

# REFLECTIONS ON SYMMETRIC POLYNOMIALS AND ARITHMETIC FUNCTIONS

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ABSTRACT. In an isomorphic copy of the ring of symmetric polynomials we study some families of polynomials which are indexed by rational weight vectors. These families include well known symmetric polynomials, such as the elementary, homogeneous, and power sum symmetric polynomials. We investigate properties of these families and focus on constructing their rational roots under a product induced by convolution. A direct application of the latter is to the description of the roots of certain multiplicative arithmetic functions (the core functions) under the convolution product.

## 1. INTRODUCTION

This paper is concerned with a certain isomorphic copy of the ring of symmetric polynomials, namely the ring of isobaric polynomials denoted by  $\Lambda'_k$ , where the isomorphism is given by a polynomial map involving the elementary symmetric polynomials. This ring is a polynomial ring with coefficients taken to be either the integers  $\mathbb{Z}$  or the rationals  $\mathbb{Q}$ , and the image of a symmetric polynomial under the isomorphism mentioned above will be called an isobaric reflect. An isobaric<sup>3</sup> polynomial is one of the form  $P_n = \sum_{\alpha} A(\alpha)t_1^{\alpha_1} \dots t_k^{\alpha_k}$ , where  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_i \geq 0$  are integers with  $\sum_{j=1}^k j\alpha_j = n$ . As for the ring of symmetric polynomials, we can allow either a finite number  $k$  of variables or we can work in  $\Lambda' = \bigoplus_{k \geq 0} \Lambda'_k$  with infinitely many variables.

Families of isobaric polynomials occur in many contexts in mathematics. In [5] it was shown that the reflects of the complete symmetric polynomials (CSP) determine the multiplicative arithmetic functions locally. In [6] it was shown that the reflects of the power sum symmetric polynomials (PSP) determine the lattice of root fields of quadratic extensions. Properties of these two sequences of polynomials were discussed in [7] where the CSP-reflects are called Generalized Fibonacci Polynomials (GFP), and the PSP-reflects are called the Generalized Lucas Polynomials (GLP). Recall that the Complete Symmetric Polynomials form a  $\mathbb{Z}$ -basis for the symmetric polynomials as do the Elementary Symmetric Polynomials (ESP), while the Power Symmetric Polynomials (PSPs) form a  $\mathbb{Q}$ -basis. The analogues of these facts carry over to the isobaric polynomial algebras by way of a canonical isomorphism of the ring of symmetric polynomials  $\Lambda_k$  to the ring  $\Lambda'_k$  denoted by  $\Xi_k$ . In fact, this isomorphism is just the one that takes symmetric functions on  $k$  variables, written in terms of elementary symmetric polynomials, and rewrites each elementary polynomial  $e_j$  as  $(-1)^{j+1}t_j$ .

$$(1.1) \quad \hat{e}_i = \Xi_k(e_i) = (-1)^{j+1}t_j.$$

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<sup>1</sup>1991 *Mathematics Subject Classification*: Primary 05E05; Secondary 11N99.

<sup>2</sup>*Keywords and phrases*: symmetric functions, isobaric polynomials, multiplicative arithmetic functions.

<sup>3</sup>the term isobaric is due to Pólya [10]; the cycle index of a finite group appearing in Pólya's Counting Theorem is an isobaric polynomial.

It is well-known that the Schur Symmetric Polynomials (SSP) determine the complex character table of the finite symmetric groups using the Littlewood-Richardson rule and the Frobenius Character Theorem. Thus the SSPs for a given  $n$  can be regarded as an encoding of the complex character table of  $Sym(n)$ . The Frobenius Character Theorem can be written in terms of isobaric polynomials, namely in terms of GLPs. Using this fact, the complex characters of  $Sym(n)$  can be easily calculated from the isobaric reflects of the SSP( $n$ ), the Schur polynomials, for a given  $n$ .

The families GFP and GLP have the additional useful property that each satisfies recursion relations (Newton identities). It will turn out that  $\Lambda'$  contains a large class of recursively defined families (Theorem 2.3). These are the families of what we have called *weighted isobaric polynomials* (WIPs), and they are the main subject of this paper. Such polynomials are determined by assigning a weight to each of the variables  $t_j$ , i.e. by assigning a weight vector to the set of variables  $\{t_j\}$ . Such families will be called weighted isobaric families. It turns out that the union of all such families does not exhaust the ring of isobaric polynomials. In fact, in this paper we show that among the Schur polynomials, exactly those Schur reflects which represent hook partitions can belong to a sequence of weighted isobaric polynomials (Theorem 3.1 and Theorem 3.4).

Families of WIPs, multi-indexed by their weights, form in a natural way a free abelian group induced by addition of their weight vectors (Theorem 2.3). The weighted families GFP and GLP, i.e. the CSP and PSP reflects, are the weighted families determined by weight vectors  $(1, 1, \dots)$  in the case of GFP, and  $(1, 2, 3, \dots)$  in the case of GLP. The Schur-hook reflects have weight vectors of the form  $(0, 0, \dots, 1, 1, \dots)$  (Theorem 3.2).

The coefficients of the monomials in an isobaric polynomial are uniquely determined by the exponents of the variables and the weight vector of the family (Theorem 2.1). In order to prove Theorem 2.1 we use the fact that each monomial determines a lattice whose nodes are the exponent partitions of the monomial obtained by derivation. However, this lattice is not the well-known Young's lattice in symmetric functions theory, but instead it is a lattice partially ordered by the pointwise inequality of the exponents of the constituent nodes. It assumes a major role in this paper in understanding the construction of the WIPs and certain other structures associated to them. This lattice is in fact isomorphic with the divisor lattice of natural numbers.<sup>4</sup>

In 1988, Carroll and Gioia [1] gave a numerical description of the  $q$ -th roots ( $q \in \mathbb{Q}$ ) of the multiplicative arithmetical functions in the group of units of the ring of arithmetical functions. In [5] it was shown that under convolution, these functions form a free abelian group generated by the completely multiplicative functions, as mentioned above; it was also shown in that paper that the GFPs give a generic set of generating functions for this group of arithmetic functions in the following sense: each multiplicative function in the core group<sup>5</sup> of the group of units of a multiplicative arithmetic function together with its convolution inverse is uniquely determined locally by a monic polynomial (over the complex field), the generating polynomial. What is called a negative element is a multiplicative function whose local values are just the coefficients of this generating polynomial, while the inverse of this negative core function, a positive element, is a multiplicative function whose local values are given by evaluating the series of GFPs truncated at the degree of the generating polynomial at these coefficients. In Section 4, we produce a sequence of isobaric polynomials which are the  $q$ -th roots for any  $q \in \mathbb{Q}$  of the generic generating functions for these roots, that is, the  $q$ -th roots of the Generalized Fibonacci and Lucas polynomials (the CSP and PSP reflects). Thus, we have embedded the core group into its divisible closure.

Moreover, our construction is more far-reaching than this. It produces a set of  $q$ -th roots with respect to a product induced by convolution, which we have called the level product (so-called

<sup>4</sup>We thank the referee for pointing out this isomorphism.

<sup>5</sup>The Core group is the subgroup of the group of units in the ring of arithmetic functions generated by the completely multiplicative arithmetic functions.

because it acts on polynomials of the same level and conserves level) for every isobaric polynomial in any weighted family (Theorem 5.7). Theorem 5.7 also implies that under the level product, an element has a level product inverse.

It is not the case that the isobaric roots of weighted functions are necessarily weighted. However, they are determined by specifying a weight vector. The appropriate structure to look at here, then, is the ring generated by the level product. This is a graded ring  $\mathcal{H}$  containing WIPs. However, there is still more algebraic structure. Since we have inversion under level-product, each weighted family and its divisible closure has the structure of the rationals. Moreover, because of Theorem 2.3 each weighted family is acted on by translations, and so we finally have that all of this together with the derivation operation give a differential graded abelian group acted on by an affine group.

## 2. WEIGHTED ISOBARIC POLYNOMIALS

Before we proceed we introduce a few notations used often in the paper.

**Notation.** For a non-negative integer vector  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_{\geq 0}^k$  we denote  $\alpha \vdash n$  if  $\sum_{j=1}^k j\alpha_j = n$ , and by  $|\alpha|$  the sum  $\sum_{j=1}^k \alpha_j$ . The partition associated to  $\alpha$  is  $(1^{\alpha_1}, 2^{\alpha_2}, \dots, k^{\alpha_k})$  which is a partition of  $n$ .

We first define the weight of a monomial and then define the weighted isobaric polynomial.

**Definition 1.** Given a rational vector  $\omega = (\omega_1, \omega_2, \dots, \omega_k)$  we define the weight of each variable  $t_i$  as  $wt(t_i) = \omega_i$ , for each  $i = 1, \dots, k$ . The weight  $wt(t^\alpha)$  is now defined inductively as

$$(2.1) \quad wt(t^\alpha) = \sum_{i=1, \alpha_i \geq 1}^k wt(t_1^{\alpha_1} \dots t_i^{\alpha_i-1} \dots t_k^{\alpha_k}).$$

A *weighted isobaric polynomial (WIP)* of weight  $\omega$  is defined to be

$$(2.2) \quad P_{k,n,\omega} = \sum_{\alpha \vdash n} wt(t^\alpha) t^\alpha.$$

**Remark 1.** 1. The definition of the weights can be best understood using a particular lattice which will now be described. The nodes of the lattice are the monomials  $\{t_1^{\alpha_1} \dots t_k^{\alpha_k}\}$ , where  $\alpha = (\alpha_1, \dots, \alpha_k)$ ,  $\alpha_j \in \mathbb{Z}$ ,  $\alpha_j \geq 0$ . The relation

$$t^\beta = (t_1^{\beta_1} \dots t_k^{\beta_k}) \leq t^\alpha = (t_1^{\alpha_1} \dots t_k^{\alpha_k}), \text{ if } \beta_j \leq \alpha_j, \text{ for every } j,$$

imposes a lattice structure on the set of  $\{t^\alpha\}$ . The *depth* of a monomial in the lattice is  $|\alpha|$ , and its *level* is  $n = \sum_j j\alpha_j$ . Let 1 be the bottom element. After we assign weight  $\omega_j$  to each  $t_j$ ,  $t^\alpha$  will be assigned the coefficient equal to the sum of the coefficients of all  $t^\beta$  that have depth  $(|\alpha| - 1)$  and for which  $t^\beta < t^\alpha$ . Thus each monomial involving  $t^\alpha$  can be associated with a (finite) sublattice  $\mathcal{L}(t^\alpha)$  with top element  $t^\alpha$ , the lattice of all those monomials whose coefficients contribute to the coefficient of  $t^\alpha$ . For any two monomials there is a monomial whose lattice contains their lattices<sup>6</sup>. Note also that the lengths of all maximal chains of the sublattice  $\mathcal{L}(t^\beta)$  with  $t^\beta < t^\alpha$  are the same and clearly equal to the corresponding *depth*.

2. Another interpretation of this lattice can be drawn by identifying  $t_j^{\alpha_j}$  with  $p_j^{\alpha_j}$ , where  $p_j$  is the  $j$ -th prime number. In this way, the order relation becomes the divisibility relation for natural numbers.

<sup>6</sup>This lattice can be thought as a lattice of Young diagrams  $(1^{\alpha_1}, \dots, k^{\alpha_k})$  in which a “smaller” diagram is one with one less row; it is clearly not a Young lattice. As far as we know these lattices have not been introduced before into the study of symmetric functions.

**Example 1.** The lattice  $\mathcal{L}(t_1^2 t_2 t_3)$  is

We can now state

**Theorem 2.1.** *If  $\omega$  is a weight vector, then the WIP of degree  $n$  has at most  $\mathcal{P}(n)$  terms, where  $\mathcal{P}(n)$  is the number of partitions of  $n$ , and the coefficients are given by*

$$(2.3) \quad \text{weight}(t^\alpha) = A_{k,n,\omega}(\alpha) = \binom{|\alpha|}{\alpha_1, \dots, \alpha_k} \frac{\sum_i \alpha_i \omega_i}{|\alpha|}.$$

In particular, the coefficients of the families  $\mathcal{F}_k$ , the GFP and  $\mathcal{G}_k$ , the GLP, are given by  $\binom{|\alpha|}{\alpha_1, \dots, \alpha_k}$  and  $n \frac{(|\alpha| - 1)!}{\prod_{j=1}^k (\alpha_j)!}$ , respectively, where the weight vector of the GFPs is given by  $(1, 1, \dots)$  and the weight vector of the GLPs is given by  $(1, 2, 3, \dots)$ . It is of interest that, when  $\omega$  is an integer vector, the numbers  $A_{k,n,\omega}(\alpha)$  are integers. This will be a trivial consequence of the proof of Theorem 2.1.

**Proof of Theorem 2.1.**

Let  $\omega = (\omega_1, \dots, \omega_k)$  with  $\omega_j \in \mathbb{Z}$  be a weight assignment to the indeterminates  $t_1, \dots, t_k$ . This assignment, together with the inductive rule for determining the coefficient of a monomial in the lattice, will define a family of WIPs, denoted by  $\mathfrak{F}_{k,\omega}$  or just  $\mathfrak{F}_\omega$ .

To see that the coefficients are as stated in the theorem, we proceed by induction on the depth to compute the coefficient of  $t^\gamma$ , where  $t^\gamma = t_1^{\gamma_1} \dots t_k^{\gamma_k}$ . The monomials that contribute to the coefficient of  $t^\gamma$  are just  $t^{\gamma^{(j)}}$ , where  $\gamma^{(j)} = (\gamma_1, \dots, \gamma_j - 1, \dots, \gamma_k)$ . Then by induction

$$A_{k,n,\omega}(\gamma^{(j)}) = \frac{(|\gamma| - 2)!}{\prod_{i \neq j} (\gamma_i)! (\gamma_j - 1)!} \left[ \sum_{i \neq j} \gamma_i \omega_i + (\gamma_j - 1) \omega_j \right]$$

and so

$$\begin{aligned} A_{k,n,\omega}(\gamma) &= \sum_{j=1}^k \frac{(|\gamma| - 2)!}{\prod_{i \neq j} (\gamma_i)! (\gamma_j - 1)!} \left[ \sum_{i \neq j} \gamma_i \omega_i + (\gamma_j - 1) \omega_j \right] = \\ &= \sum_{j=1}^k \frac{(|\gamma| - 1)! (\gamma_j)! \left[ \sum_i \gamma_i \omega_i - \omega_j \right]}{\prod_i (\gamma_i)! \sum_i (\gamma_i - 1) (\gamma_j - 1)!} = \frac{(|\gamma| - 1)!}{\prod_i (\gamma_i)!} \left( \frac{\sum_i \gamma_i \omega_i}{(\sum_i \gamma_i) - 1} |\gamma| - \frac{\sum_i \gamma_i \omega_i}{|\gamma| - 1} \right) \\ &= \frac{(|\gamma| - 1)!}{\prod_i (\gamma_i)!} \left( \sum_i \gamma_i \omega_i \right). \end{aligned}$$

□

Note that a family  $\mathfrak{F}_{k,\omega}$  is a *sequence* of polynomials, one for each degree. For example the first four of these in any sequence are of the form:

$$\begin{aligned} P_{1,\omega} &= \omega_1 t_1 \\ P_{2,\omega} &= \omega_1 t_1^2 + \omega_2 t_2 \\ P_{3,\omega} &= \omega_1 t_1^3 + (\omega_1 + \omega_2) t_1 t_2 + \omega_3 t_3 \\ P_{4,\omega} &= \omega_1 t_1^4 + (2\omega_1 + \omega_2) t_1^2 t_2 + \omega_2 t_2^2 + (\omega_1 + \omega_3) t_1 t_3 + \omega_4 t_4 \end{aligned}$$

Since all of the operations involved in computing the coefficients are ring operations, we have the following.

**Corollary 2.2.** *If  $\alpha$  and  $\omega$  are integer vectors, then  $A_{k,n,\omega}(\alpha)$  is an integer.*

**Remark 2.** For a weight  $\omega$  with  $\omega_i \neq 0$ , for any  $i$ , using the jacobian criterion we have that  $Jacob(P_{i,\omega}) = \prod_i \omega_i \neq 0$  and thus the family  $\mathcal{F}_{k,\omega}$  consists of algebraically independent polynomials. This allow us to construct a new basis for  $\Lambda'_k$  for each such weight vector  $\omega$ , whose elements are

$$P_{\lambda,\omega} = \prod_j P_{\lambda_j,\omega_j}, \quad \text{for every partition } \lambda.$$

Moreover, if  $P_{k,\omega} \in \mathfrak{F}_{k,\omega}$  and,  $P_{k,\omega'} \in \mathfrak{F}_{k,\omega'}$  then  $P_{k,\omega} + P_{k,\omega'} \in \mathfrak{F}_{k,\omega+\omega'}$ . If we define addition on these classes by  $\mathfrak{F}_{k,\omega} + \mathfrak{F}_{k,\omega'} = \mathfrak{F}_{k,\omega+\omega'}$  we have

**Theorem 2.3.**  *$\{\mathfrak{F}_{k,\omega}\}$  is a free  $\mathbb{Z}$ -module under this operation.*

**Proof** First notice that  $A_{k,n,\omega}(\alpha) = A_{k,n,\omega'}(\alpha)$  if and only if  $\omega = \omega'$ , for by Theorem 2.1 this implies that  $\sum_j \alpha_j \omega_j = \sum_j \alpha_j \omega'_j$ , and this in turn implies, by taking different values for the  $\alpha$ 's, that  $\omega_j = \omega'_j$ . Thus equality of families implies identity of weights. Clearly the set of families is a  $\mathbb{Z}$ -module under the defined operation. Since there is a homomorphic image of the  $\mathbb{Z}$ -module of families onto the  $\mathbb{Z}$ -module of integer weight vectors, the assertion follows. □

While the WIPs form a graded (and, as we shall see, a differential graded) group, they are not closed under multiplication, so the best we can do is speak of the subring of  $\Lambda'_k$  generated by WIPs. We shall see later that this is a proper subring of  $\Lambda'_k$ . The lattice  $\mathcal{L}(t^\gamma)$  is generated by derivations  $\frac{1}{\alpha_i} \partial_i$ , where  $\partial_i : t^\alpha \rightarrow \alpha_i t^\beta$ , and where  $\beta = (\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_k)$ . Thus  $\Lambda'_k$  becomes a differential graded ring. We also shall need the *total* differential operator  $D_j$  where  $D_j = D_1(D_{j-1})$  and  $D_1(t^\alpha) = \sum_i \partial_i(t^\alpha)$ . We are interested in the total differential when it is evaluated at a weight vector  $\omega = (\omega_i)_i$ . We write this as  $D_j(\omega^\alpha)$ . We shall need the following lemma later.

**Lemma 2.4.**

$$D_{|\alpha|}(\omega_1^{\alpha_1} \dots \omega_k^{\alpha_k}) = (|\alpha| - 1)! (\alpha_1 \omega_1 + \dots + \alpha_k \omega_k).$$

(Note that the right hand side of the Equation above is just  $\prod(\alpha_i)!$  times the coefficient of a monomial term in a weighted isobaric polynomial given in Theorem 2.1.)

**Proof of Lemma 2.4** Let  $u = |\alpha|$ , then

$$D_u(\omega_1^{\alpha_1} \dots \omega_k^{\alpha_k}) = \sum_j \alpha_j D_{u-1}(\omega_1^{\alpha_1} \dots \omega_j^{\alpha_j-1} \dots \omega_k^{\alpha_k})$$

and by induction it is

$$\begin{aligned} &= (u-2)! \left( \sum_j \alpha_j \left( \sum_i ((\alpha_i \omega_i) - \omega_j) \right) \right) = (u-2)! \left( \sum_i (\alpha_i \omega_i) \left( \sum_j \alpha_j \right) - \left( \sum_i \alpha_i \right) \right) \\ &= (u-2)! (u-1) \left( \sum_i \alpha_i \omega_i \right) = (|\alpha| - 1)! \left( \sum_i \alpha_i \omega_i \right). \end{aligned}$$

□

### 3. WEIGHTED ISOBARIC POLYNOMIALS AND SCHUR FUNCTIONS

In this section we are interested in the question of which isobaric polynomials (viewed as reflects of symmetric polynomials) belong to some WIP sequence. We have already seen that two families of polynomials, namely the Generalized Fibonacci Polynomials and Generalized Lucas Polynomials, which are the reflects of the complete and power symmetric polynomials. However, the rich structure of the ring of symmetric polynomials leads to ask if other well known families of symmetric polynomials are endowed with this property.

Before we proceed let us recall some facts about symmetric polynomials. For a basic overview of the subject the reader is advised to consult [2], while a more comprehensive treatment is given in [4].

The ring of symmetric polynomials in  $k$  variables  $x_1, x_2, \dots, x_k$  is formed by those polynomials  $f \in \mathbb{Z}[x_1, \dots, x_k]$  invariant under permutations of variables. This ring, denoted by  $\Lambda_k$  possesses a series of linear bases indexed by partitions. Three bases we encountered already, the elementary symmetric functions  $e_n = \sum_{1 \leq i_1 < i_2 < \dots < i_n \leq k} x_{i_1} \dots x_{i_n}$  and  $e_\lambda = \prod e_{\lambda_i}$  with  $\lambda$  partitions such that

$\lambda_1 \leq k$ . The others are the complete homogeneous polynomials  $h_n = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq k} x_{i_1} \dots x_{i_n}$

with  $h_\lambda$  defined similarly and the power sum symmetric polynomial  $p_n = \sum_{i=1}^k x_i^n$ . The  $p_\lambda$ 's are a  $\mathbb{Q}$ -basis.

However the basis that plays the most important role in the theory of symmetric functions and its connections with other areas of mathematics such as geometry and representation theory is the Schur polynomials. The polynomial  $S_\lambda$  is in fact the character function associated to the irreducible character of type  $\lambda$  of the symmetric group  $Sym(n)$ . We define them here just by the formulas below. Note:  $S_{(n)} = h_n$  and  $S_{(1^n)} = e_n$ .

Formulas that express one basis in terms of another are both remarkable and extremely useful. We will make use of them in the sequel. We will need two. The first is the Jacobi-Trudi formula that expresses the Schur basis in terms of the complete homogeneous symmetric polynomials:

$$S_\lambda = \det[h_{\lambda_j - i + j}]_{1 \leq i, j \leq k}.$$

The other is the Frobenius Character Theorem, which expresses the Schur basis in terms of the power sum symmetric polynomials:

$$S_\lambda = \frac{1}{n!} \sum_{\mu} C(\mu) \chi_\lambda^\mu p_\mu,$$

where  $C(\mu)$  is the size of the conjugacy class of  $\mu$ , and  $\chi_\lambda^\mu$  is the character of  $Sym(n)$  afforded by  $\lambda$  applied to  $\mu$ . Applying the reflection map  $\mathcal{E}_k$  to these formula we obtain  $\hat{S}_\lambda = \det[F_{\lambda_j - i + j}]_{1 \leq i, j \leq k}$  and  $\hat{S}_\lambda = \frac{1}{n!} \sum_{\mu} C(\mu) \chi_\lambda^\mu G_\mu$  respectively.

The rest of this section is concerned with which Schur reflects (i.e. reflects of the Schur polynomials) belong to a WIP family. We show that that Schur reflects determined by partitions  $\lambda$

which are hooks, i.e. of type  $\lambda = (p, 1^q)$  verify this property and that for no other partition is  $\hat{S}_\lambda$  a member of a WIP family.

The hook Schur polynomials can be expressed in terms of the homogeneous and elementary symmetric polynomials as  $S_{(n-r, 1^r)} = \sum_{j \geq r+1} (-1)^{j-r-1} e_j h_{(n-j)}$  [[4] Chap 3]. Taking the isobaric reflect we obtain

**Theorem 3.1.**

$$(3.1) \quad \hat{S}_{(n-r, 1^r)} = (-1)^r \sum_{j=r+1}^n t_j \hat{S}_{(n-j)} = (-1)^r \sum_{j \geq r+1}^n t_j F_{(n-j)}, \quad 0 \leq r \leq n.$$

The reflects of the Schur polynomials determined by hooks form families of weighted isobaric polynomials. If  $\lambda = (p, 1^q)$  is a hook partition we call  $p$  the *arm* and  $q$  the *leg* of the hook. The Schur polynomial reflects determined by hooks with legs of the same length belong to the same weighted family, where the length is the number of boxes. More precisely we have

**Theorem 3.2.** *The Schur reflects indexed by hooks with leg length  $(r+1)$  belong to the weighted family determined by the weights  $\omega_r = (-1)^r (0, 0, \dots, 1, 1, \dots)$ , with  $r$  0's, the rest 1's.*

**Proof of Theorem 3.2** We will first show that each term on the right hand side of (3.1) is a weighted polynomial and describe its weight.

**Lemma 3.3.** *The weight vector for  $t_j F_{n-j}$  is  $(0, 0, \dots, 1, 0, \dots)$ , where the  $j$ -th component is 1.*

**Proof** Exponents of the monomials in  $F_{n-j}$  satisfy  $|\alpha| = n - j$ . So the coefficients of  $F_{n-j}$  are by Theorem 2.1,  $A_{k, n-j, (1, 1, \dots)}(\alpha) = \frac{|\alpha|!}{\prod_{i=1}^k (\alpha_i)!}$ , where  $\sum i \alpha_i = n - j$ . But the coefficient of the monomial in  $t_j F_{n-j}$  with exponent  $\beta = (\alpha_1, \dots, \alpha_j + 1, \dots, \alpha_k)$ , that is the coefficients in  $A_{k, n, \omega}(\beta)$ , is the same, i.e  $A_{k, n-j, (1, 1, \dots)}(\alpha) = A_{k, n, \omega}(\beta)$ . On the other hand

$$A_{k, n, \omega}(\beta) = \frac{|\alpha|!}{(\alpha_j + 1) \prod_{i \neq j} (\alpha_i)!} \left[ \sum_{i \neq j} \alpha_i \omega_i + (\alpha_j + 1) \omega_j \right].$$

From these considerations we get that  $\sum_{i \neq j} \alpha_i \omega_i + (\alpha_j + 1) \omega_j = \alpha_j + 1$ . Since this is true for all exponents such that  $\sum i \alpha_i = n - j$  we must have that the weight vector of  $t_j F_{n-j}$  is  $(0, 0, \dots, 1, 0, \dots)$ .  $\square$

We complete the proof of Theorem 3.2 by using the result of Theorem 2.3 which shows that a sum of weighted polynomials is a weighted polynomial with weight the sum of weights of each term. Therefore the proof is complete.  $\square$

Next we give a complete answer to the question of which Schur reflects belong to weighted families.

**Theorem 3.4.** *If  $\lambda$  is a partition of  $n$  which is not a hook, then  $\hat{S}_\lambda \neq P_{n, \omega}$ , for any weight vector  $\omega$ . The Schur reflect cannot be a weighted polynomial.*

Before we proceed with the proof we need

**Definition 2** (lexicographic order on  $\mathcal{P}(n)$ ). Let  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$  and  $\mu = (\mu_1 \geq \mu_2 \geq \dots)$  be two partitions of  $n$ . We say that  $\lambda < \mu$  if and only if, for some index  $i$  we have  $\lambda_j = \mu_j$  for  $j < i$  and  $\lambda_i < \mu_i$ .

**Example 2.** *The lexicographic order on  $\mathcal{P}(4)$ .*

$$(1^4) < (1^2, 2) < (2^2) < (1^1, 3^1) < (4^1).$$

This order induces a corresponding order on the monomials  $t^\alpha$  with  $(1^{\alpha_1}, \dots, k^{\alpha_k}) \vdash n$ . Furthermore we shall write the WIPs by ordering its monomials starting with the smallest. For example  $P_{4,\omega} = \omega_1 t_1^4 + (2\omega_1 + \omega_2) t_1^2 t_2 + \omega_3 t_2^2 + (\omega_1 + \omega_3) t_1 t_3 + \omega_4 t_4$ .

Next we place in a  $\boxed{\text{box } i}$  all monomials  $t^\alpha = t_1^{\alpha_1} \dots t_i^{\alpha_i}$  such that  $\alpha_i \neq 0$  and  $\alpha_j = 0$ , for  $j > i$ .

**Example 3.** *The arrangement of boxes in  $\mathcal{P}(4)$ .*

It is easy to see that ordering the boxes according to their indices, gives a saturated chain under the lexicographic order of all monomials  $t^\alpha$ , with  $(1^{\alpha_1}, \dots, k^{\alpha_k}) \vdash n$ . We note that the smallest monomial in each box corresponds to a hook and only to a hook. This follows from the way we defined the boxes: the smallest monomial in, say  $\boxed{\text{box } i}$  is  $t_1^{n-i} t_i$  which corresponds to the hook  $(i, 1^{n-i})$ .

**Proof of Theorem 3.4** We use a formula which expresses the Schur polynomial basis in terms of the elementary symmetric polynomial basis of the ring of symmetric polynomials. This is the famous Jacobi-Trudi identity:  $S_\lambda = \det(e_{\lambda'_i - i + j})$  where  $\lambda'$  is the conjugate partition of  $\lambda$  obtained by transposing the Young diagram. Under the reflection isomorphism, i.e.,  $\hat{e}_i = (-1)^{i-1} t_i$ , we get  $\hat{S}_\lambda = \det((-1)^{\lambda'_i - i + j - 1} t_{\lambda'_i - i + j})$ . In the expression above the smallest monomial is obtained from the main diagonal of the determinant (as the transition matrix from the bases  $S_\lambda$  and  $e_\lambda$  is upper triangular [Chap 6 in [4]] and also the Appendix). The smallest monomial is  $t^\delta$  where  $(1^{\delta_1}, \dots, s^{\delta_s}) = \lambda'$  and its coefficient is  $(-1)^{n-\lambda_1} \neq 0$ .

Assume now that  $\hat{S}_\lambda$  is a weighted polynomial, that is, there exists a weight  $\omega$  such that  $\hat{S}_\lambda = P_{n,\omega}$ . Recall that from Theorem 2.1, in  $P_{n,\omega}$  the coefficient of  $t^\alpha$  is  $A(\alpha) = \frac{|\alpha|}{\alpha_1 \dots \alpha_k} \sum_i \alpha_i \omega_i$ . Assume that the smallest monomial  $t^\delta$  belongs to  $\boxed{\text{box } s}$ , i.e,  $\delta_s \neq 0$  and  $\delta_i = 0$  for  $i > s$ . Since  $\lambda$  is not a hook,  $\lambda'$  is not a hook either and so  $t^\delta$  is not the first monomial in box  $s$ . Moreover, we must have that  $A(\beta) = 0$  for any  $\beta$  such that  $(1^{\beta_1}, \dots, k^{\beta_k}) < (1^{\delta_1}, \dots, s^{\delta_s})$  in the lexicographic order. In particular  $A(\beta^{(i)}) = 0$ , for  $\beta^{(i)} = (n-i, 0, \dots, 1, \dots, 0)$ , with 1 in the  $i$ -th place, for  $i = 1, 2, \dots, s$ , that is, the first monomials in boxes  $1, 2, \dots, s$ . This is to say

$$\binom{n-i+1}{1} \frac{(n-i)\omega_1 + 1\omega_i}{n-i+1} = 0, \quad i = 1, \dots, s.$$

From this system of equation we get

$$\omega_1 = \omega_2 = \dots = \omega_s = 0,$$

which in turn gives  $A(\delta) = \frac{|\delta|}{\delta_1 \delta_2 \dots \delta_s} \sum_{i=1}^s \delta_i \omega_i = 0$ , a contradiction.  $\square$

Theorem 3.2 and 3.4 tell us that a Schur reflect is a WIP iff it is indexed by a hook. On the other hand every WIP can be written as an expression in the Schur reflect basis. It turns out that these expressions are both remarkable and simple and involve only hook Schur reflects.

**Theorem 3.5.**

$$(3.2) \quad P_{n,\omega} = \sum_{i=0}^{n-1} (-1)^{i+1} (\omega_i - \omega_{i+1}) \hat{S}_{(n-i, 1^i)}, \quad \text{where } \omega_0 = 0.$$

**Proof** Since each  $\hat{S}_{(n-i, 1^i)}$  is a weighted polynomial of weight  $(-1)^i(0, 0, \dots, 1, 1, \dots)$  with the first 1 in the  $(i+1)$ -th position, we obtain that the right hand side is a weighted polynomial of level  $n$  with weight vector:

$$\sum_i (-1)^{i+i+1} (\omega_i - \omega_{i+1}) (0, 0, \dots, 1, 1, \dots) = (\omega_1, \omega_2, \dots) = \omega.$$

□

#### 4. RECURSION PROPERTIES AND GENERATING FUNCTIONS

In [7] it was shown that the CSP reflects and the PSP reflects satisfy Newton identities, that is, are recursive. This is in fact a property possessed by all WIPs.

**Theorem 4.1.** *Let  $P_{n,\omega} = P_{k,n,\omega} \in \mathfrak{F}_{k,\omega}$ , then*

$$(4.1) \quad P_{n,\omega} = t_1 P_{n-1,\omega} + \dots + t_{n-1} P_{1,\omega} + t_n \omega_n$$

**Proof** This is, essentially, the lattice definition that assigns coefficients to monomials in a weighted family. Let  $A(\alpha)t^\alpha$  be a monomial in  $P_{n,\omega}$  and consider all monomials  $t^\beta$  in  $P_{j,\omega}$ 's on the right hand side such that  $t_{n-j}t^\beta = t^\alpha$ . In the lattice  $\mathcal{L}(t^\beta)$  we need only consider the nodes that contain  $t_j$  such that  $j = 1, \dots, k$  and only those nodes  $\partial_j t^\alpha$  for which  $t_i \partial_j t^\alpha = t^\alpha$ . But these are just the nodes of depth  $(|\alpha| - 1)$ , which by Theorem 2.1, are those whose coefficient sum is the coefficient of  $t^\alpha$ .

On the other hand, every vector  $\alpha$  that occurs in a monomial  $P_{j,\omega}$  in  $\sum t_{n-j} P_{j,\omega}$  occurs as a vector for some monomial in  $P_{n,\omega}$ . □

**Theorem 4.2.** *A generating function for the WIPs in the family  $\mathfrak{F}_\omega$ ,  $\Omega(y) = \sum_{n \geq 0} P_{n,\omega} y^n$  with  $P_{0,\omega} = 1$  is given by*

$$\Omega(y) = 1 + \frac{\omega_1 t_1 y + \omega_2 t_2 y^2 + \omega_3 t_3 y^3 + \dots + \omega_k t_k y^k}{1 - p(y)}, \quad \text{where } p(y) = t_1 y + t_2 y^2 + t_3 y^3 + \dots + t_k y^k.$$

The polynomial  $f(1/y) = x^k - t_1 x^{k-1} - \dots - t_k$ , with  $x = 1/y$  will be called the *core polynomial*. The significance of the core polynomial will be discussed below and in the next section.

We shall organize the proof of Theorem 4.2 around the following two lemmas.

**Lemma 4.3.** *A generating function for the family  $\mathfrak{F}_{\omega_{(0)}}$  where  $\omega_{(0)} = (1, 1, \dots, 1, \dots)$  is the function  $H(y) = \frac{1}{1 - p(y)}$ , where  $p(y) = (t_1 y + t_2 y^2 + t_3 y^3 + \dots + t_k y^k)$ .*

**Proof** This is a consequence of Theorem 3.1 in [5] where it was shown that  $\{F_n(t)\}$  is the “positive” multiplicative function in the core group (under the standard convolution product in the ring of arithmetic functions<sup>7</sup>) of the group of units of the ring of arithmetic functions, which is determined by the polynomial

$$x^k - t_1 x^{k-1} - \dots - t_k,$$

which itself is the generating function for the “negative” sequence with respect to the positive sequence. Thus letting  $x = 1/y$  gives the result. □

**Lemma 4.4.** *Let  $\omega$  be an arbitrary weight vector, then  $P_{n,\omega} = \omega_n t_n * F_n$ , where  $\omega_n t_n * F_n = \sum_{j=0}^n \omega_j t_j F_{n-j}$  and  $F_n \in \mathfrak{F}_{\omega_{(0)}}$ .*

<sup>7</sup>If  $\phi$  and  $\theta$  are two arithmetic functions their convolution product is defined by  $\phi * \theta(n) = \sum_{d|n} \phi(d)\theta(n/d)$ .

(The  $*$ -product is discussed in the next section where it is called the “level product”).

**Proof** An easy induction using Theorem 4.1 gives the result.  $\square$

**Proof of Theorem 4.2** It is clear that  $\sum \omega_n t_n y^n$  is the generating function for  $\omega_n t_n$ . Using Lemma 4.3 and Lemma 4.4, we obtain the result we want by multiplying the generating functions.  $\square$

## 5. ROOT POLYNOMIALS AND CONVOLUTIONS

This section is motivated by work which appeared in [1] and [7] and [12]. In [5] the subgroup generated by the completely multiplicative arithmetic functions in the group of units of the ring of arithmetic functions (the core subgroup), where multiplication is convolution, is discussed. Each multiplicative function in this group is uniquely determined locally (at primes) by a particular polynomial  $f(x, \mathbf{a}) = x^k - a_1 x^{k-1} - \dots - a_k$ . This is the *core polynomial* (see Section 3, particularly Theorem 4.2), where the parameters are evaluated at  $(a_1, a_2, \dots, a_k)$ . What was referred to as the “negative” of the multiplicative function in that paper is an arithmetic function whose values are just the coefficients of the core polynomial. It was proved in [5] that the “positive” part was determined locally by the values of the isobaric reflects of the CSPs, that is by the GFPs at these coefficients, or rather by “truncations” of the GFPs, that is, GFPs parametrized by partitions of  $n$  into parts no one of which is greater than a fixed  $k$ . (Truncation is equivalent to setting the isobaric generators  $t_{k+1}, t_{k+2}, \dots$  equal to zero in the GFPs, or in any isobaric family). This gives a far-reaching generalization of the results in the 1988 paper of T. Carroll and A. A. Gioia [1] which applies to theory of arithmetic functions as well as to the theory of symmetric polynomials.

Carroll and Gioia [1] gave a numerical description of the rational roots of the core functions. In this part of the paper, we consider isobaric polynomials with rational coefficients and find among them polynomials which play the same role for the rational roots of the multiplicative arithmetic functions in the core group as the GFPs play for the core group itself, embedding the core group into a divisible group. We do more. Given a family of WIPs, we shall provide each of its members with unique rational roots induced by convolution. We shall call the products that produce these roots, *level products*, since they are products of polynomials of the same level which preserve level.

The property of GFPs described above with respect to the core group in the convolution ring of arithmetic functions can be stated as follows. Let  $p$  be a fixed prime, and let  $\chi$  be a positive element in the core, then  $\chi(p^n) = F_{k,n}(a_1, \dots, a_k)$  where  $1, a_1, \dots, a_k$  are the coefficients of the  $k$ -th degree determining polynomial. We can think of the determining polynomial in the case of  $F_{k,n}(t_1, \dots, t_k)$  as the generic  $k$ -th degree polynomial  $x^k - t_1 x^{k-1} - \dots - t_k$ . More is true; the entire group of units in the ring of arithmetic functions is determined by the sequence  $\{F_n(t)\}_n$ . The ‘negative’ elements are determined by the coefficients of the core polynomial. The ‘positive’ elements are determined by the sequence  $\{F_{k,n}\}_n$  itself, as just pointed out; while the complement of the core group in the group of units consists of functions determined by the sequence  $\{F_{k,n}\}_n$ . The role of the core polynomial is played by a power series, and so the remaining units can be regarded as limits of core functions, giving an analogy with the relationship between rational and irrational numbers.

Recall that the convolution product of two multiplicative function is given locally by

$$\chi_1 * \chi_2(p^n) = \sum_{i=0}^n \chi_1(p^i) \chi_2(p^{n-i}).$$

By induction, the  $s$ -th convolution power of a multiplicative arithmetic function  $\chi$  is given by

$$\chi^{*s} = \sum_{\alpha \vdash_k n} C_s(\alpha) \prod_{i=1}^k \chi(p^i)^{\alpha_i}, \text{ where } C_s(\alpha) = \binom{s}{\alpha_1, \dots, \alpha_k, (s - |\alpha|)}.$$

We extend the definition of the convolution product for two sequence of polynomials  $\{P_n\}_{n \geq 0}$  and  $\{Q_n\}_{n \geq 0}$  yielding another sequence  $\{R_n\}_{n \geq 0}$  with

$$(5.1) \quad R_n := \sum_{i=0}^n P_i Q_{n-i}.$$

We will call the convolution product of two sequences of isobaric polynomials a *level product*, since the level is preserved. Let  $P_n = \sum_{\alpha} A(\alpha) t^{\alpha}$  be an isobaric polynomial. If  $P_n$  belongs to a weighted family the coefficients  $A(\alpha)$  are given by Theorem 2.1.

Let  $q \in \mathbb{Q}$  (the group of rationals). Define the sequences  $B_j^q = q(q+1) \dots (q+j)$  and  $B_{-(j)}^q = q(q-1) \dots (q-j)$ , for  $j \geq 0$ , otherwise both  $B_j^q$ ,  $B_{-(j)}^q$  are zero.

**Theorem 5.1.** *Let  $H_{k,n}(t, q)$  denote the  $q$ -th convolution root of  $F_{k,n}(t)$ , where  $F_{k,n}(t) \in GFP$ , then*

$$(5.2) \quad H_{k,n}(t, q) = \sum_{\alpha \vdash n} \frac{1}{|\alpha|!} B_{(|\alpha|-1)}^q \binom{|\alpha|}{\alpha_1, \dots, \alpha_k} t^{\alpha}.$$

**Corollary 5.2.**

$$H_{k,n}(t, 1) = F_{k,n}(t)$$

**Proof of Corollary** When  $q = 1$ , then  $B_{(|\alpha|-1)}^1 = |\alpha|!$ . The Corollary then follows from Theorem 2.1 (see remark following that theorem).  $\square$

**Corollary 5.3.** *If  $\chi$  belongs to the Core of the group of units of the convolution ring of arithmetic functions, then  $H_{k,n}(\mathbf{a}, q) = \chi^{*1/q}(p^n)$ , where  $\mathbf{a} = (a_1, \dots, a_k)$  is the set of coefficients of the core polynomial of  $\chi$ .*

$\square$

The proof of Theorem 5.1 follows from a more general result. Namely that each polynomial in a weighted family of isobaric polynomials has a unique  $q$ -root for every rational number  $q$ .

**Theorem 5.4.**  $H_{k,n,\omega} = \sum_{\alpha \vdash_k n} L_{k,n,\omega}(\alpha) t^{\alpha}$  where

$$L_{k,n,\omega}(\alpha) = \sum_{j=0}^{|\alpha|-1} \frac{1}{(\prod \alpha_i)!} \binom{|\alpha|-1}{j} B_{-(j)}^q D_{(|\alpha|-j-1)}(\omega_1^{\alpha_1} \dots \omega_k^{\alpha_k})$$

is the  $q$ -th level root of  $P_{k,n,\omega} \in \mathfrak{F}_{k,\omega}$  and  $H_{k,0,\omega} = 1$ .

Theorem 5.1 follows from Theorem 5.4. The following lemmas will be used in proving this.

**Lemma 5.5.**

$$D_j(\omega^{\alpha})|_{\omega=(1,1,\dots)} = \frac{|\alpha|!}{(|\alpha|-j)!}.$$

**Proof** At depth 0 the value is 1. If after  $(j-1)$  derivations the value is  $\frac{|\alpha|!}{(|\alpha|-j+1)!}$ , then in the  $j$ -th step the exponent sum is decreased by 1, so by derivation,  $(|\alpha|-j)$  appears as the only new factor in the value of  $D_j \omega^{\alpha}$ .  $\square$

**Lemma 5.6.**

$$\sum_{j=0}^{|\alpha|-1} \frac{(|\alpha|-1)!}{(j+1)!} \binom{|\alpha|-1}{j} B_{-(j)}^q = B_{(|\alpha|-1)}^q.$$

**Proof** Consider the following Stirling functions  $[x]_n = x(x-1)\dots(x-n+1)$ , and  $[x]^n = x(x+1)\dots(x+n-1)$ . From the theory of Stirling numbers of 1st and 2nd kind we have the relation

$$[x]^p = \sum_{j=1}^p \binom{p-1}{j-1} \frac{p!}{j!} [x]_j \text{ (e.g., [11], p15, problem 3). This translates into}$$

$$B_{(p)}^q = \sum_{j=0}^p \binom{p}{j} \frac{(p+1)!}{(j+1)!} B_{-(j)}^q.$$

Letting now  $p = |\alpha| - 1$  in the Equation above gives the result we wanted.  $\square$

Theorem 5.4 is a consequence of the following theorem, which shows an interesting closure property.

**Theorem 5.7.**

$$H_{k,n,\omega}(t, q) * H_{k,n,\omega}(t, q') = H_{k,n,\omega}(t, q + q')$$

Before we proceed we need these lemmas.

**Lemma 5.8.**

$$\sum_{j=0}^{n+1} \binom{n+1}{j} B_{-(n-j)}^q B_{-(j-1)}^{q'} = B_{-(n)}^{q+q'}$$

**Proof** As in Lemma 5.6 this is a consequence of the theory of Stirling numbers. Here the relevant result is that  $[x+y]_{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} [x]_{n+1-j} [y]_j$ . An analogous formula for  $[x+y]^{n+1}$  shows that  $\sum_{j=0}^{n+1} \binom{n+1}{j} B_{(n-j)}^q B_{(j-1)}^{q'} = B_{(n)}^{q+q'}$ .  $\square$

**Lemma 5.9.** *Let  $\alpha = (\alpha_1, \dots, \alpha_k)$  with  $\alpha_i \geq 0$  and  $|\alpha| = \sum \alpha_i = n$ . Then*

$$(5.3) \quad \sum_{|\beta|=m, \beta \leq \alpha} \prod_i \binom{\alpha_i}{\beta_i} D_p \omega^\beta D_q \omega^{\alpha-\beta} = \binom{n-p-q}{m-p} D_{p+q} \omega^\alpha$$

where  $m \leq n$  and  $p, q \in \mathbb{N}$ .

**Proof** We will prove this lemma by induction.

Let  $\mathcal{P}(q)$  be the following statement: “(5.3) is true for every  $p$ ”. First we will show that  $\mathcal{P}(0)$  is true, i.e.

$$(5.4) \quad \sum_{|\beta|=m, \beta \leq \alpha} \prod_i \binom{\alpha_i}{\beta_i} (D_p \omega^\beta) \omega^{\alpha-\beta} = \binom{n-p}{m-p} D_p \omega^\alpha$$

Before we proceed let us note the case where  $p = 0$ , which will be used in the sequel. In this case the identity (5.4) becomes  $\sum_\beta \prod_i \binom{\alpha_i}{\beta_i} \omega^\beta \omega^{\alpha-\beta} = \binom{n}{m} \omega^\alpha$ , that is  $\sum_\beta \prod_i \binom{\alpha_i}{\beta_i} = \binom{n}{m}$ . The latter is true since the left hand side of it is the coefficient of  $x^m$  in the product  $(1+x)^{\alpha_1} \dots (1+x)^{\alpha_k} = (1+x)^{\sum \alpha_i} = (1+x)^n$ , which is clearly  $\binom{n}{m}$ .

For the general case we may write  $D_p \omega^\beta = \sum_{|\beta-\gamma|=p} f_\gamma(\beta) \omega^\gamma$ , where  $f_\gamma(\beta)$  is a polynomial in the  $\beta$ 's. More precisely, if  $s_i = \beta_i - \gamma_i$  then  $f_\gamma(\beta) = c \cdot \prod [\beta_i]_{s_i} = c \prod \binom{\beta_i}{\gamma_i} (\beta_i - \gamma_i)!$ , where  $c$  is a

constant. This constant counts the number of paths in the lattice  $\mathcal{L}(t^\alpha)$  from  $t^\beta$  to  $t^\alpha$ . Equivalently,  $c$  is the enumeration of  $\{\alpha_1, \dots, \alpha_1 - s_1 + 1, \dots\}$  such that  $\alpha_1$  precedes  $\alpha_1 - 1$  and so on. Hence  $c = \frac{(s_1 + \dots + s_k)!}{s_1! \dots s_k!} = \frac{p!}{\prod(\beta_i - \gamma_i)!}$ . In fact we have showed that

$$(5.5) \quad D_p \omega^\beta = p! \sum_{|\beta - \gamma| = p} \prod_i \binom{\beta_i}{\gamma_i} \omega^\gamma.$$

We can now rewrite Equation 5.4 as

$$\sum_{|\beta| = m} \prod_i \binom{\alpha_i}{\beta_i} p! \sum_{|\beta - \gamma| = p} \prod_i \binom{\beta_i}{\gamma_i} \omega^\gamma \omega^{\alpha - \beta} = \binom{n - p}{m - p} p! \sum_{|\alpha - \delta| = p} \prod_i \binom{\alpha_i}{\delta_i} \omega^\delta$$

For  $\delta$  fixed on the right hand side we must have equality of the corresponding coefficients, i.e.

$$\sum_{|\beta| = m} \prod_i \binom{\alpha_i}{\beta_i} \binom{\beta_i}{\delta_i - \alpha_i + \beta_i} = \binom{n - p}{m - p} \prod_i \binom{\alpha_i}{\delta_i}$$

and if we rewrite the left hand side we obtain

$$\sum_{|\beta| = m} \prod_i \binom{\alpha_i}{\delta_i} \binom{\delta_i}{\alpha_i - \beta_i} = \binom{n - p}{m - p} \prod_i \binom{\alpha_i}{\delta_i}$$

which gives the identity we showed for  $p = 0$ .

To show the induction step we differentiate the expression in  $\mathcal{P}(q)$  ( 5.3) to get

$$\sum_{|\beta| = m} \prod_i \binom{\alpha_i}{\beta_i} D_{p+1} \omega^\beta D_q \omega^{\alpha - \beta} + \sum_{|\beta| = m} \prod_i \binom{\alpha_i}{\beta_i} D_p \omega^\beta D_{q+1} \omega^{\alpha - \beta} = \binom{n - p - q}{m - p} D_{p+q+1} \omega^\alpha$$

and by the induction step

$$\begin{aligned} \sum_{|\beta| = m} \prod_i \binom{\alpha_i}{\beta_i} D_p \omega^\beta D_{q+1} \omega^{\alpha - \beta} &= \left[ \binom{n - p - q}{m - p} - \binom{n - p - q - 1}{m - p - 1} \right] D_{p+q+1} \omega^\alpha \\ &= \binom{n - p - q - 1}{m - p} D_{p+q+1} \omega^\alpha \end{aligned}$$

which is exactly  $\mathcal{P}(q + 1)$ , and thus the proof of Lemma 5.9 is complete.  $\square$

We need one more lemma which is a binomial identity.

**Lemma 5.10.** *Let  $p, q$  and  $n$  such that  $p + q < n$ . Then*

$$\sum_{i=0}^{n-p-q} \binom{p+i}{p} \binom{n-p-i}{q} = \binom{n+1}{p+q+1}$$

**Proof** If we make the convention that  $\binom{a}{b} = 0$  if  $a < b$ , we can think of the left hand side as being the sum  $\sum_{i=0}^n \binom{i}{p} \binom{n-i}{q}$ . This is in fact the coefficient of  $x^p y^q$  in

$$\sum_{i=0}^n (1+x)^i (1+y)^{n-i} = \frac{(1+x)^{n+1} - (1+y)^{n+1}}{x-y}.$$

Let us denote by  $a(i, j)$  the coefficient of  $x^i y^j$  in the expression above, i.e.

$$\frac{(1+x)^{n+1} - (1+y)^{n+1}}{x-y} = \sum_{i,j} a(i, j) x^i y^j.$$

We have that

$$(1+x)^{n+1} - (1+y)^{n+1} = \sum_{i,j} a(i, j) (x^{i+1} y^j - x^i y^{j+1}) = \sum_{i,j} [a(i-1, j) - a(i, j-1)] x^i y^j.$$

Here  $a(i, j) = 0$  if one of  $i, j$  is negative. By equating coefficients, we obtain

$$a(i, 0) = a(0, i) = \binom{n+1}{i+1},$$

and if both  $i, j > 0$ , then we get  $a(i-1, j) = a(i, j-1)$ . An easy inductive argument shows that  $a(i, j) = a(i+1, j-1) = \dots = a(i+j, 0) = \binom{n+1}{i+j+1}$  and the proof of the lemma is complete.  $\square$

**Proof of Theorem 5.7** By the definition of the level product we have that

$$\begin{aligned} H_{n,\omega}(t, q) * H_{n,\omega}(t, q') &= \sum_{i=0}^n H_{n-i,\omega}(t, q) H_{i,\omega}(t, q') = \sum_{i=0}^n \left( \sum_{\beta \vdash n-i} L^q(\beta) t^\beta \right) \left( \sum_{\gamma \vdash i} L^{q'}(\gamma) t^\gamma \right) \\ &= \sum_{\alpha \vdash n} \sum_{\beta \leq \alpha} L^q(\beta) L^{q'}(\alpha - \beta) t^\alpha, \end{aligned}$$

where  $\beta \leq \alpha$  means  $\beta_i \leq \alpha_i$ , for every  $i$ . Therefore we need to show that

$$\sum_{\beta \leq \alpha} L^q(\beta) L^{q'}(\alpha - \beta) = L^{q+q'}(\alpha).$$

By replacing  $L$ 's with their formulas (see definition in the statement of Theorem 5.4), we therefore need to show that

$$\begin{aligned} \sum_{\beta \leq \alpha} \prod_i \binom{\alpha_i}{\beta_i} \sum_{s=0}^{r-1} \sum_{t=0}^{p-r-1} \binom{r-1}{s} \binom{p-r-1}{t} B_{-(s)}^q B_{-(t)}^{q'} D_{(r-s-1)} \omega^\beta D_{(p-r-t-1)} \omega^{\alpha-\beta} \\ (5.6) \qquad \qquad \qquad = \sum_{j=0}^{p-1} \binom{p-1}{j} B_{-(j)}^{q+q'} D_{(p-j-1)} \omega^\alpha, \end{aligned}$$

where we denote for simplicity  $|\alpha| = p$  and  $|\beta| = r$ . Fix now an index  $j$  in the right hand side above. An important fact is that for each such  $j$  the expression of  $D_{p-j-1} \omega^\alpha$  gives an homogeneous polynomial of degree  $(j+1)$  in  $\omega_1, \dots, \omega_k$ . So it suffices to show that the two corresponding homogeneous polynomials of the same degree on both sides coincide. To obtain the homogeneous polynomial of degree  $(j+1)$  on the left hand side we need to pick indices  $s$  and  $t$  such that  $(s+1) + (t+1) = j+1$ , i.e.,  $t = j - s - 1$ . The homogeneous polynomial of degree  $(j+1)$  on the left hand side is therefore

$$\sum_{\beta \leq \alpha} \prod_i \binom{\alpha_i}{\beta_i} \sum_{s=0}^{r-1} \binom{r-1}{s} \binom{p-r-1}{j-s-1} B_{-(s)}^q B_{-(j-s-1)}^{q'} D_{(r-s-1)} \omega^\beta D_{(p-r-j+s)} \omega^{\alpha-\beta}.$$

As before we consider  $\binom{a}{b} = 0$  if  $a < b$ . We can rewrite  $\sum_{\beta \leq \alpha}$  as  $\sum_{r=0}^p \sum_{|\beta|=r, \beta \leq \alpha}$ , and the expression above becomes

$$\begin{aligned} & \sum_{r=0}^p \sum_{|\beta|=r, \beta \leq \alpha} \prod_i \binom{\alpha_i}{\beta_i} \sum_{s=0}^{r-1} \binom{r-1}{s} \binom{p-r-1}{j-s-1} B_{-(s)}^q B_{-(j-s-1)}^{q'} D_{(r-s-1)} \omega^\beta D_{(p-r-j+s)} \omega^{\alpha-\beta} \\ &= \sum_{r=0}^p \sum_{s=0}^{r-1} \binom{r-1}{s} \binom{p-r-1}{j-s-1} B_{-(s)}^q B_{-(j-s-1)}^{q'} \sum_{|\beta|=r, \beta \leq \alpha} \prod_i \binom{\alpha_i}{\beta_i} D_{(r-s-1)} \omega^\beta D_{(p-r-j+s)} \omega^{\alpha-\beta}, \end{aligned}$$

which by using Lemma 5.9 is

$$\begin{aligned} &= \sum_{r=0}^p \sum_{s=0}^{r-1} \binom{r-1}{s} \binom{p-r-1}{j-s-1} B_{-(s)}^q B_{-(j-s-1)}^{q'} \binom{j+1}{s+1} D_{p-j-1} \omega^\alpha \\ &= \sum_{s=0}^{p-1} \binom{j+1}{s+1} B_{-(s)}^q B_{-(j-s-1)}^{q'} D_{p-j-1} \omega^\alpha \sum_{r=0}^p \binom{r-1}{s} \binom{p-r-1}{j-s-1}, \end{aligned}$$

and by Lemma 5.10 is

$$= \sum_{s=0}^{p-1} \binom{j+1}{s+1} B_{-(s)}^q B_{-(j-s-1)}^{q'} D_{p-j-1} \omega^\alpha \binom{p-1}{j},$$

which finally by Lemma 5.8 is

$$= \binom{p-1}{j} B_{-(j)}^{q+q'} D_{p-j-1} \omega^\alpha.$$

This is exactly the homogeneous polynomial of degree  $(j+1)$  in the right hand side of (5.6) and thus the proof of Theorem 5.7 is complete.  $\square$

## 6. ALGEBRAIC STRUCTURE

Let  $\mathcal{H}_{n,\omega}$  denote the algebra generated by all  $H_{n,\omega}$  under the addition and the level product. As a consequence of Theorem 5.7 each  $H_{n,\omega}$  has a level-product inverse in  $\mathcal{H}$ .

**Theorem 6.1.**

$$H_{n,\omega}^{-1}(t, q) = H_{n,\omega}(t, -q)$$

$\square$

So from Theorems 5.7 and 6.1 we have that for a fixed weight  $\omega$  and a given level  $n$  the polynomials  $\mathcal{H}_{n,\omega} = \{H_{n,\omega}(t, q)\}_{q \in \mathbb{Q}}$  form an abelian group under the level product isomorphic to the rationals,  $\mathbb{Q}$ , under addition. The group  $\mathcal{T} = \{\mathfrak{F}_\omega\}_\omega$  acts on this group by translation in the following way:  $\mathcal{T}$  acts on a family of WIPs by (say) a right translation (Theorem 2.3) and in a natural way the  $q$ -th roots follow along. Theorem 6.1 applies to a family of WIPs, as well, giving the subgroup determined by a weighted isobaric family under the level operation. All of this together with the derivation operators  $\partial_j$  give a structure of differential graded group to  $\mathcal{H} = \bigoplus_\omega \bigoplus_n \mathcal{H}_{n,\omega}$  acted on by an affine group.

## 7. APPENDIX

*Schur reflects for  $Sym(n)$ , with  $n = 1, 2, \dots, 6$ .*

$$\hat{S}_{(1)} = t_1$$

$$\hat{S}_{(2)} = t_1^2 + t_2$$

$$\hat{S}_{(1^2)} = -t_2$$

$$\hat{S}_{(3)} = t_1^3 + 2t_1t_2 + t_3$$

$$\hat{S}_{(2,1)} = -t_1t_2 - t_3$$

$$\hat{S}_{(1^3)} = t_3$$

$$\hat{S}_{(4)} = t_1^4 + 3t_1^2t_2 + t_2^2 + 2t_1t_3 + t_4$$

$$\hat{S}_{(3,1)} = -t_1^2t_2 - t_2^2 - t_1t_3 - t_4$$

$$\hat{S}_{(2^2)} = t_2^2 - t_1t_3$$

$$\hat{S}_{(2,1^2)} = t_1t_3 + t_4$$

$$\hat{S}_{(1^4)} = -t_4$$

$$\hat{S}_{(5)} = t_1^5 + 4t_1^3t_2 + 3t_1t_2^2 + 3t_1^2t_3 + 2t_2t_3 + 2t_1t_4 + t_5$$

$$\hat{S}_{(4,1)} = -t_1^3t_2 - 2t_1t_2^2 - t_1^2t_3 - 2t_2t_3 - t_1t_4 - t_5$$

$$\hat{S}_{(3,2)} = t_1t_2^2 - t_1^2t_3 + t_2t_3 - t_1t_4$$

$$\hat{S}_{(3,1^2)} = t_1^2t_3 + t_2t_3 + t_1t_4 + t_5$$

$$\hat{S}_{(2^2,1)} = -t_2t_3 + t_1t_4$$

$$\hat{S}_{(2,1^3)} = -t_1t_4 - t_5$$

$$\hat{S}_{(1^5)} = t_5$$

$$\hat{S}_{(6)} = t_1^6 + 5t_1^4t_2 + 6t_1^2t_2^2 + t_2^3 + 4t_1^3t_3 + t_3^2 + 6t_1t_2t_3 + 3t_1^2t_4 + 2t_2t_4 + 2t_1t_5 + t_6$$

$$\hat{S}_{(5,1)} = -t_1^4t_2 - 3t_1^2t_2^2 - t_2^3 - t_1^3t_3 - t_3^2 - 4t_1t_2t_3 - t_1^2t_4 + 2t_2t_4 - 2t_1t_5 - t_6$$

$$\hat{S}_{(4,2)} = t_1^2t_2^2 + t_2^3 - t_1^3t_3 - t_1^2t_4 + t_2t_4 - t_1t_5$$

$$\hat{S}_{(4,1^2)} = t_1^3t_3 + t_2^3 + 2t_1t_2t_3 + t_1^2t_4 + t_2t_4 + t_1t_5 + t_6$$

$$\hat{S}_{(3^2)} = -t_2^3 + t_2^3 + 2t_1t_2t_3 - t_1^2t_4 - t_2t_4$$

$$\hat{S}_{(3,2,1)} = -t_3^2 - t_1t_2t_3 + t_1^2t_4 + t_1t_5$$

$$\hat{S}_{(2^3)} = t_3^2 - t_2t_4$$

$$\hat{S}_{(3,1^3)} = -t_1^2t_4 - t_2t_4 - t_1t_5 - t_6$$

$$\hat{S}_{(2^2,1^2)} = t_2t_4 - t_1t_5$$

$$\hat{S}_{(2,1^4)} = t_1t_5 + t_6$$

$$\hat{S}_{(1^6)} = -t_6$$

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