(n)} \), and \( b(n) \in D_{B(n)} \), so that \( a := \sum_a a(n) \) and \( b := \sum_b b(n) \) satisfy \( \Xi_m(a + b) \not\approx_{1/d} \Phi_+(a + b) \) for all \( m \).
Fix an enumeration $\tilde{X} = \{x_n : n \in \mathbb{N}\}$ and write $C(n) := \tilde{X} \setminus \bigcup_{j \leq n} (A(j) \cup B(j))$ for $n \in \mathbb{N}$ such that $A(j)$ and $B(j)$ have been chosen for $j \leq n$. We will recursively choose $A(j)$, $B(j)$, $a(j)$, and $b(j)$ so that for every $n \in \mathbb{N}$, every $c \in D_{C(n)}$ satisfies

$$\Xi_n(\sum_{j \leq n} a(j) + \sum_{j \leq n} b(j) + c)q_{A(n)} \not\equiv_{1/d}^X \Phi_n(a(n)).$$ \hspace{1cm} (17.5)

Choose any $A(0) \subseteq \tilde{X}$ such that $x_0 \in A(0)$ and $\tilde{X} \setminus A(0)$ is infinite. By our assumption and Lemma 17.7.1, there exist $a(0) \in D_{A(0)}$, $B(0) \subseteq X \setminus A(0)$, and $b(0) \in D_B$ such that $X \setminus (A(0) \cup B(0))$ is infinite and

$$\Xi_0(a(0) + b(0) + c)q_{A(0)} \not\equiv_{1/d}^X \Phi_n(a(0))$$

for all $c \in D_{C(0)}$.

Now suppose that $A(j)$, $B(j)$, $a(j)$, and $b(j)$ as required have been chosen for all $j \leq n$. Choose $A(n+1) \subseteq C(n)$ so that $x_n \in \bigcup_{j \leq n+1} A(j) \cup \bigcup_{j \leq n} B(j)$ and both $A(n+1)$ and $C(n)$ are infinite. The set

$$\{(a, c) \in D_n : (\forall j \leq n)(a \upharpoonright A(j) = a(j), a \upharpoonright B(j) = b(j))\}$$

is analytic as an intersection of an analytic set with a closed set. Again Lemma 17.7.1 implies that there are $a(n+1), B(n+1)$, and $b(n+1)$ as in (17.5).

Since every $x_n$ belongs to $\bigcup_{j \leq n} A(j) \cup \bigcup_{j \leq n} B(j)$, the sets $A(j)$ and $B(j)$ form a partition of $\tilde{X}$. Let $a := \sum_n a(n)$ and $b := \sum_n b(n)$. By our assumption, there is $m$ such that $(a + b, \Phi_n(a + b)) \in D_m$. This implies $\Xi_m(a + b) \approx_{1/d}^X \Phi_m(a + b)$, hence $\Xi_m(a + b)q_{A(m)} \approx_{1/d}^X \Phi_m(a(m))$; contradiction.

Clause (2) follows from (1) and Lemma 17.5.3 (3). $\square$

17.8 Aaçai, V. Coherent Families of Unitaries

In the present section we use OCA$_T$ to prove that every automorphism given by a coherent family of unitaries is inner, completing the proof that OCA$_T$ implies all automorphisms of $\mathcal{L}(H)$ are inner (Theorem 17.8.5). In addition to the usual suspects, this section relies on §17.1, §9.5, and §9.7.

As in §17.1, we identify $U(\ell_1)$ with $\mathbb{T}^\mathbb{N}$. From this section we also import the notation $\Delta_t(u, v)$ (see (17.3)), $\sim_E$ (Definition 17.1.3), and $E$ (see (17.1)) for $E \in \text{Part}\mathbb{N}$. This happens to be the cast of characters of Lemma 17.1.4, which provides the motivation for the following definition (for a commutative analog, see Exercise 9.10.31).

**Definition 17.8.1.** A family $\mathcal{F}$ of pairs $(E, x)$ for $E \in \text{Part}\mathbb{N}$ and $x \in \mathbb{T}^\mathbb{N}$ is a **coherent family of unitaries** if $\{E : (E, x) \in \mathcal{F}\}$ is $\leq_t$-cofinal in $\text{Part}\mathbb{N}$ and $u \sim_E v$ whenever $(E, u)$ and $(F, v)$ belong to $\mathcal{F}$ and $E \leq_t F$. 
By Lemma 17.1.4, every coherent family of unitaries \( \mathcal{F} \) defines a unique automorphism \( \Phi = \Phi_\mathcal{F} \) of \( \mathcal{L}(H) \) such that the restriction of \( \Phi \) to \( \mathcal{F}[E] \) agrees with \( \text{Ad} u \) for every pair \( (E, u) \in \mathcal{F} \). In Theorem 17.1.12 we used a weakening of the Continuum Hypothesis to construct an outer automorphism of the form \( \Phi_\mathcal{F} \); this cannot happen under OCA\(_T\).

**Theorem 17.8.2.** If OCA\(_T\) holds then the automorphism \( \Phi_\mathcal{F} \) is inner for every coherent family of unitaries \( \mathcal{F} \).

**Proof.** Fix \( d \geq 1 \) and define a partition \( [\mathcal{F}]^2 = L_0^d \sqcup L_1^d \) by \( \{(E, u), (F, v)\} \in L_0^d \) if \( (L_0^d) \) For some \( m, n, l := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1}) \) satisfies \( \Delta_l(u, v) > 2^{-d} \).

This is an open partition if \( \mathcal{F} \) is equipped with the subspace topology inherited from \( \mathcal{F} \times \mathbb{T}^N \).

**Claim.** All \( L_0^d \)-homogeneous subsets of \( \mathcal{F} \) are countable.

**Proof.** Suppose otherwise and let \( \mathcal{H} \subseteq \mathcal{F} \) be \( L_0^d \)-homogeneous and uncountable. Since OCA\(_T\) implies \( b > \aleph_1 \) (Proposition 9.5.7) and \( \mathcal{P}_{\aleph_1} \) is Tukey-equivalent to \( (\mathbb{N}^\mathbb{N}, \leq^*) \) (Theorem 9.7.8), there are \( (M, w) \in \mathcal{F} \) and an uncountable \( \mathcal{H}_1 \subseteq \mathcal{H} \) such that \( E \leq^* M \) for \( (E, u) \in \mathcal{H}_1 \).

Therefore for each \( (E, u) \in \mathcal{H}_1 \) there is \( i := i(E, u) \) such that for every \( j \geq i \) there exists \( l \) which satisfies \( \Delta_{E_j(E_{i+1})}(u, v) < 2^{-d-1} \) and \( E_j \cup E_{j+1} \subseteq M_l \cup M_{l+1} \). By replacing \( \mathcal{H}_1 \) with an uncountable subset, we may assume that neither \( i \) nor \( k := \min(E_i) \) depend on the choice of \( (E, u) \in \mathcal{H}_1 \).

Since \( \mathbb{T}^k \) is separable and metrizable, by Lemma 6.6.5 there are \( w \in \mathbb{T}^k \) and an uncountable \( \mathcal{H}_2 \subseteq \mathcal{H}_1 \) such that \( |w|_l - v|_l| < 2^{-d-1} \) for all \( l \leq k \) and \( (F, v) \in \mathcal{H}_2 \). Since \( \mathcal{H}_2 \) is \( L_0^d \)-homogeneous, we can fix distinct \( (E, u) \) and \( (F, v) \) in \( \mathcal{H}_2 \) and a pair \( m, n \) such that \( l := (E_m \cup E_{m+1}) \cap (F_n \cup F_{n+1}) \) satisfies \( \Delta_l(u, v) > 2^{-d} \). By the proximity of \( w \) to both \( u \mid k \) and \( v \mid k \), we have \( \min(m, n) \geq i \), hence \( I \subseteq M_l \cup M_{l+1} \) for some \( l \). Therefore \( \Delta_l(u, v) \leq \Delta_l(u, w) + \Delta_l(v, w) < 2^{-d} \); contradiction. \( \square \)

For \( \mathbb{X} \subseteq \mathcal{F} \) write \( X_0 := \{E : (E, u) \in \mathbb{X}\} \). Since \( \mathcal{P}_{\aleph_1} \) is \( \sigma \)-directed, Lemma 9.5.2 implies that if \( X_0 \) is \( \leq^* \)-cofinal in \( \mathcal{P}_{\aleph_1} \) and \( \mathbb{X} \) is partitioned into countably many pieces, then for one of these pieces, \( \mathbb{Y} \), the set \( Y_0 \) is \( \leq^* \)-cofinal in \( \mathcal{P}_{\aleph_1} \). By Claim and OCA\(_T\), \( \mathcal{F} \) can be covered by countably many \( L_1^d \)-homogeneous sets. We can now recursively choose \( \mathbb{F}(d) \subseteq \mathcal{F} \) for \( d \geq 1 \) so that \( [\mathbb{F}(d)]^2 \subseteq L_1^d \), \( \mathbb{F}(d) \subseteq \mathbb{F}(d+1) \), and \( \mathbb{F}(d_0) \) is \( \leq^* \)-cofinal in \( \mathcal{P}_{\aleph_1} \) for all \( d \).

By Lemma 9.7.9, for every \( d \) there are infinitely many \( k \) such that the set \( \bigcup \{E_n : E \in \mathbb{F}(d), n \in \mathbb{N}, k = \min(E_n)\} \) is infinite. We can therefore recursively choose an increasing sequence \( k(d) \), for \( d \in \mathbb{N} \), such that for every \( d \) and \( l > k(d) \) there are \( (E(d, l), u(d, l)) \in \mathbb{F}(d) \) with \( E(d, l) \supseteq [k(d), l] \) for some \( j \). Since \( \mathbb{T}^\mathbb{N} \) is compact and metrizable, by going to a subsequence if necessary we can find \( u(d) \in \mathbb{T}^\mathbb{N} \) such that \( \lim u(d, l) = u(d) \). Since \( \Delta_k(\cdot, \cdot) \) is jointly continuous for a fixed \( F \) and \( L_1^d \) is closed, the \( L_1^d \)-homogeneity of \( \mathbb{F}(d) \) implies that if \( (F, v) \in \mathbb{F}(d) \) and \( m \) is such that \( k(d) \leq m \) then \( \Delta_{F_m}(v, u(d)) \leq 2^{-d} \). Since \( \mathbb{F}(d) \subseteq \mathbb{F}(d+1) \), we have...
Δ_{k(d+1),∞}(u(d), u(d+1)) ≤ 2^{-d}.

By (6) of Lemma 17.1.5, for every $d$ there exists $λ_d ∈ \mathbb{T}$ such that

$$\sup_{i ≥ k(d+1)} |u(d) - λ_d u(d+1)| ≤ 2^{-d}. $$

Let $w(0) := u(0)$ and $w(d) := \prod_{r=0}^{d} λ_r u(r)$ for $d ≥ 1$. Then for all $d < d'$ we have $\sup_{i ≥ k(d)} |w(d) - w(d')| < 2^{-d+1}$. By the second inequality in (6) of Lemma 17.1.5, this implies $Δ_{k(d'),∞}(w(d), w(d')) < 2^{-d+2}$.

Define $w ∈ \mathbb{T}^\mathbb{N}$ by $w_j := w(d)$, if $k(d) < j ≤ k(d+1)$, using $k(-1) := 0$. We claim that $w$ implements $Φ_2$. Fix $d$ for a moment. Then

$$Δ_{k(d),∞}(w, w(d)) ≤ 2 \sup_{i ≥ k(d)} |w_i - w(d)_i| ≤ 2 \sup_{d' ≥ k(d)} \sup_{d' ≥ d} |w(d) - w(d')| ≤ 2^{-d+2}. $$

Using the triangle inequality for $Δ_{F_{m+1} \cup F_{m+1}, (v, w)}$, for $(F, v) ∈ \mathcal{F}(d)$ and $k(d) ≤ \min F_m$ we have

$$Δ_{F_m \cup F_{m+1}, (v, w)} ≤ Δ_{F_m \cup F_{m+1}, (v, w(d))} + 2^{-d+2} ≤ 5 \cdot 2^{-d}$$

and $\|Φ_2(F, v) - \text{Ad} w\| ≤ 10 \cdot 2^{-d}$. The uniqueness of $Φ_2$ implies $Φ_2 = Φ_2(F, v)$ for all $d$, and therefore $Φ = \text{Ad} w$.

Together with Corollary 17.1.13, Theorem 17.8.2 implies the following.

**Corollary 17.8.3.** The assertion ‘Every automorphism of $\mathcal{L}(H)$ determined by some coherent family of unitaries is inner’ is independent from ZFC.

**Corollary 17.8.4.** Assume OCA_{\mathbb{T}}. Then an automorphism $Φ$ of $\mathcal{L}(H)$ is inner if and only if the restriction of $Φ$ to $\mathcal{D}_{\mathbb{X}}[E]$ is implemented by a partial isometry for some $\mathbb{X}$ and $E$ such that $\sup_{n ∈ \mathbb{N}} |E_n| = \infty$.

**Proof.** Only the converse implication requires a proof. By Lemma 17.5.1, the restriction of $Φ$ to $\mathcal{D}[F]$ is implemented by a partial isometry $v_F$ for every $F ∈ \text{Part}_{\mathbb{N}}$. Since $\mathcal{D}[F]$ is a unital C*-subalgebra of $\mathcal{L}(H)$, $v_F$ is a Fredholm operator and $Φ_1 := \text{Ad}(v_F^*) \circ Φ$ is an automorphism of $\mathcal{L}(H)$. It will suffice to prove that $Φ_1$ is inner. For every $E$, $Φ_1$ is implemented by a partial isometry $u_E$ on $\mathcal{D}[E]$, because $Φ$ is. Moreover, the restriction of $Φ_1$ to the atomic masa $A$ diagonalized by $(ξ_k)$ is equal to the identity. Equivalently, $π(u_E)$ belongs to $A' \cap \mathcal{L}(H)$ and by Theorem 12.3.2 we can choose $u_E$ in $A$. Therefore $u_E$ has Fredholm index 0, and we can choose it to be a unitary. Since $Φ_1 = Φ_2$ for $F = \{(E, u_E) : E ∈ \text{Part}_{\mathbb{N}}\}$, OCA_{\mathbb{T}} and Theorem 17.8.2 together imply that $Φ_1$ is inner.

**Theorem 17.8.5.** Assume OCA_{\mathbb{T}}. Then every automorphism of $\mathcal{L}(H)$ is inner.

Let’s relax and let the whole proof flash before our eyes.

**Proof.** Fix an automorphism $Φ$ of $\mathcal{L}(H)$. For a moment fix $d ≥ 1$. By OCA_{\mathbb{T}} and Lemma 17.6.3, $Φ$ has a $σ$-narrow $1/d$-approximation on $\mathcal{D}_{\mathbb{X}}[E]$ for some $E ∈ \text{Part}_{\mathbb{N}}$. For a moment fix $d$.
and X such that \( \lim_n |E_n| = \infty \). By Lemma 17.7.2, there is an infinite \( A(d) \subseteq X \) such that \( \Phi \) has a C-measurable \( 1/d \)-approximation on \( D_{A(d)}[E] \). By Lemma 17.5.3 (3), \( \Phi \) has a C-measurable \( 1/d \)-approximation on \( D[E] \) for all \( d \geq 1 \).

By Lemma 17.4.5, \( \Phi \) has a continuous lifting on \( D_Y[E] \) for some infinite \( Y \). By Proposition 17.5.4, this lifting can be chosen to be of product type so that its \( n \)-th component \( \Xi_n \) is a unital \( 1/n \)-approximate \( * \)-homomorphism for every \( n \) and \( \Xi_m(1) \) is orthogonal to \( \Xi_n(1) \) for all \( m \neq n \). By Corollary 17.5.5, for every \( F \in \text{Part}_N \), some \( * \)-homomorphism serves as a lifting of \( \Phi \) on \( D[F] \). By Proposition 17.5.7, this restriction is implemented by a partial isometry for every \( F \in \text{Part}_N \). By OCA and Corollary 17.8.4, \( \Phi \) is inner. \( \square \)

17.9 Exercises

**Exercise 17.9.1.** Find separable C*-subalgebras \( A \subseteq B \) of the Calkin algebra and an injective unital \( * \)-homomorphism \( \Phi: A \to \mathcal{D}(H) \) that cannot be extended to an injective unital \( * \)-homomorphism \( \Phi: B \to \mathcal{D}(H) \).

**Exercise 17.9.2.** Prove that for every separable C*-subalgebra \( A \) of \( \mathcal{D}(H) \) and every automorphism \( \Phi \) of \( \mathcal{D}(H) \) constructed in Theorem 17.1.12, the restriction of \( \Phi \) to \( A \) is implemented by a unitary in \( \mathcal{D}(H) \). Conclude that none of these automorphisms sends the image of the unilateral shift to its adjoint.

**Exercise 17.9.3.** Suppose that \( A \) is a unital C*-algebra with an outer automorphism. Prove that \( \mathcal{D}(A \otimes c_0[\mathbb{N}]) \) has an outer automorphism.

**Exercise 17.9.4.** Suppose that \( A \) is a unital C*-algebra. Prove that \( \mathcal{D} = \mathbb{R}_1 \) implies that the corona of \( A \otimes \mathcal{K}(H) \) has \( 2^{\mathbb{R}_1} \) automorphisms.

The following two exercises show that the result of Exercise 17.9.4 can be, to some extent, improved.

**Exercise 17.9.5.** Suppose that \( A \) is a C*-algebra with an approximate unit consisting of projections, \( r_n \), for \( n \in \mathbb{N} \). Suppose \( \mathcal{D} = \mathbb{R}_1 \).

1. Suppose that for all \( m < n \) we have \( r_m A(1 - r_n) \neq 0 \). Prove that the corona of \( A \) has \( 2^{\mathbb{R}_1} \) automorphisms.
2. Suppose instead that for all \( m \) we have \( r_m A(1 - r_m) = \{0\} \). Prove that the corona of \( A \) has \( 2^{\mathbb{R}_1} \) automorphisms.

**Exercise 17.9.6.** Prove that there exists a separable C*-algebra with an approximate unit \( q_m \), for \( m \in \mathbb{N} \), consisting of projections but with no approximate unit \( r_m \), for \( m \in \mathbb{N} \) that consists of projections and satisfies the assumptions of (1) or (2) from Exercise 17.9.5.

The following exercise promised after Definition 17.3.3 is for the readers sufficiently familiar with descriptive set theory and Shoenfield’s Absoluteness Theorem (Theorem B.2.12).
Exercise 17.9.7. Fix separable, nonunital, $C^*$-algebras $A$ and $B$. Prove the following.

1. For a given Borel-measurable map $f : \mathcal{M}(A)_1 \to \mathcal{M}(B)_2$, the assertion ‘$f$ is a lifting of a $^*$-homomorphism between $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$’ is Π₁¹.

2. The assertion that there exists a topologically trivial isomorphism between $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ is Σ₁². It is therefore absolute between transitive models of ZFC that contain all countable ordinals.

Exercise 17.9.8. Show that Claim in the proof of Proposition 17.1.11 cannot be extended to uncountable cofinalities: There exists a $\ll^*$-increasing sequence $E(\alpha)$, for $\alpha < \aleph_1$, in Part II such that for every upper bound $F$ of this sequence the map $x \mapsto (\pi_{E(\alpha)}(x) : \alpha < \aleph_1)$ is not surjective onto $\varprojlim \mathcal{G}(E(\alpha))$.

Exercise 17.9.9. Assume OCA_T. Prove that every $^*$-homomorphism between coronas of the form $\prod_{n \in X} M_n(\mathbb{C}) / \bigoplus_{n \in X} M_n(\mathbb{C})$ can be lifted by a $^*$-homomorphism on $\prod_{n \in Y} M_n(\mathbb{C}) / \bigoplus_{n \in Y} M_n(\mathbb{C})$ for some infinite $Y$.

Exercise 17.9.10. Prove (in ZFC) that every topologically trivial automorphism of the Calkin algebra is inner.

Exercise 17.9.11. 1. Prove that the surjective endomorphism $\Psi$ of $\ell_\infty / c_0$ defined in Example 17.6.2 has a $\sigma$-narrow lifting, but not a continuous, or even a Baire-measurable, lifting.

2. Modify the definition of $\Psi$ to obtain an injective endomorphism of $\ell_\infty / c_0$ with a $\sigma$-continuous lifting, but not a continuous lifting.

Exercise 17.9.12. Consider $\mathcal{P}(\mathbb{N})$ as a group with respect to the symmetric difference; then Fin is a subgroup.

1. Prove (in ZFC) that $\mathcal{P}(\mathbb{N}) / \text{Fin}$ has $2^\mathbb{N}$ automorphisms.

2. Prove (in ZFC) that $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N}) / \text{Fin}$ are isomorphic as groups.

3. Suppose that a group automorphism $\mathcal{P}(\mathbb{N}) / \text{Fin}$ has a Baire-measurable lifting. Prove that it has a continuous lifting.

Exercise 17.9.13. Prove that there exists small enough $\tilde{\epsilon} > 0$ such that for every $\epsilon < \tilde{\epsilon}$ and every Borel-measurable $\epsilon$-representation $\Theta : G \to \text{GL}(A)$ of a compact group $G$ that satisfies $\sup_{x \in G} \| \Theta(x)^{-1} \| \leq 2$ there exists a unitary representation $\Lambda : G \to \text{U}(A)$ such that $\| \Lambda - \Theta \| \leq 3\epsilon$.

Exercise 17.9.14. Use Exercise 17.9.9 and Exercise 16.8.27 to find infinite subsets $X$ and $Y$ of $\mathbb{N}$ such that CH implies that the $C^*$-algebras $\prod_{n \in X} M_n(\mathbb{C}) / \bigoplus_{n \in X} M_n(\mathbb{C})$ and $\prod_{n \in Y} M_n(\mathbb{C}) / \bigoplus_{n \in Y} M_n(\mathbb{C})$ are isomorphic, but OCA_T implies that they are not.

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11 If this were not the case, it would then be possible to construct an outer automorphism of $\mathcal{Q}(H)$ assuming $\mathfrak{c} \leq \aleph_2$, contradicting Theorem 17.8.5. There is, however, a direct proof.

12 If OCA_T is boosted with Martin’s Axiom, then $\Phi$ can be lifted by a $^*$-homomorphism ([182]).
Notes for Chapter 17

§17.1 The first construction of an outer automorphism of the Calkin algebra assuming the Continuum Hypothesis was given in [200] (Philips and Weaver also constructed $2^\aleph_1$ outer automorphisms). This proof uses the full power of the Continuum Hypothesis and the highly nontrivial machinery of $K$-homology and $E$-theory in particular. Exercise 15.6.16 is an instance of some of these arguments simple enough to have an ‘elementary’ proof. Our elementary proof was adapted from [82, §1] and [48].

The assumption $c < 2^{\aleph_1}$ used in Corollary 17.1.13 in conjunction with $d = \aleph_1$ is known as the weak Continuum Hypothesis and it has surprisingly strong consequences (subsumed in the ‘weak diamond,’ see [55]). The weak Continuum Hypothesis is not a consequence of the equality $d = \aleph_1$. The models of ZFC in which the former fails and the latter holds include the Laver model and both iterated and product Sacks models (see [20]). It is not known whether the weak Continuum Hypothesis is a necessary addition to $d = \aleph_1$ in Corollary 17.1.13.

Constructions of automorphisms of quotient structures related to $\mathcal{D}(H)$ using the Continuum Hypothesis go at least as far back as W. Rudin’s construction of $2^{\aleph_1}$ automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ ([211]). This is Corollary 16.7.3; see the introduction to [200] for a discussion. This being the commutative version of the problem about automorphisms of the Calkin algebra discussed in this section, it may be worth noting that the two problems are different. It is relatively consistent with ZFC that all automorphisms of $\mathcal{P}(\mathbb{N})/\text{Fin}$ are trivial while the Calkin algebra has $2^{\aleph_1}$ outer automorphisms ([97, Corollary 2]). In this model the assumptions of Corollary 17.1.13 hold, while all quotients of the form $\mathcal{P}(\mathbb{N})/\mathcal{J}$ for an analytic ideal $\mathcal{J}$ have only topologically trivial automorphisms. (An automorphism of $\mathcal{P}(\mathbb{N})/\mathcal{J}$ is topologically trivial if it has a continuous lifting, where $\mathcal{P}(\mathbb{N})$ is considered with the Cantor set topology.) This result was refined in [114], where it was proved that in a closely related model of ZFC (still satisfying the assumptions of Corollary 17.1.13) all coronas of the form $\prod M_n(\mathbb{C})/\bigoplus M_n(\mathbb{C})$ (see §16.7) for an analytic ideal $\mathcal{J}$ have only inner automorphisms. Exercise 17.9.9 is related to a weak form of a result from [182].

Additional evidence that the two problems are unrelated comes from the fact that if $\Phi$ is an automorphism of $\mathcal{D}(H)$ which sends the atomic masa $A$ to itself, then the restriction of $\Phi$ to $A$ is trivial. This can be proved using the main result of [111].

§17.2 It was Ulam who first asked under what conditions an approximate homomorphism between metric groups can be uniformly approximated by a true homomorphism. The relevance of Ulam-stability of approximate homomorphisms in a given category to the existence of liftings of homomorphisms between quotients was apparently first observed in [77] and [78]. The proof of Theorem 17.2.3 has been adapted from the proof of [37, Theorem 3.1]. A form of this ‘Newton ap-

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13 As elementariness is in the eye of the beholder, I rephrase: Quite elementary, to a set theorist.

14 Or equivalently, using the Gelfand–Naimark and Stone dualities, an autohomeomorphism of $\beta\mathbb{N}\setminus\mathbb{N}$ or an automorphism of $\ell_\infty/c_0$.
proximation’ argument appeared as early as [140, Theorem 3.1] in the context of Banach algebras as an application of an approximate diagonal. The analog of Theorem 17.2.3 is true even when the target is a C*-algebra; this is the main result of [183]. A weaker version appeared in [82, §5].

§17.3 It has been conjectured in [48, Conjecture 1.2] that the Continuum Hypothesis implies that every corona of a separable and nonunital C*-algebra has \(2^{\aleph_1}\) nontrivial automorphisms. Exercises 17.9.4 and 17.9.5 show that this is true in the case of the coronas of some separable C*-algebras. This conjecture has also been confirmed in [98] for \(A = C_0(\mathbb{R})\) and in [250] for \(A = C_0(X)\) for every noncompact, locally compact, metrizable manifold \(X\). Another sweeping conjecture, [48, Conjecture 1.3], asserts that sufficiently strong forcing axioms imply that all automorphisms of every corona of a separable C*-algebra are topologically trivial. In a reverse to the historical trend (the CH results have usually been proved decades before the corresponding forcing axiom results), Vignati confirmed this conjecture in [251]. In general, the ‘isometry trick’ does not apply and because of that OCA_T often has to be replaced by a strengthening, OCA_\infty, and supplemented with Martin’s Axiom. In [251], Vignati also proved that every isomorphism between coronas of separable abelian C*-algebras is induced by a homeomorphism between co-compact subsets of their spectra, thus confirming a strong form of the abelian case of [48, Conjecture 1.3].

Analogous result for not necessarily abelian C*-algebras is still out of reach. It would require (among other things) a proof of the Ulam-stability for approximate *-homomorphisms between arbitrary C*-algebras.

The remaining parts of §17.2–§17.8 are based largely on [82], with a few details borrowed from [182], [251], and [183]. Every corner has been cut, resulting in a proof that may be over-optimized (cf. the intriguing discussion of Roman aqueducts in [237, p. 223]). This result belongs to a long line of proofs of rigidity statements, starting with Shelah’s groundbreaking proof that all automorphisms of \(\mathcal{P}(N) / \text{Fin}\) are trivial ([219]). See also [144], [247] (the partition used in Lemma 17.6.3 essentially comes from there), [77], and the references thereof.

While the assertion ‘Every automorphism of \(\mathcal{L}(H)\) determined by some coherent family of unitaries is inner’ is independent from ZFC, it is not known whether the conclusion of Lemma 17.6.3 can be proved in ZFC.

Example 17.6.2 is essentially [77, Example 3.2.3]. A much more impressive example of a highly nontrivial endomorphism of \(\ell^\infty / c_0\) (presented in the dual, topological form) can be found in [63]. Example 17.3.5 is due to N.C. Phillips. Exercise 17.9.1 appears in [200] where it was attributed to John McCarthy. Exercise 17.9.9 and Exercise 17.9.14 are due to Ghasemi ([114]).
Part IV
Appendices
Appendix A
Axiomatic Set Theory

Nothing will come of nothing.  
Shakespeare, King Lear

Standard sources include [166] (with its prequel, [165]), [134], and [218].

In first-order logic quantification is allowed only over the elements of the domain of discourse, and in the second-order logic quantification is allowed over subsets, relations, and functions on the domain of discourse. The Zermelo–Fraenkel Set theory with the Axiom of Choice (ZFC) is a theory of first-order logic in language whose only non-logical symbol is binary relation symbol, $\in$. Its models are structures $(M, E)$, where $E$ is the interpretation of $\in$. It is a distinguished partial ordering on $M$. A model of ZFC satisfies every theorem of ZFC (and therefore all of mathematics, as we presently know it). Since ZFC is a first-order theory, it is consistent if and only if each of its finite subsets is consistent.  

A.1 The Axioms of ZFC

The first axiomatization of set theory was introduced by Zermelo in order to formalize his proof that the Axiom of Choice is equivalent to the assertion that every set can be well-ordered. The Axiom of Replacement has been added by Fraenkel (see [149]).

We will use the logical connectives $\neg$ (negation), $\land$ (and), $\lor$ (or), $\rightarrow$ (implies), $\iff$ (if and only if), and quantifiers $\forall$ (for all) and $\exists$ (there exists). The axioms are more conveniently stated using the following common abbreviations:

$$(\exists!x)\varphi(x) \text{ stands for } \text{‘there exists a unique } x \text{ that satisfies } \varphi(x) \text{’ (in symbols, } \exists!x)(\varphi(x) \land (\forall y)(\varphi(y) \rightarrow x = y)).$$

1 After ZFC has been developed in a rudimentary language, it is used to properly define syntax, semantics, and all of model theory ($\S$D.1). Only then one can consider ZFC as a formal first-order theory. King Lear may not have been referring to this issue, but his statement stands nonetheless.
(∀x ∈ y)ϕ stands for ‘every x in y satisfies ϕ,’ i.e., (∀x)(x ∈ y → ϕ), and
(∃x ∈ y)ϕ stands for ‘some x in y satisfies ϕ,’ i.e., (∃x)(x ∈ y ∧ ϕ).

Another common abbreviation is x ⊆ y, for (∀z)(z ∈ x → z ∈ y).

The universal closure of a formula ϕ is the sentence ( ∀x₀)( ∀x₁)...( ∀xₙ₋₁)ϕ, where x₀,...,xₙ₋₁ is the list of all variables freely occurring in ϕ. ZFC consists of universal closures of the following axioms and axiom schemes.²

1. (Extensionality) (∀z)(z ∈ x ↔ z ∈ y) → x = y.
2. (Foundation, or Regularity) (∃y)(y ∈ x) → (∃y)(y ∈ x ∧ (∀z)(z /∈ x ∨ z /∈ y).
3. (Comprehension Scheme) For every formula ϕ of the language of ZFC in which y does not occur freely ( ∃y)(∀x)(x ∈ y ↔ (x ∈ z ∧ ϕ(x))).
4. (Pairing) (∃z)(∃z)(∀z)(∀t)(t ∈ z ∧ z ∈ x → t ∈ y).
5. (Union) (∃y)(∀z)(∀t)(∀t)(t ∈ z ∧ z ∈ x → t ∈ y).
6. (Replacement Scheme) For every formula ϕ of the language of ZFC in which t does not occur freely ( ∀y ∈ x)( ∃!z)ϕ(y, z) → (∃z)(∀y ∈ x)(∃z ∈ t)ϕ(y, z).
7. (Infinity) (∃x)(∃φ)(∀x)(∀y)(y ∈ x ∨ y = x).
8. (Power Set) (∃y)(∀z)(∀z)(z ⊆ x ∧ x ∈ y).
9. (The Axiom of Choice, AC) For every set x all of whose elements are nonempty sets there exists a function f with domain x such that f(y) ∈ y for all y ∈ x.

Translations of selected axioms of ZFC into the English language follow. ‘Formula’ stands for ‘first-order formula in the language of ZFC.’

10. Comprehension Scheme: Every subset of a set that is definable by a formula is also a set.
11. Union: If x is a set, then there exists a set that includes ∪x, the union of all elements of x.³
12. Replacement Scheme: If x is a set and a formula ϕ defines a function f with domain x via f(y) = z if and only if ϕ(y, z), then the image ϕ(x) := {f(y) : y ∈ x} exists.
13. Infinity: There exists a set whose elements are {0, 1, 2,...,n, n+1,...}.
14. Power Set: It literally states that for every set x there exists a set y that includes the power set ℘(x) of x.⁴
15. AC: For every set x all of whose elements are nonempty sets there exists a function f with domain x such that f(y) ∈ y for all y ∈ x.

By ZF we denote the axioms of ZFC without the Axiom of Choice. Since by Gödel’s theorem one cannot prove the consistency of ZFC within ZFC, metamathematical considerations usually start from a ‘sufficiently large finite fragment of ZFC.’ This fragment depends on the context and it is denoted ZFC⁺.

A proof of the following can be found in every introductory book on set theory and most sufficiently old introductory functional analysis texts.

² Each of (3) and (6) is not a single axiom, but a scheme that associates an axiom to every formula of the language of set theory. It is known that ZFC is not finitely axiomatizable.
³ By applying the Comprehension Scheme one proves that ∪x exists.
⁴ By applying the Comprehension Scheme one proves that ℘(x) itself exists.
Theorem A.1.1. The following statements are provably equivalent in ZF.²

1. The Axiom of Choice.
2. Zorn’s Lemma: If a partially ordered set \( P \) contains an upper bound for every linearly ordered subset, then it has at least one maximal element.
3. Hausdorff’s maximality principle: If \( P \) is a partially ordered set, then every linearly ordered subset of \( P \) is contained in a maximal (under the inclusion) linearly ordered subset.
4. Tychonoff’s theorem: the product of compact topological spaces is compact.
5. The Cartesian product of any family of nonempty sets is nonempty.

With the axioms of ZFC in place, one proceeds to define ordered pairs, Cartesian products, functions, and all other mathematical objects, as sets. Using the axioms of ZFC, one can then prove the existence of these objects. The details can be found in any basic text on axiomatic set theory.

A.2 Well-foundedness, Transfinite Induction, and Transfinite Recursion

Let \( i \) be an element of \( j \) and let \( j \) be an element of \( i \).³

Anonymous (as shared by Alan Dow)

Suppose \( E \) is an ordering (i.e., a transitive, antisymmetric, and irreflexive binary relation) and \( Y \) is a subset of its domain. Some \( a \in Y \) is a minimal element of \( Y \) if there is no \( b \in Y \) such that \( bEa \). An ordering on \( X \) is well-founded if every nonempty subset of \( X \) has a minimal element. An ordering which is both linear and well-founded is a well-ordering. The Axiom of Choice implies that \( E \) is well-founded if and only if there is no infinite decreasing sequence of elements of its domain. The Axiom of Foundation implies that \( \in \) is well-founded. The utility of well-foundedness is most obvious from the principles of transfinite induction and transfinite recursion. We state only a weak variant of each one of them (see Remark A.3.1).

Proposition A.2.1 (Transfinite Induction). If \( E \) is a well-founded relation on \( X \), \( \varphi(x) \) is a formula, and \( (\forall x \in X)(\forall y E x) \varphi(y) \rightarrow \varphi(x) \), then \( (\forall x \in X)\varphi(x) \).

Every object in a model of ZFC is a set and any formula of the language of ZFC can be evaluated at any set. An example of the effect of this flexibility on the readability of formulas is (inadvertently) provided in the Definition by Transfinite Recursion stated below. The formula \( \varphi \) occurring in it has two parameters. One of them is ‘the function defined up to \( x \)’ and the other is ‘the value of the function at \( x \).’ By \( f \rvert X \) we denote the restriction of a function \( f \) to a subset \( X \) of its domain.

² Notably, the justification of this theorem was Zermelo’s motivation for introducing axiomatic set theory; see [186].
³ This is a set theorist’s way of saying ‘Let \( \epsilon < 0 \),’ only worse.
Proposition A.2.2 (Definition by Transfinite Recursion). Suppose $E$ is a well-founded relation on a set $X$. $\varphi(x, y)$ is a formula, and for every $x$ there exists a unique $y$ such that $\varphi(x, y)$ holds. Then there exists a unique function $f : X \to Z$, for some set $Z$, such that $\varphi(f \restriction \{y \in X : y \in x\}, f(x))$ holds for all $x \in X$.

Some applications of this theorem are the Mostowski Collapsing Theorem (Theorem A.6.2) and the definition of the rank of a well-founded tree (Proposition B.2.8). Both Proposition A.2.1 and Proposition A.2.2 are theorems of ZF. All of the recursive constructions presented in this book rely on the following variant of the latter. Its proof requires the Axiom of Choice.

Proposition A.2.3. Suppose $E$ is a well-founded relation on a set $X$. $\varphi(x, y)$ is a formula, and for every $x$ there exists $y$ such that $\varphi(x, y)$ holds. Then there exists a function $f : X \to Z$, for some set $Z$, such that $\varphi(f \restriction \{y \in X : y \in x\}, f(x))$ holds for all $x \in X$.

If the set $\{y : (\exists x)\varphi(x, y)\}$ as in Proposition A.2.3 is equipped with a well-ordering $\leq$, then $\varphi'(x, y) := \varphi(x, y) \land ((\forall z)\varphi(x, z) \to y \leq z)$ satisfies the assumptions of Proposition A.2.2, and the particular instance of this proposition does not require the Axiom of Choice.

### A.3 Transitive Sets. Ordinals.

For a set $X$ define $\bigcup X := \{z : (\exists y \in X)z \in y\}.$ (Some authors use $\bigcup_{y \in X} y$ instead.) A set $X$ is transitive if every element of $X$ is a subset of $X$. Every set $X$ has the transitive closure: the smallest transitive set $Y$ that includes $X$. For $n \geq 1$ define the operation $\bigcup^n X$ by recursion on $n$ as follows: $\bigcup^1 X := X$ and $\bigcup^{n+1} X := \bigcup(\bigcup^n X)$ for $n \geq 1$. The transitive closure of $X$ is equal to $\bigcup_{n \in \mathbb{N}} \bigcup^n X$.

An ordinal is a set which is both transitive and linearly ordered by $\in$. The examples include $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$ . . . The class of all ordinals is denoted OR. (There is no consensus; some authors use ORD, some authors use ON.) If $\alpha$ and $\beta$ are ordinals, then so is $\alpha \cap \beta$. The Extensionality axiom implies that for all ordinals we have $\alpha \cap \beta = \alpha$ or $\alpha \cap \beta = \beta$. Therefore $\alpha \in \beta$, $\beta \in \alpha$, or $\alpha = \beta$. Since $\alpha$ and $\beta$ were arbitrary, OR is linearly ordered by $\in$. For ordinals $\alpha$ and $\beta$ one usually writes $\alpha < \beta$ in place of $\alpha \in \beta$, and we have $\alpha = \{\beta : \beta < \alpha, \beta \in \text{OR}\}$ for every $\alpha \in \text{OR}$. The successor function $S$ is defined on OR by $S(\alpha) := \alpha \cup \{\alpha\}$. If $\alpha \in \text{OR}$ then $S(\alpha) \in \text{OR}$ is the least ordinal greater than $\alpha$; it is often denoted $\alpha + 1$. The intersection of all sets satisfying the Axiom of Infinity is an ordinal. It is the least infinite ordinal, denoted $\omega$. The ordering $(\omega, \in)$ is isomorphic to $(\mathbb{N}, <)$ and the set $\mathbb{N}$ is interpreted in ZFC as the set of all finite ordinals. An ordinal $\alpha$ is a successor ordinal if $\alpha = \beta + 1$ for some ordinal $\beta$. It is otherwise a limit ordinal. Two ordinals are equal if and only if they are order-isomorphic. Hence for every well-ordering

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7 This makes sense only if $X$ is a set of sets, but in this appendix every object is a set.
There exists a unique ordinal $\alpha$ isomorphic to it; this is the order type of $(X, <)$, denoted $\text{otp}(X, <)$. In particular,

$$\omega := \text{otp}(\mathbb{N}, <).$$

The sum of two ordinals is $\alpha + \beta := \text{otp}(\{0\} \times \alpha \cup \{1\} \times \beta)$, where the set on the right-hand side is taken with the lexicographical ordering. Similarly, the product of two ordinals is $\alpha \cdot \beta := \text{otp}(\alpha \times \beta)$, the Cartesian product being taken with the lexicographical ordering. Hence $2 \cdot \omega = \omega + \omega$ but $\omega \cdot 2 = \omega$.

**Remark A.3.1.** Since $\alpha < S(\alpha)$, there is no largest ordinal and OR is not a set but a proper class. It is, however, set-like: for every $\alpha \in \text{OR}$ the class $\{\beta \in \text{OR} : \beta < \alpha\}$ is a set. The analogs of Proposition A.2.1 and Proposition A.2.2 hold for set-like classes.

### A.4 Cardinals. Cardinal Arithmetic

Twenty-four is the highest number.

Bob Odenkirk, Mr. Show with Bob and David, episode 7

Two sets $X$ and $Y$ are equinumerous if there exists a bijection between them. Therefore $X$ is equinumerous with a subset of $Y$ if and only if there exists an injection from $X$ into $Y$. This easily implies that there exists a surjection from $Y$ onto $X$.9

The Axiom of Choice is not needed in the proof of the following fundamental result. The cardinality of $X$ is not greater than the cardinality of $Y$ ($|X| \leq |Y|$) if $X$ is equinumerous with a subset of $Y$. Equinumerosity is the symmetrization of this relation.

**Theorem A.4.1 (Schöder–Bernstein).** There is a bijection between $X$ and $Y$ if and only if there is an injection from $X$ into $Y$ and an injection from $Y$ into $X$. □

The Axiom of Choice implies that every set can be well-ordered, and is therefore equinumerous to an ordinal. The least such ordinal, called the cardinality of $X$, is denoted $|X|$. An ordinal is a cardinal if it is not equinumerous to any smaller ordinal. By Theorem 8.1.2 and the Axiom of Choice, there is no largest cardinal. Since the supremum of any set of cardinals is, by Replacement, a cardinal, the class $\text{CARD}$ of all cardinals is not a set. (It can also be proved that $\text{CARD}$ is cofinal in OR without using the Axiom of Choice.)

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8 ZFC does not handle proper classes well. There are alternative axiomatizations of set theory which do allow proper classes as parameters in formulas. Every one of these theories has other issues, and we stick with ZFC. See e.g., [166].

9 The Axiom of Choice implies the converse, that if there exists a surjection from $Y$ onto $X$ then there exists an injection from $Y$ to $X$. However, in the context in which the Axiom of Choice does not hold this breaks down. Quotient maps associated with Borel equivalence relations rarely have Borel selectors (see however Theorem B.2.13), and taking quotients in general leads to an increase in complexity (see Notes to §3.10).
The smallest cardinality of a cofinal subset of $\alpha$ is its cofinality, denoted $\operatorname{cof}(\alpha)$. A cardinal $\kappa$ is regular if $\operatorname{cof}(\kappa) = \kappa$ and singular otherwise. The successor of a cardinal $\kappa$ is the least cardinal greater than $\kappa$, denoted $\kappa^+$. A cardinal $\kappa$ is limit if it is not a successor and a strong limit if $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$.

Cardinals $\mathbf{K}_\alpha$ for $\alpha \in \mathbb{O}$ are defined by transfinite recursion on $\mathbb{O}$: $\mathbf{K}_0 := |\mathbb{N}|$ and $\mathbf{K}_\beta := \sup_{\alpha < \beta} \mathbf{K}_\alpha$ for $\beta > 0$. Some authors distinguish between the cardinal $\mathbf{K}_\alpha$ and the corresponding ordinal $\omega_\alpha$, and write e.g., $|X| = \mathbf{K}_\alpha$ (cardinality) and $x_\alpha$, for $\alpha < \omega_\alpha$ (order-type). Other authors use $\omega_\alpha$ to denote the cardinal $\mathbf{K}_\alpha$. Due to several factors (such as the shortage of fonts, $\omega$ already being overused in operator algebras, and my personal bias), I will use $\mathbf{K}_\alpha$ to denote both cardinals and ordinals, with one exception. The symbol $\omega$ is used to denote the least infinite ordinal in situations in which any other symbol would look awkward. (Needless to say, I reserve the right to decide what is awkward-looking in a given context.)

The Axiom of Choice implies that every successor cardinal is regular. The least limit cardinal greater than $\mathbf{K}_0$ is $\mathbf{K}_\omega = \sup_{n < \omega} \mathbf{K}_n$. Since $\operatorname{cof}(\mathbf{K}_\alpha) = \operatorname{cof}(\alpha)$ if $\alpha$ is a limit ordinal, $\operatorname{cof}(\mathbf{K}_\omega) = \omega$. A regular limit cardinal, if there is one, is called weakly inaccessible. The limit cardinals are ‘typically’ singular. The Replacement Scheme implies that there exist arbitrarily large strong limit cardinals. A cardinal $\kappa$ is strongly inaccessible if it is a regular, strong limit, cardinal.

**Definition A.4.2 (Cardinal Arithmetic).** The sum of two cardinals $\kappa + \lambda$ is defined to be $|\kappa \times \{0\} \cup \lambda \times \{1\}|$. It is equal to the cardinality of the disjoint union of any two sets whose cardinalities are $\kappa$ and $\lambda$. The product of two cardinals $\kappa$ and $\lambda$ is defined to be $|\kappa \times \lambda|$. The power $\kappa^\lambda$ is defined to be $|\{f : f : \lambda \rightarrow \kappa\}|$. In particular $2^\kappa = |\mathcal{P}(\kappa)|$. We write $\mathfrak{c} := 2^{\mathbf{K}_0}$.

Cantor’s Continuum Hypothesis (CH) asserts that $\mathfrak{c} = \mathbf{K}_1$ (§8.1). The Generalized Continuum Hypothesis (GCH) asserts that $2^\kappa = \kappa^+$ for every cardinal $\kappa$.

**Theorem A.4.3.** Suppose that $2 \leq \kappa$ and $\lambda$ is infinite.

1. $\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$.
2. $\lambda < 2^{\mathbf{K}_0} = \lambda$.
3. If $\kappa \leq 2^\lambda$ then $\kappa^\lambda = 2^\lambda$.
4. (Konig’s Theorem) If $2 \leq \kappa$ then $\lambda < \operatorname{cof}(\kappa^\lambda)$.

At a deeper level it may appear that almost every nontrivial statement of cardinal arithmetic is independent from ZFC (see [166, Corollary IV.7.18]). At an even deeper level, a beautiful theory of cardinal arithmetics provable in ZFC emerges (see [221] and [1]).

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10 In $\mathcal{L}(\mathbb{R})$, the most important model of ZF in which the Axiom of Choice fails, the standard large cardinal assumptions imply that $\mathbf{K}_n$ is singular for $3 \leq n < \omega$ (see [149, §28]).

11 There is no proof in ZFC that regular limit cardinals exist; see the discussion following Theorem A.5.2.
A.5 The Cumulative Hierarchy and the Constructible Hierarchy

Von Neumann’s *cumulative hierarchy* is defined by transfinite recursion on ordinals. The sets $V_\alpha$, for $\alpha \in \text{OR}$, are defined recursively as follows: $V_0 := \emptyset$, $V_{\alpha+1} := \mathcal{P}(V_\alpha)$, and $V_\beta := \bigcup_{\alpha < \beta} V_\alpha$ if $\beta$ is a limit ordinal.

**Example A.5.1.**  
1. The standard definitions of $\mathbb{Z}$ and $\mathbb{Q}$ as quotients of $\mathbb{N}^2$ show that $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ belong to $V_{\omega + 1}$.
2. As the set of Dedekind cuts in $\mathbb{Q}$, $\mathbb{R}$ belongs to $V_{\omega + 2}$ and $\mathbb{C}$ is in $V_{\omega + 3}$. Most of mathematics can be interpreted within $V_{\omega + n}$ for a large enough $n$.
3. By using codes defined in §7.1.2, every metric structure of density character $\kappa$ has an isomorphic copy in $V_{\kappa + 3}$. Therefore the theory of $C^*$-algebras of density at most $\kappa$ can be developed in $V_{\kappa + n}$ for a large enough $n$.\(^{12}\)

The Axiom of Foundation implies that every set belongs to $V_\alpha$ for some $\alpha$. No mathematical application of this axiom is known and it is unlikely that there will ever be one. This is because the restriction of $\in$ to $V$ is well-founded by definition and every concrete mathematical object can be constructed within $V$ (or at least as a subclass of $V$, like for example in the case of categories). Metamathematically, pinning down every set to some $V_\alpha$ comes very handy. If $x \in V$ then the rank of $x$ is the minimal $\alpha$ such that $x \in V_\alpha$. One can therefore apply transfinite induction and transfinite recursion to the $\in$ relation. The sets $V_\alpha$ are the *rank-initial segments* of the universe.

Within any model of ZF one constructs Gödel’s *constructible universe*, $L$. The hierarchy $L_\alpha$, for $\alpha \in \text{OR}$, is a pared-down analog of von Neumann’s hierarchy. Gödel’s $L$ is the smallest (under the inclusion) transitive model of ZF containing all ordinals.

**Theorem A.5.2 (Gödel).** The constructible universe $L$ satisfies the following.

1. The Axiom of Choice.
2. The Generalized Continuum Hypothesis.
3. (Jensen) $\Diamond_\kappa$ for every uncountable regular cardinal $\kappa$.
4. There exists a projective,\(^{13}\) and even $\Delta^*_2$, well-ordering of the reals. \(\square\)

Every cardinal remains a cardinal in $L$, and every weakly inaccessible cardinal is strongly inaccessible in $L$. For a strongly inaccessible $\kappa$ both $V_\kappa$ and $L_\kappa$ are models of all axioms of ZFC (and $L_\kappa$ is also a model of $V = L$). Therefore Gödel’s Incompleteness Theorem implies that the existence of strongly inaccessible cardinals cannot be proved within ZFC either.\(^{14}\) Every weakly inaccessible cardinal is

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\(^{12}\) This should not be interpreted as saying that there is no reason to study higher realms of von Neumann’s hierarchy, see p. xi.

\(^{13}\) The family of projective subsets of $\mathbb{R}^n$ is the smallest family containing all Borel sets that is closed under continuous images and complements.

\(^{14}\) Unless ZFC is inconsistent. But in the unlikely case that ZFC was inconsistent, this would hardly be the only book that required a complete rewrite.
strongly inaccessible in the constructible universe, and therefore the existence of weakly inaccessible cardinals cannot be proved in ZFC. Inaccessible cardinals are at the bottom of the large cardinal hierarchy (also known as the hierarchy of strong axioms of infinity); see e.g., [149].

A.6 Transitive Models of ZFC*

All models are wrong; some models are useful.\footnote{Whether the first part of Box’s aphorism applies to models of ZFC* is besides the point. These models are useful regardless of whether they are right or wrong, or whether the question of their correctness is meaningful at all.}

George E. P. Box

A structure \((M, \in)\) is a transitive model if its domain is a transitive set and \(E\) coincides with \(\in\). We shall be interested in transitive models of ZFC and its large enough finite fragments. Given \(\varphi\), \(\varphi^M\) denotes the relativization of \(\varphi\) to \(M\), obtained by relativizing all quantifiers occurring in \(\varphi\) to \(M\) (one can think of \(\varphi^M\) as the interpretation of \(\varphi\) in model \(M\), as defined in §D.1). As pointed out in §A.1 ‘a sufficiently large finite fragment of ZFC∗’ is commonly denoted ZFC∗.

ZFC is sufficiently strong to prove the existence of a model of any of its finite fragments. The following is proved by a closing-off argument similar to that in the proof of the Löwenheim–Skolem Theorem formalized within ZFC.

**Theorem A.6.1 (Reflection).** If \(\varphi\) is a formula (possibly with parameters) of ZFC then there exists an ordinal \(\alpha\) such that \(\varphi^V \iff \varphi^{V_\alpha}\). We say that \(\varphi\) reflects to \(V_\alpha\). For a fixed \(\varphi\) the class of all \(\alpha\) such that \(\varphi\) reflects to \(V_\alpha\) is a closed proper class. \(\square\)

**Theorem A.6.2 (Mostowski’s Collapsing Theorem).** Suppose \((M, E)\) is a well-founded model of a sufficiently large fragment of ZFC. Then \((M, E)\) is isomorphic to a unique transitive model.

**Proof.** Since \(E\) is well-founded, the collapsing function \(\pi(x) := \{\pi(y) : y \in E \cdot x\}\) on \(M\) is well-defined by Proposition A.2.2. Since \(E\) is extensional, \(\pi\) is an isomorphism. The facts that \(X := \pi(M)\) is transitive and that \(\pi\) is an isomorphism are proved by transfinite induction (Proposition A.2.1). \(\square\)

A.7 The Structure \(H_\kappa\)

All of mathematics as we know it can be interpreted within ZFC. One of the consequences of this overarching assertion is the fact that the directed and \(\sigma\)-complete poset (Definition 6.2.3) of countable elementary submodels of a large enough model of ZFC∗ ordered by inclusion is, in a certain sense, universal. All we need is a model
A.7 The Structure $H_\kappa$

of a large enough fragment of ZFC, like ones ZFC provided by the following definition.

**Definition A.7.1.** If $\kappa$ is a cardinal then $H_\kappa$ is the set of all sets $X$ whose transitive closure ($\S A.3$) has cardinality smaller than $\kappa$.

**Example A.7.2.**
1. The set of all hereditarily finite sets is $H_{\aleph_0}$. The structure $(H_{\aleph_0}, \in)$ satisfies all the axioms of ZFC except the axiom of infinity. The elements of $H_{\aleph_0}$ can be recursively identified with the natural numbers (Exercise [166, I.14.14]), and the study of $H_{\aleph_0}$ is the subject of number theory.
2. The set of all hereditarily countable sets is $H_{\aleph_1}$. Every countable structure in a fixed countable language has an isomorphic copy in $H_{\aleph_1}$.
3. The Löwenheim–Skolem and Mostowski Collapsing Theorems together imply that for every $\kappa$ there exists a hereditarily countable and transitive set $M$ such that $(M, \in)$ is elementarily equivalent to $(V_\kappa, \in)$. Therefore all large cardinal axioms expressible in $V_\kappa$ reflect to $H_{\aleph_1}$. This has far-reaching consequences to the structure of definable sets of real numbers ([149]).
4. Every separable metric structure in a fixed countable language can be identified with an element of $H_{\aleph_1}$ and much of contemporary mathematics takes place in $H_{\aleph_1}$ (but see p. xi).

If $\kappa$ is an uncountable cardinal, the structure $H_\kappa$ considered with respect to the membership relation $\in$ is a model of a significant fragment of ZFC (see §A.1 and Exercise 6.7.21).

**Example A.7.3.** Suppose $\kappa$ is an infinite cardinal.
1. Since each ordinal is transitive, $\kappa$ is the least ordinal that does not belong to $H_\kappa$.
2. By the Axiom of Choice and (1), every structure (discrete or metric) of cardinality $\kappa$ in a language of cardinality $\kappa$ has an isomorphic copy in $H_{\aleph_1}$.
3. Suppose in addition that $\lambda^{\aleph_0} < \kappa$ for all cardinals $\lambda < \kappa$. Then the small category of metric structures in a fixed countable language of density character strictly smaller than $\kappa$ belongs to $H_{\aleph_1}$. Since the cardinality of a complete metric space of density character $\lambda$ is at most $\lambda^{\aleph_0}$, this is a consequence of (2).
4. If $\kappa$ is an inaccessible cardinal, then $V_\kappa = H_\kappa$. Also, both $\{V_\lambda\}$ and $\{H_\lambda\}$ are increasing families of sets indexed by all cardinals, continuous under limits, whose union is a proper class including all of the universe.\(^{16}\) Any two such hierarchies have to agree for a closed proper class of ordinals by an analog of Proposition 6.2.9).

See Exercise 6.7.22.

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\(^{16}\) The last bit is a consequence of the Axiom of Regularity.
Appendix B
Descriptive Set Theory

I always thought that the descriptive set theory was this thing about counting quantifiers, but then Boban [Veličković] told me ‘No, descriptive set theory is group theory!’

Simon Thomas

The standard references include [152] and [112].

Definition B.0.1. A topological space is Polish if it is separable and completely metrizable. If considering a space which carries more than one natural topology is given, we may talk about Polish topology on this space. Descriptive Set Theory is the study of definable subsets of Polish spaces.

B.1 Trees

In order to remove the ‘noise’ not relevant to set-theoretic considerations of Polish spaces one considers trees and spaces of their branches (see Remark B.1.6).

A poset \((T, \leq_T)\) is a tree if the set of predecessors \(\cdot, t\) \(=\) \(\{s \in T : s \leq_T t\}\) of \(t \in T\) is well-ordered by \(\leq_T\). The elements of a tree are called nodes. A node is terminal if it has no successors in \(T\).

Suppose \(Z\) is a set. By \(Z^\omega\) and \(Z^\infty\) we denote sets of all finite (infinite, respectively) sequences of elements of \(Z\) (\(Z^\omega\) is sometimes denoted \(Z^{<\omega}\) or \(\text{Seq}(Z)\), and \(Z^\infty\) is sometimes denoted \(Z^\omega\)). Notation \(Z^\infty\) is used for the union of these two sets whenever convenient. We have \(Z^\infty = \bigcup_{n \in \mathbb{N}} Z^n\) with the convention that \(Z^0\) has exactly one element, the empty sequence \(\langle\rangle\). The length of a sequence \(s\), denoted \(lh(s)\), is \(n\) such that \(s \in Z^n\) or \(\infty\) if \(s \in Z^\infty\). For \(s \in Z^{<\infty}\) and \(t \in Z^\infty\) we write \(s \subset t\) (or \(t \supset s\)) if \(s\) is an initial segment of \(t\) and say that \(t\) is an end-extension of \(s\), and that \(s\) is an initial segment of \(t\). If \(t \in Z^{<\infty}\) and \(n \leq lh(t)\) or if \(t \in Z^\infty\) then \(t | n\) denotes the unique initial segment of \(t\) of length \(n\). A subset \(T\) of \(Z^{<\infty}\) is a tree on \(Z\) if the set of all initial segments of each of its nodes is included in \(T\). If \(T\) is a tree on \(Z\) then

\footnote{If this does not seem to make sense, see e.g., [239].}
an \( x \in \mathbb{Z}^n \) is a branch of \( T \) if \( x \upharpoonright n \in T \) for all \( n \in \mathbb{N} \). The set of all branches of \( T \) is denoted \([T]\).

An immediate successor of \( s \in T \) is \( t \supseteq s \) such that \( \text{lh}(t) = \text{lh}(s) + 1 \). A tree \( T \) is finitely branching if the set of immediate successors of every \( s \in T \) is finite.

**Theorem B.1.1 (König's Lemma).** If \( T \) is a finitely branching infinite tree then it has an infinite branch. \( \square \)

**Definition B.1.2.** For \( x \) and \( y \) in \( \mathbb{Z}^n \) let \( \Delta(x,y) := \max\{n \in \mathbb{N} : x \upharpoonright n = y \upharpoonright n\} \). Then \( d(x,y) := \frac{1}{\Delta(x,y)+1} \) defines a metric on \( \mathbb{Z}^n \). If \( s \in \mathbb{Z}^n \) then \([s] := \{x \in \mathbb{Z}^n : s \subseteq x\}\) is the clopen \( 1/(n+1) \)-ball centered at any of its elements. If \( s \in \mathbb{Z}^n \) and \( z \in \mathbb{Z} \) then \( s^{-1}z \) is \( t \in \mathbb{Z}^{n+1} \) such that \( t(j) = s(j) \) if \( j < n \) and \( t(n) = z \).

The space \( \mathbb{Z}^n \) is a complete metric space, and it is compact if and only if \( Z \) is finite. The two most important examples of spaces of the form \( \mathbb{Z}^n \) follow.

**Example B.1.3 (The Cantor space).** By mapping \( x \subseteq \mathbb{N} \) to its characteristic function \( \chi_x \in \{0,1\}^\mathbb{N} \), \( \chi_x(n) = 1 \) if \( n \in x \) and \( \chi_x(n) = 0 \) if \( n \notin x \) we obtain a bijection between \( \mathcal{P}(\mathbb{N}) \) and \( \{0,1\}^\mathbb{N} \). This equips \( \mathcal{P}(\mathbb{N}) \) with compact metric. Also, the map \( \{0,1\}^\mathbb{N} \ni x \to \sum_{n \in x} 3^{-n} \in [0,1] \) is a homeomorphism between \( \{0,1\}^\mathbb{N} \) and Cantor's ternary set.

**Example B.1.4 (The Baire space).** Consider \( \mathbb{N} \) with the discrete topology. Then \( \mathbb{N}^\mathbb{N} \) considered with respect to the product topology is the Baire space. Some authors use \( \mathcal{N} \) to denote the Baire space. The Baire space is homeomorphic to the space of the irrationals, via identifying \( x \in [0,1] \setminus \mathbb{Q} \) with the infinite sequence \( n_j \), for \( j \in \mathbb{N} \), corresponding to the continued fraction \( x = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \ldots}} \) \([152]\).  

**Theorem B.1.5.** Suppose that \( X \) is a zero-dimensional Polish space without isolated points. If \( X \) is compact, then it is homeomorphic to the Cantor space. If every compact subset of \( X \) is countable, then \( X \) is homeomorphic to the Baire space. \( \square \)

**Remark B.1.6.** A customary (and innocuous, see Theorem B.2.4) practice in set theory is to refer to \( \mathcal{P}(\mathbb{N}) \), \( \mathbb{N}^\mathbb{N} \), \( \mathbb{R} \), or any other uncountable Polish space as the reals.

### B.2 Polish Spaces

Many familiar topological spaces are Polish.

1. A metric space is compact if and only if it is totally bounded and complete. Therefore every compact metric space is separable, and in particular Polish.
2. All of the spaces \( \mathbb{R} \), \( \mathbb{C} \), \( T := \{z \in \mathbb{C} : |z| = 1\} \), \( \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\} \) are Polish.
3. Every separable C*-algebra is Polish. Therefore any closed ball in \( A \), and any norm-closed subset of \( A \) or \( \tilde{A} \), such as \( \mathcal{P}(A) \), \( \mathcal{U}(\tilde{A}) \), or \( \mathcal{U}_0(\tilde{A}) \), is Polish.
4. If \( H \) is a Hilbert space then the unit ball \( \mathcal{B}(H) \leq 1 \) is WOT-compact. It is metrizable, and therefore Polish, if \( H \) is separable.
5. More generally, if $M$ is a von Neumann algebra with separable predual $M_*$, then its unit ball is, by the Banach–Alaoglu Theorem, compact and metrizable in the weak$^*$-topology obtained by identifying $M$ with the dual space of $M_*$. 

6. If $K$ is a compact metrizable space, then the space $\text{Exp}(K)$ of closed subsets of $K$ with respect to the Vietoris (also known as the exponential) topology is compact and metrizable. Given a compatible metric $d$ on $K$, the Hausdorff distance $d_H(X, Y) := \max(\sup_{a \in X} \inf_{b \in Y} d(a, b), \sup_{b \in Y} \inf_{a \in X} d(a, b))$ is a metric on $\text{Exp}(K)$ compatible with the Vietoris topology.

The following theorem dates back to the times when set theory and functional analysis were one subject (see also §C.2 and §8.5).

**Theorem B.2.1 (Baire Category Theorem).** Assume $(X, d)$ is a complete metric space or a compact Hausdorff space. An intersection of a countable family of dense open subsets of $X$ is dense in $X$. $\square$

A subset of a topological space is *meager* (or of the first category) if it can be covered by countably many nowhere dense sets. The Baire Category Theorem asserts that if $X$ is a complete metric space or a compact Hausdorff space then $X$ is not meager in itself.

**Definition B.2.2.** A subset of a topological space $X$ is *Borel* if it belongs to the $\sigma$-algebra generated by the open subsets of $X$. A subset of $X$ is $G_\delta$ if it is an intersection of a countable sequence of open sets. It is $F_\sigma$ if it is a union of a countable sequence of closed sets. A subset of $X$ is $F_{\sigma\delta}$ if it is the union of a countable sequence of $G_\delta$ sets, $G_{\delta\sigma}$ if it is the intersection of a countable sequence of $F_\sigma$ sets, and so on.

While this old-fashioned terminology works just fine in handling the finite-level Borel sets, one clearly needs a better notation to handle pointclasses of higher complexity. The recursive definition of $\Sigma^0_\alpha$ and $\Pi^0_\alpha$ pointclasses for all $\alpha < \aleph_1$ can be found in [152]; we need not go that far.

A function $f$ between Polish spaces is called *Borel-measurable* (or *Borel*) if the preimage of every Borel set is Borel. It is a *Borel isomorphism* if $f$ is a bijection such that both the $f$ and $f^{-1}$ are Borel.

**Lemma B.2.3.** Suppose $A$ is Borel set in a Polish space $X$.

1. Then $A$ is countable or it contains a homeomorphic copy of the Cantor space.
2. There is a Borel injection from $A$ into the Cantor space.

**Proof.** There is nothing new to say about the proof (1) but here is an elegant proof of (2) due to Jenna Zomback.² Let $U_n$, for $n \in \mathbb{N}$, be an enumeration of the basis for $X$. Define $f(x) \in \mathcal{P}(\mathbb{N})$ by $f(x)(n) = 1$ if $x \in U_n$ and $f(x)(n) = 0$ otherwise. $\square$

² I am grateful to Anush Tserunyan for communicating this proof and to Jenna Zomback for her kind permission to include it.
Theorem B.2.4 (Kuratowski). Every two uncountable Polish spaces are Borel-isomorphic.

Proposition B.2.5. A subset of a Polish space is Polish in the relative topology if and only if it is $G_δ$.

Together with the relevant bit of Theorem B.1.5, this implies the following.

Corollary B.2.6. If $X$ is a Polish space without isolated points, then it has a dense $G_δ$ subset homeomorphic to the Baire space.

At this point it will be convenient to turn all trees upside down—the trees in descriptive set theory are commonly visualized as growing downwards.

Definition B.2.7. A tree $T$ is well-founded if it has no infinite branches. Thus $T$ is well-founded if and only if the poset $(T, \geq_T)$ is well-founded. A tree with an infinite branch is said to be ill-founded.

If $X \subseteq \mathbb{Z}^\mathbb{N}$ then $T_X := \{ x \mid n : x \in X, n \in \mathbb{N} \}$, the set of all initial segments of elements of $X$, is a tree. It is equal to $\mathbb{Z}^{<\mathbb{N}}$ if and only if $X$ is dense in $\mathbb{Z}^\mathbb{N}$. In general, the set $|T_X|$ is equal to the topological closure of $X$. Since $T|_{|T|} = T$, the map $T \mapsto |T|$ is a bijection between trees on $\mathbb{Z}^{<\mathbb{N}}$ and closed subsets of $\mathbb{Z}^\mathbb{N}$.

Proposition B.2.8. A tree $T$ is well-founded if and only if there is a function $\rho_T : T \to |T|^+$ such that $t <_T s$ implies $\rho_T(t) > \rho_T(s)$ for all $s$ and $t$.

Proof. Suppose $\rho_T : T \to \mathbb{OR}$ is such that $s <_T t$ implies $\rho_T(s) > \rho_T(t)$. If $T$ has an infinite branch $b$ then $\rho_T(b \upharpoonright n)$, for $n \in \mathbb{N}$, is an infinite decreasing sequence of ordinals; contradiction. If $T$ is well-founded then $\rho_T(s) := \sup \{ \rho_T(t) + 1 : s <_T t \}$ defines a function from $T$ into $\mathbb{OR}$ by Proposition A.2.2. By induction on $s \in T$ one proves $|\rho_T(s)| \leq |T|$ and therefore $\rho_T|T| \subseteq |T|^+$.

The function $\rho_T$ defined in the proof of Proposition B.2.8 is the rank function on $T$. If $T$ is well-founded then the range of $\rho_T$ is an ordinal, called the rank of $T$. For every ordinal $\alpha$ the tree of all finite decreasing sequences of ordinals less than $\alpha$ has rank $\alpha$ and cardinality $|\alpha|$.

B.2.1 Analytic Sets and the Property of Baire

Suppose $X$ is a Polish space. A subset of $X$ is analytic if it is the range of a continuous function $f : \mathbb{N}^\mathbb{N} \to X$ and coanalytic if it is a complement of an analytic set.

The projection of a tree $T$ on $\mathbb{Z} \times \mathbb{N}$, $p|T|$, is defined as

$$p|T| := \{ x \in \mathbb{Z}^\mathbb{N} : (\exists f \in \mathbb{N}^\mathbb{N}) (\forall n) (x \upharpoonright n, f \upharpoonright n) \in T \}.$$
Theorem B.2.9. A subset of $\mathbb{Z}^\mathbb{N}$ is analytic if and only if it is equal to the projection of some tree $T$ on $\mathbb{Z} \times \mathbb{N}$.

A subset $A$ of a Polish space $X$ has the Property of Baire if there exists an open $U \subseteq \mathcal{P}$ such that $A \Delta U$ is meager. Subsets of $X$ that have the property of Baire form a $\sigma$-algebra that includes all analytic sets, and therefore all $\mathcal{C}$-measurable functions have the Property of Baire. A function between Polish spaces is Baire-measurable if it is measurable with respect to this $\sigma$-algebra.

Proposition B.2.10. A function $f : X \to Y$ between Polish spaces is Baire-measurable if and only if there exists a dense $G_\delta$ subset of $X$ such that the restriction of $f$ to $X$ is continuous.

In the following Proposition, $\text{ZFC}^*$ stands for ‘a fragment of $\text{ZFC}$ large enough to imply Proposition B.2.8.’

Proposition B.2.11. Suppose $M$ is a transitive model of a large enough fragment of $\text{ZFC}$, $Z$ is a countable set, and $T$ is a tree on $\mathbb{Z} \times \mathbb{N}$ that belongs to $M$. Then $p[T]^M = p[T] \cap M$. In other words, for every $x \in \mathbb{Z}^\mathbb{N} \cap M$, $(x \in p[T])^M$ if and only if $x \in p[T]$.

Proof. Suppose that $x \in \mathbb{Z}^\mathbb{N} \cap M$. If $(x \in p[T])^M$ then there exists $f \in \mathbb{N} \cap M$ such that $(x \restriction n, f \restriction n) \in T$ for all $n$, and therefore $x \in p[T]$. Conversely, if $(x \notin p[T])^M$ then the tree $T_x := \{ t : \langle x \restriction |t|, t \rangle \in T \}$ is well-founded. This tree belongs to $M$, and by applying Proposition B.2.8 within $M$ we conclude that there exists a function $\rho_T : T \to |T|^+$ in $M$ such that $s <_T t$ implies $\rho_T(s) > \rho_T(t)$ for all $s$ and $t$ in $T_x$. This function witnesses that $T_x$ is well-founded, and therefore $x \notin p[T]$.

Proposition B.2.11 implies that $M$ correctly computes every analytic set ‘coded’ in $M$. In order to properly formulate a question that begs itself, we introduce some terminology.

The projective hierarchy $\Sigma^1_n$, $\Pi^1_n$, for $n \geq 1$, of subsets of any Polish space is defined by recursion. A set is $\Sigma^1_1$ if it is analytic. For $n \geq 1$, a set is $\Pi^1_n$ if it is a complement of a $\Sigma^1_n$ set. A set is $\Sigma^1_{n+1}$ if it is a continuous image of a $\Pi^1_n$ set. A statement is $\Sigma^1_n$ if it asserts that some $\Sigma^1_n$ set is nonempty. Every $\Sigma^1_n$ statement depends on a parameter, a tree $T$ on $\mathbb{Z} \times \mathbb{N}^\alpha$. The space of all trees on a countable set is a closed subset of the power set of this set, and therefore a compact metric space. Proposition B.2.11 implies that every transitive model $M$ of a sufficiently large fragment of $\text{ZFC}$ is correct about every $\Sigma^1_1$ statement with parameters in $M$. This is the $\Sigma^1_1$-absoluteness Theorem.

Every $\Pi^1_1$ subset of $\mathbb{Z}^\mathbb{N}$ for a countable set $Z$ is the projection of a tree $T$ on $\mathbb{Z} \times \mathbb{R}$, called the Shoenfield tree. Using the argument of Proposition B.2.11, this implies the following (see e.g., [149, Theorem 13.15]).
Theorem B.2.12 (Shoenfield’s Absoluteness Theorem). Every transitive model of a large enough fragment of ZFC that contains all countable ordinals is correct about all $\Sigma^1_2$ statements with parameters in $M$. □

One consequence of this theorem is that $\Sigma^1_2$ statements are immune to the standard methods for proving independence from ZFC. One instance of this phenomenon, the absoluteness of trivial automorphisms of coronas of separable $C^*$-algebras, has already been discussed in Notes to Chapter 17. Many famous open problems in operator algebras, such as the free group factor isomorphism problem and the Connes Embedding Problem, are subject to this theorem. This does not mean that the answers to these problems cannot be independent from ZFC, but it means that if they are independent then proving their independence would require novel methods.

This is not the end of the story. Given the right assumptions, it is possible to present $\Sigma^1_3$ sets, and even $\Sigma^1_n$ sets for all $n$, as projections of a tree on $\mathbb{Z} \times \kappa$ for some $\kappa$ and obtain analogous absoluteness result; see e.g., [149]. More substantial large cardinal assumptions imply that the pointclass of subsets of Polish spaces that can be represented as projections of trees in a manner that implies absoluteness between sufficiently closed countable transitive models of ZFC forms a $\sigma$-algebra closed under the continuous images of its elements (see [102]).

### B.2.2 Uniformization

For $A \subseteq X \times Y$ we write $\pi_X[A] := \{x : (\exists y)(x,y) \in A\}$. A uniformization of $A \subseteq X \times Y$ is a function $f : \pi_X[A] \to X$ whose graph is included in $A$. While a uniformization can be found using the Axiom of Choice, not every Borel subset of a product of Polish spaces can be uniformized by a Borel function. A function between Polish spaces is $C$-measurable if the preimage of every open set belongs to the $\sigma$-algebra generated by analytic sets. (Some authors call such functions $\sigma(\Sigma^1_1)$-measurable.) Since analytic sets are universally measurable, $C$-measurable functions share many regularity properties of Borel functions. In particular, they are Baire-measurable.

Theorem B.2.13 (Jankov, von Neumann). If $X$ and $Y$ are Polish spaces then every analytic $A \subseteq X \times Y$ can be uniformized by a $C$-measurable function. □

The following is not a uniformization theorem, but it will be used in §17.7 in conjunction with Theorem B.2.13. It is proved as [152, Theorem 29.22].

Theorem B.2.14 (Novikov). If $X$ and $Y$ are Polish spaces and $A \subseteq X \times Y$ is analytic, then the set $\{x \in X : A_x \text{ is nonmeager}\}$ is analytic. □
Appendix C
Functional Analysis

The standard texts include [196], [267], [212], and [10]. More information on specific topics can be found in [16], [229], and [228].

C.1 Topological Vector Spaces

A vector space $X$ over a field $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$, with the standard Polish topology) is a topological vector space if it is equipped with a Hausdorff topology such that both vector addition and scalar multiplication are jointly continuous.

If $X$ is a topological vector space and $Y \subseteq X$ then the closed linear span of $Y$, denoted $\text{span}(Y)$, is the closure of the set of all linear combinations of elements of $Y$. Every subset of $X$ of the form $\text{span}(Y)$ is a closed linear subspace, and $Y$ is a closed linear subspace if and only if $Y = \text{span}(Y)$.

Morphisms in the category of topological vector spaces are continuous linear maps. A linear map between topological vector spaces is continuous if and only if it is continuous at 0.

A seminorm on a vector space $X$ is a function $\| \cdot \| : X \to [0, \infty)$ such that $\|x\| \geq 0$, $\|x + y\| \leq \|x\| + \|y\|$, and $\|sx\| = |s|\|x\|$ for all $x, y$ in $X$ and every scalar $s$. It is a norm if $\|x\| = 0$ implies $x = 0$.

A normed space $(X, \| \cdot \|)$ is a metrizable topological vector space, with respect to the metric $d(x, y) = \|x - y\|$.

The $n$-ball of a normed space $X$ is $X_{\leq n} := \{x \in X : \|x\| \leq n\}$. The 1-ball is called the unit ball. A subset $A$ of a topological vector space $X$ is bounded if for every open neighbourhood $U$ of 0 there is $n$ such that $nU \supseteq A$ (where $nU := \{nx : x \in U\}$). Therefore a subset of a normed space is bounded if and only if it is included in the $n$-ball for a large enough $n$. An $m$-dimensional topological vector space is linearly homeomorphic to $\mathbb{K}^m$. Therefore the closed $n$-ball of a finite-dimensional normed space is compact by the Heine–Borel theorem.
Proposition C.1.1 (Riesz Lemma). If X is a normed space and Y is a proper closed subspace of X, then for every ε > 0 there exists x ∈ X such that ∥x∥ = 1 and dist(x, Y) > 1 − ε. □

Definition C.1.2. Banach space is a complete normed vector space (X, ∥·∥).

A linear space is a pre-Hilbert space if it is equipped with the inner product ⟨·,·⟩ which is sesquilinear, i.e., linear in the first coordinate and conjugate linear in the second. A norm on a pre-Hilbert space is defined by ∥ξ∥ := (∥ξ∥2)1/2. A pre-Hilbert space is a Hilbert space if it is complete.

Example C.1.3. The space ℓ2(N) := {α ∈ ℓ∞(N) : ∑n |αn|2 < ∞} is a Hilbert space with respect to the inner product ⟨α|β⟩ := ∑n αnβn.

A family of vectors ξj, for j ∈ J in a Hilbert space is orthonormal if ⟨ξi|ξj⟩ = δij (where δij = 0 if i ≠ j and δii = 1). It is an orthonormal basis if H = span{ξj : j ∈ J}. By the (transfinite) Gram–Schmidt orthogonalization process, every Hilbert space has an orthonormal basis (or shortly, basis).

Since the rational linear combinations of vectors in an orthonormal basis are dense in H, H has a basis of an infinite cardinality κ if and only if its density character is κ. Such κ is also called the dimension of H. (The Hilbert space is quite exceptional: there are Banach spaces, even separable Banach spaces, without a basis.) The dimension of a finite-dimensional Hilbert space is the cardinality of any orthonormal basis of H. Two Hilbert spaces of the same dimension (and over the same scalar field) are isometrically isomorphic. In particular, up to isomorphism there is only one complex, separable, infinite-dimensional Hilbert space. (For most practical purposes, this is the Hilbert space.)

Example C.1.4. 1. The space ℓ∞(N) of all bounded sequences in C with respect to the supremum norm, ∥α∥∞ := supn |αn| is a Banach space.
2. The space c0(N) := {α ∈ ℓ∞(N) : limn |αn| = 0} is a closed subspace of ℓ∞(N), and therefore a Banach space with respect to ∥·∥∞.
3. The space c00(N) := {α ∈ ℓ∞(N) : (∃ωn)αn = 0} is norm-dense in c0(N).
4. The space ℓ1(N) := {α ∈ ℓ∞(N) : ∑n |αn| < ∞} is norm-dense in c0(N). It is a Banach space with respect to the norm ∥α∥1 := ∑n |αn|.

Given an arbitrary index set J, the spaces c00(J), c0(J), ℓ∞(J), and ℓ1(J) are defined analogously. It is common to write ℓ∞, c0, c00, ℓ1, . . . for ℓ∞(N), c0(N), c00(N), or ℓ1(N).

We move on to consider the function spaces.

Example C.1.5. Suppose X is a locally compact Hausdorff space and K is R or C.

1. A continuous function f : X → C vanishes at infinity if for every ε > 0 the set {x ∈ X : |f(x)| ≥ ε} is compact. The space C0(X, K) of all continuous K-valued functions vanishing at infinity is a Banach space with respect to the sup norm,

\[ ||f||_∞ := \sup\{x ∈ X : |f(x)|\}. \]
By the compactness of $X$, for every $f \in C_0(X, \mathbb{K})$ the norm of $f$ is attained at some $x \in X$. In this book $C_0(X)$ stands for $C_0(X, \mathbb{C})$.

2. The vector space of all complex Radon measures over $X$ is denoted $M(X)$. A norm on $M(X)$ defined by $\|\mu\| := \sup\{|\int f \, d\mu : f \in C_0(X), \|f\|_\infty \leq 1\}$ is complete. The space of real Radon measures on $X$ is defined analogously.

The standard analog of a basis for a Banach space, Schauder basis, is defined and discussed in Definition 7.4.5. Not every Banach space has a Schauder basis, but every infinite-dimensional Banach space has an infinite-dimensional subspace with a Schauder basis.

Suppose $(X, \| \cdot \|)$ is a normed space and $Y$ is a closed subspace of $X$. On the quotient vector space $X/Y$ the norm is given by $\|x + Y\| := \inf_{y \in Y} \|x + y\|$. The quotient $(X/Y, \| \cdot \|)$ normed space is a Banach space if $X$ is a Banach space. This construction is used to prove the following.

**Theorem C.1.6.** Suppose that $(X, \| \cdot \|)$ is a normed vector space. Then there is an isometric isomorphism of $X$ into a dense subspace of a Banach space, unique up to the isomorphism. This space is the completion of $X$. \hfill \Box

**Example C.1.7.** If $\mu$ is a positive Radon measure on a locally compact Hausdorff space $X$ the Hilbert space $L_2(X, \mu)$ (and any other $L_p$ space) can be presented in two different ways. The space $\{f \in C_0(X) : \int |f|^2 \, d\mu < \infty\}$ is a pre-Hilbert space with respect to the inner product $(f|g) := \int f \overline{g} \, d\mu$, and $L_2(X, \mu)$ is isomorphic to its completion.

Alternatively, denoting the $\sigma$-algebra of all $\mu$-measurable sets by $\Sigma$, consider the space of all $\Sigma$-measurable functions $f : X \to \mathbb{C}$, such that $|f|^2$ is $\mu$-integrable. On this space $\|f\|_2$ is a seminorm, and the quotient is isomorphic to $L_2(X, \mu)$.

**C.2 Consequences of the Baire Category Theorem**

Every Banach space is completely metrizable and therefore subject to the Baire Category Theorem (Theorem B.2.1). Since every finite-dimensional subspace of a topological vector space is closed, the Baire Category Theorem implies that the vector space dimension of an infinite-dimensional Banach space is uncountable.

When combined with clever geometric arguments, the Baire Category Theorem has far-reaching consequences to the structure of Banach spaces.

**Theorem C.2.1 (Open Mapping Theorem).** Suppose $X$ and $Y$ are Banach spaces and $f : X \to Y$ is a bounded linear map such that $f[X]$ is of second category (i.e., nonmeager) in $Y$. Then $f$ is open. \hfill \Box

**Corollary C.2.2.**

1. Every bounded bijection between Banach spaces has a bounded inverse.

2. If $f : X \to Y$ is a bounded linear map between Banach spaces then $f[X]$ is either equal to $Y$ or it is of the first category in $Y$. \hfill \Box
Corollary C.2.3. Suppose $X$ is a vector space. $\| \cdot \|$ and $\| \cdot \|$ are two Banach space norms on $X$, and there is $r < \infty$ such that $\|a\| \leq r\|a\|$ for all $a$. Then there is $s > 0$ such that $\|a\| \geq s\|a\|$ for all $a$. □

If $X$ and $Y$ are normed spaces, then $X \times Y$ is normed by $\|(x, y)\| := \max(\|x\|, \|y\|)$. If $X$ and $Y$ are Banach spaces, so is $X \times Y$. The graph $\Gamma_f := \{(x, f(x)) : x \in X\}$ of a linear map is a vector subspace of $X \times Y$.

Theorem C.2.4 (Closed Graph Theorem). If $X$ and $Y$ are Banach spaces and $f : X \to Y$ is a linear map then $f$ is continuous if and only if $\Gamma_f$ is a closed subspace of $X \times Y$. □

If $X$ and $Y$ are normed spaces, let $B(X, Y)$ denote the vector space of all bounded linear operators from $X$ to $Y$. On $B(X, Y)$, $\|f\| := \inf \{r \geq 0 : f(B_X) \subseteq rB_Y\}$ defines the operator norm.

Proposition C.2.5. If $X$ and $Y$ are normed spaces so is $B(X, Y)$. If in addition $Y$ is a Banach space then $B(X, Y)$ is a Banach space.

The following is also known as the Uniform Boundedness Principle.

Theorem C.2.6 (Banach–Steinhaus Theorem). If $X$ and $Y$ are Banach spaces, $\mathcal{F} \subseteq B(X, Y)$, and for every $x \in X$ the set $\mathcal{O}_x := \{f(x) : f \in \mathcal{F}\}$ is bounded then

$$\sup \{\|f\| : f \in \mathcal{F}\} < \infty.$$ □

C.3 Duality

Definition C.3.1. Suppose $X$ is a normed space. A linear functional is a linear map $\varphi$ from $X$ into the field of scalars $\mathbb{K}$. It is bounded if $\|\varphi\| := \sup_{x \in X, \|x\| \leq 1} |\varphi(x)|$ is finite.

The dual space of a topological vector space $X$, denoted $X^*$, is the space of all continuous linear functionals on $X$.

A linear functional between normed spaces is bounded if and only if it is continuous. Therefore the dual $X^*$ of a normed space $X$ is naturally isomorphic to $B(X, \mathbb{K})$ and it is a Banach space by Proposition C.2.5. In addition to the norm topology (also known as the uniform topology) $X^*$ carries other important topologies ($\S$C.4).

A proof of the following theorem requires a fragment of the Axiom of Choice. If all sets of reals have the Property of Baire, then the quotient Banach space $\ell_\infty/c_0$ has no nonzero bounded linear functionals (cf. Exercise 9.10.15 and Exercise 12.6.6).

Theorem C.3.2 (Hahn–Banach Extension Theorem). Suppose $X$ is a normed space, $Y$ is a subspace of $X$, and $\varphi$ is a bounded linear functional on $Y$. Then $\varphi$ can be extended to a functional $\psi$ on $X$ such that $\|\psi\| = \|\varphi\|$.

Corollary C.3.3. Assume $X$ is a normed space and $x \in X$. Then there exists $\varphi \in X$ such that $|\varphi(x)| = \|x\|$ and $\|\varphi\| = 1$. □
The functional \( \varphi \) as in Corollary C.3.3 is called the norming functional for \( x \).

**Corollary C.3.4.** Assume \( X \) is a normed space and \( Y \) is a proper closed subspace of \( X \). Then for every vector \( x \in X \setminus Y \) there exists a functional \( \varphi \in X^* \) such that \( \| \varphi \| = 1 \), \( Y \subseteq \ker(\varphi) \), and \( \varphi(x) = \dist(x,Y) \). \( \square \)

If \( X \) is a normed space then every \( x \in X \) defines a linear functional \( x^{**} \) on \( X^* \) by \( x^{**}(\varphi) := \varphi(x) \). By Corollary C.3.3, \( \| x^{**} \| = \| x \| \). Therefore \( x \mapsto x^{**} \) is an isometry of \( X \) into a subspace of \( X^{**} \) and we have the following.

**Corollary C.3.5.** If \( X \) is a normed space, then the completion of \( X \) is isometrically isomorphic to a subspace of the second dual \( X^{**} \) of \( X \). \( \square \)

**Definition C.3.6.** Two linear spaces \( X \) and \( Y \) over \( K \) are in algebraic duality if there is a bilinear form \( (\cdot,\cdot) : X \times Y \to K \) such that the functionals \( (\cdot,y) \) for \( y \in Y \) separate points in \( X \) and the functionals \( (x,\cdot) \) for \( x \in X \) separate points in \( Y \). If in addition \( X \) and \( Y \) are normed spaces, and all functionals \( (\cdot,y) \) and \( (x,\cdot) \) are bounded, then the spaces \( X \) and \( Y \) are said to be in duality.

**Example C.3.7.** 1. Any pair \( X,X^* \) is in duality with the bilinear form given by the functional evaluation, \( (x,\varphi) := \varphi(x) \).
2. By the Fréchet–Riesz Theorem, if \( H \) is a Hilbert space then every \( \varphi \in H^* \) is of the form \( \varphi(\xi) = (\xi,\eta_\varphi) \) for some \( \eta_\varphi \in H \). The function \( \varphi \mapsto \eta_\varphi \) is a conjugate-linear isometry of \( H^* \) onto \( H \).
3. The dual of \( c_0(\mathbb{N}) \) is isomorphic to \( \ell_1(\mathbb{N}) \), and the dual of \( \ell_1(\mathbb{N}) \) is isomorphic to \( \ell_\infty(\mathbb{N}) \). Both dualities are implemented by \( (\bar{a},\bar{b}) := \sum_n a(n)b(n) \).

The spaces \( C_0(X) \) and \( M(X) \) were defined in Example C.1.5. Note that \( c_0 \) is isometrically isomorphic to \( C_0(\mathbb{N}) \) (where \( \mathbb{N} \) is taken with the discrete topology), and its dual is \( \ell_1 \). If \( X \) does not have a dense set of isolated points, then the dual of \( C_0(X) \) is considerably richer.

**Theorem C.3.8 (Riesz Representation Theorem).** If \( X \) is a locally compact Hausdorff space then \( C_0(X)^* \) is isometric to \( M(X) \) via \( (f,\mu) := \int_X f \, d\mu \). \( \square \)

The annihilator of a subset \( Y \) of a normed space \( X \) is defined as \( Y^\perp := \{ \varphi \in X^* | Y \subseteq \ker(\varphi) \} \). The annihilator of \( Z \subseteq X^* \) is \( Z^\perp := \{ x \in X | \varphi(x) = 0 \text{ for all } \varphi \in Z \} \).

**Proposition C.3.9.** Suppose \( Y \) is a closed subspace of a Banach space \( X \). Then \( Y^* \cong X^*/Y^\perp \) and \((X/Y)^* \cong Y^\perp \). \( \square \)

**Proposition C.3.10.** If \( X \) is a normed space and \( Y \) is a subspace of \( X \) then \((Y^\perp)^\perp \) is the norm-closure of \( Y \). In particular, if \( Y \) is a closed subspace then \((Y^\perp)^\perp = Y \). \( \square \)

**Example C.3.11.** It is not true that if \( Z \) is a subspace of \( X^* \) then \((Z^\perp)^\perp \) is the norm-closure of \( Z \). For example, take \( X = \ell_1 \) and \( Z = c_0 \), as identified with a subspace of \( \ell_\infty \). Then \( Z \) is norm-closed, but \((Z^\perp)^\perp = \ell_\infty \) (but there is a more suitable topology for this situation; see Proposition C.4.11).
A Banach space is **reflexive** if every functional in $X^{**}$ is implemented by a vector in $X$. Every Hilbert space is reflexive (Example C.3.7), and Hilbert spaces are the only reflexive Banach spaces that appear in this book.

The **shift** on $\ell_\infty(\mathbb{N})$ is a linear operator defined by $S(x)_n := x_{n+1}$ for all $n$. The linear functional $\text{Lim}$ whose existence is guaranteed by the following theorem, proved using the Hahn–Banach extension theorem, is called the **Banach limit**.

**Theorem C.3.12.** On the real $\ell_\infty(\mathbb{N})$ there is a bounded linear functional $\text{Lim}$ such that $\text{Lim}(S(x)) = \text{Lim}(x)$ and $\liminf_n x_n \leq \text{Lim}(x) \leq \limsup_n x_n$ for all $x \in \ell_\infty$. □

### C.4 Weak Topologies

In this subsection we introduce a family of vector space topologies hinted at in Example C.3.11.

**Definition C.4.1.** A family $\mathcal{F}$ of seminorms or functionals on a vector space $X$ is **separating** if for all nonzero $x$ in $X$ there is $\mu \in \mathcal{F}$ such that $\mu(x) \neq 0$. The **weak topology** on $X$ induced by a separating family $\mathcal{F}$ is the weakest topological vector space topology on $X$ with respect to which all $\mu \in \mathcal{F}$ are continuous.

With $\mathcal{F}$ as in Definition C.4.1, we can identify $x \in X$ with the evaluation function $\hat{x}$ on $\mathcal{F}$, $\hat{x}(\phi) := \phi(x)$. If $\mathcal{F}$ is sufficiently rich, the weak topology induced by $\mathcal{F}$ is the subspace topology on $X$ when identified with $\{\hat{x} : x \in X\} \subseteq \prod_{\phi \in \mathcal{F}} \mathbb{K}$.

A vector $x$ in a Banach space $X$ is identified with the linear functional $x^{**}$ on $X^*$.

**Example C.4.2.** The weak topology on $X^*$ induced by $\{x^{**} : x \in X\}$ is called the **weak* topology**. Therefore a net $\{\phi_\lambda\}$ in $X^*$ converges to $\phi$ if $\lim_\lambda \phi_\lambda(x) = \phi(x)$ for every $x \in X$. If we identify elements of $X^*$ with functions on $X$ the weak* topology coincides with the topology of pointwise convergence. The weak topology on a topological vector space $X$ is the weak topology induced by $X^*$.

**Theorem C.4.3 (Hahn–Banach Separation Theorem).** Suppose $X$ is a topological vector space and $W$ and $A$ are its disjoint convex subsets. Suppose moreover that $W$ is open. Then there exist a continuous linear functional $\phi \in X^*$ and $r \in \mathbb{R}$ such that $\Re(x, \phi) < r \leq \Re(a, \phi)$ for all $x \in W$ and all $a \in A$. □

The following lemma is known to logicians as the assertion that logic-continuous functions on the type space correspond to formulas (see §16.1).

**Lemma C.4.4.** Suppose $Y$ is a vector space. For $n \in \mathbb{N}$, functionals $\phi_j$, for $j < n$, and $\psi$ on $Y$ the following are equivalent.

1. The functional $\psi$ is a linear combination of $\phi_j$, for $j < n$.
2. There exists $r > 0$ such that $|\psi(y)| \leq \max_{j \leq n} r|\phi_j(y)|$ for all $y \in Y$.
3. $\ker \psi \supseteq \bigcap_{j<n} \ker \phi_j$. □
Corollary C.4.5. Suppose that $X$ is a topological vector space and $W$ and $A$ are disjoint convex subsets of $X^*$ and $W$ is open. Then there exist $x \in X$ and $r \in \mathbb{R}$ such that $\Re(x, \phi) < r \leq \Re(x, \psi)$ for all $\phi \in W$ and all $\psi \in A$.

Corollary C.4.6. If $X$ is a normed space. If $X^*$ is taken with respect to the weak$^*$-topology, then its dual is naturally isomorphic to $X$.

A clever application of Tychonoff’s Theorem gives the following.

Theorem C.4.7 (Banach–Alaoglu). Suppose $X$ is a topological vector space and $U$ is a neighbourhood of 0 in $X$. Then the polar of $U$, $Z := \{ \phi \in X^* : \sup_{x \in U} |\phi(x)| \leq 1 \}$ is weak$^*$-compact.

Corollary C.4.8. The unit ball of the dual of a normed space is compact in the weak$^*$-topology.

Definition C.4.9. Given normed spaces $X$ and $Y$, the transpose of a linear map $T : X \to Y$ is $T^* : Y^* \to X^*$ defined by $T^*(\phi) := \phi \circ T$.

Proposition C.4.10. If $X$ and $Y$ are Banach spaces and $T \in B(X,Y)$ then $T^*$ is bounded and weak$^*$-weak$^*$ continuous. Every weak$^*$-weak$^*$-continuous linear map $S \in B(Y^*,X^*)$ is of this form, and the map $T \mapsto T^*$ from $B(X,Y)$ into $B(Y^*,X^*)$ is a linear isometry.

A consequence of the Hahn–Banach separation theorem complements Proposition C.3.10 and Example C.3.11.

Proposition C.4.11. If $X$ is a normed space and $Z$ is a subspace of $X^*$, then $(Z^\bot)^\bot$ is equal to the weak$^*$ closure of $Z$.

C.5 Convexity

Convex is good (a smiley), concave is bad (a frowny).

Nassim Nicholas Taleb, Antifragile: Things That Gain from Disorder

A subset $K$ of a topological vector space $X$ is convex if it includes the line segment $\{ ra + (1-r)b : 0 \leq r \leq 1 \}$ for all $a$ and $b$ in $K$. A convex combination of a finite set of elements $a_j$, for $j < n$, of $X$ is a vector of the form $\sum_{j<n} r_j a_j$ where $0 \leq r_j \leq 1$ for all $j$ and $\sum_j r_j = 1$. By induction on $n$ one can prove that a set is convex if and only if it contains all convex combinations of its elements. The convex closure of $Y \subseteq X$, denoted $\text{conv}(Y)$, is the topological closure of the set of all convex combination of elements of $Y$. It is equal to the intersection of all closed convex sets that include $Y$.

A topological vector space is locally convex if it has a basis consisting of convex sets.
Theorem C.5.1. A topological vector space is locally convex if and only if its topology is a weak topology induced by a family of seminorms.

A face of a convex set $K$ is a nonempty $L \subseteq K$ such that a convex combination of two points of $K$ belongs to $L$ if and only if both points belong to $L$. In particular every face is convex. A point $x$ of a convex set $K$ is an extreme point if $\{x\}$ is a face.

Lemma C.5.2. A face of a face is a face. An extreme point of a face is an extreme point of the original convex set.

Theorem C.5.3 (Krein–Milman). Suppose $A$ is a compact convex subset of a locally convex topological vector space $X$. Then the set of all convex combinations of extreme points of $A$ is dense in $A$.

A measure $\mu$ on $X$ is a point mass measure concentrated at some $y \in X$ if $\mu(\{y\}) = 1$ if $y \in A$ and $\mu(\{a\}) = 0$ otherwise.

Example C.5.4. Suppose that $X$ is a compact Hausdorff space and consider the space $M_{+1}(X)$ of all probability Radon measures on $X$. By Theorem C.3.8, it can be identified with a subset of the unit ball of $C(X)^\ast$. Since it is clearly closed, it is compact by the Banach–Alaoglu Theorem. It is also clearly convex, and therefore abundant in extreme points by the Krein–Milman Theorem. It is good to know that the extreme points are exactly the point mass measures. Every point mass measure is clearly an extreme point of $M_{+1}(X)$. Conversely, if $\mu$ is not a point mass measure, then there is a measurable $A \subseteq X$ such that $\mu(A) \neq 0$ and $\mu(X \setminus A) \neq 0$. Then $\mu$ can be expressed as a convex combination of $\nu_0(B) := \mu(A)^{-1}\mu(B \cap A)$ and $\nu_1(B) := \mu(X \setminus A)^{-1}\mu(B \setminus A)$, and it is therefore not an extreme point.

Definition C.5.5. An algebra over a field $K$ is a $K$-vector space $(X, +)$ equipped with multiplication $\cdot$ such that $(X, -, +)$ is a ring. It is a Banach algebra if it carries a Banach space norm that is submultiplicative, i.e., $\|xy\| \leq \|x\|\|y\|$ for all $x$ and $y$.

Theorem C.5.6 (Stone–Weierstrass). Suppose $X$ is a compact Hausdorff space and $A$ is a subalgebra of $C(X, \mathbb{R})$ containing all constant functions. Then $A$ separates points in $X$ if and only if it is dense in $C(X)$.

The complex version of the Stone–Weierstrass theorem requires an additional assumption. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $A \subseteq C(D)$ be the set of all analytic functions. Then $A$ is a complex algebra that separates the points of $D$. However, the complex conjugation, $z \mapsto \overline{z}$, is a continuous function that does not belong to the uniform closure of $A$.

Theorem C.5.7 (Stone–Weierstrass, complex version). Suppose $X$ is a compact Hausdorff space and $A$ is a self-adjoint subalgebra of $C(X)$ containing all constant functions. Then $A$ separates points in $X$ if and only if it is dense in $C(X)$.

We will also need the lattice version of the Stone–Weierstrass theorem in §16.3.
Theorem C.5.8 (Stone-Weierstrass, lattice version). Suppose \( X \) is a compact Hausdorff space and \( A \) is a sublattice of \( C(X, \mathbb{R}) \) containing all constant functions. Then \( A \) separates points in \( X \) if and only if it is dense in \( C(X) \). \( \square \)

Definition C.5.9. A convex cone, or simply a cone, is a convex subset of a vector space closed under multiplication by positive scalars.

A convex cone is automatically additive, i.e., closed under addition of its elements.

C.6 Operator Theory and Spectral Theory

Throughout this section \( H \) denotes an infinite-dimensional Hilbert space. Since \( H \) is reflexive, its weak topology coincides with the weak* topology and, by the Banach–Alaoglu theorem, the closed unit ball is weakly compact.

Conditions (3) and (4) of the following theorem are not necessarily equivalent for an arbitrary Banach space.

Theorem C.6.1. For an operator \( a \in B(H) \) the following are equivalent.

1. \( a \) is in the norm-closure of \( \mathcal{B}_f(H) \), the algebra of finite-rank operators on \( H \).
2. The restriction of \( a \) to the unit ball of \( H \) is weak-norm continuous.
3. The a-image of the unit ball of \( H \) is norm-compact.
4. The norm-closure of the a-image of the unit ball of \( H \) is norm-compact. \( \square \)

An operator satisfying any of the equivalent conditions in Theorem C.6.1 is called compact and the algebra of compact operators on \( H \) is denoted \( \mathcal{K}(H) \). A strengthening of the following standard fact is proved in Proposition 12.3.4.

Proposition C.6.2. The algebra \( \mathcal{K}(H) \) is a self-adjoint, two-sided, norm-closed ideal of \( B(H) \). If \( H \) is separable then the only other self-adjoint, two-sided, norm-closed ideals of \( B(H) \) are \( \{0\} \) and \( B(H) \). \( \square \)

The quotient \( \mathscr{L}(H) := B(H)/\mathcal{K}(H) \), called the Calkin algebra, is a C*-algebra that plays a major role in this text. An operator \( a \in B(H) \) is Fredholm if both \( \ker(a) \) and \( \ker(a^*) \) are finite-dimensional and its range \( a[H] \) is a closed subspace of \( H \). The Fredholm index of a Fredholm operator is defined to be

\[
\text{index}(a) := \dim(\ker(a)) - \dim(\ker(a^*)).
\]

Theorem C.6.3 (Atkinson’s Theorem). An operator \( a \) is Fredholm if and only if \( \pi(a) \) is invertible in the Calkin algebra \( \mathscr{L}(H) \). \( \square \)

Example C.6.4. 1. The unilateral shift \( s \) (Example 1.1.1) is a Fredholm operator of index \(-1\).
2. If \( a \) is normal then \( \ker(a) = \ker(a^*a) = \ker(aa^*) = \ker(a^*) \), and therefore a normal Fredholm operator has index 0.

**Proposition C.6.5.** The function \( \pi(a) \mapsto \text{index}(a) \) from \( GL(\mathcal{D}(H)) \) into \( \mathbb{Z} \) is a continuous and surjective group homomorphism.

**Definition C.6.6.** A bounded linear operator \( a \) on \( H \) is normal if \( aa^* = a^*a \) and self-adjoint if \( a = a^* \). The spectrum of \( a \) is \( \text{sp}(a) := \{ \lambda \in \mathbb{C} : a - \lambda \text{ is not invertible} \} \).

If \( H \) is finite-dimensional, then \( \text{sp}(a) \) is the set of eigenvalues of \( a \). The multiplication operator \( M_f \) on \( L^2([0,1], \text{Lebesgue}) \) associated with \( f(t) = t \) is self-adjoint and it has no eigenvectors. Its spectrum is equal to \([0,1] \).

**Proposition C.6.7.** For every \( a \in B(H) \), \( \text{sp}(a) \) is a nonempty and closed subset of \( \{ z \in \mathbb{C} : |z| \leq \|a\| \} \).

**Theorem C.6.8.** If \( a \in B(H) \) is normal and compact, then \( H \) has an orthonormal basis consisting of eigenvectors for \( a \).

**Theorem C.6.9.** If \( A \subseteq B(H) \) is an abelian \( C^* \)-algebra then some probability measure space \((X, \mu)\) and a unitary \( u : H \to L^2(X, \mu) \) satisfy \( uAu^* \subseteq L^\infty(X, \mu) \).

The following is Theorem 3.1.10.

**Theorem C.6.10.** Suppose \( M \) is a von Neumann algebra and \( a \in M \) is normal. Then \( W^*(a) \) is isomorphic to \( L_\infty(\text{sp}(a), \mu) \) for some Radon probability measure \( \mu \) on \( \text{sp}(a) \). The isomorphism sends \( a \) to the equivalence class of the identity function and \( f \) to \( f(a) \).

A proof of the following theorem can be found e.g., in [196, Theorem 4.7.7] (masa in a \( C^* \)-algebra is a maximal abelian subalgebra).

**Theorem C.6.11.** 1. If \( (X, \mu) \) is a measure space, then \( L_\infty(X, \mu) \) is a masa in \( B(L^2(X, \mu)) \).

2. Conversely, if \( D \) is a masa in \( B(H) \) then there are a measure space \( (X, \mu) \) and a unitary \( u : H \to L^2(X, \mu) \) such that \( uDu^* = L_\infty(X, \mu) \).

**Theorem C.6.12 (Fuglede).** If \( a \) and \( b \) are in \( B(H) \), \( ab = ba \), and \( a \) is normal, then \( a^*b = ba^* \).

**Corollary C.6.13.** If \( a \in B(H) \) is normal, then \( \{a\}' \cap B(H) \) is a \( C^* \)-algebra.

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### C.7 Ultraproducts in Functional Analysis

Ultraproducts have been used by functional analysts to study \( \text{II}_1 \) factors, Banach spaces, and \( C^* \)-algebras for decades.\(^1\) Variants of the following definition exist for Banach spaces, \( \text{II}_1 \) factors, and other Banach space-based structures.

\(^1\) Functional analysts used ultrapowers of \( \text{II}_1 \) factors before Loš’s seminal paper. See the introduction to [227].
Definition C.7.1. Suppose $U$ is an ultrafilter on an index set $J$ and $A_j$, for $j \in J$, are $C^*$-algebras. The elements $a$ of $\prod_{j \in J} A_j$ are norm-bounded indexed families $(a_j : j \in J)$. Then $c_U = \{ a \in \prod_j A_j : \lim_{j \to U} \|a_j\| = 0 \}$ is a two-sided, self-adjoint, norm-closed ideal of $\prod_j A_j$, and the quotient $\prod_U A_j := \prod_j A_j / c_U$ is the ultraproduct associated to $U$. If all $A_j$ are equal to some $A$, the ultraproduct is denoted $A^U$ and called ultrapower.

One identifies $A$ with its diagonal image in the ultrapower. The relative commutant of $A$ in its ultrapower is $A' \cap A^U := \{ b \in A^U : ab = ba \text{ for all } a \in A \}$. 
Appendix D
Model Theory

As logicians we do our subject a disservice by convincing others that logic is first-order logic and then convincing them that almost none of the concepts of modern mathematics can really be captured in first-order logic.

J. Barwise ([21])

In this section we review some model theory with an emphasis on model theory of metric structures ([22]) and model theory of C*-algebras ([87, §2]). We will consider only some rudimentary model-theoretic notions (elementary submodels, types, and saturation). The classical (‘discrete’) model theory is used only briefly in this book. Standard sources include [39], [131], and [178] for classical theory, [22] for model theory of metric structures, and [87] for model theory of C*-algebras.

D.1 The Classical (Discrete) Theory

In the discrete case we treat only the single-sorted languages. A language $\mathcal{L}$ is a triple $(F, R, C)$ which contains the following data:

1. The set of function symbols, $F$. For each function symbol $f \in F$ we specify its \textit{arity}, $n(f) \geq 1$.
2. The set of relation symbols, $R$. For each $R \in R$ we specify its \textit{arity}, $n(R) \geq 1$.
3. The set of constant symbols, $C$.

Given a language $\mathcal{L}$, an $\mathcal{L}$-structure is a quadruple $\mathfrak{A} = (A, F, R, C)$ which consists of the following:

4. The set $A$ which is the \textit{domain} of $\mathfrak{A}$.
5. For each $f \in F$ a function $f^A : A^{n(f)} \rightarrow A$.
6. For each $R \in R$ a relation $R^\mathfrak{A} \subseteq A^{n(R)}$.

\footnote{It should not be necessary to emphasize (but I will) that model theoretic methods have been invaluable in some of the most sophisticated applications of logic to the mainstream mathematics. The use of model theory in the present book is reduced to a convenient language.}
7. For each $c \in C$ an element $c^A \in A$.

We say that $f^A$, $R^A$, and $c^A$ are the interpretations of $f$, $R$, and $c$, respectively, in $A$.

The language $\mathcal{L}$ is equipped with an infinite supply of variables $x_i$, for $i \in \mathbb{N}$.

**Definition D.1.1.** Terms and formulas in $\mathcal{L}$ are defined inductively:

8. A variable $x_i$ is a term; a constant $c$ is a term;

9. If $f \in F$ and $\tau_j$, for $j < n(f)$, are terms then $f(\tau_0, \ldots, \tau_{n(f)-1})$.

10. If $R \in R$ and $\tau_j$, for $j < n(R)$, are terms, then $R(\tau_0, \ldots, \tau_{n(R)-1})$ is a formula. If $\tau_0$ and $\tau_1$ are terms, then $\tau_0 \equiv \tau_1$ is a formula. These are the atomic formulas.

11. If $\varphi$ and $\psi$ are formulas then $\varphi \lor \psi$ (disjunction), $\varphi \land \psi$ (conjunction), $\neg \varphi$ (negation), $\varphi \rightarrow \psi$ (implication), and $\varphi \leftrightarrow \psi$ (equivalence) are formulas.

The relation ‘$\varphi$ is a subformula of $\psi$’ is defined recursively in a natural way. This relation is well founded, enabling one to prove statements about all formulas by using Proposition A.2.1. This is the indiction on the complexity of a formula.

Suppose $\varphi(\bar{x})$ is a formula with free variables among $\bar{x} = (x_0, \ldots, x_{n-1})$ and $A$ is an $\mathcal{L}$-structure. If $\bar{a}$ is an $n$-tuple in $A$ then the interpretation of $\varphi$ in $A$ at $\bar{a}$, $\varphi^A(\bar{a})$, is the truth value defined by induction on the complexity of $\varphi$.

A formula with no free variables is called a sentence and the set of all $\mathcal{L}$-sentences is denoted $\text{Sent}_{\mathcal{L}}$. If $\varphi$ is a sentence and the value $\varphi^A$ is ‘true’, we write $A \models \varphi$ and say that $\varphi$ is satisfied in $A$ or that $A$ is a model of $\varphi$.

**Definition D.1.2.** The theory of $A$ is $\text{Th}(A) := \{ \varphi \in \text{Sent}_{\mathcal{L}} : A \models \varphi \}$. If $T \subseteq \text{Th}(A)$ then we say $A$ satisfies $T$, or that $A$ is a model of $T$ and write $A \models T$. The class of all models of a theory $T$ is denoted $\text{Mod}(T)$.

A theory $T$ is consistent if it has a model. Two $\mathcal{L}$-structures $A$ and $B$ are elementarily equivalent, $A \equiv B$, if $\text{Th}(A) = \text{Th}(B)$.

A substructure $B$ of $A$ is an elementary submodel of $A$, $B \preceq A$, if for every formula $\varphi$ of the language of $A$ and every $\bar{b}$ in $B$ we have $\varphi^B(\bar{b}) = \varphi^A(\bar{b})$. If $B \preceq A$ then we say that $A$ is an elementary extension of $B$. If $f : B \rightarrow A$ and $f[B] \preceq A$ then we say that $f$ is an elementary embedding.

**Theorem D.1.3** (Tarski–Vaught). If $B$ is a substructure of $A$ then $B \preceq A$ if and only if for every formula $\varphi(x, \bar{z})$ and every $\bar{b}$ in $B$, if $\varphi(\{a, \bar{b}\})^A$ for some $a \in A$ then $\varphi(a', \bar{b})^A$ for some $a' \in B$.

**Remark D.1.4.** Constant symbols can be considered as functions of zero arity, and therefore assuming that $\mathcal{C} = \emptyset$ does not cause a loss of generality. An $n$-ary function $f : A^n \rightarrow A$ can be identified with its graph $R_f := \{(\bar{a}, f(\bar{a}) : \bar{a} \in A^n\}$, considered as an $(n+1)$-ary relation. Since the assertion that an $(n+1)$-ary relation $R(\bar{x}, y)$ is a graph of an $n$-ary function is expressed by a first-order axiom, $\mathcal{L}$-structures are in a bijective correspondence with models of a first-order theory in a relational language.

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2 This is a theorem, but it works fine as a definition.

3 $(\forall \bar{x})(\exists y)R(\bar{x}, y) \land (\forall \bar{x})(\forall y)(\forall z)(R(\bar{x}, y) \land R(\bar{x}, z) \rightarrow y = z)$. 

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A function \( f : A^n \to A \) is definable in \( A \) if there exists a formula \( \varphi(\vec{x}, y) \) such that for all \( \vec{a} \) and \( b \) in \( A \) we have \( f(\vec{a}) = b \) if and only if \( A \models \varphi(\vec{a}, b) \).

**Lemma D.1.5.** If a function \( f \) is definable in \( A \) and \( B \preceq A \) then \( B \) is closed under \( f \).

**Definition D.1.6.** An \( L \)-structure \( A \) with domain \( X \) can be identified with a subset of \( \bigsqcup_{R \in R} X^{\mathbb{N}(R)} \). The space of all \( L \)-structures with domain \( X \) is therefore construed as \( \text{Struct}(L, X) := \mathcal{P}(\bigsqcup_{R \in R} X^{\mathbb{N}(R)}) \). The cardinality of a language \( L \), \( |L| \), is the cardinality of its set of symbols.

**Lemma D.1.7.** If \( |L| = \kappa \) then the cardinality of the set of all \( L \)-formulas is equal to \( \kappa^{<\mathbb{R}^0} = \max(\mathbb{R}_0, \kappa) \). \( \square \)

**Example D.1.8.** If a language \( L \) has cardinality not greater than a cardinal \( \kappa \) then by fixing a bijection between \( \kappa \) and \( \bigsqcup_{R \in R} X^{\mathbb{N}(R)} \) we can identify \( \text{Struct}(L, \kappa) \) with the power set of \( \kappa \). In particular the set of \( L \)-structures for a countable language \( L \) is naturally identified with the Cantor space (Example B.1.3) and is therefore equipped with a compact metric topology.

For an \( L \)-theory \( T \) the space of all models of \( T \) with domain \( X \) is

\[
\text{Mod}(T, X) := \{ \mathfrak{A} \in \text{Struct}(L, X) : \mathfrak{A} \models T \}.
\]

We stop here. Types, saturation, and axiomatizability are treated only in the case of logic of metric structures.

### D.2 Model Theory of Metric Structures and C*-algebras

#### D.2.1 Metric Structures

We review the basics of logic of metric structures, also known as continuous logic.

A metric structure is a triple \( \langle S, \mathcal{F}, \mathcal{R} \rangle \) which consists of the following.

1. The set of sorts \( S \) is an indexed family of metric spaces \( (S, d_S) \) where \( d_S \) is a complete, bounded metric on \( S \).
2. The set of functions \( \mathcal{F} \) is a set of uniformly continuous functions such that for \( f \in \mathcal{F} \), its domain \( \text{dom}(f) \) is a finite product of sorts and its range \( \text{rng}(f) \) is a sort.
3. The set of relations, \( \mathcal{R} \), is a set of uniformly continuous functions such that for \( R \in \mathcal{R} \), \( \text{dom}(R) \) is a finite product of sorts and its range \( \text{rng}(R) \) is a bounded subset of \( \mathbb{R} \).
4. The set of constants, \( \mathcal{C} \), such that for \( c \in \mathcal{C} \) a sort \( S_c \) is specified.
**Definition D.2.1.** To a $C^*$-algebra $A$ we associate the metric structure $\mathbb{M}(A)$ defined as follows. The sorts correspond to the closed $n$-balls $A_{\leq n}$, for $n \in \mathbb{N}$. Formally, for every $m$ we have separate function symbols corresponding to the standard functions, $+, \cdot$, and $\ast$, with the domain $A_{\leq 2m}$ or $A_{\leq m}$ and the range $A_{\leq 2m}$, $A_{\leq m2}$, or $A_{\leq m}$, respectively. This formality is routinely suppressed. There are no relations and the constants are 0 and, if $A$ is unital, 1. In the case of a $C^*$-algebra with additional predicates or functions, such as traces or automorphisms, the language is expanded by additional relation or function symbols corresponding to the restriction of these predicates and functions to each $n$-ball.

A $^*$-homomorphism between $C^*$-algebras $\Phi: A \rightarrow B$ uniquely defines a homomorphism between metric structures $\mathbb{M}(A)$ and $\mathbb{M}(B)$, and we have a functor from the category of $C^*$-algebras to the category of metric structures.

**D.2.2 Syntax: Language, Terms and Formulas**

A language $\mathcal{L}$ is a quadruple $\langle \mathcal{S}, F, R, C \rangle$ which contains the following data:

5. The set of sorts, $\mathcal{S}$. For each $S \in \mathcal{S}$ there is a symbol $d_S$ meant to be interpreted as a metric together with a positive number $M_S$ meant to be the bound on $d_S$.

6. The set of function symbols, $F$: for each $f \in F$ we specify $\text{dom}(f)$ as a sequence $(S_1, \ldots, S_{n-1})$ from $\mathcal{S}$ and $\text{rng}(f) = S$ for some $S \in \mathcal{S}$. We will want $f$ to be interpreted as a uniformly continuous function. To this effect we specify, as part of the language, functions $\delta_f^i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, for $i \leq n$. These functions are called uniform continuity moduli.

7. The set of relation symbols, $R$: for each $R \in R$ we specify $\text{dom}(R)$ as a sequence $(S_1, \ldots, S_n)$ of sorts and $\text{rng}(R) = K_R$ for some compact interval $K_R$ in $\mathbb{R}$. As with function symbols, we additionally specify, as part of the language, functions $\delta_f^i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, for $i \leq n$, called uniform continuity moduli.

8. The set of constant symbols, $C$: for each $c \in C$ we specify a sort $S_c$.

For each sort $S \in \mathcal{S}$, we have an infinite supply of variables $x_i^S$, for $i \in \mathbb{N}$, for which we will almost always omit the superscript.

**Definition D.2.2.** Terms and formulas are defined recursively, similarly to the corresponding definition in §D.1.

1. A variable $x_i^S$ is a term with domain and range $S$; a constant $c$ is a term with domain and range $S_c$.

2. If $f \in F$, $\text{dom}(f) = (S_0, \ldots, S_{n-1})$ and $t_0, \ldots, t_{n-1}$ are terms with $\text{rng}(t_i) = S_i$ for $i < n$ then $f(t_0, \ldots, t_{n-1})$ is a term with range the same as $f$ and domain determined by the $t_i$'s.

---

4 Some authors (including this one) use $\mathcal{M}(A)$, but in this text $\mathcal{M}(A)$ denotes the multiplier algebra.
3. If $R \in \mathbb{R}$, $\text{dom}(R) = (S_0, \ldots, S_{n-1})$, and $t_0, \ldots, t_{n-1}$ are terms with $\text{rng}(t_i) = S_i$ for all $i < n$, then $R(t_0, \ldots, t_{n-1})$ is a formula. Both the domain and uniform continuity moduli of $R(t_0, \ldots, t_{n-1})$ can be determined naturally from $R$ and $t_0, \ldots, t_{n-1}$.

These are the atomic formulas.

4. If $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function and $\varphi_0, \ldots, \varphi_{n-1}$ are formulas then $f(\varphi_0, \ldots, \varphi_{n-1})$ is a formula.

5. If $\varphi$ is a formula and $x$ is a variable of sort $S$ then both $\inf_{x \in S} \varphi$ and $\sup_{x \in S} \varphi$ are formulas. Their domains are the same as that of $\varphi$ except that the sort $S$ is removed.

The collection of all formulas of $\mathcal{L}$ is denoted $\mathcal{F}_\mathcal{L}$. A formula with no free variables is called a sentence. The set of all $\mathcal{L}$-sentences is denoted $\text{Sent}_\mathcal{L}$. The set of all $\mathcal{L}$-formulas whose free variables are included in $\bar{x} = (x_0, \ldots, x_{n-1})$ is denoted $\mathcal{F}^\bar{x}_\mathcal{L}$.

Therefore $\mathcal{F}_\mathcal{L} = \bigcup_{\bar{x}} \mathcal{F}^\bar{x}_\mathcal{L}$ where $\bar{x}$ ranges over all tuples of variables. We write $\sup_x \varphi$ and $\inf_x \varphi$ whenever the sort of $x$ is clear from the context. In the case of $\mathbb{C}^*$-algebras it suffices to consider quantification over the unit ball.

Functions $f$ as in (4) corresponds to logical connectives, $\lor, \land$, and $\leftrightarrow$. (There are no analogs of negation or implication in logic of metric structures.) The quantifiers $\sup$ and $\inf$ in (5) correspond to the quantifiers $\forall$ and $\exists$.

### D.2.3 Semantics: Interpretation of Formulas and Theories

Given a metric language $\mathcal{L}$, an $\mathcal{L}$-structure is a multi-sorted structure $M$ whose sorts correspond to the sorts of $\mathcal{L}$. In addition, all relations, functions, and constants in $\mathcal{L}$ are interpreted as relations, functions, and constants of $M$ of the appropriate arities, sort, and moduli of uniform continuity.

Suppose $\varphi(\bar{x})$ is a formula with free variables $\bar{x} = (x_0, \ldots, x_{n-1})$ and $M$ is an $\mathcal{L}$-structure. If $\bar{a}$ is an $n$-tuple in $M$ and $a_i$ is in the sort associated with $a_i$ then the interpretation of $\varphi$ in $M$ at $\bar{a}$, $\varphi^M(\bar{a})$, is the real number defined naturally and inductively according to the construction of $\varphi$. Quantification as in (5) is interpreted as taking suprema and infima over the sort associated with the variable being quantified.

**Lemma D.2.3.** To every $\mathcal{L}$-term $\tau(\bar{x})$ and every $\mathcal{L}$-formula $\varphi(\bar{x})$ one can associate a uniform continuity modulus so that in every $\mathcal{L}$-structure $M$ the interpretations $\tau^M$ and $\varphi^M$ satisfy this uniform continuity modulus.

**Definition D.2.4.** For a fixed $\mathcal{L}$-theory $T$ and $\varphi \in \mathcal{F}^\bar{x}_\mathcal{L}$, let

$$\|\varphi\|_T := \sup\{\|\varphi^M(\bar{a})\| : M \text{ satisfies } T, \bar{a} \in M\}.$$  

(This is a well-defined seminorm on $\mathcal{F}^\bar{x}_\mathcal{L}$ by Lemma D.2.3.) The density character of $T$ is defined to be the supremum of density characters of $(\mathcal{F}^\bar{x}_\mathcal{L}, \|\cdot\|_T)$ as $\bar{x}$ ranges over all tuples of variables.
We also consider $F_{\mathcal{L}}^x$ with respect to the norm $\| \cdot \|$ in which no theory $T$ is specified.

The density character of $(F_{\mathcal{L}}^x, \| \cdot \|_T)$ is at most $|\mathcal{L}| + \aleph_0$. If $\mathcal{L}$ has only countably many sorts and $(F_{\mathcal{L}}^x, \| \cdot \|_T)$ is separable for all $\bar{x}$ then we say that $T$ is separable.

An $\mathcal{L}$-formula is in \textit{prenex normal form} (PNF) if it is of the form

$$\sup_{x_1} \inf_{x_2} \sup_{x_3} \ldots \inf_{x_k} g(\|p_1(\bar{x})\|, \ldots, \|p_k(\bar{x})\|)$$

for some $k \geq 1$, *-polynomials $p_j(\bar{x})$ for $j \leq k$ in non-commuting variables, and a continuous function $g$.

\textbf{Definition D.2.5.} A formula $\varphi$ is called \textit{$\mathcal{P}_0$-restricted} if it is obtained from the atomic formulas by recursively applying the functions $t \mapsto t/2$ and $(s,t) \mapsto s^{-t}$ (where $s^{-t} = \max(s-t,0)$), the constant functions, and the quantifiers $\sup_x$ and $\inf_t$ (see [22, Definition 6.5–Proposition 6.9]).

\textbf{Proposition D.2.6.} \textit{The set of PNF $\mathcal{P}_0$-restricted formulas is dense in $(F_{\mathcal{L}}^x, \| \cdot \|)$.}

\textit{Proof.} By Theorem C.5.8 and [22, Theorem 6.3], every $\mathcal{L}$-formula can be uniformly approximated by $\mathcal{P}_0$-restricted formulas. Now combine [22, Theorem 6.3, Theorem 6.6, and Theorem 6.9].

\textbf{Definition D.2.7 (Theory as a set).} \textit{The theory of an $\mathcal{L}$-structure} $M$ \textit{is the set}

$$\text{Th}(M) := \{ \varphi \in \text{Sent}_{\mathcal{L}} : \varphi^M = 0 \}.$$

A \textit{theory} is any set of sentences. If $T \subseteq \text{Th}(M)$ then we say that $M$ \textit{satisfies} $T$, or that $M$ is a \textit{model} of $T$, or in symbols, $M \models T$. Equivalently, $M \models T$ if $\varphi^M = 0$ for all $\varphi \in T$. The class of all models of $T$ is

$$\text{Mod}(T) := \{ A : A \text{ is an } \mathcal{L} \text{-structure and } A \models T \}.$$

A theory $T$ is \textit{consistent} if it has a model. Two $\mathcal{L}$-structures $A$ and $B$ are \textit{elementarily equivalent}, $A \equiv B$, if $\text{Th}(A) = \text{Th}(B)$.

Since every real scalar is a formula, $\text{Th}(M)$ uniquely determines the functional on $\text{Sent}_{\mathcal{L}}$ given by $\varphi \mapsto \varphi^M$. The space $\text{Sent}_{\mathcal{L}}$ is an algebra over $\mathbb{R}$, and this functional is a character of that algebra. The theory as defined in Definition D.2.7 is the kernel of this character.

\textbf{Definition D.2.8 (Theory as a character).} \textit{The theory of an $\mathcal{L}$-structure} $M$ \textit{is the character} $\varphi \mapsto \varphi^M$ \textit{on} $\text{Sent}_{\mathcal{L}}$.

Types can also be construed both as sets and as characters (§16).

A substructure $M$ of $N$ is an \textit{elementary submodel} of $N$, $M \preceq N$, if for every formula $\varphi$ of the language of $N$ and every $\bar{a}$ in $M$ we have $\varphi^M(\bar{a}) \equiv \varphi^N(\bar{a})$. If $M \preceq N$ then we say that $N$ is an \textit{elementary extension} of $M$. If $f : M \to N$ is an embedding of $M$ into $N$ and the image of $M$ is an elementary submodel of $M$ then we say that $f$ is an \textit{elementary embedding}. The Tarski–Vaught test has a metric analog.
Theorem D.2.9 (Tarski–Vaught). If $B$ is a substructure of $A$ then $B \preceq A$ if and only if for every $r \in \mathbb{R}$ and every formula $\varphi(x, \bar{b})$ where $x$ is of sort $S$ and $\bar{b}$ in $B$, if there is $a \in S^A$ such that $\varphi^B(a, \bar{b}) < r$ then there is $c \in S^B$ such that $\varphi^A(c, \bar{b}) < r$. \hfill \Box

We will sometimes need to handle more than one language at a time.

Definition D.2.10. If $L_0$ and $L_1$ are languages such that $L_0 \subseteq L_1$, then we have the forgettable functor from the category of $L_1$-structures to the category of $L_1$ structures that sends an $L_1$-structure $M_1$ to the $L_0$-structure $M_0$ with the same domain and the same interpretations of the symbols in $L_0$. We say that $M_0$ is the reduct of $M_1$ and that $M_1$ is an expansion of $M_0$.

D.2.4 Axiomatizability

If $T$ is an $\mathcal{L}$-theory, a class of $\mathcal{L}$-structures is axiomatizable if it is the set of all models of some $\mathcal{L}$-theory. A category is axiomatizable if it is equivalent to the category of models of a theory in logic of metric structures.

Example D.2.11. 1. Both Banach spaces and Banach algebras are axiomatizable.
   2. $C^*$-algebras are axiomatizable ([90]).
   3. The II$_1$ factors are also axiomatizable ([90]).

Essentially all presently known axiomatizable classes of $C^*$-algebras are collected in [87, Theorem 2.5.1]. Definability by uniform families of formulas was defined in [87, §5.7a]. We do not need the exact definition, only the following fact.

Lemma D.2.12. If a class $\mathcal{C}$ of metric structures is definable by uniform families of formulas, $B \preceq A$ are metric structures, and $A \in \mathcal{C}$, then $B \in \mathcal{C}$.

Proof. By [87, Definition 5.7.1], belonging to $\mathcal{C}$ is characterized by omission of a sequence of types. To complete the proof, it suffices to note that if some $\bar{x}$ in $B$ realizes a type and $B \preceq A$, then $\bar{x}$ realizes the same type in $A$. \hfill \Box

A list of the properties of $C^*$-algebras presently known to be definable by uniform families of formulas is given in [87, Theorem 5.7.3].

D.2.5 Reduced Products and Ultraproducts in Model Theory

We include a general definition of a reduced product of metric structures.

Definition D.2.13. Fix a language $\mathcal{L}$, an ideal $\mathcal{J}$ on an index set $\mathcal{J}$, and $\mathcal{L}$-structures $M_j$, for $j \in \mathcal{J}$. We will describe $M := \prod_j M_j / \mathcal{J}$, the reduced product of the $M_j$’s with respect to $\mathcal{J}$.
For each sort $S \in \mathcal{L}$ with metric $d^S_j$ on $S(M_j)$ take $\prod_j S(M_j)$ together with the pseudo-metric
\[ d^S(a, b) := \inf_{X \in J} \sup_{j \in J \setminus X} d^S_j(a_j, b_j) \]
and let $S(M)$ be the quotient of $\prod_j S(M_j)$ by $d^S$. For each function symbol $f \in \mathcal{L}$ define $f_M$ coordinatewise on the appropriate sorts. The uniform continuity requirements on $f$ imply that this is well-defined and that $f^M$ has the necessary continuity modulus. For each relation symbol $R \in \mathcal{L}$ define
\[ R^M(\bar{a}) := \inf_{X \in J} \sup_{j \in J \setminus X} R^M_j(\bar{a}_j). \]
All of these functions are well-defined and uniformly continuous by the uniform continuity requirements of the language.

If $J$ is a maximal ideal, then $\mathcal{U} := J^*$ is an ultrafilter, and the corresponding special case of Definition D.2.13 is worth singling out.

**Definition D.2.14.** Suppose $\mathcal{U}$ is an ultrafilter and $J = \mathcal{U}^*$. Then the pseudo-metric $d^S$ on a sort $S$ as in Definition D.2.13 reduces to $d^S := \lim_{J \to \mathcal{U}} d^S_j$. The interpretation of the relation symbols is given analogously, and the interpretation of the function symbols is unchanged. In this case we write $\prod \mathcal{U} M_j$ for $\prod M_j / J$ and call it the ultraproduct of the $M_j$’s with respect to $\mathcal{U}$.

If all $M_j$ are equal to a fixed metric structure $M$ then the ultraproduct is called ultrapower and denoted $M^\mathcal{U}$. 
But I now leave my cetological System standing thus unfinished, even as the great Cathedral of Cologne was left, with the crane still standing upon the top of the uncompleted tower. For small erections may be finished by their first architects; grand ones, true ones, ever leave the copestone for posterity. God help me from ever completing anything. This whole book is but a draught – nay, but the draught of a draught. Oh, Time, Strength, Cash, and Patience!

Herman Melville, Moby Dick
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