On k-groups and Tychonoff k_R-spaces

(*Category theory for topologists, topology for group theorists, and group theory for categorical topologists*)

Gábor Lukács
Universität Bremen, Germany

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$\mathscr{H}(G,L)$ is the space of continuous homomorphisms between the groups G and L, equipped with the compact-open topology.

If A is abelian, one puts $\hat{A} = \mathscr{H}(A,\mathbb{T})$, where $\mathbb{T}$ is the unit circle; $\hat{A}$ is a topological group again. Important questions:

— What is the relationship between A and $\hat{A}$?

— What is the relationship between A and $\hat{\hat{A}}$?

— Is $\mathscr{H}(A,\mathbb{T})$ the “right” dual object?

Every $g \in A$ gives rise to a (continuous) character ω_g of $\hat{A}$, by evaluation $\omega_g(\zeta) = \zeta(g)$. This defines an algebraic homomorphism $\omega_A \colon A \longrightarrow \hat{\hat{A}}$ ($g \longmapsto \omega_g$); it is natural in A, but it is **not necessarily continuous**!

A is ***Pontryagin-reflexive*** if ω_A is an isomorphism of topological groups.

When A is LCA (locally compact and abelian), then the famous Pontryagin duality holds:

— ω_A is an isomorphism of topological groups;

— Compact groups have discrete duals, and vice versa;

— If $B \leq A$ is a closed subgroup, then $\hat{B} = \hat{A}/B^{\perp}$ and $\widehat{A/B} = B^{\perp}$, where $B^{\perp}$ is the annihilator of B in $\hat{A}$;

— For $c(A)$ the connected component of A, and $B(A)$ the subgroup of elements g such that $\overline{\langle g \rangle}$ is compact, $\widehat{c(A)} = \hat{A}/B(\hat{A})$ and $\widehat{B(A)} = \hat{A}/c(\hat{A})$.

Trouble: Infinite products of LCA groups are not LC anymore.

Recall that a category $\mathcal{C}$ is ***cartesian closed*** if for every $a \in \mathcal{C}$, the functor $a \times - : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint (which we denote by $[a, -]$).

One approach to overcoming the problem of the continuity of ω_A is working in a (complete) cartesian closed category $\mathcal{C}$. It can be immediately seen that in such categories, for every $x, y \in \mathcal{C}$, the natural map $x \to [[x, y], y]$ given by "evaluation at x" is a morphism.

If a and b are abelian group objects in $\mathcal{C}$, then it is possible to define the internal group homomorphism-functor $\{a, b\}$ (which is going to be an abelian group object again!), and one obtains that $a \to \{\{a, b\}, b\}$ is a morphism of group objects in $\mathcal{C}$.

Another interesting feature one gets "for free" from cartesian closure is that if d is a fixed abelian group object (that we think of as the "dualizing" object), then $\{a, \{b, d\}\} \longleftrightarrow \{b, \{a, d\}\}$ is an isomorphism in $\mathcal{C}$ (natural in a and b); in other words, **we get a dual adjunction for free.**

Candidate I: k-spaces

A *test function* for $X \in \mathsf{Top}$ is a continuous map $t\colon K \to X$ with K compact Hausdorff. A map $f\colon X \to Y$ is *k-continuous* if for every test function t for X, $f \circ t$ is continuous. (When X is Hausdorff, it suffices to require that $f\big|_K$ is continuous for every compact subset K of X.) X is a ***k-space*** if every k-continuous map of X is continuous. Basic properties of kTop and kHaus (due to Vogt and Brown, respectively):

— kTop and kHaus are complete, cocomplete, and cartesian closed;

— kTop and kHaus are coreflective in Top and Haus, respectively (the coreflector is the k-ification, k);

— The product of X, Y in kTop or kHaus is given by $\mathsf{k}(X \times Y)$;

— Internal hom-functor is given by the coreflection $\mathsf{k}\mathscr{C}(X, Y)$ of the compact-open topology.

Abelian group objects in kHaus behave very nicely in terms of ω_A (in fact, the continuity of ω_A for LCA groups can be deduced from here).

Drawbacks:

1. The addition is only k-continuous, and it is not necessarily continuous in both variables;
2. As a result, group objects in kHaus need not be Tychonoff;
3. If B is a closed subgroup of A, A/B need not be Hausdorff!

Recall that a space X is *weakly Hausdorff* (t_2*-space*) if for every test function t for X, the image of $t \times t$ is closed in $X \times X$; such spaces are clearly T_1. Although by switching to the category of weakly Hausdorff k-spaces, as Lamartin (1977) did, difficulty no. 3 can be eliminated, the two other troubles nevertheless remain.

Another problem: $\mathbb{T}$ can capture only the Hausdorff part.

Remark. The success of Dubuc and Porta (1971) in describing topological algebras in the category kHaus (i.e., the operations are k-continuous) was due to a dual adjunction similar to what is described above. Often the source of such dual adjunctions is cartesian closure.

The incompatibility of k-spaces with the Tychonoff property is worse than one would imagine:

Example(s). For topological spaces X and Y, put $\mathcal{P}$ for the topology of separate continuity on $X \times Y$, and set $\mathcal{Q}$ to be the cross-topology: $V \in \mathcal{Q}$ if the intersection of V with every fiber $\{x\} \times Y$ and $X \times \{y\}$ is an open subset of the fiber.

Henriksen and Woods (1999) proved that $\mathsf{k}(X \times Y, \mathcal{P}) = (X \times Y, \mathcal{Q})$ and $\tau(X \times Y, \mathcal{Q}) = (X \times Y, \mathcal{P})$ for every Tychonoff k-space X and Y. (Here $\tau\colon \mathsf{Top} \longrightarrow \mathsf{Tych}$ is the Tychonoff reflection.)

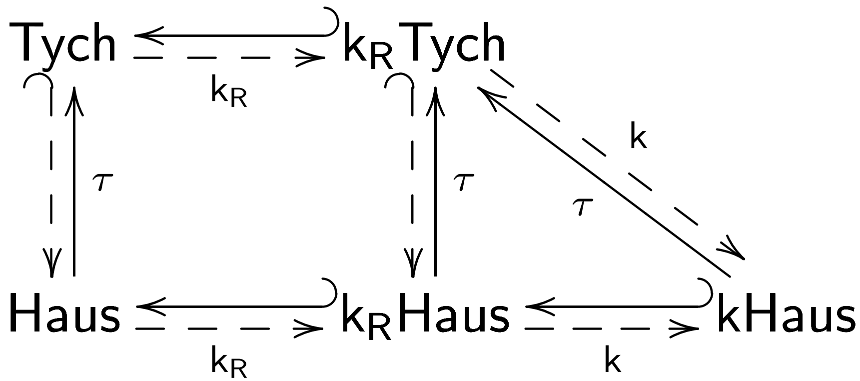
Candidate II: k_R-spaces

X is a ***k_R-space*** if every k-continuous function from X to a Tychonoff space is continuous. The k_R-ification $\mathsf{k_R}X$ always exists. **Features** (GL):

— If X is a k_R-space, then so is τX; if X is Tychonoff, then so is $\mathsf{k_R}X$;

— $\mathsf{k_R}$Haus is coreflective in Haus; $\mathsf{k_R}$Tych is coreflective in Tych;

— $\mathsf{k_R}$Tych is reflective in $\mathsf{k_R}$Haus;



(The dashed arrows are right adjoints.)

— $\mathsf{k_R Tych}$ is equivalent to a proper epireflective subcategory of kHaus;

— $\mathsf{k_R Tych}$ is cartesian closed, and the internal hom-functor is $\mathsf{k_R}\mathscr{C}(X, Y)$;

— If P is a Tychonoff space that contains a path (i.e., a homeomorphic image of $\mathbb{I}$), then $\eta_X : X \to \mathsf{k_R}\mathscr{C}(\mathsf{k_R}\mathscr{C}(X, P), P)$ is an embedding.

Pros and cons:

1. Nice dual adjunction $(+)$;
2. The trouble with forming quotients remains $(-)$;
3. Tychonoff k_R-spaces turn out to be useful $(+)$.

Candidate III: Convergence groups & Binz-Butzmann duality

A ***convergence space*** is a set X together with a relation $\to$ between filters on X and points of X such that:

(Conv1) $\mathcal{F} \to x$ and $\mathcal{G} \to x \Longrightarrow \mathcal{F} \wedge \mathcal{G} \to x$;

(Conv2) $\mathcal{F} \to x$ and $\mathcal{F} \leq \mathcal{G} \Longrightarrow \mathcal{G} \to x$;

(Conv3) $\dot{x} \to x$, where $\dot{x} = \{A \subset X \mid x \in A\}$.

Remark. Every convergence structure on X defines also a topology, but the convergence with respect to that topology need not be the same as the original convergence.

Conv is cartesian closed, and the internal hom-functor is given by the *continuous convergence structure*: It is the coarsest convergence structure that makes the evaluation map $e : \mathsf{Conv}(X,Y) \times X \to Y$ continuous: $\mathcal{F} \to \zeta \in \mathsf{Conv}(X,Y)$ if $e(\mathcal{F} \times \mathcal{H}) \to \zeta(x)$ for every filter $\mathcal{H} \to x \in X$.

A ***convergence group*** is a group object in the category of convergence spaces and their continuous maps (Conv).

If A is an abelian convergence group, one puts $\Gamma_c A$ to be subspace of $\mathsf{Conv}(A, \mathbb{T})$ consisting of the homomorphisms.

A is BB*-reflexive* if the continuous homomorphism $\kappa_A \colon A \to \Gamma_c \Gamma_c A$ is an isomorphism of convergence groups. **Main features** (due to Butzmann):

— If $\{A_i\}_{i \in I}$ are BB-reflexive, then so are $\prod_{i \in I} A_i$ and $\bigoplus_{i \in I} A_i$;

— If A is a topological group, then $\Gamma_c A$ is locally compact (as a convergence group), and $\Gamma_c \Gamma_c A$ is a topological group again;

— κ_A is an embedding if and only if A is "locally quasi-convex";

— If ω_A is continuous, then $\hat{\hat{A}} \subseteq \Gamma_c \Gamma_c A$; in this special case, BB-reflexivity implies P-reflexivity (Chasco & Martín-Peinador).

$\boldsymbol{k}$-groups (of Noble)

A group G is a ***k*-group** if every k-continuous homomorphism $\varphi\colon G \to H$ is continuous. **Features:**

— kGrp is a coreflective subcategory of $\mathsf{Grp}(\mathsf{Haus})$, with coreflector $\mathsf{k_G}$;

— Quotients of k-groups are k-groups;

— If H is an open subgroup of G, then H is a k-group iff G is so;

— If $\{G_i\}_{i\in I}$ is a family of k-groups, then $\prod\limits_{i\in I} G_i$, $\bigoplus\limits_{i\in I} G_i$, $\bigoplus\limits_{i\in I}{}^{\Sigma} G_i$, and $\sum\limits_{i\in I} G_i$ are k-groups [$\sum\limits_{i\in I} G_i$ is $\bigoplus\limits_{i\in I} G_i$ equipped with the final topology];

— Groups that are k-spaces are also k-groups; in particular, LC and metrizable groups are k-groups;

— An arbitrary product $\prod\limits_{i\in I} M_i$ of metrizable groups is a k-group.

Examples and remarks:

1. Let H be a non-LC k-group such that $\mathscr{C}(H,\mathbb{R})$ is metrizable, and hence a k-group (Warner, 1958). Put $G = H \times \mathscr{C}(H,\mathbb{R})$; G is a k-group.

The evaluation $e\colon H \times \mathscr{C}(H,\mathbb{R}) \longrightarrow \mathbb{R}$ is k-continuous, but it can be continuous only if H is LC (Arens, 1946). Hence, G is not a k_R-space.

2. A group G admits a *quasi-invariant basis* if for every nbhd U of e there exists a countable family $\mathcal{V}$ of nbhds of e such that for any $g \in G$ there exists $V \in \mathcal{V}$ such that $gVg^{-1} \subset U$. Groups with this property are precisely the subgroups of products $\prod\limits_{i\in I} M_i$ of metrizable groups.

Since not every complete group admitting a quasi-invariant basis is a k-group, such groups provide a large number of examples for closed subgroups of k-groups that are not k-groups themselves.

Examples and remarks (continued):

3. The only compact subgroup of $\mathbb{R}$ is the trivial one, and since $\mathbb{R}$ has many non-continuous homomorphisms, this shows that kGrp is not the coreflective hull of the compact Hausdorff groups.

4. kGrp is the coreflective hull of the groups that are generated by a compact subset. In fact, it is also the coreflective hull of the class of free groups on compact Hausdorff spaces.

By "free group" on a Tychonoff space X we mean the group FX, where $F\colon \mathsf{Tych} \longrightarrow \mathsf{Grp}(\mathsf{Haus})$ is the left adjoint to the forgetful functor $U\colon \mathsf{Grp}(\mathsf{Haus}) \longrightarrow \mathsf{Tych}$. We note that the unit $\iota_X\colon X \to FX$ of this adjunction is a closed embedding for every $X \in \mathsf{Tych}$ and that FX is algebraically generated by X.

Examples and remarks (continued):

5. The projective limit of k-groups need not be a k-group:

Put $A = \mathbb{Z}_2^{\oplus\omega_1}$ (only finite number of non-zero coordinates).

Set $B_\alpha = \{(g_\beta) \mid g_\beta = 0 \text{ for } \beta < \alpha\}$ for $\alpha < \omega_1$.

Topologize A such that each B_α is an open subgroup.

— A is complete, with base $\{B_\alpha\}$ at 0;

— Thus, $A = \varprojlim A/B_\alpha$;

— Each A/B_α, being countable and discrete, is a k-group;

— A is a P-space (G_δ-sets are open), so its compact subspaces are finite;

— Therefore, $\mathsf{k_G} A$ is discrete.

Since A is not discrete, this shows that A is not a k-group.

$\mathsf{k_R Tych}$ comes to the rescue:

Example no. 5 shows that the forgetful functor $U_0\colon \mathsf{kGrp} \longrightarrow \mathsf{Tych}$ does not preserve limits, because $\varprojlim_{\mathsf{kGrp}} A/B_\alpha = \mathsf{k_G} A \neq A = \varprojlim A/B_\alpha$. In particular, it is hopeless to construct a free k-group functor on Tych.

For every Tychonoff k_R-space X, the free group FX is a k-group.

Since the k_R-ification "absorbs" $\mathsf{k_G}$ in the sense that $\mathsf{k_R k_G} = \mathsf{k_R}$, our conclusion is that the "right" forgetful functor is $\mathsf{k_R} U_0\colon \mathsf{kGrp} \longrightarrow \mathsf{k_R Tych}$.

Indeed, the restricted free group functor $F_0\colon \mathsf{k_R Tych} \longrightarrow \mathsf{kGrp}$ has a right adjoint, namely $\mathsf{k_R} U_0$.

Remark. $\mathsf{k_R}$ is an embedding of kGrp into $\mathsf{Grp}(\mathsf{k_R Tych})$. Furthermore, $\mathsf{k_R} \mathscr{H}(\mathsf{k_R} G, \mathsf{k_R} L) = \mathsf{k_R} \mathscr{H}(G, L)$ for $G, L \in \mathsf{kGrp}$.

Abelian groups:

Let A be an abelian k-group. Noble proved:

— ω_A is continuous;

— A is complete $\&$ admits a base of open subgroups $\Rightarrow$ A is P-reflexive;

— A is an open subgroup of a P-reflexive k-group $\Rightarrow$ A is P-reflexive;

A is a kk-*group* if every k-continuous homomorphism of A into a compact group is continuous. The next result is due to Deaconu.

Let Σ be a subgroup of $\mathrm{Hom}(A, \mathbb{T})$. There exists an LCA topology $\mathcal{T}$ on A such that $\Sigma \cong \hat{G}$ (algebraically) if and only if both conditions are fulfilled:

(i) Σ is dense in $\mathrm{Hom}(G, \mathbb{T})$ in the topology of pointwise-convergence;

(ii) Σ is a kk-group and $\mathsf{k}\Sigma$ is LC.

kAb is monoidal closed: Let A, B, and C be abelian k-groups.

$$A \longrightarrow \mathscr{H}(B, C) \tag{1}$$

$$A \longrightarrow \mathsf{k_G}\mathscr{H}(B, C) \tag{2}$$

$$\mathsf{k_R}A \longrightarrow \mathsf{k_G}\mathscr{H}(B, C) \tag{3}$$

$$\mathsf{k_R}A \longrightarrow \mathsf{k_R}\mathscr{H}(B, C) = \mathsf{k_R}\mathscr{H}(\mathsf{k_R}B, \mathsf{k_R}C) \tag{4}$$

$$\mathsf{k_R}(A \times B) \xrightarrow{\text{bil}} \mathsf{k_R}C \tag{5}$$

$$\mathsf{k_R}(A \times B) \xrightarrow{\text{bil}} C \tag{6}$$

$$F_0\mathsf{k_R}(A \times B) \xrightarrow{\text{bil}} C \tag{7}$$

$$F_0\mathsf{k_R}(A \times B)/R(A, B) \longrightarrow C \tag{8}$$

Here $R(A, B)$ stands for the closed subgroup generated by the commutator and the usual bilinear relations.

Put $A \otimes_{\mathsf{k}} B = F_0 \mathsf{k}_{\mathsf{R}}(A \times B)/R(A, B)$; it is clearly an abelian k-group, and $A \otimes_{\mathsf{k}} -\colon \mathsf{kGrp} \longrightarrow \mathsf{kGrp}$ is the left adjoint to $\mathsf{k}_{\mathsf{G}}\mathscr{H}(B, -)\colon \mathsf{kGrp} \longrightarrow \mathsf{kGrp}$.

Since this definition is very inconvenient for computations, all computations have to be done using the right adjoints. In particular, that is the way to show that

(i) $\mathbb{Z}$ is the neutral object with respect to $\otimes_{\mathsf{k}}$;

(ii) $\otimes_{\mathsf{k}}$ satisfies the "pentagon condition" (with correctly chosen maps).

For (ii), one uses the cartesian closure of $\mathsf{k}_{\mathsf{R}}\mathsf{Tych}$.

Consequently, $\mathsf{k}_{\mathsf{G}}\mathscr{H}(A \otimes_{\mathsf{k}} B, C) \cong \mathsf{k}_{\mathsf{G}}\mathscr{H}(A, \mathsf{k}_{\mathsf{G}}\mathscr{H}(B, C))$; in particular $\mathsf{k}_{\mathsf{G}}\widehat{A \otimes_{\mathsf{k}} B} \cong \mathsf{k}_{\mathsf{G}}\mathscr{H}(A, \mathsf{k}_{\mathsf{G}}\hat{B}) \cong \mathsf{k}_{\mathsf{G}}\mathscr{H}(B, \mathsf{k}_{\mathsf{G}}\hat{A})$. For $\tilde{A} = \mathsf{k}_{\mathsf{G}}\hat{A}$, it follows from the above that $\gamma_A\colon A \to \tilde{\tilde{A}}$ is continuous. (Notice that $\tilde{\tilde{A}} = \mathsf{k}_{\mathsf{G}}\hat{\hat{A}}$.)

Open questions:

1. How to characterize the image $U_0(\mathsf{kGrp})$ of U_0 in Tych?

2. Is $\tilde{A}$ the "right" dual object for k-groups?

For gourmands:

3. Is there a totally minimal h-complete group that is not a k-group?