Invariant Mean Value Property and Harmonic Functions

Jinman Kim and M. W. Wong

Department of Mathematics and Statistics
York University
4700 Keele Street
Toronto, Ontario M3J 1P3
Canada

Abstract We give conditions on the functions \(\sigma\) and \(u\) on \(\mathbb{R}^n\) such that if \(u\) is given by the convolution of \(\sigma\) and \(u\), then \(u\) is harmonic on \(\mathbb{R}^n\).

Keywords: Harmonic functions; Mean value property, Convolution transform; Heat kernel; Fourier transforms; Paley-Wiener theorem; Fourier hyperfunctions, Gelfand–Shilov spaces; Weyl’s lemma

2000 Mathematics Subject Classifications: Primary 46F12, 46F15; Secondary 30D15

1 Introduction

A function \(u\) on \(\mathbb{R}^n\) is said to be harmonic if it is in \(C^2(\mathbb{R}^n)\) and such that

\[
(\Delta u)(x) = 0, \quad x \in \mathbb{R}^n,
\]

*Corresponding author. Email: mwwong@mathstat.yorku.ca
where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$. One of the best characterizations of harmonic functions is perhaps the mean value property to the effect that

$$\frac{1}{\mu(S^{n-1})} \int_{S^{n-1}} u(x + r\omega) \, d\mu(\omega) = u(x), \quad x \in \mathbb{R}^n,$$

(1.1)

where $\mu$ is the Lebesgue measure on the unit sphere $S^{n-1}$ with center at the origin in $\mathbb{R}^n$. Now, let $\sigma \in L^1(\mathbb{R}^n)$ be a radial function given by

$$\sigma(x) = \Sigma(|x|), \quad x \in \mathbb{R}^n.$$

Let $u$ be a harmonic function for which the convolution transform $K_\sigma u$ of $u$ with kernel $\sigma$ given by

$$K_\sigma u = \sigma * u$$

is defined. Then

$$(K_\sigma u)(x) = \mu(S^{n-1}) \left( \int_0^\infty r^{n-1} \Sigma(r) \, dr \right) u(x), \quad x \in \mathbb{R}^n.$$  

(1.2)

If $\mu(S^{n-1}) \int_0^\infty r^{n-1} \Sigma(r) \, dr = 1$, then $K_\sigma u = u$. In other words, $u$ is invariant under the convolution transform with kernel $\sigma$. This prompts us to look at the following problem in this paper.

**Problem** Find conditions on the kernel $\sigma$ and, if necessary, on the function $u$ such that $K_\sigma u = u$ implies that $u$ is harmonic.


One of the main results in this paper is the following theorem.

**Theorem 1.1** Let $\sigma$ be a nonnegative and radial function in $C_0^2(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \sigma(x) \, dx = 1.$$

Then for all tempered functions $u$ on $\mathbb{R}^n$ with

$$K_\sigma u = u,$$

$u$ is harmonic on $\mathbb{R}^n$. 

2
For tempered functions and tempered distributions used in this paper, see, for instance, Chapter 4 of the book [7] by Wong.

The following example, due to M. Engliš, points out the need for conditions on $u$ for some kernels $\sigma$.

**Example 1.2** Let $\sigma(x) = \frac{1}{2\pi}e^{-|x|^2/2}$, $x \in \mathbb{R}^2$. Then $\int_{\mathbb{R}^2} \sigma(x) \, dx = 1$. Let $u$ be the function on $\mathbb{R}^2$ defined by

$$u(x_1, x_2) = e^{ax_1 + bx_2}, \quad x_1, x_2 \in \mathbb{R},$$

where $a$ and $b$ are complex numbers such that $a^2 + b^2 = 4\pi i$. Then $\Delta u = 4\pi iu$ and an easy contour integration gives $\sigma \ast u = u$.

It is important to realize that the kernel $\sigma$ in Example 1.2 is the heat kernel evaluated at time $t = \frac{1}{2}$. We give in the following theorem a condition on the function $u$ to ensure that the harmonicity of $u$ is a consequence of the invariance of $u$ under the convolution transform with the heat kernel at time $t = \frac{1}{2}$.

**Theorem 1.3** Let $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ be such that

$$\|ue^{-\varepsilon|\cdot|}\|_{L^\infty(\mathbb{R}^n)} < \infty$$

for every positive number $\varepsilon$ and

$$K_{\sigma}u = u,$$

where

$$\sigma(x) = (2\pi)^{-n/2}e^{-|x|^2/2}, \quad x \in \mathbb{R}^n.$$

Then $u$ is harmonic on $\mathbb{R}^n$.

We conclude this introduction with a genesis of this paper. Theorem 1.1 is proved in Section 2. This is achieved using the Paley–Wiener theorem for the Fourier transform. The Fourier transform $\hat{f}$ of a function $f$ in $L^1(\mathbb{R}^n)$ is taken to be the one defined by

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) \, dx, \quad \xi \in \mathbb{R}^n.$$

Fourier hyperfunctions, which are used in the proof of Theorem 1.3, will be recalled with proofs in Section 3. The proof of Theorem 1.3 is given in Section 4.
2 Proof of Theorem 1.1

Lemma 2.1 Let $\sigma$ be as in Theorem 1.1. Then

$$(I - K_\sigma)\mathcal{S} = \Delta \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz space on $\mathbb{R}^n$.

Proof Without loss of generality, we assume that $n = 1$. Since $\wedge: \mathcal{S} \rightarrow \mathcal{S}$ is a homeomorphism, it is sufficient to prove that the multiplication operator $q: \mathcal{S} \rightarrow \mathcal{S}$ is surjective, where

$$q(\xi) = \frac{1 - (2\pi)^{1/2}\hat{\sigma}(\xi)}{-\xi^2}, \quad \xi \in \mathbb{R}.$$  

In view of the Paley-Wiener theorem, $\hat{\sigma}$ is an even entire function on $\mathbb{C}$ and

$$\hat{\sigma}(0) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \sigma(x) \, dx = (2\pi)^{-1/2}.$$ 

Using the Taylor series expansion of $\hat{\sigma}$, we get

$$\hat{\sigma}(z) = \sum_{n=0}^{\infty} a_{2n} z^{2n}, \quad z \in \mathbb{C},$$

where

$$a_{2n} = \frac{1}{(2n)!} (2\pi)^{-1/2} \int_{-\infty}^{\infty} (-ix)^{2n} \sigma(x) \, dx, \quad n = 1, 2, \ldots.$$ 

Since $a_0 = (2\pi)^{-1/2}$, it follows that 0 is a removable singularity of the function $q$ defined on the punctured plane $\mathbb{C} \setminus \{0\}$ by

$$q(z) = -(2\pi)^{1/2} \sum_{n=1}^{\infty} a_{2n} z^{2n-2}, \quad z \neq 0.$$ 

To prove that $q\varphi \in \mathcal{S}$ for all $\varphi \in \mathcal{S}$, it is enough to prove that for every nonnegative integer $m$, there exist positive constants $C_m$ and $N_m$ such that

$$|q^{(m)}(x)| \leq C_m (1 + |x|)^{N_m}, \quad x \in \mathbb{R}. $$
But, by Cauchy’s integral formula,

\[ q^{(m)}(x) = \frac{m!}{2\pi i} \int_{|z-x|=1} \frac{q(z)}{(z-x)^{m+1}} dz, \quad x \in \mathbb{R}, \quad (2.1) \]

for \( m = 0, 1, 2, \ldots \). If we assume that \( \text{supp}(\sigma) \subseteq [-R, R] \), then, by the Paley-Wiener theorem, there exists a positive constant \( C \) such that

\[ |q(z)| \leq C e^{R|\text{Im} z|}, \quad z \in \mathbb{C}. \quad (2.2) \]

Thus, by (2.1) and (2.2), there exists a positive constant \( C' \) such that

\[ |q^{(m)}(x)| \leq C'm!, \quad m = 0, 1, 2, \ldots \]

Next,

\[ \frac{\varphi}{q} \in \mathcal{S}, \quad \varphi \in \mathcal{S}. \]

Indeed, using the fact that \( \sigma \) is even, we get for all nonzero \( x \) in \( \mathbb{R} \),

\[
1 - (2\pi)^{1/2} \tilde{\sigma}(x) = \int_{-\infty}^{\infty} \sigma(\xi) d\xi - \int_{-\infty}^{\infty} e^{-ix\xi} \sigma(\xi) d\xi \\
= \int_{-\infty}^{\infty} (1 - e^{-ix\xi}) \sigma(\xi) d\xi \\
= 2 \int_{0}^{\infty} (1 - \cos x\xi) \sigma(\xi) d\xi > 0.
\]

By Leibniz’ formula (see, for instance, the formula (0.4) in the book [7] by Wong), we get for \( m = 0, 1, 2, \ldots \),

\[
\left( \frac{1}{q} \right)^{(m)} = \sum_{m_1 + m_2 + \cdots + m_k = m} \frac{q^{(m_1)}q^{(m_2)} \ldots q^{(m_k)}}{q^{k+1}},
\]

where \( m_1, m_2, \ldots, m_k \) form a partition of \( m \). Therefore for all \( m = 0, 1, 2, \ldots \), there exists a positive constant \( C_m \) such that

\[
\left| \left( \frac{1}{q} \right)^{(m)} \right| \leq C_m
\]
and we are done. \qed

**Proof of Theorem 1.1** Let \( \varphi \in S \). Then using \( K_\sigma u = u \) and Lemma 2.1, there exists a function \( \psi \) in \( S \) such that

\[
(\Delta u)(\varphi) = u(\Delta \varphi) = u((I - K_\sigma)\psi) = (u - K_\sigma)(\psi) = 0.
\]

Therefore \( \Delta u = 0 \), and by Weyl’s lemma, \( u \) is harmonic on \( \mathbb{R}^n \).

## 3 Fourier Hyperfunctions

Let \( \mathcal{F} \) be the set of all functions \( \varphi \) in \( C^\infty(\mathbb{R}^n) \) such that there exist positive constants \( C, h \) and \( k \) for which

\[
|(|\partial^\alpha \varphi)(x)|e^{k|x|} \leq Ch^{\alpha!}
\]

for all \( x \) in \( \mathbb{R}^n \) and all multi-indices \( \alpha \).

Let \( \{\varphi_j\}_{j=1}^\infty \) be a sequence of functions in \( \mathcal{F} \). Then we say that \( \varphi_j \to 0 \) in \( \mathcal{F} \) as \( j \to \infty \) if there exist positive constants \( h \) and \( k \) such that

\[
\sup_{x,\alpha} \frac{|(|\partial^\alpha \varphi_j)(x)|e^{k|x|}}{h^{\alpha!}} \to 0
\]

as \( j \to \infty \).

Let \( u \) be a linear functional on \( \mathcal{F} \) such that \( u(\varphi_j) \to 0 \) as \( j \to \infty \) for every sequence \( \{\varphi_j\}_{j=1}^\infty \) in \( \mathcal{F} \) with \( \varphi_j \to 0 \) in \( \mathcal{F} \) as \( j \to \infty \). Then we call \( u \) a Fourier hyperfunction on \( \mathbb{R}^n \). The set of all Fourier hyperfunctions on \( \mathbb{R}^n \) is denoted by \( \mathcal{F}' \).

It is obvious that \( \mathcal{F} \subseteq S \). Thus, tempered distributions on \( \mathbb{R}^n \) are contained in \( \mathcal{F}' \).

Using the fact proved by Chung, Chung and Kim in \([3]\), \( \mathcal{F} \) is the same as the Gelfand–Shilov space \( S^1_1 \) in Chapter 4 of the book \([5]\) by Gelfand and Shilov.

**Lemma 3.1** Let \( \varphi \in C^\infty(\mathbb{R}) \) be such that we can find a positive constant \( H \) and a positive constant \( C_\varepsilon \) for every positive number \( \varepsilon \) for which

\[
\sup_{m\in\mathbb{N}_0} \frac{|\varphi^{(m)}(x)|}{H^m m!} \leq C_\varepsilon e^{\varepsilon|x|}, \quad x \in \mathbb{R},
\]

where \( \mathbb{N}_0 \) is the set of all nonnegative integers. Then \( \varphi \psi \in \mathcal{F} \) for all \( \psi \in \mathcal{F} \).
Proof Let $\psi \in \mathcal{F}$. Then there exist positive constants $A$ and $k$ such that

$$\sup_{m \in \mathbb{N}_0, x \in \mathbb{R}} \left| \frac{\psi^{(m)}(x)e^{k|x|}}{A^m m!} \right| < \infty.$$ 

In other words, there exists a positive constant $C$ such that

$$|\psi^{(m)}(x)| \leq CA^m m! e^{-k|x|}, x \in \mathbb{R}, m \in \mathbb{N}_0.$$ 

So, using Leibniz’ formula and the hypothesis on $\varphi$, we get a positive constant $H$ and a positive number $C_\varepsilon$ for every positive number $\varepsilon$ such that

$$|\varphi \psi^{(m)}(x)| \leq C \sum_{j=0}^{m} C_{\varepsilon} \binom{m}{j} H^j j! A^{m-j} (m-j)! e^{\varepsilon|x|} e^{-k|x|}$$

for all $x$ in $\mathbb{R}$ and all $m$ in $\mathbb{N}_0$. Let $\varepsilon = \frac{k}{2}$. Then

$$|\varphi \psi^{(m)}(x)| \leq C \sum_{j=0}^{m} C_{\varepsilon} \binom{m}{j} H^j j! A^{m-j} (m-j)! e^{-k|x|/2}$$

for all $x$ in $\mathbb{R}$ and all $m$ in $\mathbb{N}_0$. So, there exists a positive constant $C'$ such that

$$|\varphi \psi^{(m)}| \leq C' \sum_{j=0}^{m} m! \{\max(H, A)\}^m e^{-k|x|/2}, \quad x \in \mathbb{R}, m \in \mathbb{N}_0.$$ 

This shows that

$$\sum_{m \in \mathbb{N}_0, x \in \mathbb{R}} \frac{|\varphi \psi^{(m)}(x)e^{k|x|/2}|}{K^m m!} < \infty,$$

where $K = \max(H, A)$. This completes the proof. \qed

Lemma 3.2 For $t > 0$, let $k_t$ be the heat kernel given by

$$k_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \quad x \in \mathbb{R}^n.$$ 

Then for all $\varphi \in \mathcal{F}$, $\varphi_t \rightarrow \varphi$ in $\mathcal{F}$ as $t \rightarrow 0^+$, where $\varphi_t = k_t \ast \varphi$. 

7
Proof Since \( \varphi \in \mathcal{F} \), we can find positive constants \( C, h \) and \( k \) such that
\[
|\langle \partial^\alpha \varphi \rangle(x)| \leq C h |\alpha|! e^{-k|x|}
\]
for all \( x \in \mathbb{R}^n \) and all multi-indices \( \alpha \). Let \( \delta \) be any positive number. Then for all \( x \in \mathbb{R}^n \) and all multi-indices \( \alpha \),
\[
(\partial^\alpha \varphi_t)(x) - (\partial^\alpha \varphi)(x) = I_1 + I_2 - I_3,
\]
where
\[
I_1 = \int_{|y| \leq \delta} k_t(y) \{ (\partial^\alpha \varphi)(x - y) - (\partial^\alpha \varphi)(x) \} \, dy,
\]
\[
I_2 = \int_{|y| \geq \delta} k_t(y) (\partial^\alpha \varphi)(x - y) \, dy
\]
and
\[
I_3 = \int_{|y| \geq \delta} k_t(y) (\partial^\alpha \varphi)(x) \, dy.
\]
Now, for \( |y| \leq \delta \), we can use the mean value theorem with integral remainder (see Theorem 6.3 in the book [7] by Wong) to get for all \( x \in \mathbb{R}^n \) and all multi-indices \( \alpha \),
\[
(\partial^\alpha \varphi)(x - y) - (\partial^\alpha \varphi)(x) = \sum_{|\gamma| = 1} (-y)^\gamma \int_0^1 (\partial^\gamma \partial^\alpha \varphi)(x - \theta y) \, d\theta,
\]
and hence there exist positive constants \( C' \) and \( H \) such that
\[
|(\partial^\alpha \varphi)(x - y) - (\partial^\alpha \varphi)(x)| \leq C' e^{k\delta |y|} h |\alpha|! + (|\alpha| + 1) |\alpha|! e^{-k|x|}.
\]
Let \( \varepsilon \) be a positive number. Then we can find a positive number \( \delta \) such that
\[
C'\delta < \frac{\varepsilon}{3}.
\]
Hence
\[
\frac{|I_1| e^{k|x|}}{h |\alpha|!} < \frac{\varepsilon}{3}.
\]
Next, for all \( x \in \mathbb{R}^n \) and all multi-indices \( \alpha \),
\[
\frac{|I_2| e^{k|x|}}{h |\alpha|!} \leq \int_{|y| \geq \delta} h_t(y) \frac{|(\partial^\alpha \varphi)(x - y)| e^{k|x-y|} e^{k|y|}}{h |\alpha|!} \, dy
\]
\[
\leq C \int_{|y| \geq \delta} h_t(y) e^{k|y|} \, dy < \frac{\varepsilon}{3}.
\]
provided that \( t \) is small enough. Finally, for all \( x \) in \( \mathbb{R}^n \) and all multi-indices \( \alpha \),

\[
\frac{|I_3|e^{k|x|}}{h^{(|\alpha|\alpha)!}} \leq \int_{|y| \geq \delta} h_t(y) \frac{|(\partial^\alpha \varphi)(x)|e^{k|x|}}{h^{(|\alpha|\alpha)!}} dy 
\leq C \int_{|y| \geq \delta} h_t(y) dy < \frac{\varepsilon}{3}
\]

provided that \( t \) is small enough. \( \Box \)

We need the following version of Weyl’s lemma for Fourier hyperfunctions.

**Theorem 3.3** Let \( u \in \mathcal{F}' \) be such that \( \Delta u = 0 \). Then \( u \) is harmonic on \( \mathbb{R}^n \).

**Proof** For \( t > 0 \), the convolution \( u \ast k_t \) is a \( C^\infty \) function given by

\[
(u \ast k_t)(x) = u(k_t(x - \cdot)), \quad x \in \mathbb{R}^n, \quad t > 0.
\]

Now, for \( t > 0 \),

\[
0 = (\Delta u) \ast k_t = \Delta(u \ast k_t) = u \ast (\Delta k_t) = u \ast (\partial_t k_t) = \partial_t (u \ast k_t) = 0.
\]

So, for every \( x \) in \( \mathbb{R}^n \),

\[
(u \ast k_t)(x) = v(x), \quad t \in (0, \infty),
\]

where \( v \) is a \( C^\infty \) function on \( \mathbb{R}^n \). Since \( k_t \to \delta \) in \( \mathcal{F}' \) as \( t \to 0 \), it follows that

\[
v = u \ast k_t \to u \ast \delta = u
\]

as \( t \to 0 \). Therefore \( u \in C^\infty(\mathbb{R}^n) \) and the proof is complete. \( \Box \)

### 4 Proof of Theorem 1.3

We begin with a careful look at M. Engliš’ counterexample with respect to the growth rate at infinity. Indeed,

\[
|u(x)| = |e^{ax_1 + bx_2}| \leq e^{(|a|+|b|)|x|}, \quad x \in \mathbb{R}^2.
\]
Thus, \( u \) is of an exponential growth at infinity. Now, we show that this
growth condition is in fact an optimal criterion for the implication

\[ K_\sigma u = u \Rightarrow \Delta u = 0. \]

To do this, we make use of Fourier analysis on Gelfand–Shilov spaces. For nonnegative real numbers \( \alpha \) and \( \beta \), we define the subspace \( S_{\alpha}^{\beta} \) to be the set of all functions \( \varphi \) in \( S(\mathbb{R}) \) for which there exist positive constants \( A, B \) and \( C \) such that for all nonnegative integers \( m \) and \( n \),

\[ \sup_{x \in \mathbb{R}} |x^m \varphi^{(n)}(x)| \leq C A^m B^n (m!)^\alpha (n!)^\beta. \]

As is noted in Section 3, \( S_1^1 = \mathcal{F} \) by a result in Chung, Chung and Kim [3]. So, we can denote \( S_1^1 \) by \( \mathcal{F} \) and let \( \mathcal{F}' \) be its dual space. Let \( u \in \mathcal{F}' \). Then we define the Fourier transform of \( u \) by

\[ \hat{u}(\varphi) = u(\hat{\varphi}), \quad \varphi \in \mathcal{F}. \]

Lemma 4.1 The Fourier transforms \( \wedge : \mathcal{F} \to \mathcal{F} \) and \( \wedge : \mathcal{F}' \to \mathcal{F}' \) are homeomorphisms.

According to a result in Kaneko [6], a function \( u \) in \( L^1_{\text{loc}}(\mathbb{R}^n) \) is a Fourier hyperfunction if and only if for every positive number \( \varepsilon \),

\[ \| u e^{-|\cdot|} \|_{L^\infty(\mathbb{R}^n)} < \infty. \]

We are now ready for a proof of Theorem 1.3.

Proof of Theorem 1.3 Again, we assume that \( n = 1 \). First we prove that

\[ (I - K_\sigma)\mathcal{F} = \Delta \mathcal{F}. \]

By Lemma 3.1, this is the same as proving that the multiplication operator

\[ q : \mathcal{F} \to \mathcal{F} \]

is surjective, where \( q \) is given by

\[ q(x) = \frac{1 - (2\pi)^{1/2} \hat{\sigma}(x)}{-x^2} = \frac{1 - e^{-x^2/2}}{-x^2}, \quad x \in \mathbb{R}. \]

As in the complex-analytic proof of Lemma 1.2, we get positive constants \( A \) and \( C \) such that

\[ |q^{(m)}(x)| = \frac{m!}{2\pi i} \left| \int_{|z-x|=A^{-1}} \frac{q(z)}{(z-x)^{m+1}} dz \right| \leq C m! A^m, \quad m = 0, 1, 2, \ldots. \]
Thus, $q\varphi \in \mathcal{F}$ for all $\varphi$ in $\mathcal{F}$. This follows from a fact in [6], which states that if a function $\eta$ in $C^\infty(\mathbb{R})$ is such that for every positive number $\varepsilon$, there exist positive constants $A$ and $C_\varepsilon$ satisfying
\[ \frac{|\eta^{(m)}(x)|}{A^m m!} \leq C_\varepsilon e^{\varepsilon|x|}, \quad x \in \mathbb{R}, \]
for $m = 0, 1, 2, \ldots$. Then $\eta\varphi \in \mathcal{F}$ for all $\varphi \in \mathcal{F}$. To estimate $(1/q)^{(m)}$, we first note that
\[ 1 - e^{-z^2/2} = 0 \iff e^{-z^2/2} = 1, \]
where $z = x + iy$. To solve the equation for $z$, we write the equation as
\[ e^{(y^2 - x^2)/2} e^{-ixy} = 1. \]
Therefore
\[ \left| e^{(y^2 - x^2)/2} \right| = 1, \]
and we get $x = \pm y$. So,
\[ -ix(\pm x) = 2k\pi i, \quad k \in \mathbb{Z}. \]
Hence
\[ x = \pm \sqrt{2k\pi}, \quad k = 0, 1, 2, \ldots. \]
Thus, on the strip $R$ given by
\[ R = \{x + iy : x \in \mathbb{R}, |y| \leq \sqrt{2\pi}/4\}, \]
\[ e^{-z^2/2} = 1 \iff z = 0. \]
Since the function
\[ \frac{1}{q(z)} = \frac{-z^2}{1 - e^{-z^2/2}} \]
has a removable singularity at $z = 0$, Cauchy’s integral formula gives
\[ \left( \frac{1}{q} \right)^{(m)}(x) = \frac{m!}{2\pi i} \int_{|z-x|=\sqrt{2\pi}/4} \frac{(1/q)(z)}{(z-x)^{m+1}} dz, \quad x \in \mathbb{R}. \]
Now, to estimate \( \left( \frac{1}{q} \right)(x) \), we note that for \( |x| \) sufficiently large, say, \( |x| \geq \eta \),

\[
|1 - e^{-x^2/2}| \geq \frac{1}{2},
\]

and so,

\[
\left| \left( \frac{1}{q} \right)(z) \right| = \left| \frac{-z^2}{1 - e^{-z^2/2}} \right| \leq 2|z|^2 \leq 4x^2.
\]

Hence there exists a positive constant \( C \) such that

\[
\left| \left( \frac{1}{q} \right)(z) \right| \leq C(x^2 + 1), \quad z \in \mathbb{R}.
\]

So, using Leibniz’ formula as in the proof of Theorem 1.1, we get positive constants \( B \) and \( C' \) such that

\[
\left| \left( \frac{1}{q} \right)(x) \right| \leq C'm!B^m(x^2 + 1), \quad x \in \mathbb{R}.
\]

Since for every positive number \( \varepsilon \), we can choose a positive constant \( C_{\varepsilon} \) such that

\[
x^2 + 1 \leq C_{\varepsilon}e^{\varepsilon|x|}, \quad x \in \mathbb{R},
\]

it follows from Lemma 3.1 that \( \frac{1}{q} \varphi \in \mathcal{F} \) for all \( \varphi \) in \( \mathcal{F} \). As in the proof of Theorem 2.1, we get \( \Delta u = 0 \) in the sense of Fourier hyperfunctions. Using the analytic hypoellipticity of Fourier hyperfunctions, \( u \) is a classical harmonic function, and this completes the proof.

**Acknowledgment** This research has been partially supported by the Natural Sciences and Engineering Research Council of Canada.

**References**


