THE PRIME NUMBER RACE AND ZEROS OF DIRICHLET L-FUNCTIONS OFF THE CRITICAL LINE. III

KEVIN FORD, SERGEI KONYAGIN, AND YOUNESS LAMZOURI

Abstract. We show, for any \( q \geq 3 \) and distinct reduced residues \( a, b \pmod{q} \), the existence of certain hypothetical sets of zeros of Dirichlet L-functions lying off the critical line implies that \( \pi(x; q, a) < \pi(x; q, b) \) for a set of real \( x \) of asymptotic density 1.

1 Introduction

For \( (a, q) = 1 \), let \( \pi(x; q, a) \) denote the number of primes \( p \leq x \) with \( p \equiv a \pmod{q} \). The study of the relative magnitudes of the functions \( \pi(x; q, a) \) for a fixed \( q \) and varying \( a \) is known colloquially as the “prime race problem” or “Shanks-Rényi prime race problem”. For a survey of problems and results on prime races, the reader may consult the papers [4] and [5].

One basic problem is the study of \( P_{q,a_1,\ldots,a_r} \), the set of real numbers \( x \geq 2 \) such that \( \pi(x; q, a_1) > \cdots > \pi(x; q, a_r) \). It is generally believed that all sets \( P_{q,a_1,\ldots,a_r} \) are unbounded. Assuming the Generalized Riemann Hypothesis for Dirichlet L-functions modulo \( q \) (GRH), and that the nonnegative imaginary parts of zeros of these L-functions are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown for any \( r \)-tuple of reduced residue classes \( a_1, \ldots, a_r \pmod{q} \), that \( P_{q,a_1,\ldots,a_r} \) has a positive logarithmic density (although it may be quite small in some cases). We recall that the logarithmic density of a set \( E \subset (0, +\infty) \) is defined as

\[
\delta(E) = \lim_{X \to \infty} \frac{1}{\log X} \int_{[2, X] \cap E} \frac{dt}{t}
\]

provided that the limit exists.

In [2] and [3], Ford and Konyagin investigated how possible violations of the Generalized Riemann Hypothesis (GRH) would affect prime number races. In [2], they proved that the existence of certain sets of zeros off the critical line would imply that some of the sets \( P_{q,a_1,a_2,a_3} \) are bounded, giving a negative answer to the prime race problem with \( r = 3 \). Paper [3] was devoted to similar questions for \( r \)-way prime races with \( r > 3 \). One result from [3] states that for any \( q, r \leq \phi(q) \) and set \( \{a_1, \ldots, a_r\} \) of reduced residues modulo \( q \), the existence of certain hypothetical sets of zeros of Dirichlet L-functions modulo \( q \) implies that at most \( r(r-1) \) of the sets \( P_{q,\sigma(a_1),\ldots,\sigma(a_r)} \) are unbounded, \( \sigma \) running over all permutations of \( \{a_1, \ldots, a_r\} \).

In this paper, we investigate the effect of zeros of L-functions lying off the critical line for two way prime races. This case is harder, since it is unconditionally proved that for certain races \( \{q; a, b\} \) the set \( P_{q,a,b} \) is unbounded. For example, Littlewood [11] proved that \( P_{4,3,1}, P_{4,1,3}, P_{3,1,2} \) and \( P_{3,2,1} \) are unbounded. Later Knapowski and Turán ([9], [10]) proved for many \( q, a, b \) that \( \pi(x; q, b) - \pi(x; q, a) \) changes sign infinitely often and more recently Sneed [13] showed that \( P_{q,a,b} \) is unbounded for every \( q \leq 100 \) and all possible pairs \((a, b)\).
Nevertheless, we prove that the existence of certain zeros off the critical line would imply that the set $P_{q,a,b}$ has asymptotic density zero, in contrast with a conditional result of Kaczorowski [7] on GRH, which asserts that $P_{q,1,b}$ and $P_{q,b,1}$ have positive lower densities for all $(b,q) = 1$.

Let $q \geq 3$ be a positive integer and $a, b$ be distinct reduced residues modulo $q$. Moreover, for any set $S$ of real numbers we define $S(X) = S \cap [2, X]$.

**Theorem 1.1.** Let $q \geq 3$ and suppose that $a$ and $b$ are distinct reduced residues modulo $q$. Let $\chi$ be a nonprincipal Dirichlet character with $\chi(a) \neq \chi(b)$, and put $\xi = \text{arg}(\chi(a) - \chi(b)) \in [0, 2\pi)$. Suppose $\frac{1}{2} < \sigma < 1$, $0 < \delta < \sigma - \frac{1}{2}$, $A > 0$, and $B = B(\xi, \sigma, \delta, A)$ is a multiset of complex numbers satisfying the conditions listed in Section 2. If $L(\rho, \chi) = 0$ for all $\rho \in B$, $L(s, \chi)$ has no other zeros in the region $\{s : \text{Re}(s) \geq \sigma - \delta, \text{Im}(s) \geq 0\}$, and for all other nonprincipal characters $\chi'$ modulo $q$, $L(s, \chi') \neq 0$ in the region $\{s : \text{Re}(s) \geq \sigma - \delta, \text{Im}(s) \geq 0\}$, then

$$\lim_{X \to \infty} \frac{\text{meas}(P_{q,a,b}(X))}{X} = 0.$$

**Remarks.** A character $\chi$ with $\chi(a) \neq \chi(b)$ exists whenever $a$ and $b$ are distinct modulo $q$. The sets $B$ have the property that any $\rho \in B$ has real part in $[\sigma - \delta, \sigma]$, imaginary part greater than $A$, and multiplicity $O((\log \text{Im}(\rho))^{3/4})$ (that is, the multiplicities are much smaller than known bounds on the multiplicities of zeros of Dirichlet $L$-functions). The number of elements of $B$ (counted with multiplicity) with imaginary part less than $T$ is $O((\log T)^{5/4})$, and thus $B$ is quite a “thin” set. Also, we note that if $L(\beta + i\gamma, \chi) = 0$ then $L(\beta - i\gamma, \overline{\chi}) = 0$, which is a consequence of the functional equation for Dirichlet $L$-functions (See e.g. Ch. 9 of [1]). The point of Theorem 1.1 is that proving

$$\limsup_{X \to \infty} \frac{\text{meas}(P_{q,a,b}(X))}{X} > 0$$

requires showing that the multiset of zeros of $L(s, \chi)$ cannot contain any of the multisets $B$. This is beyond what is possible with existing technology (see e.g. [6] for the best known estimates for multiplicities of zeros). In other words, Theorem 1.1 claims that under certain suppositions the set $P_{q,a,b}(X)$ has the zero asymptotic density. This implies that its logarithmic density is also zero, in contrast to conditional results from [12].

Our method works as well for the difference $\pi(x) - \text{li}(x)$, the error term in the prime number theorem. Littlewood [11] established that this quantity changes sign infinitely often. Let $P_1$ be the set of real numbers $x \geq 2$ such that $\pi(x) > \text{li}(x)$. In [8] Kaczorowski proved, assuming the Riemann Hypothesis, that both $P_1$ and $\overline{P}_1$ have positive lower densities. Assuming the Riemann Hypothesis and that the nonnegative imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$ are linearly independent over the rationals, Rubinstein and Sarnak [12] have shown that $P_1$ has a positive logarithmic density $\delta_1 \approx 0.00000026$. In contrast with these results we prove that the existence of certain zeros of $\zeta(s)$ off the critical line would imply that the set $P_1$ has asymptotic density zero (or asymptotic density 1).

**Theorem 1.2.** Suppose $\frac{1}{2} < \sigma < 1$, $0 < \delta < \sigma - \frac{1}{2}$ and $A > 0$. (i) If $\xi = 0$, $B = B(\xi, \sigma, \delta, A)$ satisfies the conditions of Section 2, $\zeta(\rho) = 0$ for all $\rho \in B$, and $\zeta(s)$ has no other zeros in the region $\{s : \text{Re}(s) \geq \sigma - \delta, \text{Im}(s) \geq 0\}$, then

$$\lim_{X \to \infty} \frac{\text{meas}(P_1(X))}{X} = 0.$$
(ii) If $\xi = \pi$, $B$ satisfies the conditions of Section 2, $\zeta(\rho) = 0$ for all $\rho \in B$, and $\zeta(s)$ has no other zeros in the region $\{s : \Re(s) \geq \sigma - \delta, \Im(s) \geq 0\}$, then
\[
\lim_{X \to \infty} \frac{\text{meas}(P_1(X))}{X} = 1.
\]

We omit the proof of Theorem 1.2, as it is nearly identical to the proof of Theorem 1.1 in the case $q = 4$.

2 The construction of $B$

For $j \geq 1$, we fix any real numbers $\gamma_j, \delta_j$ and $\theta_j$ satisfying
\[
\exp\left(\frac{1}{2}\right) \leq \gamma_j \leq 2 \exp\left(\frac{1}{2}\right), \quad \left|\delta_j - \frac{1}{j^8}\right| \leq \frac{1}{j^9},
\]
and
\[
\left|\theta_j - \frac{\xi - \pi/2}{j^{16}}\right| \leq \frac{1}{j^{17}}.
\]

We choose $j_0$ so large that for all $j \geq j_0$, $\gamma_j > A$ and $\sigma - \delta \leq \sigma - \delta_j$. Then we take $B$ to be the union, over $j \geq j_0$ and $1 \leq k \leq j^3$, of $m(k, j) = k(j^3 + 1 - k)$ copies of $\rho_{j,k}$, where
\[
\rho_{j,k} = \sigma - \delta_j + i(k\gamma_j + \theta_j).
\]

3 Preliminary Results

The following classical-type explicit formula was established in Lemma 1.1 of [2] when $x' = x$. The slightly more general result below, which is more convenient for us, is proved in exactly the same way.

**Lemma 3.1.** Let $\beta \geq 1/2$ and for each non-principal character $\chi \mod q$, let $B(\chi)$ be the sequence of zeros (duplicates allowed) of $L(s, \chi)$ with $\Re(s) > \beta$ and $\Im(s) > 0$. Suppose further that all $L(s, \chi)$ are zero-free on the real segment $\beta < s < 1$. If $(a, q) = (b, q) = 1$, $x$ is sufficiently large and $x' \geq x$, then
\[
\phi(q)(\pi(x; q, a) - \pi(x; q, b)) = -2\Re\left(\sum_{\chi \neq \chi_0 \mod q} \left(\chi(a) - \overline{\chi(b)}\right) \sum_{\rho \in B(\chi), |\Im(\rho)| \leq x'} f(\rho)\right) + O\left(x^3 \log^2 x\right),
\]
where
\[
f(\rho) := \frac{x^\rho}{\rho \log x} + \frac{1}{\rho} \int_2^x \frac{t^\rho}{t \log^2 t} dt = \frac{x^\rho}{\rho \log x} + O\left(\frac{x^{\Re(\rho)}}{|\rho|^2 \log^2 x}\right)
\]

**Remark.** For Theorem 1.2, we use a similar explicit formula for $\pi(x)$ in terms of the zeros $B(\zeta)$ of the Riemann zeta function which satisfy $\Re(\rho) > \beta$ and $\Im(\rho) > 0$:
\[
\pi(x) = \text{li}(x) - 2\Re\sum_{\rho \in B(\zeta), |\Im(\rho)| \leq x'} f(\rho) + O(x^3 \log^2 x).
\]

Using properties of the Fejér kernel we prove the following key proposition.
Proposition 3.2. Let $\gamma \geq 1$, $L \geq 4$ and $X \geq 2$. Define

$$F_{\gamma,L}(x) = \sum_{k=1}^{L-1} (L - k) \cos(k\gamma \log x).$$

Then

$$\text{meas}\left\{ x \in [1, X] : F_{\gamma,L}(x) \geq -\frac{L}{4} \right\} \ll \frac{X}{\sqrt{L}}.$$

Proof. The Fejér kernel satisfies the following identity

$$\frac{1}{L} \left( \frac{\sin \left( \frac{L\theta}{2} \right)}{\sin \left( \frac{\theta}{2} \right)} \right)^2 = 1 + 2 \sum_{k=1}^{L-1} \left( 1 - \frac{k}{L} \right) \cos(k\theta).$$

This yields

$$F_{\gamma,L}(x) = \frac{\sin^2 \left( \frac{L\gamma \log x}{2} \right)}{2 \sin^2 \left( \frac{\gamma \log x}{2} \right)} - \frac{L}{2}.$$

Therefore, if $F_{\gamma,L}(x) \geq -L/4$ then

$$\sin^2 \left( \frac{\gamma \log x}{2} \right) \leq \frac{2}{L} \sin^2 \left( \frac{L\gamma \log x}{2} \right) \leq \frac{2}{L}.$$

Hence,

$$\left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon := \frac{1}{\sqrt{2L}},$$

where $\|t\|$ denotes the distance to the nearest integer. We observe that the condition $\left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon$ means that for some integer $k$ we have

$$k - \varepsilon \leq \frac{\gamma \log x}{2\pi} \leq k + \varepsilon,$$

or equivalently $e^{2\pi(k-\varepsilon)/\gamma} \leq x \leq e^{2\pi(k+\varepsilon)/\gamma}$. Thus,

$$\text{meas}\left\{ x \in [1, X] : F_{\gamma,L}(x) \geq -\frac{L}{4} \right\} \leq \text{meas}\left\{ x \in [1, X] : \left\| \frac{\gamma \log x}{2\pi} \right\| \leq \varepsilon \right\} \leq \sum_{0 \leq k \leq \frac{\gamma \log X}{2\pi} + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} - e^{2\pi(k-\varepsilon)/\gamma} \ll \frac{\varepsilon}{\gamma} \sum_{0 \leq k \leq \frac{\gamma \log X}{2\pi} + \varepsilon} e^{2\pi(k+\varepsilon)/\gamma} \ll \varepsilon X. \qquad \square$$

4 Proof of Theorem 1.1

Suppose $X$ is large and $\sqrt{X} \leq x \leq X$. For brevity, let

$$\Delta = \phi(q) \left( \pi(x; q, a) - \pi(x; q, b) \right).$$
It follows from Lemma 3.1 with \( x' = \max\{x, \max\{j^3 \gamma_j : \gamma_j \leq x\}\} \) that

\[
\Delta = -\frac{2}{\log x} \Re \left( (\chi(a) - \chi(b)) \sum_{\gamma_j \leq x} \sum_{k=1}^{j^3} \frac{x^{\sigma - \delta_j + i(k \gamma_j + \theta_j)} m(k, j)}{\gamma_j k^2} + x^{\sigma - \delta} \log^2 x \right) + O \left( \frac{x^{\sigma - \delta}}{\log^2 x} \sum_{\gamma_j \leq x} \frac{j^3 m(k, j)}{\gamma_j^2} + x^{\sigma - \delta} \log^2 x \right).
\]

(4.1)

This is

\[
= 2x^{\sigma - \delta} \Re \left( i(\chi(a) - \chi(b)) \sum_{\gamma_j \leq x} \frac{x^{\delta_j}}{\gamma_j} \sum_{k=1}^{j^3} x^{i(k \gamma_j + \theta_j)} (j^3 + 1 - k) \right) + O \left( \frac{x^{\sigma - \delta}}{\log x} \sum_{\gamma_j \leq x} \frac{j^4 x^{\delta_j}}{\gamma_j^2} + x^{\sigma - \delta} \log^2 x \right).
\]

Note that

\[
\frac{x^{\delta_j}}{\gamma_j} = \exp \left( -\frac{\log x}{j^8} \left( 1 + O \left( \frac{1}{j} \right) \right) - j^8 + O(1) \right).
\]

The maximum of this function over \( j \) occurs around \( J = J(x) := \lfloor (\log x)^{1/16} \rfloor \). In this case we have \( \log x = J^{16}(1 + O(1/J)) \) so that

(4.2) \quad \frac{x^{\delta_j}}{\gamma_j} = \exp \left( -2J^8 + O \left( J^7 \right) \right) = \exp \left( -2(\log x)^{1/2} + O((\log x)^{7/16}) \right).

We will prove that most of the contribution to the main term on the right hand side of (4.1) comes for the \( j \)'s in the range \( J - J^{3/4} \leq j \leq J + J^{3/4} \). First, if \( j \geq 3J/2 \) or \( j \leq J/2 \) then

\[
\frac{x^{\delta_j}}{\gamma_j} \ll \exp \left( -4J^8 \right) \ll \exp \left( -\log x \right)^{1/2} \frac{x^{\delta_j}}{\gamma_j}.
\]

Now suppose that \( J/2 < j < J - J^{3/4} \) or \( J + J^{3/4} < j < 3J/2 \). Write \( j = J + r \) with \( J^{3/4} < |r| < J/2 \). Then \( x > 0, x + 1/x = 2 + (x - 1)^2/x \), hence

\[
\left( 1 + \frac{r}{J} \right)^8 + \left( 1 + \frac{r}{J} \right)^{-8} \geq \left( 1 + \left| \frac{r}{J} \right| \right)^8 + \left( 1 + \left| \frac{r}{J} \right| \right)^{-8} \geq 2 + \frac{(8r/J)^2}{1 + 8r/J} \geq 2 + 12(r/J)^2.
\]

We infer from (4.2) that

\[
\frac{x^{\delta_j}}{\gamma_j} = \exp \left( -\frac{J^{16}}{J^8} \left( 1 + O \left( \frac{1}{J} \right) \right) - j^8 \right) = \exp \left( -J^8 \left( \left( 1 + \frac{r}{J} \right)^8 + \left( 1 + \frac{r}{J} \right)^{-8} \right) + O(J^7) \right) \leq \exp \left( -2J^8 \left( \left( 1 + \frac{6}{\sqrt{J}} \right) + O(J^7) \right) \right) \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{\delta_j}}{\gamma_j}.
\]
Since $\gamma_j \leq x$ implies that $j \ll (\log x)^{1/8}$, the contribution of the terms $1 \leq j < J - J^{3/4}$ or $J + J^{3/4} < j$ to the main term of (4.1) is

$$
(4.3) \quad \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j} \sum_{j \in (\log x)^{1/4}} \sum_{k=1}^{j^{3}} (j^{3} + 1 - k) \ll \exp \left( -(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j}.
$$

Similarly, we have

$$
\frac{x^{-\delta_j}}{\gamma_j^2} = \exp \left( -\frac{\log x}{j^8} \left( 1 + O \left( \frac{1}{j} \right) \right) - 2j^8 + O(1) \right)
$$

$$
\ll \exp \left( -2\sqrt{2}(\log x)^{1/2} (1 + o(1)) \right)
$$

$$
\ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{-\delta_j}}{\gamma_j},
$$

which follows from (4.2) along with the fact that the maximum of $f(t) = -\log x/t^8 - 2t^8$ occurs at $t = (\log x/2)^{1/16}$. Hence, using (4.2), the contribution of the error term of (4.1) is

$$
(4.4) \quad \ll \exp \left( -2(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j} \sum_{j \in (\log x)^{1/4}} j^{4} + x^{\sigma - \delta} \log^2 x \ll \exp \left( -(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j}.
$$

Therefore, inserting the bounds (4.3) and (4.4) in (4.1) we deduce that

$$
\Delta = \frac{2x^{\sigma}}{\log x} \Re \left( \frac{i(\overline{\chi}(a) - \overline{\chi}(b))}{|j - J|^{3/4}} \sum_{j \in (\log x)^{1/4}} \frac{x^{-\delta_j}}{\gamma_j} \sum_{k=1}^{j^{3}} \exp \left( i(k \gamma_j + \theta_j) \log x \right) (j^{3} + 1 - k) \right)
$$

$$
+ O \left( \exp \left( -(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j} \right).
$$

Let $J - J^{3/4} \leq j \leq J + J^{3/4}$. Then $j^{16} = J^{16} \left( 1 + O(J^{-1/4}) \right)$. Hence we get

$$
\theta_j \log x = \left( \arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) \frac{\log x}{j^{16}} + O \left( \frac{\log x}{j^{17}} \right)
$$

$$
= \left( \arg(\chi(a) - \chi(b)) - \frac{\pi}{2} \right) + O \left( \frac{1}{j^{1/4}} \right).
$$

This implies

$$
i(\overline{\chi}(a) - \overline{\chi}(b)) \exp \left( i\theta_j \log x \right) = \left| \chi(a) - \chi(b) \right| \left( 1 + O \left( \frac{1}{j^{1/4}} \right) \right),
$$

since $e^{i\arg z} = z/|z|$. Inserting this estimate in (4.5) we obtain

$$
\Delta = \left( 1 + O \left( \frac{1}{\log^{1/64} x} \right) \right) 2|\chi(a) - \chi(b)| \sum_{|j - J| \leq j^{3/4}} \frac{x^{\sigma - \delta_j}}{\gamma_j \log x} F_{\gamma_j,j^{3}+1}(x)
$$

$$
+ O \left( \exp \left( -(\log x)^{1/3} \right) \frac{x^{\sigma - \delta_j}}{\gamma_j} \right).
$$
For \( x \in [\sqrt{X}, X] \) we have \( \frac{1}{4}(\log X)^{1/16} \leq J - J^{3/4} \) and \( J + J^{3/4} \leq 4(\log X)^{1/16} \) if \( X \) is sufficiently large, since \( J = (\log x)^{1/16} + O(1) \). We define
\[
\Omega := \left\{ x \in [\sqrt{X}, X] : F_{\gamma, j^3}(x) \leq -\frac{j^3}{4} \text{ for all } \frac{1}{4}(\log X)^{1/16} \leq j \leq 4(\log X)^{1/16} \right\}.
\]
Then it follows from Proposition 3.2 that
\[
\text{meas}\Omega = X + O \left( \sum_{\frac{1}{4}(\log X)^{1/16} \leq j \leq 4(\log X)^{1/16}} \frac{1}{j^{3/2} + \sqrt{X}} \right).
\]
Furthermore, if \( x \in \Omega \) then we infer from (4.6) that
\[
\Delta \leq -\frac{1}{3}\left| \chi(a) - \chi(b) \right| \sum_{|j - J| \leq J^{3/4}} \frac{j^3 x^{\sigma - \delta_j}}{\gamma_j \log x} + O \left( \exp \left( -\frac{1}{3} \frac{x^{\sigma - \delta_j}}{\gamma_j} \right) \right).
\]
\[
\leq -\frac{1}{3}\left| \chi(a) - \chi(b) \right| \frac{j^3 x^{\sigma - \delta_j}}{\gamma_j \log x} (1 + o(1)) \leq -x^\sigma / \exp((2 + o(1))\sqrt{x}) < 0
\]
as \( X \to \infty \), which completes the proof.

5 Acknowledgements

The research of K. F. was partially supported by National Science Foundation grant DMS-0901339.

The research of S. K. was partially supported by Russian Fund for Basic Research, Grant N. 11-01-00329.

The research of Y. L. was supported by a Postdoctoral Fellowship from the Natural Sciences and Engineering Research Council of Canada.

References


http://www.math.uiuc.edu/~ford/papers.html


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, 1409 W. GREEN STREET, URBANA, IL, 61801

*E-mail address*: ford@math.uiuc.edu

STEKLOV MATHEMATICAL INSTITUTE, 8, GUBKIN STREET, MOSCOW, 119991, RUSSIA

*E-mail address*: konyagin@mi.ras.ru

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, 1409 W. GREEN STREET, URBANA, IL, 61801

*E-mail address*: lamzouri@math.uiuc.edu