Positive Definite Temperature Functions on the Heisenberg Group

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Abstract We characterize positive definite temperature functions, i.e., positive definite solutions of the heat equation, on the Heisenberg group in terms of the initial values. We also obtain an integral representation for positive definite and $U(n)$-invariant temperature functions with polynomial growth, where $U(n)$ is the group of all $n \times n$ unitary matrices.

Keywords: Heisenberg group, sub-Laplacian, positive definite temperature functions

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1 Introduction

There has been considerable progress in the study of subelliptic operators
given by sums of squares of vector fields on Lie groups. See, for instance,
most important case is perhaps the sub-Laplacian $\Delta_{H^n}$ on the Heisenberg
group $H^n$.

This paper is devoted to a study of positive definite temperature func-
tions on $H^n$, which are positive definite solutions of the heat equation gov-
erned by the sub-Laplacian $\Delta_{H^n}$ on $H^n \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$. Results
on positive definite solutions of the classical heat equation on $\mathbb{R}^n \times \mathbb{R}^+$ can
be found in, for instance, the book [12] by Widder.

After a recall of the basic analysis on the Heisenberg group $H^n$ in this
section, we give in Section 2 a criterion for the existence and uniqueness
of solutions of the heat equation on $H^n \times \mathbb{R}^+$. In Section 3, we give a
characterization of positive definite temperature functions on $H^n$. We give
an integral representation for positive definite and $U(n)$-invariant temper-
ature functions with polynomial growth on $H^n$ in Section 4, and we give
an integral transform that gives a one-to-one correspondence between posi-
tive definite temperature functions and positive Radon measures with some
growth condition in Section 5.

We define the binary operation $\cdot$ on $\mathbb{R}^{2n} \times \mathbb{R}$ by

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(x' \cdot y - x \cdot y'))$$

(1.1)

for all $(x, y, t)$ and $(x', y', t')$ in $\mathbb{R}^{2n} \times \mathbb{R}$, where $x \cdot y$ is the Euclidean inner
product of $x$ and $y$ in $\mathbb{R}^n$. With respect to the binary operation $\cdot$ defined
by (1.1), $\mathbb{R}^{2n} \times \mathbb{R}$ is a non-abelian group in which $(0, 0, 0)$ is the identity
element and the inverse of $(x, y, t)$ is $(-x, -y, -t)$ for all $(x, y, t)$ in $\mathbb{R}^{2n} \times \mathbb{R}$. The
group $\mathbb{R}^{2n} \times \mathbb{R}$ with respect to the binary operation $\cdot$ defined by (1.1) is
called the Heisenberg group and is denoted by $H^n$. It is a locally compact
and Hausdorff group on which the left (and right) Haar measure is the
Lebesgue measure $dx dy dt$. 
For \( j = 1, 2, \ldots, n \), we let \( X_j \) and \( X_{n+j} \) be the left-invariant vector fields on \( \mathbb{H}^n \) defined by

\[
X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad X_{n+j} = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}.
\]

A basis for the Lie algebra \( \mathfrak{h}^n \) of the left-invariant vector fields on \( \mathbb{H}^n \) is then given by \( \{X_1, X_2, \ldots, X_{2n}, T\} \), where \( T = \frac{\partial}{\partial t} \). The sub-Laplacian \( \Delta_{\mathbb{H}^n} \) on \( \mathbb{H}^n \) is defined by

\[
\Delta_{\mathbb{H}^n} = \sum_{j=1}^{2n} X_j^2
\]

and \( \frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n} \) is the heat operator on \( \mathbb{H}^n \times \mathbb{R} \).

We identify \( \mathbb{R}^{2n} \) with \( \mathbb{C}^n \) and identify any point \((x, y)\) in \( \mathbb{R}^{2n} \) with the point \( z = x + iy \) in \( \mathbb{C}^n \). In the paper [7] by Gaveau, a fundamental solution \( P_s \), \( s \in \mathbb{R} \), of the heat operator \( \frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n} \) with a pole at the origin is constructed. Such a fundamental solution is explicit only up to the partial Fourier transform with respect to the center of \( \mathbb{H}^n \) and is given by

\[
P_s(z, t) = \begin{cases} 
(4\pi s)^{-\frac{1}{2}(n+1)} & \int_{-\infty}^{\infty} \left( \frac{2\tau}{\sinh 2\tau} \right)^n \exp \left( \frac{i\tau}{2s} - \frac{2|x|^2\tau}{4s\tanh 2\tau} \right) d\tau, \ s > 0, \\
0, & s \leq 0,
\end{cases}
\]

(1.2)

for all \((z, t) \in \mathbb{H}^n\).

An important group of automorphisms of \( \mathbb{H}^n \) is given by the Heisenberg dilations \( \delta \lambda \), \( \lambda > 0 \), defined by

\[
\delta \lambda (x, y, t) = (\lambda x, \lambda y, \lambda^2 t), \quad (x, y, t) \in \mathbb{H}^n, \ \lambda > 0.
\]

The homogeneous dimension \( Q \) of \( \mathbb{H}^n \) (see Folland and Stein [6]) is given by

\[
Q = 2n + 2.
\]

A function \( u : \mathbb{H}^n \to \mathbb{C} \) is said to be Heisenberg-homogeneous of degree \( k \in \mathbb{Z} \) if \( u \circ \delta \lambda = \lambda^k u, \ \lambda > 0 \). A distinguished Heisenberg-homogeneous
function of degree one is the distance function (see Folland [3]) \( d \) on \( \mathbb{H}^n \) given by
\[
d(x, y, t) = ((x^2 + y^2)^2 + t^2)^{1/4}, \quad (x, y, t) \in \mathbb{H}^n.
\]
The distance \( d((x, y, t), (x', y', t')) \) between two points \((x, y, t)\) and \((x', y', t')\) in \( \mathbb{H}^n \) is given by
\[
d((x, y, t), (x', y', t')) = d((x', y', t')^{-1} \cdot (x, y, t)).
\]

Let \( S(\mathbb{H}^n) = S(\mathbb{R}^{2n} \times \mathbb{R}) \) be the Schwartz space on \( \mathbb{H}^n \). Let \( I = (i_1, i_2, \ldots, i_{2n}) \in \mathbb{N}_0^{2n} \), where \( \mathbb{N}_0 \) is the set of all nonnegative integers. Then we define the function \( X_I \varphi \) on \( \mathbb{H}^n \) for every \( \varphi \) in \( S(\mathbb{H}^n) \) by
\[
(X_I \varphi)(x, y, t) = (X_1^{i_1} X_2^{i_2} \cdots X_{2n}^{i_{2n}} \varphi)(x, y, t), \quad (x, y, t) \in \mathbb{H}^n.
\]

Let \( \varphi \) and \( \psi \) be in \( L^1(\mathbb{H}^n) \). Then we define the convolution \( \varphi \ast \psi \) of \( \varphi \) and \( \psi \) by
\[
(\varphi \ast \psi)(x, y, t) = \int_{\mathbb{H}^n} \varphi(x', y', t') \psi((x', y', t')^{-1} \cdot (x, y, t)) dx' dy' dt'
\]
for all \((x, y, t)\) in \( \mathbb{H}^n \).

An infinitely differentiable function \( F \) on \( \mathbb{H}^n \times (0, S), S > 0 \), is said to be a temperature function on \( \mathbb{H}^n \times (0, S) \) if
\[
\left( \left( \frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n} \right) F \right)(z, t, s) = 0, \quad (z, t, s) \in \mathbb{H}^n \times (0, S).
\]

Let \( \mathbb{R}^\times = \mathbb{R} \setminus \{0\} \). Then for \((m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}^\times \), we let \( \psi_{m, \lambda} \) be the function on \( \mathbb{H}^n \) defined by
\[
\psi_{m, \lambda}(z, t) = e^{i\lambda t} L_m^{n-1} \left( \frac{|\lambda|}{2} |z|^2 \right) e^{-|\lambda| |z|^2}, \quad (z, t) \in \mathbb{H}^n,
\]
where \( L_m^{n-1} \) is the Laguerre polynomial of order \( n - 1 \) and degree \( m \) with \( L_m^{n-1}(0) = 1 \). The functions \( \psi_{m, \lambda}, (m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}^\times \), are studied in some
detail in the books [5] and [13] by Folland and Wong respectively. For all \( \varphi \) in \( L^1(\mathbb{H}^n) \) and \((m, \lambda) \) in \( \mathbb{N}_0 \times \mathbb{R}^\times \), we define \( \hat{\varphi}(\psi_{m,\lambda}) \) by

\[
\hat{\varphi}(\psi_{m,\lambda}) = \int_{\mathbb{H}^n} \psi_{m,\lambda}(z, t) \varphi(z, t) dz \, dt.
\]

We sometimes denote \( \hat{\varphi}(\psi_{m,\lambda}) \) by \( \hat{\varphi}(m, \lambda) \) for all \((m, \lambda) \) in \( \mathbb{N}_0 \times \mathbb{R}^\times \). Let \( f \in L^\infty(\mathbb{H}^n) \). Then we say that \( f \) is positive definite on \( \mathbb{H}^n \) if

\[
\int_{\mathbb{H}^n} f(z, t)(\varphi \ast \varphi^*)(z, t) dz \, dt \geq 0, \quad \varphi \in \mathcal{S}(\mathbb{H}^n),
\]

where

\[
\varphi^*(z, t) = \overline{\varphi(-z, -t)}, \quad (z, t) \in \mathbb{H}^n.
\]

It can be shown that \( \psi_{m,\lambda} \) is positive definite on \( \mathbb{H}^n \) for all \((m, \lambda) \) in \( \mathbb{N}_0 \times \mathbb{R}^\times \).

Let \( U(n) \) be the group of all \( n \times n \) unitary matrices. A continuous function \( f \) on \( \mathbb{H}^n \) is said to be \( U(n) \)-invariant if

\[
f(\sigma z, t) = f(z, t), \quad (z, t) \in \mathbb{H}^n,
\]

for all \( \sigma \) in \( U(n) \). Let \( f \in \mathcal{S}'(\mathbb{H}^n) \). Then for all \( \sigma \) in \( U(n) \), we define \( f_\sigma \) by

\[
f_\sigma(\varphi) = f(\varphi_\sigma^{-1}), \quad \varphi \in \mathcal{S}(\mathbb{H}^n),
\]

where

\[
\varphi_\sigma^{-1}(z, t) = \varphi(\sigma^{-1}z, t), \quad (z, t) \in \mathbb{H}^n.
\]

We say that a function \( \varphi \) in \( \mathcal{S}(\mathbb{H}^n) \) is \( U(n) \)-invariant if

\[
\varphi_\sigma = \varphi, \quad \varphi \in \mathcal{S}(\mathbb{H}^n),
\]

for all \( \sigma \) in \( U(n) \). In fact, it can be proved easily that a distribution \( f \) in \( \mathcal{S}'(\mathbb{H}^n) \) is \( U(n) \)-invariant if and only if \( f = f^\natural \), where

\[
f^\natural(\varphi) = f(\varphi^\natural), \quad \varphi \in \mathcal{S}(\mathbb{H}^n),
\]

where

\[
\varphi^\natural(z, t) = \int_{U(n)} \varphi(\sigma z, t) d\sigma, \quad (z, t) \in \mathbb{H}^n,
\]
and $d\sigma$ is the Haar measure on $U(n)$ for which
$$\int_{U(n)} d\sigma = 1.$$ Thus, a function $\varphi$ in $S(\mathbb{H}^n)$ is $U(n)$-invariant if and only if $\varphi^k = \varphi$.

Let $f \in S'(\mathbb{H}^n)$ and let $\varphi \in S(\mathbb{H}^n)$. Then we define the convolution of $f \ast \varphi$ of $f$ and $\varphi$ by
$$(f \ast \varphi)(z, t) = f(\varphi((.,.)^{-1}\cdot(z, t))), \ (z, t) \in \mathbb{H}^n.$$ It is well-known that $f \ast \varphi$ is a smooth function on $\mathbb{H}^n$ and the linear mapping
$$S'(\mathbb{H}^n) \times S(\mathbb{H}^n) \ni (f, \varphi) \mapsto f \ast \varphi \in S'(\mathbb{H}^n)$$ is separately continuous.

## 2 The Temperature Function

We begin with three technical lemmas, which will be useful to us.

**Lemma 2.1** Let $\{P_s : s \in \mathbb{R}\}$ be the heat kernel associated to the sub-Laplacian $\Delta_{\mathbb{H}^n}$ given by (1.2). Then for all $I$ in $\mathbb{N}_0^{2n}$ and nonnegative integers $k$, there exist positive constants $C$ and $C_{1k}$ such that
$$\left|\left(\left(\frac{\partial}{\partial s}\right)^k X_I P_s\right)(z, t)\right| \leq C_{1k}s^{-k-|I|-\frac{2}{2}d(z, t)^2/s}, \ (z, t) \in \mathbb{H}^n, \ (2.1)$$ where $|I|$ is the length of the multi-index $I$.

**Lemma 2.2** For all $I$ and $J$ in $\mathbb{N}_0^{2n}$, and nonnegative integers $k$, we can find a constant $C_1$ and positive constants $C_{1Jk}$ such that
$$\left|\left(\left(\frac{\partial}{\partial s}\right)^k X_I X_J P_s\right)(z', t')^{-1} \cdot (z, t)\right|$$
\[ C_{IJK} s^{-\frac{Q}{2} - k - \frac{|I| + |J|}{2}} e^{-C_1 d((z',t')^{-1}, (z,t))^2/s} \]

for \( 0 < s < 1 \).

Lemma 2.1 and Lemma 2.2 can be found in the paper [9] by Jerison and Sánchez-Calle. The following lemma can be proved using the semigroup property of the heat kernel on the Heisenberg group given in the paper [4] by Folland.

**Lemma 2.3** The heat kernel \( \{ P_s : s > 0 \} \) associated to the sub-Laplacian \( \Delta_{\mathbb{H}^n} \) given by (1.2) satisfies the following conditions.

(i) \( P_s \in \mathcal{S}(\mathbb{H}^n) \), \( s > 0 \).

(ii) For every \( \psi \) in \( \mathcal{S}(\mathbb{H}^n) \), \( P_s \psi \to \psi \) in \( \mathcal{S}(\mathbb{H}^n) \) as \( s \to 0^+ \).

The following theorem is the main result in this section.

**Theorem 2.4** Let \( f \in \mathcal{S}'(\mathbb{H}^n) \) and \( S \) be a positive number. Then the function \( F \) on \( \mathbb{H}^n \times (0, S) \) defined by

\[ F(z, t, s) = (f * P_s)(z, t), \quad (z, t, s) \in \mathbb{H}^n \times (0, S), \]

satisfies the following conditions.

(i) \( \left( \frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n} \right) F \left( z, t, s \right) = 0, \quad (z, t, s) \in \mathbb{H}^n \times (0, S) \).

(ii) There exist positive constants \( M, N, C \) and \( S' \) such that

\[ |F(z, t, s)| \leq Cs^{-M} (1 + d(z, t))^N, \quad (z, t, s) \in \mathbb{H}^n \times (0, S'). \]

(iii) For all \( \varphi \) in \( \mathcal{S}(\mathbb{H}^n) \),

\[ \lim_{s \to 0^+} \int_{\mathbb{H}^n} F(z, t, s) \varphi(z, t) dz dt = f(\varphi). \]
Conversely, every temperature function \( F \) on \( \mathbb{H}^n \times (0, S) \) satisfying (ii) can be uniquely expressed in the form

\[
F(z, t, s) = (f * P_s)(z, t), \quad (z, t, s) \in \mathbb{H}^n \times (0, S),
\]

for some \( f \) in \( S'(\mathbb{H}^n) \).

**Proof:** Let \( f \in S'(\mathbb{H}^n) \). Then by part (i) of Lemma 2.3 and Theorem 2.1.3 in Hörmander [8], we can easily see that the function \( F \) on \( \mathbb{H}^n \times (0, S) \) defined by

\[
F(z, t, s) = (f * P_s)(z, t), \quad (z, t, s) \in \mathbb{H}^n \times (0, S),
\]
is a smooth function on \( \mathbb{H}^n \times (0, S) \). Since \( \Delta_{\mathbb{H}^n} \) is a left-invariant vector field on \( \mathbb{H}^n \), it follows that

\[
\left( \left( \frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n} \right) F \right)(z, t, s) = 0, \quad (z, t, s) \in \mathbb{H}^n \times (0, S).
\]

Since \( f \in S'(\mathbb{H}^n) \), we can find positive constants \( C, N \) and \( K \) such that

\[
|f(\varphi)| \leq C \sup_{(z', t') \in \mathbb{H}^n} \sum_{|I| \leq K} (1 + d(z', t'))^N |X_I \varphi(z', t')| \tag{2.2}
\]

for all \( \varphi \) in \( S(\mathbb{H}^n) \). Therefore, by (2.1),

\[
|F(z, t, s)| = |(f * P_s)(z, t)| \leq C \sup_{(z', t') \in \mathbb{H}^n} \sum_{|I| \leq K} (1 + d(z', t'))^N |X_I (z', t') P_s((z', t')^{-1}(z, t))| \tag{2.3}
\]

for all \((z, t, s)\) in \( \mathbb{H}^n \times (0, S) \). By Lemma 2.2 and (2.2), we get a constant \( C_1 \) and a positive constant \( C_I \) such that

\[
|F(z, t, s)| \leq C \sup_{(z', t') \in \mathbb{H}^n} \sum_{|I| \leq K} (1 + d(z', t'))^N C_I \times s^{-\frac{Q}{2} - \frac{|I|}{2}} e^{-C_1 d((z', t')^{-1}(z, t))^2/s}
\]

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for $0 < s < 1$. If we let $(z'', t'') = (z', t')^{-1} \cdot (z, t)$ in (2.3), then we get

$$|F(z, t, s)| \leq C \sup_{(z'', t'') \in \mathbb{H}^n} \sum_{|I| \leq K} (1 + d((z, t) \cdot (z'', t'')^{-1}))^N C_I$$

for $0 < s < 1$. Since $d((z, t) \cdot (z'', t'')^{-1}) \leq d(z, t) + d(z'', t''')$ and

$$\sup_{(z'', t'') \in \mathbb{H}^n, 0 < s < 1} d(z'', t''')e^{-C_1d(z'', t''')}^2/s < \infty,$$

$m = 0, 1, 2, \ldots$, we can use an argument in Folland [3] to get a positive constant $C'$ such that

$$|F(z, t, s)| \leq C'(1 + d(z, t))^N s^{-\frac{Q}{2} - \frac{|I|}{2}}$$

for $0 < s < 1$. To prove part (iii), let $\varphi \in \mathcal{S}(\mathbb{H}^n)$. Then

$$\int_{\mathbb{H}^n} F(z, t, s) \varphi(z, t) dz \, dt = f(P_s * \varphi)$$

by taking the limit of Riemann sums of the integral on the left hand side as in the proof of Lemma 4.1.3 in Hörmander [8]. To prove the converse, let $F$ be a smooth function on $\mathbb{H}^n \times (0, S)$ such that (i) and (ii) are satisfied, and let $\eta$ be the function on $\mathbb{R}$ defined by

$$\eta(s) = \begin{cases} 
 s^{l-1}, & s \geq 0, \\
 0, & s < 0.
\end{cases}$$

Multiplying $\eta$ by a suitable smooth function with compact support, we get, by Theorem 7.1.22 in Hörmander [8], a function $\nu$ on $\mathbb{R}$ such that $\nu^{(l)} = \delta + \omega$, where

$$\nu(s) = \begin{cases} 
 \eta(s), & s \leq \frac{S}{2}, \\
 0, & s \geq \frac{S}{2},
\end{cases}$$

$\omega$ is a smooth function with $\text{supp}(\omega) \subseteq \left[\frac{S}{4}, \frac{S}{2}\right]$ and $\delta$ is the Dirac measure supported at the origin.
Now, we take \( l = \lceil N \rceil + 2 \), where \([N]\) is the greatest integer less than or equal to \( N \), and consider the function \( F^* \) on \( (0, \frac{S}{2}) \) given by

\[
F^*(z, t, s) = \int_0^\infty F(z, t, s + s')\nu(s')ds', \quad 0 < s < \frac{S}{2}.
\]

Using the definition of \( \nu \) and part (ii), we see that \( F^* \) is a smooth function on \( \mathbb{H}^n \times (0, \frac{S}{2}) \) and we can find a positive constant \( C'' \) such that

\[
|F^*(z, t, s)| \leq C''(1 + d(z, t))N, \quad (z, t, s) \in \mathbb{H}^n \times \left(0, \frac{S}{2}\right).
\]

Thus, \( F^* \) can be extended to a continuous function, again denoted by \( F^* \), on \( \mathbb{H}^n \times \left[0, \frac{S}{2}\right] \). Furthermore, we have

\[
\left(\left(-\frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n}\right)F^*\right)(z, t, s) = 0, \quad (z, t, s) \in \mathbb{H}^n \times \left(0, \frac{S}{2}\right).
\]

Therefore

\[
\left(\left(-\Delta_{\mathbb{H}^n}\right)^l F^*\right)(z, t, s) = \left(\left(-\frac{\partial}{\partial s}\right)^l F^*\right)(z, t, s) = F(z, t, s) + \int_0^\infty F(z, t, s + s')\omega(s')ds' \tag{2.4}
\]

for all \((z, t, s)\) in \( \mathbb{H}^n \times (0, \frac{S}{2}) \). Now, we let \( G \) be the function on \( \mathbb{H}^n \times (0, \frac{S}{2}) \) defined by

\[
G(z, t, s) = -\int_0^\infty F(z, t, s + s')\omega(s')ds', \quad (z, t, s) \in \mathbb{H}^n \times \left(0, \frac{S}{2}\right).
\]

Then \( G \) is a smooth temperature function on \( \mathbb{H}^n \times (0, \frac{S}{2}) \), which can be extended to a continuous function, again denoted by \( G \), on \( \mathbb{H}^n \times \left[0, \frac{S}{2}\right] \). Now, we let \( g \) and \( h \) be functions on \( \mathbb{H}^n \) defined by

\[
g(z, t) = G(z, t, 0), \quad h(z, t) = F^*(z, t, 0) \tag{2.5}
\]
for all \((z, t)\) in \(\mathbb{H}^n\). Then \(g\) and \(h\) are continuous on \(\mathbb{H}^n\) and have polynomial growth with respect to the distance function \(d\) on the Heisenberg group \(\mathbb{H}^n\). It follows then from the uniqueness of the solution of the heat equation on the Heisenberg group that

\[
G(z, t, s) = (g * P_s)(z, t), \quad F^*(z, t, s) = (h * P_s)(z, t) \quad (2.6)
\]

for all \((z, t, s)\) in \(\mathbb{H}^n \times (0, \frac{S^2}{2})\). Let \(f \in \mathcal{S}'(\mathbb{H}^n)\) be defined by

\[
f = (-\Delta_{\mathbb{H}^n})^l h + g. \quad (2.7)
\]

Then for all \(\varphi\) in \(\mathcal{S}(\mathbb{H}^n)\), we get, by (2.4), (2.5) and (2.7),

\[
\lim_{s \to 0+} \int_{\mathbb{H}^n} F(z, t, s) \varphi(z, t) dz dt = f(\varphi).
\]

Finally, by (2.4), (2.6) and (2.7),

\[
(f * P_s)(z, t) = (-\Delta_{\mathbb{H}^n})^l (h * P_s)(z, t) + (g * P_s)(z, t) \\
= (-\Delta_{\mathbb{H}^n})^l F^*(z, t, s) + G(z, t, s) = F(z, t, s)
\]

for all \((z, t, s)\) in \(\mathbb{H}^n \times (0, \frac{S^2}{2})\) and the proof is complete. \(\square\)

We end this section with a corollary.

**Corollary 2.5** The initial value problem

\[
\begin{cases}
\frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n} F(z, t, s) = 0, & (z, t, s) \in \mathbb{H}^n \times (0, S), \\
F(z, t, 0+) = f(z, t), & f \in \mathcal{S}'(\mathbb{H}^n),
\end{cases}
\]

can be uniquely solved for the temperature function \(F\) satisfying part (ii) in Theorem 2.4.

### 3 Positive Definite Temperature Functions

The following characterization of positive definite distributions on a unimodular Lie group will be used in the sequel.
Theorem 3.1 Let $G$ be a unimodular Lie group and let $f \in \mathcal{D}'(G)$ be a positive definite distribution on $G$. Then $f$ can be expressed as a finite sum

$$f = \sum_{j=1}^{N} D^j E^j f_j,$$

where $f_j \in L^\infty(G)$, and $D^j$ and $E^j$ are left-invariant and right-invariant vector fields on $G$ respectively.

A proof of Theorem 3.1 can be found in Barker [1].

We can now give a characterization of positive definite temperature functions on the Heisenberg group $\mathbb{H}^n$.

Theorem 3.2 Let $F$ be a smooth temperature function on $\mathbb{H}^n \times \mathbb{R}^+$ satisfying part (ii) of Theorem 2.4 and such that

$$\int_{\mathbb{H}^n} F(z, t, s)(\varphi * \varphi^*)(z, t) dz dt \geq 0, \quad s > 0,$$

for all $\varphi$ in $\mathcal{S}(\mathbb{H}^n)$. Then there exists a unique positive definite distribution $f$ in $\mathcal{S}'(\mathbb{H}^n)$ such that

$$f(\varphi) = \lim_{s \to 0^+} \int_{\mathbb{H}^n} F(z, t, s) \varphi(z, t) dz dt, \quad \varphi \in \mathcal{S}(\mathbb{H}^n).$$

Conversely, every positive definite distribution $f$ in $\mathcal{D}'(\mathbb{H}^n)$ defines a positive definite temperature function $F$ on $\mathbb{H}^n \times \mathbb{R}^+$ satisfying part (ii) of Theorem 2.4.

Proof: It follows from Theorem 2.4 that there exists a unique distribution $f$ in $\mathcal{S}'(\mathbb{H}^n)$ such that

$$f(\varphi) = \lim_{s \to 0^+} \int_{\mathbb{H}^n} F(z, t, s) \varphi(z, t) dz dt, \quad \varphi \in \mathcal{S}(\mathbb{H}^n).$$
Thus, using the assumption that $F(\cdot, s)$ is positive definite on $\mathbb{H}^n$ for every positive number $s$, we get

$$f(\varphi \ast \varphi^*) = \lim_{s \to 0} \int_{\mathbb{H}^n} F(z, t, s)(\varphi \ast \varphi^*)(z, t)dz dt \geq 0$$

for all $\varphi$ in $S(\mathbb{H}^n)$. To prove the converse, let $f$ be a positive definite distribution on $\mathbb{H}^n$. Then as a consequence of Theorem 3.1, $f \in S'(\mathbb{H}^n)$. Let $F$ be the function on $\mathbb{H}^n \times \mathbb{R}^+$ defined by

$$F(z, t, s) = (f * P_s)(z, t), (z, t, s) \in \mathbb{H}^n \times \mathbb{R}^+.$$

Then, using a simple computation given on pages 20 and 21 of the book [11] by Taylor, we have

$$(f * P_s)(\varphi \ast \varphi^*) = P_s * (\hat{f}(\varphi \ast \varphi^*)), \varphi \in S(\mathbb{H}^n),$$

where $\hat{f}(\psi) = f(\hat{\psi})$, $\psi \in S(\mathbb{H}^n)$, and $\hat{\psi}(z, t) = \psi((z, t)^{-1})$, $(z, t) \in \mathbb{H}^n$. Since $P_s$ is a positive function on $\mathbb{H}^n$ (see Gaveau [7]) and $\hat{f}$ is a positive definite distribution in $S'(\mathbb{H}^n)$, we get

$$(f * P_s)(\varphi \ast \varphi^*) = P_s * (\hat{f} * (\varphi \ast \varphi^*)) \geq 0, \quad s > 0.$$

This completes the proof. $\square$

4 Positive Definite and $U(n)$-Invariant Temperature Functions

We obtain in this section an integral representation for positive definite and $U(n)$-invariant temperature functions with polynomial growth.

**Theorem 4.1** Let $F$ be a positive definite and $U(n)$-invariant temperature function on $\mathbb{H}^n \times \mathbb{R}^+$ satisfying part (ii) of Theorem 2.4. Then there exists a unique positive and tempered Radon measure $\mu$ on $\mathbb{N}_0 \times \mathbb{R}^n$ such that

$$F(z, t, s) = \int_{\mathbb{N}_0 \times \mathbb{R}^n} \psi_{m, -\lambda}(z, t)e^{-|\lambda|(2m+n)s}d\mu(m, \lambda) \quad (4.1)$$
for all \((z,t,s)\) in \(\mathbb{H}^n \times \mathbb{R}^+\), and

\[
\int_{\mathbb{N}_0 \times \mathbb{R}^\times} (1 + |\lambda|)(2m + n)^{-\kappa} d\mu(m, \lambda) < \infty
\]

for some positive number \(\kappa\). Conversely, a continuous function \(F\) defined by (4.1) is a smooth, positive definite and \(U(n)\)-invariant temperature function on \(\mathbb{H}^n \times \mathbb{R}^+\).

**Proof:** Suppose that \(F\) is a positive definite and \(U(n)\)-invariant temperature function on \(\mathbb{H}^n \times \mathbb{R}^+\) satisfying part (ii) of Theorem 2.4. Then, by Theorems 2.4 and 3.2, there exists a unique positive definite distribution \(f\) in \(\mathcal{S}(\mathbb{H}^n)\) such that

\[
F(z, t, s) = (f \ast P_s)(z, t), \quad (z, t, s) \in \mathbb{H}^n \times \mathbb{R}^+.
\]

(4.2)

It follows from the \(U(n)\)-invariance of \(F\) that

\[
F^\flat(z, t, s) = (f \ast P_s)(z, t), \quad (z, t, s) \in \mathbb{H}^n \times \mathbb{R}^+.
\]

Thus, we get, for all \(\varphi \in \mathcal{S}(\mathbb{H}^n)\),

\[
\lim_{s \to 0^+} \int_{\mathbb{H}^n} F^\flat(z, t, s) \varphi(z, t) dz dt = \int_{\mathbb{H}^n} F(z, t) \varphi^\flat(z, t) dz dt = f(\varphi^\flat).
\]

(4.3)

Using (4.3) and the fact that \(F^\flat = F\), we get

\[
f(\varphi) = f^\flat(\varphi), \quad \varphi \in \mathcal{S}(\mathbb{H}^n),
\]

(4.4)

and hence the \(U(n)\)-invariance of \(f\). Let \(s\) be a fixed positive number. Then it follows from (4.2) and (4.4) that the spherical transform \(\hat{F}(\cdot, \cdot, s)\) of \(F(\cdot, \cdot, s)\) satisfies

\[
\hat{F}(m, \lambda, s) = \hat{P}_s(m, \lambda) \hat{f}(m, \lambda), \quad (m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}_\times.
\]

(4.5)

By Theorem 3.1 in Kim and Wong [10], there exists a positive number \(\kappa\) and a positive Radon measure \(\nu\) such that

\[
\int_{\mathbb{N}_0 \times \mathbb{R}^\times} (1 + |\lambda|(2m + n))^{-\kappa} d\nu(m, \lambda) < \infty,
\]

(4.6)
where \( \nu(m, \lambda) = \hat{f}(m, \lambda), (m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}^\times \). By Godement’s theorem (see Kim and Wong [10]), the spherical transform of \( F(\cdot, \cdot, s) \) is a positive and finite Radon measure for each positive number \( s \). Since \( F \) is a temperature function on \( \mathbb{H}^n \times \mathbb{R}^+ \), it follows that

\[
\left( \frac{\partial}{\partial s} + |\lambda|(2m + n) \right) \hat{F}(m, \lambda, s) = 0 \tag{4.7}
\]

for all \((m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}^\times \) and all positive numbers \( s \), if we take the spherical transform of \( F(\cdot, \cdot, s) \) and use the fact that

\[
\Delta_{\mathbb{H}^n} \psi_{m, \lambda} = -|\lambda|(2m + n) \psi_{m, \lambda}, \quad (m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}^\times.
\]

By (4.7), we get

\[
\frac{\partial}{\partial s} \left( e^{\lambda(2m+n)s} \hat{F}(m, \lambda, s) \right) = 0 \tag{4.8}
\]

for all \((m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}^\times \) and all positive numbers \( s \). This implies that

\[
\mu(m, \lambda) = e^{\lambda(2m+n)s} \hat{F}(m, \lambda, s)
\]

is a positive Radon measure on \( \mathbb{N}_0 \times \mathbb{R}^\times \) by (4.5), which is independent of \( s \). Hence by Godement’s theorem (see Kim and Wong [10] again),

\[
F(z, t, s) = \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \psi_{m, -\lambda}(z, t) d\hat{F}(m, \lambda, s)
= \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \psi_{m, -\lambda}(z, t) e^{-|\lambda|(2m+n)s} e^{\lambda(2m+n)s} d\hat{F}(m, \lambda, s)
= \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \psi_{m, -\lambda}(z, t) e^{-|\lambda|(2m+n)s} d\mu(m, \lambda) \tag{4.9}
\]

for all \((z, t, s) \in \mathbb{H}^n \times \mathbb{R}^+ \).

Now, we prove that \( \mu \) is a tempered measure on \( \mathbb{N}_0 \times \mathbb{R}^\times \). Since \( F(\cdot, \cdot, s) \) is a positive definite function on \( \mathbb{H}^n \) for every positive number \( s \), we get, by (4.9) and the fact that

\[
|\psi_{m, \lambda}(z, t)| \leq \psi_{m, \lambda}(0, 0) = 1, \quad (z, t) \in \mathbb{H}^n,
\]
\[ |F(z, t, s)| \leq F(0, 0, s), \quad s > 0. \]

Therefore for every positive number \( \epsilon \),
\[
\int_{\mathbb{N}_0 \times \mathbb{R}^\times} e^{-|\lambda|(2m+n)^r} \, d\mu(m, \lambda) < \infty.
\]

Thus, by Fubini’s theorem, we get
\[
\int_{\mathbb{H}^n} F(z, t, s) \varphi(z, t) \, dz \, dt = \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \hat{\varphi}(m, -\lambda) e^{-|\lambda|(2m+n)s} \, d\mu(m, \lambda) \tag{4.10}
\]
for all \( \varphi \) in \( \mathcal{S}(\mathbb{H}^n) \) and all positive numbers \( s \). By (4.5),
\[
\int_{\mathbb{H}^n} F(z, t, s) \varphi(z, t) \, dz \, dt = \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \hat{\varphi}(m, \lambda) \hat{P}_s(m, \lambda) \, d\nu(m, \lambda) \tag{4.11}
\]
for all \( \varphi \) in \( \mathcal{S}(\mathbb{H}^n) \) and all positive numbers \( s \). Letting \( s \to 0^+ \), we get, by (4.6) and (4.11),
\[
f(\varphi) = \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \hat{\varphi}(m, \lambda) \, d\nu(m, \lambda), \quad \varphi \in \mathcal{S}(\mathbb{H}^n), \tag{4.12}
\]

because
\[
(1 + |\lambda|(2m+n))^r |\hat{\varphi}(m, \lambda)| < \infty, \quad r \in \mathbb{N}_0, \ (m, \lambda) \in \mathbb{N}_0 \times \mathbb{R}^\times,
\]
and \( \hat{P}_s \to 1 \) uniformly as \( s \to 0^+ \). So, by (4.10) and (4.12),
\[
\int_{\mathbb{N}_0 \times \mathbb{R}^\times} \hat{\varphi}(m, \lambda) \, d\mu(m, -\lambda) = \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \hat{\varphi}(m, \lambda) \, d\nu(m, \lambda), \quad \varphi \in \mathcal{S}(\mathbb{H}^n).
\]

This proves that \( \mu \) is tempered. To prove the converse, let us note that the \( U(n) \)-invariance of \( F \) is obvious by (4.1). We only need to prove that \( F \) is a smooth and positive definite temperature function on \( \mathbb{H}^n \times \mathbb{R}^\times \). By Proposition 4.1 and Theorem 4.16 in Folland [4], the smoothness of \( F \) on \( \mathbb{H}^n \) is equivalent to
\[
(1 - \Delta_{\mathbb{H}^n})^r F \in L^\infty(\mathbb{H}^n)
\]
for all \( r \in \mathbb{N}_0 \) and all positive numbers \( s \). Now, we note that for \( \kappa = 0, 1, 2, \ldots \),
\[
\int_{\mathbb{N}_0 \times \mathbb{R}^\times} (1 + |\lambda|(2m + n))^\kappa e^{-|\lambda|(2m+n)s}d\mu(m, \lambda) < \infty
\]
for all \( s \in \mathbb{R}^+ \). Thus, we can interchange the order of differentiation and integration and use the formula
\[
(\Delta_{\mathbb{H}^n} \psi_{m,-\lambda})(z, t) = -|\lambda|(2m + n)\psi_{m,-\lambda}(z, t)
\]
for all \((m, \lambda)\) in \( \mathbb{N}_0 \times \mathbb{R}^+ \) and \((z, t)\) in \( \mathbb{H}^n \times \mathbb{R}^+ \) to get
\[
\left( (1 - \Delta_{\mathbb{H}^n})^n F \right)(z, t, s) = \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \psi_{m,-\lambda}(z, t)e^{-|\lambda|(2m+n)s}(1 + |\lambda|(2m + n))^\kappa d\mu(m, \lambda)
\]
is a continuous function of \((z, t, s)\) on \( \mathbb{H}^n \times \mathbb{R}^+ \). Hence, by Proposition 4.1 and Theorem 4.16 in Folland [4], we see that for every positive number \( s \), \( F(\cdot, \cdot, s) \) is smooth. By an easy computation, we can also see that for every \((z, t)\) in \( \mathbb{H}^n \), \( F(z, t, \cdot) \) is smooth on \( \mathbb{R}^+ \). So, we can see easily that
\[
\left( \left( \frac{\partial}{\partial s} - \Delta_{\mathbb{H}^n} \right) F \right)(z, t, s) = 0, \quad (z, t, s) \in \mathbb{H}^n \times \mathbb{R}^+.
\]
That \( F \) is \( U(n) \)-invariant for each positive number \( s \) is obvious. So, we only need to show that for every positive number \( s \), \( F(\cdot, \cdot, s) \) is positive definite on \( \mathbb{H}^n \). But, for every \( \varphi \) in \( S(\mathbb{H}^n) \), we get
\[
\int_{\mathbb{H}^n} F(z, t, s)(\varphi \ast \varphi^*)(z, t)dz \, dt
\]
\[
= \int_{\mathbb{N}_0 \times \mathbb{R}^\times} \int_{\mathbb{H}^n} \psi_{m,-\lambda}(z, t)e^{-|\lambda|(2m+n)s}(\varphi \ast \varphi^*)(z, t)dz \, dt \, d\mu(m, \lambda)
\]
for every positive number \( s \) because for any such an \( s \),
\[
\int_{\mathbb{N}_0 \times \mathbb{R}^\times} e^{-|\lambda||z|^2} \left| L_m^{n-1} \left( \frac{|\lambda||z|^2}{2} \right) \right| e^{-|\lambda|(2m+n)s}d\mu(m, \lambda) < \infty.
\]
Now, for every \((m, \lambda)\) in \(\mathbb{N}_0 \times \mathbb{R}^+\), the function \(\psi_{m,\lambda}\) is positive definite on \(\mathbb{H}^n\). Hence
\[
\int_{\mathbb{H}^n} \psi_{m,-\lambda}(z,t)(\varphi \ast \varphi^*)(z,t)dzdt \geq 0, \quad \varphi \in \mathcal{S}(\mathbb{H}^n).
\]
This completes the proof. \(\square\)

## 5 The Temperature Transform

Let \(B(\mathbb{H}^n, U(n))\) be the set of all positive and tempered Radon measures on \(\mathbb{N}_0 \times \mathbb{R}^+\) and let \(H(\mathbb{H}^n, U(n))\) be the set of all positive definite and \(U(n)\)-invariant temperature functions on \(\mathbb{H}^n \times \mathbb{R}^+\) satisfying part (ii) of Theorem 2.4. Then we define \(F : B(\mathbb{H}^n, U(n)) \to H(\mathbb{H}^n, U(n))\) by
\[
F(\mu)(z,t,s) = \int_{\mathbb{N}_0 \times \mathbb{R}^+} e^{-|\lambda|(2m+n)s} \psi_{m,-\lambda}(z,t)d\mu(m, \lambda), \quad (z,t,s) \in \mathbb{H}^n \times \mathbb{R}^+.
\]
Then \(F : B(\mathbb{H}^n, U(n)) \to H(\mathbb{H}^n, U(n))\) is a one-to-one correspondence, which we call the temperature transform. Moreover, with a suitable topology on \(\mathbb{N}_0 \times \mathbb{R}^+\) specified in, say, the papers [2] by Benson, Jenkins and Ratcliff and [10] by Kim and Wong, we have the following result.

**Theorem 5.1** The temperature transform \(F : B(\mathbb{H}^n, U(n)) \to H(\mathbb{H}^n, U(n))\) is a homeomorphism.

**Remark 5.2** If we let \(s = 0\), then \(F(\mu)(z,t,0)\) is the inverse spherical transform of \(\mu\) in \(\dot{\mathcal{S}}(U(n), \mathbb{H}^n)\). See the paper [2] by Benson, Jenkins and Ratcliff for the definition of \(\dot{\mathcal{S}}(U(n), \mathbb{H}^n)\).

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References


