The Heat Equation for the Hermite Operator on the Heisenberg Group

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Abstract

We give a formula for the one-parameter strongly continuous semigroup $e^{-tL}$, $t > 0$, generated by the Hermite operator $L$ on the Heisenberg group $\mathbb{H}^1$ in terms of Weyl transforms, and use it to obtain an $L^2$ estimate for the solution of the initial value problem for the heat equation governed by $L$ in terms of the $L^p$ norm of the initial data for $1 \leq p \leq \infty$.

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1 The Hermite Semigroup on $\mathbb{R}$

As a prologue to the Hermite semigroup on the Heisenberg group $\mathbb{H}^1$, we give an analysis of the Hermite semigroup on $\mathbb{R}$.

For $k = 0, 1, 2, \ldots$, the Hermite function of order $k$ is the function $e_k$ on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where $H_k$ is the Hermite polynomial of degree $k$ given by

$$H_k(x) = (-1)^k e^{x^2} \left( \frac{d}{dx} \right)^k (e^{-x^2}), \quad x \in \mathbb{R}.$$

It is well-known that $\{e_k : k = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Let $A$ and $\overline{A}$ be differential operators on $\mathbb{R}$ defined by

$$A = \frac{d}{dx} + x$$

and

$$\overline{A} = -\frac{d}{dx} + x.$$ 

In fact, $\overline{A}$ is the formal adjoint of $A$. The Hermite operator $H$ is the ordinary differential operator on $\mathbb{R}$ given by

$$H = -\frac{1}{2}(AA + \overline{A}A).$$

A simple calculation shows that

$$H = -\frac{d^2}{dx^2} + x^2.$$

The spectral analysis of the Hermite operator $H$ is based on the following result, which is easy to prove.

**Theorem 1.1** For all $x$ in $\mathbb{R}$,

$$(Ae_k)(x) = 2k e_{k-1}(x), \quad k = 1, 2, \ldots,$$

and

$$(\overline{A}e_k)(x) = e_{k+1}(x), \quad k = 0, 1, 2, \ldots.$$
Remark 1.2 In view of Theorem 1.1, we call $A$ and $\overline{A}$ the annihilation operator and the creation operator, respectively, for the Hermite functions $e_k$, $k = 0, 1, 2, \ldots$, on $\mathbb{R}$.

An immediate consequence of Theorem 1.1 is the following theorem.

Theorem 1.3 $He_k = (2k + 1)e_k$, $k = 0, 1, 2, \ldots$.

Remark 1.4 Theorem 1.3 says that for $k = 0, 1, 2, \ldots$, the number $2k + 1$ is an eigenvalue of the Hermite operator $H$, and the Hermite function $e_k$ on $\mathbb{R}$ is an eigenfunction of $H$ corresponding to the eigenvalue $2k + 1$.

We can now give a formula for the Hermite semigroup $e^{-tH}$, $t > 0$.

Theorem 1.5 Let $f$ be a function in the Schwartz space $S(\mathbb{R})$. Then for $t > 0$,

$$e^{-tH}f = \sum_{k=0}^{\infty} e^{-k(2k+1)t} (f, e_k) e_k,$$

where the convergence is uniform and absolute on $\mathbb{R}$.

Theorem 1.6 For $t > 0$, the Hermite semigroup $e^{-tH}$, initially defined on $S(\mathbb{R})$, can be extended to a unique bounded linear operator from $L^p(\mathbb{R})$ into $L^2(\mathbb{R})$, which we again denote by $e^{-tH}$, and there exists a positive constant $C$ such that

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq C^2 \frac{1}{2 \sinh t} \|f\|_{L^p(\mathbb{R})}$$

for all $f$ in $L^p(\mathbb{R})$, $1 \leq p \leq 2$.

Remark 1.7 In fact, by a well-known asymptotic formula for Hermite functions,

$$\sup \{ \|e_k\|_{L^\infty(\mathbb{R})} : k = 0, 1, 2, \ldots \} < \infty$$

and hence $C$ can be any positive constant such that

$$C \geq \sup \{ \|e_k\|_{L^\infty(\mathbb{R})} : k = 0, 1, 2, \ldots \}.$$
Proof of Theorem 1.6 Let $f \in \mathcal{S}(\mathbb{R})$. Then, by Theorem 1.5 and Minkowski’s inequality,

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq \sum_{k=0}^{\infty} e^{-(2k+1)t} |(f,e_k)|. \quad (1.1)$$

Now, for $k = 0, 1, 2, \ldots$, by Schwarz’ inequality,

$$|(f,e_k)| \leq \|f\|_{L^2(\mathbb{R})} \quad (1.2)$$

and

$$|(f,e_k)| \leq \|f\|_{L^1(\mathbb{R})}\|e_k\|_{L^\infty(\mathbb{R})}. \quad (1.3)$$

But, using an asymptotic formula in the book [4] by Szegö for Hermite functions, we can find a positive constant $C$, which can actually be estimated, such that

$$\|e_k\|_{L^\infty(\mathbb{R})} \leq C \quad (1.4)$$

for $k = 0, 1, 2, \ldots$. So, by (1.3) and (1.4),

$$|(f,e_k)| \leq C\|f\|_{L^1(\mathbb{R})}. \quad (1.5)$$

Hence, by (1.1), (1.2) and (1.5), we get

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq \frac{1}{2 \sinh t} \|f\|_{L^2(\mathbb{R})} \quad (1.6)$$

and

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq \frac{1}{2 \sinh t} C\|f\|_{L^1(\mathbb{R})}. \quad (1.7)$$

Hence, by (1.6), (1.7) and the Riesz-Thorin theorem, we get

$$\|e^{-tH}f\|_{L^2(\mathbb{R})} \leq C^{2^p-1} \frac{1}{2 \sinh t} \|f\|_{L^p(\mathbb{R})}$$

for $1 \leq p \leq 2$. 

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2 The Hermite Operator on the Heisenberg Group

Let $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ be linear partial differential operators on $\mathbb{R}^2$ given by

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i \frac{\partial}{\partial y}.$$  

Then we define the linear partial differential operator $L$ on $\mathbb{R}^2$ by

$$L = -\frac{1}{2}(Z\bar{Z} + \bar{Z}Z),$$

where

$$Z = \frac{\partial}{\partial z} + \frac{1}{2}z, \quad \bar{z} = x - iy,$$

and

$$\bar{Z} = \frac{\partial}{\partial \bar{z}} - \frac{1}{2}z, \quad z = x + iy.$$  

The vector fields $Z$ and $\bar{Z}$, and the identity operator $I$ form a basis for a Lie algebra in which the Lie bracket of two elements is their commutator. In fact, $-\bar{Z}$ is the formal adjoint of $Z$ and $L$ is an elliptic partial differential operator on $\mathbb{R}^2$ given by

$$L = -\Delta + \frac{1}{4}(x^2 + y^2) - i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right),$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$  

Thus, $L$ is the ordinary Hermite operator $-\Delta + \frac{1}{4}(x^2 + y^2)$ perturbed by the partial differential operator $-iN$, where

$$N = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}.$$  

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is the rotation operator. We can think of $L$ as the Hermite operator on $\mathbb{H}^1$. The vector fields $Z$ and $\overline{Z}$, and the Hermite operator $L$ are studied in the books [5, 6] by Thangavelu and [7] by Wong. The connection of $L$ with the sub-Laplacian on the Heisenberg group $\mathbb{H}^1$ can be found in the book [6] by Thangavelu. The heat equations for the sub-Laplacians on Heisenberg groups are first solved explicitly and independently in [1] by Gaveau and in [2] by Hulanicki.

In this paper, we compute the Hermite semigroup on $\mathbb{H}^1$, i.e., the one-parameter strongly continuous semigroup $e^{-tL}$, $t > 0$, generated by $L$ using an orthonormal basis for $L^2(\mathbb{R}^2)$ consisting of special Hermite functions on $\mathbb{R}^2$, which are eigenfunctions of $L$. We give a formula for the Hermite semigroup on $\mathbb{H}^1$ in terms of pseudo-differential operators of the Weyl type, i.e., Weyl transforms. The Hermite semigroup on $\mathbb{H}^1$ is then used to obtain an $L^2$ estimate for the solution of the initial value problem of the heat equation governed by $L$ in terms of the $L^p$ norm of the initial data for $1 \leq p \leq \infty$.

The results in this paper are valid for the Hermite operator $L$ on $\mathbb{H}^n$ given by

$$L = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \overline{Z}_j + \overline{Z}_j Z_j),$$

where, for $j = 1, 2, \ldots, n$,

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2} z_j, \quad \overline{z}_j = x_j - iy_j,$$

and

$$\overline{Z}_j = \frac{\partial}{\partial \overline{z}_j} - \frac{1}{2} \overline{z}_j, \quad z_j = x_j + iy_j.$$

Of course, for $j = 1, 2, \ldots, n$,

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j}$$

and

$$\frac{\partial}{\partial \overline{z}_j} = \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j}. $$
Section 4.4 of the book [5] by Thangavelu contains some information on the $L^p-L^2$ estimates of the solutions of the wave equation governed by the Hermite operator $L$. The $L^p$ norm of the solution of the wave equation for the special Hermite operator in terms of the initial data for values of $p$ near 2 is studied in the paper [3] by Narayanan and Thangavelu.

3 Weyl Transforms

Let $f$ and $g$ be functions in the Schwartz space $S(\mathbb{R})$ on $\mathbb{R}$. Then the Fourier-Wigner transform $V(f, g)$ of $f$ and $g$ is defined by

$$V(f, g)(q, p) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{iqy} f\left(y + \frac{p}{2}\right) g\left(y - \frac{p}{2}\right) dy$$

(3.1)

for all $q$ and $p$ in $\mathbb{R}$. It can be proved that $V(f, g)$ is a function in the Schwartz space $S(\mathbb{R}^2)$ on $\mathbb{R}^2$. We define the Wigner transform $W(f, g)$ of $f$ and $g$ by

$$W(f, g) = V(f, g)\wedge,$$

(3.2)

where $\hat{F}$ is the Fourier transform of $F$, which we choose to define by

$$\hat{F}(\zeta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-iz\cdot\zeta} F(z) dz, \quad \zeta \in \mathbb{R}^n,$$

for all $F$ in the Schwartz space $S(\mathbb{R}^n)$ on $\mathbb{R}^n$. It can be shown that

$$W(f, g)(x, \xi) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\xi p} f\left(x + \frac{p}{2}\right) g\left(x - \frac{p}{2}\right) dp$$

for all $x$ and $\xi$ in $\mathbb{R}$. It is obvious that

$$W(f, g) = W(g, f), \quad f, g \in S(\mathbb{R}).$$

(3.3)

Now, let $\sigma \in L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$, and let $f \in S(\mathbb{R})$. Then we define $W_\sigma f$ to be the tempered distribution on $\mathbb{R}$ by

$$(W_\sigma f, g) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x, \xi) W(f, g)(x, \xi) dx d\xi$$

(3.4)
for all $g$ in $\mathcal{S}(\mathbb{R})$, where $(F, G)$ is defined by

$$(F, G) = \int_{\mathbb{R}^n} F(z)\overline{G(z)}dz$$

for all measurable functions $F$ and $G$ on $\mathbb{R}^n$, provided that the integral exists. We call $W_\sigma$ the Weyl transform associated to the symbol $\sigma$. It should be noted that if $\sigma$ is a symbol in $\mathcal{S}(\mathbb{R}^2)$, then $W_\sigma f$ is a function in $\mathcal{S}(\mathbb{R})$ for all $f$ in $\mathcal{S}(\mathbb{R})$.

We need the following result, which is an abridged version of Theorem 14.3 in the book [7] by Wong.

**Theorem 3.1** Let $\sigma \in L^p(\mathbb{R}^2)$, $1 \leq p \leq 2$. Then $W_\sigma$ is a bounded linear operator from $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ and

$$\|W_\sigma\|_* \leq (2\pi)^{-\frac{1}{p}}\|\sigma\|_{L^p(\mathbb{R}^2)},$$

where $\|W_\sigma\|_*$ is the operator norm of $W_\sigma : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

### 4 Hermite Functions on $\mathbb{R}^2$

For $j, k = 0, 1, 2, \ldots$, we define the Hermite function $e_{j,k}$ on $\mathbb{R}^2$ by

$$e_{j,k}(x,y) = V(e_j,e_k)(x,y)$$

for all $x$ and $y$ in $\mathbb{R}$. Then we have the following fact, which is Theorem 21.2 in the book [7] by Wong.

**Theorem 4.1** $\{e_{j,k} : j, k = 0, 1, 2, \ldots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

The spectral analysis of the Hermite operator $L$ on $\mathbb{H}^1$ is based on the following result, which is Theorem 22.1 in the book [7] by Wong.

**Theorem 4.2** For all $x$ and $y$ in $\mathbb{R}$,

$$(Ze_{j,k})(x,y) = i(2k)^{\frac{1}{2}}e_{j,k-1}(x,y), \quad j = 0, 1, 2, \ldots, \quad k = 1, 2, \ldots,$$

and

$$(\overline{Ze_{j,k}})(x,y) = i(2k+2)^{\frac{1}{2}}e_{j,k+1}(x,y), \quad j, k = 0, 1, 2, \ldots.$$
Remark 4.3 In view of Theorem 4.2, we call $Z$ and $\overline{Z}$ the annihilation operator and the creation operator, respectively, for the special Hermite functions $e_{j,k}$, $j, k = 0, 1, 2, \ldots$, on $\mathbb{R}^2$.

An immediate consequence of Theorem 4.2 is the following theorem.

Theorem 4.4 $L e_{j,k} = (2k + 1)e_{j,k}$, $j, k = 0, 1, 2, \ldots$.

Remark 4.5 Theorem 4.4 says that for $k = 0, 1, 2, \ldots$, the number $2k + 1$ is an eigenvalue of the Hermite operator $L$ on $\mathbb{H}^1$, and the Hermite functions $e_{j,k}$, $j = 0, 1, 2, \ldots$, on $\mathbb{R}^2$ are eigenfunctions of $L$ corresponding to the eigenvalue $2k + 1$.

5 The Hermite Semigroup on $\mathbb{H}^1$

A formula for the Hermite semigroup $e^{-tL}$, $t > 0$, on $\mathbb{H}^1$ is given in the following theorem.

Theorem 5.1 Let $f \in \mathcal{S}(\mathbb{R}^2)$. Then for $t > 0$,

$$e^{-tL}f = (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-t(2k+1)}V(Wf e_k, e_k),$$

where the convergence is uniform and absolute on $\mathbb{R}^2$.

Proof Let $f$ be any function in $\mathcal{S}(\mathbb{R}^2)$. Then for $t > 0$, we use Theorem 4.4 to get

$$e^{-tL}f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} e^{-t(2k+1)}(f, e_{j,k})e_{j,k},\quad (5.1)$$

where the series is convergent in $L^2(\mathbb{R}^2)$, and is also uniformly and absolutely convergent on $\mathbb{R}^2$. Now, by (3.1)–(3.4) and Plancherel’s theorem,

$$\begin{align*}
(f, e_{j,k}) &= \int_{\mathbb{R}^2} f(z)\overline{V(e_j, e_k)(z)}dz \\
&= \int_{\mathbb{R}^2} \hat{f}(\zeta)\overline{V(e_j, e_k)(\zeta)}d\zeta \\
&= \int_{\mathbb{R}^2} \hat{f}(\zeta)\overline{W(e_j, e_k)(\zeta)}d\zeta \\
&= (2\pi)^{\frac{1}{2}}(Wf e_k, e_j)\quad (5.2)
\end{align*}$$

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for \( j, k = 0, 1, 2, \ldots \) Similarly, for \( j, k = 0, 1, 2, \ldots \), and \( g \) in \( \mathcal{S}(\mathbb{R}^2) \), we get
\[
(e_{j,k}, g) = (g, e_{j,k}) = (2\pi)^{\frac{1}{2}}(W_\hat{g}e_k, e_j) = (2\pi)^{\frac{1}{2}}(e_j, W_\hat{g}e_k).
\] (5.3)

So, by (5.1)–(5.3), Fubini’s theorem and Parseval’s identity,
\[
(e^{-tL}f, g) = 2\pi \sum_{k=0}^{\infty} e^{-(2k+1)t} \sum_{j=0}^{\infty} (W_\hat{f}e_k, e_j) (e_j, W_\hat{g}e_k)
\]
\[
= 2\pi \sum_{k=0}^{\infty} e^{-(2k+1)t}(W_\hat{f}e_k, W_\hat{g}e_k)
\] (5.4)

for \( t > 0 \), where the series is absolutely convergent on \( \mathbb{R} \). But, by (3.2)–(3.4) and Plancherel’s theorem,
\[
(W_\hat{f}e_k, W_\hat{g}e_k) = (2\pi)^{\frac{1}{2}} \int_{\mathbb{R}^2} \hat{g}(z) W_\hat{f}(e_k, e_k)(z) dz
\]
\[
= (2\pi)^{\frac{1}{2}} \int_{\mathbb{R}^2} W_\hat{f}(e_k, e_k)(z) \hat{g}(z) dz
\]
\[
= (2\pi)^{\frac{1}{2}} \int_{\mathbb{R}^2} V(W_\hat{f}e_k, e_k)(z) \hat{g}(z) dz
\] (5.5)

for \( k = 0, 1, 2, \ldots \) Thus, by (5.4), (5.5) and Fubini’s theorem,
\[
(e^{-tL}f, g) = (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-(2k+1)t}(V(W_\hat{f}e_k, e_k), g)
\]
\[
= (2\pi)^{\frac{1}{2}} \left( \sum_{k=0}^{\infty} e^{-(2k+1)t}V(W_\hat{f}e_k, e_k), g \right)
\] (5.6)

for all \( f \) and \( g \) in \( \mathcal{S}(\mathbb{R}^2) \) and \( t > 0 \). Thus, by (5.6),
\[
e^{-tL}f = (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-(2k+1)t}V(W_\hat{f}e_k, e_k)
\]

for all \( f \) in \( \mathcal{S}(\mathbb{R}^2) \) and \( t > 0 \), where the uniform and absolute convergence of the series follows from (3.1) and Theorem 3.1.
\]
6 An $L^p - L^2$ Estimate, $1 \leq p \leq 2$

We begin with the following result, which is known as the Moyal identity and can be found in the book [7] by Wong.

**Theorem 6.1** For all $f$ and $g$ in $\mathcal{S}(\mathbb{R})$,

$$\|V(f, g)\|_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}.$$ 

We can now prove the following theorem as an application of the formula for the Hermite semigroup on $H^1$ given in Theorem 5.1.

**Theorem 6.2** For $t > 0$, the Hermite semigroup $e^{-tL}$ on $H^1$, initially defined on $\mathcal{S}(\mathbb{R}^2)$, can be extended to a unique bounded linear operator from $L^p(\mathbb{R}^2)$ into $L^2(\mathbb{R}^2)$, which we again denote by $e^{-tL}$, and

$$\|e^{-tL}f\|_{L^2(\mathbb{R}^2)} \leq (2\pi)^{\frac{1}{2} - \frac{1}{p}} \frac{1}{2\sinh t} \|f\|_{L^p(\mathbb{R}^2)}$$

for all $f$ in $L^p(\mathbb{R}^2)$, $1 \leq p \leq 2$.

**Proof** Let $f \in \mathcal{S}(\mathbb{R}^2)$. Then, by Theorems 5.1 and 6.1, and Minkowski’s inequality

$$\|e^{-tL}f\|_{L^2(\mathbb{R}^2)} \leq (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-(2k+1)t} \|V(W_{\hat{f}}e_k, e_k)\|_{L^2(\mathbb{R}^2)}$$

$$= (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-(2k+1)t} \|W_{\hat{f}}e_k\|_{L^2(\mathbb{R})} \|e_k\|_{L^2(\mathbb{R})}$$

$$= (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-(2k+1)t} \|W_{\hat{f}}e_k\|_{L^2(\mathbb{R})}$$

(6.1)

for $t > 0$. So, by (6.1) and Theorem 3.1, we get for $t > 0$,

$$\|e^{-tL}f\|_{L^2(\mathbb{R}^2)} \leq (2\pi)^{\frac{1}{2}} \sum_{k=0}^{\infty} e^{-(2k+1)t} (2\pi)^{\frac{1}{2}} \frac{1}{2\sinh t} \|f\|_{L^p(\mathbb{R}^2)}$$

$$= (2\pi)^{\frac{1}{2} - \frac{1}{p}} \frac{1}{2\sinh t} \|f\|_{L^p(\mathbb{R}^2)}$$

(6.2)

for all $f$ in $\mathcal{S}(\mathbb{R}^2)$. Thus, by (6.2) and a density argument, the proof is complete. \qed
Remark 6.3 Theorem 6.2 gives an \( L^2 \) estimate for the solution of the initial value problem

\[
\begin{aligned}
\frac{\partial u}{\partial t}(z,t) &= (Lu)(z,t), \quad z \in \mathbb{R}^2, \quad t > 0, \\
\quad u(z,0) &= f(z), \quad z \in \mathbb{R}^2,
\end{aligned}
\] (6.3)

in terms of the \( L^p \) norm of the initial data \( f \), \( 1 \leq p \leq 2 \).

Remark 6.4 Instead of using Weyl transforms, Theorem 6.2 can be proved using an \( L^p - L^2 \) restriction theorem such as Theorem 2.5.4 in the book [5] by Thangavelu. To wit, we note that the formula (5.1) for the special Hermite semigroup gives

\[ e^{-tL}f = \sum_{k=0}^{\infty} e^{-(2k+1)t}Q_kf, \quad f \in \mathcal{S}(\mathbb{R}^2), \]

where \( Q_k \) is the projection onto the eigenspace corresponding to the eigenvalue \( 2k + 1 \). Thus, by Theorem 2.5.4 in [5], the estimate for \( p = 1 \) follows. The estimate for \( p = 2 \) is easy. Hence the estimate for \( 1 \leq p \leq 2 \) follows if we interpolate.

7 An \( L^p-L^2 \) Estimate, \( 1 \leq p \leq \infty \)

Using the theory of localization operators on the Weyl-Heisenberg group in the paper [8] or Chapter 17 of the book [9] by Wong, we can give an \( L^p-L^2 \) estimate for \( 1 \leq p \leq \infty \). To this end, we need two results.

Theorem 7.1 Let \( \Lambda \) be the function on \( \mathbb{C} \) defined by

\[ \Lambda(z) = \pi^{-1}e^{-|z|^2}, \quad z \in \mathbb{C}. \]

Then for all \( F \in L^p(\mathbb{C}), 1 \leq p \leq \infty, \)

\[ W_{F \ast \Lambda} = L_F, \]

where \( L_F \) is the localization operator on the Weyl-Heisenberg group with symbol \( F \).

**Theorem 7.2** Let $F \in L^p(\mathbb{C})$, $1 \leq p \leq \infty$. Then

$$\|L_F\|_* \leq (2\pi)^{-\frac{1}{2}} \|F\|_{L^p(\mathbb{C})}.$$

Theorem 7.2 is Theorem 17.11 in the book [9] by Wong.

The main result in this section is the following theorem.

**Theorem 7.3** Let $g \in L^p(\mathbb{C})$, $1 \leq p \leq \infty$, and let $u$ be the solution of the initial value problem (6.3) with initial data $(g \ast \Lambda)^\vee$, where $\vee$ is the inverse Fourier transform. Then

$$\|u\|_{L^2(\mathbb{R}^2)} \leq (2\pi)^{\frac{1}{2}} \frac{1}{2 \sinh t} \|g\|_{L^p(\mathbb{R}^2)}.$$

The proof is the same as that of Theorem 6.2 if we note that, by Theorem 7.1, $W_j = W_{g \ast \Lambda} = L_g$ and hence the estimate follows from Theorem 7.2.

**References**


