Distribution of values of $L$-functions at the edge of the critical strip

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**Serre’s Theorem (1968)**

Let $S_k^p(N)$ be the set of normalized primitive holomorphic cusp forms of level $N$ and weight $k$. For $f \in S_k^p(N)$, we have

The Sato Tate conjecture holds for $f \iff L(1 + it, \text{Sym}^m f) \neq 0$, for all $m \in \mathbb{N}$. 
The distribution of these values has been extensively studied over the last decades.

Erdös and Chowla (50’s), Elliott (70’s)

Let $\tau > 0$. The proportion of fundamental discriminants $|d| \leq x$ with $L(1, \chi_d) \geq \tau$, approaches a continuous limit $F(\tau)$ as $x \to \infty$.

Montgomery-Vaughan (1999) Conjectured the shape of the tail of this distribution function as $\tau, x \to \infty$ in the full range $\tau \leq (e^\gamma + o(1)) \log \log x$, and noted that it should decay “double exponentially”.

Granville-Soundararajan (2003) Proved this conjecture and got a precise estimate for the tail of this distribution function. Moreover they proved analogous results for the families $|\zeta(1 + it)|$ for $t \in [T, 2T]$ and $|L(1, \chi)|$ where $\chi$ runs over primitive characters modulo a large prime $q$. 
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The distribution of other Families

- Cogdell and Michel (2004).
  Computed large complex moments of $L(1, \text{Sym}^m f)$ for $f \in S^p_2(q)$, in the level aspect. (Unconditionally for $m = 1, 2, 3, 4$ and assuming the automorphy of these $L$-functions for $m \geq 5$).
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  Family of $L(1, f)$ for $f \in S_k^p(1)$, in the weight aspect.
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  Family of $L(1, f)$ for $f \in S_k^p(1)$, in the weight aspect.

  The distribution function of complex values of $\zeta(1 + it)$, that is the joint distribution function of $\arg \zeta(1 + it)$ and $|\zeta(1 + it)|$. 
Consider a family of $L$-functions $\mathcal{L} = \{L(s, \pi) : \pi \in \mathcal{F}\}$. Where $\mathcal{F}$ is the set of some interesting arithmetic objects $\pi$.

$$L(1, \pi) = \prod_{p} \prod_{j=1}^{d} \left(1 - \frac{\alpha_{j, \pi}(p)}{p}\right)^{-1}.$$
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$$L(1, \pi) = \prod_{p} \prod_{j=1}^{d} \left( 1 - \frac{\alpha_{j, \pi}(p)}{p} \right)^{-1}.$$ 

As $\pi$ varies in $\mathcal{F}$ and the conductor of $\mathcal{F}$ tends to $\infty$, we expect that $\alpha_{j, \pi}(p)$ should be distributed like some random variables $X_j(p)$. 
For \( \zeta(1 + it) \): as \( t \to \infty \) the values \( p^{it} \) are expected to be distributed like random variables \( X(p) \) uniformly distributed on the unit circle \( \mathbb{U} \).
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For \( L(1, \chi_d) \): as \( d \to \infty \) the values \( \chi_d(p) \) are expected to be distributed like random variables \( X(p) \) which take values \( -1 \) and \( 1 \) with equal probabilities.

For \( L(1, f) \): the local roots are expected to be distributed like random variables \( X_1(p) = X_2(p) = e^{i \theta(p)} \) where \( \theta(p) \) are distributed in \([0, \pi]\) according to the Sato-Tate measure \( \frac{2}{\pi} \sin^2(\theta) \, d\theta \).
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Construct a random model

\[ L_{\mathcal{F}}(1, X) = \prod_{p} \prod_{j=1}^{d} \left( 1 - \frac{X_j(p)}{p} \right)^{-1}. \]

- \( X_j(p) \) are random variables having the expected distribution.
- \( X_j(p) \) and \( X_j(q) \) are independent for \( p \neq q \).
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Guess

\[ \frac{1}{|\mathcal{F}|} \sum_{\pi \in \mathcal{F}} |L(1, \pi)|^k \approx \mathbb{E} \left( |L_{\mathcal{F}}(1, X)|^k \right). \]
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A careful study of the random model

\[\downarrow\]

Information of the distribution of the “L-values”.
Let

$$\Phi_T(\tau) := \frac{1}{T} \text{meas}\{ t \in [T, 2T] : |\zeta(1 + it)| \geq e^{\gamma} \tau \}.$$
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Uniformly for \( \tau \leq \log \log T - 20 \) we have

\[ \Phi_T(\tau) = \exp \left( -\frac{e^{\tau - A_1}}{\tau} (1 + o(1)) \right), \] (1)

where \( A_1 = 1 + \int_0^1 \log l_0(t) \frac{dt}{t^2} + \int_1^{\infty} (\log l_0(t) - t) \frac{dt}{t^2} \).
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- The proportion of fundamental discriminants \( d \) with \( |d| \leq x \), for which \( L(1, \chi_d) > e^{\gamma \tau} \), has the same formula as (1) with \( A_2 = 1 + \int_0^1 \log \cosh t \frac{dt}{t^2} + \int_1^\infty (\log \cosh t - t) \frac{dt}{t^2} \).
- The proportion of \( f \) in \( S^p_k(1) \) for which \( L(1, f) > (e^{\gamma \tau})^2 \) has the same formula as (1), with some other constant \( A_3 \).
Study a large class of random models which includes all the previous ones. As an application we get an estimate for the distribution of the values $L(1, \text{Sym}^mf)$ where $f$ varies over elements of $S^p_2(q)$ as $q \to \infty$. (Using Cogdell-Michel result).

Study the distribution of “general” $L$-functions in the height aspect on the line $\text{Re}(s) = 1$.

Study the distribution of $L(\pi \otimes \chi_d, 1)$, for fundamental discriminants $|d| \leq x$, where $\pi$ is an automorphic cuspidal representation of $GL_n(\mathbb{A}_\mathbb{Q})$, as $x \to \infty$. 
Let $\theta_j(p)$ be random variables distributed on $[-\pi, \pi]$, and consider

$$L(1, X) := \prod_p \prod_{j=1}^{d} \left( 1 - \frac{e^{i\theta_j(p)}}{p} \right)^{-1}.$$
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**Condition 1.** $\mathbb{E}(e^{i\theta_j(p)}) = 0$, for all primes $p$ and $1 \leq j \leq d$. 
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**Condition 1.** $\mathbb{E}(e^{i\theta_j(p)}) = 0$, for all primes $p$ and $1 \leq j \leq d$.

**Condition 2.** $\theta_j(p)$ and $\theta_k(q)$ are independent random variables for $p \neq q$. 
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- **Condition 2.** \( \theta_j(p) \) and \( \theta_k(q) \) are independent random variables for \( p \neq q \).
- **Condition 3.** The random variables \( X(p) := \sum_{j=1}^d e^{i\theta_j(p)}/d \), are identically distributed, for every prime \( p \).
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**Condition 4.** There exists an absolute constant $\alpha > 0$ such that for all primes $p$ and all $\epsilon > 0$, we have

$$\text{Prob} \left( |\theta_1(p)| \leq \epsilon, \ldots, |\theta_d(p)| \leq \epsilon \right) \gg \epsilon^\alpha.$$
Let

$$\Phi(\tau) := \text{Prob} \left( |L(1, X)| > (e^\gamma \tau)^d \right).$$
Let
\[ \Phi(\tau) \coloneqq \text{Prob} \left( |L(1, X)| > (e^{\gamma \tau})^d \right). \]

**Theorem 1 (L-2009)**

Let \( d \) be a positive integer. For \( 1 \leq j \leq d \) and prime \( p \) let \( \theta_j(p) \) be random variables distributed on \([-\pi, \pi]\) and satisfying conditions 1-4. Then for large \( \tau \), we have

\[ \Phi(\tau) = \exp \left( -\frac{e^{\tau-A_X}}{\tau} (1 + o(1)) \right), \]

where

\[ A_X \coloneqq 1 + \int_0^1 h_X(t) \frac{dt}{t^2} + \int_1^\infty (h_X(t) - t) \frac{dt}{t^2}, \]

and \( h_X(t) = \log \mathbb{E} \left( e^{\text{Re}(X)t} \right) \), where \( X \) is a random variable having the same distribution as the \( X(p) \).
$S_2^p(q)$ is the set of normalized primitive holomorphic cusp forms of weight 2 and level $q$ ($q$ is large prime).

Assume that the $k$-th symmetric power $L$-function of $f \in S_2^p(q)$ is automorphic (hypothesis $\text{Sym}(q)$). This is true for the symmetric powers up to 4.

Let $\omega_f := 1/(4\pi \|f\|)$ be the usual harmonic weight.
Let

\[ \Phi_q(\text{Sym}^k, \tau) = \left( \sum_{f \in S_2^p(q)} \omega_f \right)^{-1} \sum_{f \in S_2^p(q)} \omega_f. \]

where \( L(1, \text{Sym}^k f) \geq (e^\gamma \tau)^{k+1} \).
Let
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\]

**Theorem 2 (L-2009)**

Let \( k \geq 1 \) be an integer and \( q \) be a large prime such that Hypothesis \( \text{Sym}(q) \) holds. Then uniformly in the region \( \tau \leq \log \log q(1 + o(1)) \) we have

\[
\Phi_q(\text{Sym}^k, \tau) = \exp \left( - \frac{e^{\tau - A_k}}{\tau} (1 + o(1)) \right),
\]

where \( A_k = 1 + \int_0^1 \frac{h_k(t)}{t^2} dt + \int_1^\infty \frac{h_k(t) - t}{t^2} dt \) and

\[
h_k(t) = \log \left( \frac{2}{\pi} \int_0^\pi \exp \left( \frac{t}{k + 1} \sum_{j=0}^k \cos(\theta(k - 2j)) \right) \sin^2 \theta d\theta \right).
\]
Let $S^p$ be the class of Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s}, \text{ for } \text{Re}(s) > 1,$$

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satisfying the following properties

1. Analyticity: $(s - 1)^r F(s)$ is an entire function of finite order for some non-negative integer $r$. 

2. Ramanujan hypothesis: $a_F(n) \ll \epsilon n^\epsilon$ for any fixed $\epsilon > 0$.

3. Functional equation: $F(s)$ satisfies the functional equation

$$\Phi(s) = \omega \Phi(1-s),$$

where $\Phi(s) = Q s^F k \prod_{i=1} \Gamma(w_i s + \mu_i) F(s)$, $|\omega| = 1$, $Q_F > 0$, $w_i > 0$ and $\text{Re}(\mu_i) \geq 0$ are parameters depending on $F$. 
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4. Euler product: for $\text{Re}(s) > 1$ we have

$$F(s) = \prod_{p} \prod_{i=1}^{d} \left( 1 - \frac{\alpha_{i,F}(p)}{p^s} \right)^{-1},$$

where $\alpha_{i,F}(p) \neq 0$ for all primes except finitely many. The $\alpha_{i,F}(p)$ are complex numbers called the local roots of $F$ at $p$ and $d \in \mathbb{N}$ is called the degree of $F$. 
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To study the distribution of the values $F(1 + it)$ we define

The random model

Let $\{X(p)\}_{p \text{ primes}}$ are independent random variables uniformly distributed on the unit circle $\mathbb{U}$, and define

$$F(1, X) = \prod_p \prod_{i=1}^d \left( 1 - \frac{\alpha_{i,F(p)}X(p)}{p} \right)^{-1}.$$
Theorem 3 (L-2009)

Let $F \in S^p$. Let $T > 0$ be large, and take $A > 0$. Then for all positive integers $k$ in the range $1 \leq k \leq \log T/(B(\log \log T)^2)$ (for a suitably large constant $B = B(A, F)$), we have

$$
\frac{1}{T} \int_T^{2T} |F(1+it)|^{2k} \, dt = \mathbb{E} \left( |F(1, X)|^{2k} \right) \left( 1 + O \left( \frac{1}{\log^A T} \right) \right).
$$

Remark 1.

This result is true (though in a slightly smaller range) in the more general case where the Ramanujan Hypothesis is replaced by the following Ramanujan Petersson bound on average

$$
\sum_{n \leq x} |a_n F(n)| \ll x \left( \log x \right)^\beta F,
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provided some zero density estimate near $\Re(s) = 1$ holds for $F$. In particular these assumptions are true for $L$-functions attached to $GL(2)$-Maass cusp forms.
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Large values of $|F(1 + it)|$

Conjecture 1
Let $F \in S_p$. Then there exists some constant $\kappa_F > 0$ such that
$$\sum_{p \leq x} |a_F(p)|^p \asymp \kappa_F \log \log x + O(1).$$

Corollary 1
Let $F \in S_p$ such that Conjecture 1 holds for $F$. Then for $T > 0$ large, there exists some $t \in [T, 2T]$ such that
$$|F(1 + it)| \gg (\log \log T)^{\kappa_F}.$$

Remark 2.
The bound provided by Corollary 1 is best possible (up to a constant). Indeed by a standard argument of Littlewood we can show, under the Generalized Riemann Hypothesis for $F$,$$|F(1 + it)| \ll (\log \log t)^{\kappa_F}.$$
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Remark 2. The bound provided by Corollary 1 is best possible (up to a constant). Indeed by a standard argument of Littlewood we can show, under the Generalized Riemann Hypothesis for $F$
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|F(1 + it)| \ll (\log \log t)^{\kappa_F}.
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A precise formula for the distribution of $|F(1 + it)|$

In general the values $a F(p)$ are expected to have some distribution as $p \to \infty$.

Hypothesis D

There exists a compactly supported distribution function $\psi(t)$ (with support in some interval $[0, U]$), such that for all continuous functions $g$ we have

$$\sum_{p \leq x} g(|a F(p)|) = \pi(x) \left( \int_{U} g(t) \psi(t) \, dt + o(1 \log x) \right),$$

as $x \to \infty$.

Let $F \in S$ and satisfies hypothesis D. Let $N := \int_{U} t \psi(t) \, dt$.

Then Conjecture 1 holds for $F$ with $\kappa_F = N$. Define

$$b_F := \prod_{p \max_{t \in [-\pi, \pi]} |d \prod_{i=1}^{p}(1 - e^{it\alpha_i}, F(p)) - 1|}.$$
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There exists a compactly supported distribution function $\psi(t)$ (with support in some interval $[0, U]$), such that for all continuous functions $g$ we have

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Let $F \in S^p$ and satisfies hypothesis $D$. Let $N := \int_0^U t\psi(t)dt$. Then Conjecture 1 holds for $F$ with $\kappa_F = N$. Define
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Let \( T > 0 \) be large. Let \( F \in S^p \), and satisfies Hypothesis D (with distribution function \( \psi \)). Then uniformly in the region \( \tau \leq \log_2 T - \log_3 T - 2 \log_4 T \), we have

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- \( M/N - \log N = 0 \) for degree one \( L \)-functions.
- \( M/N - \log N \neq 0 \) for \( L \)-functions attached to \( GL(2) \)-automorphic forms by the Sato-Tate conjecture.