11.11 Applications of Taylor Polynomials
Approximating Functions by Polynomials
Approximating Functions by Polynomials

Suppose that $f(x)$ is equal to the sum of its Taylor series at $a$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

The notation $T_n(x)$ is used to represent the $n$th partial sum of this series and we can call it as the $n$th-degree Taylor polynomial of $f$ at $a$.

Thus

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i$$
Approximating Functions by Polynomials

Since $f$ is the sum of its Taylor series, we know that $T_n(x) \to f(x)$ as $n \to \infty$ and so $T_n$ can be used as an approximation to $f(x)$:

$$f(x) \approx T_n(x).$$

Notice that the first-degree Taylor polynomial

$$T_1(x) = f(a) + f'(a)(x - a)$$

is the same as the linearization of $f$ at $a$. 
Notice also that $T_1$ and its derivative have the same values at $a$ that $f$ and $f'$ have. In general, it can be shown that the derivatives of $T_n$ at $x=a$ agree with those of $f$ up to and including derivatives of order $n$.

To illustrate these ideas let’s take another look at the graphs of $y = e^x$ and its first few Taylor polynomials, as shown in Figure 1.
Approximating Functions by Polynomials

The graph of $T_1$ is the tangent line to $y = e^x$ at $(0, 1)$, and this tangent line is the best linear approximation to $e^x$ near $(0, 1)$.

The graph of $T_2$ is the parabola $y = 1 + x + x^2/2$, and the graph of $T_3$ is the cubic curve $y = 1 + x + x^2/2 + x^3/6$, which is a closer fit to the exponential curve $y = e^x$ than $T_2$.

The next Taylor polynomial $T_4$ would be an even better approximation, and so on.
Approximating Functions by Polynomials

The values in the table give a numerical demonstration of the convergence of the Taylor polynomials $T_n(x)$ to the function $y = e^x$. We see that when $x = 0.2$ the convergence is very rapid, but when $x = 3$ it is somewhat slower.

In fact, the farther $x$ is from 0, the more slowly $T_n(x)$ converges to $e^x$. When using a Taylor polynomial $T_n$ to approximate a function $f$, we have to ask the questions: How good an approximation is it?

<table>
<thead>
<tr>
<th></th>
<th>$x = 0.2$</th>
<th>$x = 3.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_2(x)$</td>
<td>1.220000</td>
<td>8.500000</td>
</tr>
<tr>
<td>$T_4(x)$</td>
<td>1.221400</td>
<td>16.375000</td>
</tr>
<tr>
<td>$T_6(x)$</td>
<td>1.221403</td>
<td>19.412500</td>
</tr>
<tr>
<td>$T_8(x)$</td>
<td>1.221403</td>
<td>20.009152</td>
</tr>
<tr>
<td>$T_{10}(x)$</td>
<td>1.221403</td>
<td>20.079665</td>
</tr>
<tr>
<td>$e^x$</td>
<td>1.221403</td>
<td>20.085537</td>
</tr>
</tbody>
</table>
How large should we take $n$ to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three possible methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.
Approximating Functions by Polynomials

2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.

3. In all cases we can use Taylor’s Inequality which says that if $|f^{(n+1)}(x)| \leq M$, then

$$|R_n(x)| \leq \frac{M}{(n + 1)!} |x - a|^{n+1}$$
Example 1

(a) Approximate the function \( f(x) = \sqrt[3]{x} \) by a Taylor polynomial of degree 2 at \( a = 8 \).

(b) How accurate is this approximation when \( 7 \leq x \leq 9 \)?

Solution:

(a) \[
\begin{align*}
  f(x) &= \sqrt[3]{x} = x^{1/3} \\
  f'(x) &= \frac{1}{3} x^{-2/3} \\
  f''(x) &= -\frac{2}{9} x^{-5/3} \\
  f'''(x) &= \frac{10}{27} x^{-8/3}
\end{align*}
\]

\[
\begin{align*}
  f(8) &= 2 \\
  f'(8) &= \frac{1}{12} \\
  f''(8) &= -\frac{1}{144}
\end{align*}
\]
Example 1 – Solution

Thus the second-degree Taylor polynomial is

\[ T_2(x) = f(8) + \frac{f'(8)}{1!} (x - 8) + \frac{f''(8)}{2!} (x - 8)^2 \]

\[ = 2 + \frac{1}{12} (x - 8) - \frac{1}{288} (x - 8)^2 \]

The desired approximation is

\[ \sqrt[3]{x} \approx T_2(x) \]

\[ = 2 + \frac{1}{12} (x - 8) - \frac{1}{288} (x - 8)^2 \]
Example 1 – Solution

(b) The Taylor series is not alternating when $x < 8$, so we can’t use the Alternating Series Estimation Theorem in this example.

But we can use Taylor’s Inequality with $n = 2$ and $a = 8$:

$$| R_2(x) | \leq \frac{M}{3!} | x - 8 |^3 $$

where $|f'''(x)| \leq M$.

Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{8/3}} \leq \frac{10}{27} \cdot \frac{1}{7^{8/3}} < 0.0021$$
Example 1 – Solution cont’d

Therefore we can take $M = 0.0021$. Also $7 \leq x \leq 9$, so $-1 \leq x - 8 \leq 1$ and $|x - 8| \leq 1$.

Then Taylor’s Inequality gives

$$|R_2(x)| \leq \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} < 0.0004$$

Thus, if $7 \leq x \leq 9$, the approximation in part (a) is accurate to within 0.0004.
Approximating Functions by Polynomials

Let’s use a graphing device to check the calculation in Example 1. Figure 2 shows that the graphs of $y = \frac{3}{\sqrt{x}}$ and $y = T_2(x)$ are very close to each other when $x$ is near 8.
Figure 3 shows the graph of $|R_2(x)|$ computed from the expression

$$| R_2(x) | = \left| \sqrt[3]{x} - T_2(x) \right|$$

We see from the graph that

$$|R_2(x)| < 0.0003$$

when $7 \leq x \leq 9$.

Thus the error estimate from graphical methods is slightly better than the error estimate from Taylor’s Inequality in this case.
Applications to Physics
Taylor polynomials are also used frequently in physics. In order to gain insight into an equation, a physicist often simplifies a function by considering only the first two or three terms in its Taylor series.

In other words, the physicist uses a Taylor polynomial as an approximation to the function. Taylor’s Inequality can then be used to gauge the accuracy of the approximation.

The next example shows one way in which this idea is used in special relativity.
Example 3

In Einstein’s theory of special relativity the mass of an object moving with velocity $v$ is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where $m_0$ is the mass of the object when at rest and $c$ is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0c^2$$
Example 3

(a) Show that when $v$ is very small compared with $c$, this expression for $K$ agrees with classical Newtonian physics: $K = \frac{1}{2} m_0 v^2$.

(b) Use Taylor’s Inequality to estimate the difference in these expressions for $K$ when $|v| \leq 100 \ m/s$.

Solution:

(a) Using the expressions given for $K$ and $m$, we get

$$K = mc^2 - m_0 c^2 = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 = m_0 c^2 \left[ \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} - 1 \right]$$
Example 3 – Solution

With \( x = -v^2/c^2 \), the Maclaurin series for \((1 + x)^{-1/2}\) is most easily computed as a binomial series with \( k = -\frac{1}{2} \). (Notice that \(|x| < 1\) because \(v < c\).) Therefore we have

\[
(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots
\]

\[
= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots
\]

and

\[
K = m_0c^2 \left[ \left( 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \cdots \right) - 1 \right]
\]

\[
= m_0c^2 \left( \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \frac{5}{16} \frac{v^6}{c^6} + \cdots \right)
\]
Example 3 – Solution

If \( v \) is much smaller than \( c \), then all terms after the first are very small when compared with the first term. If we omit them, we get

\[
K \approx m_0 c^2 \left( \frac{1}{2} \frac{v^2}{c^2} \right) = \frac{1}{2} m_0 v^2
\]

(b) If \( x = -\frac{v^2}{c^2} \), \( f(x) = m_0 c^2 [(1 + x)^{-1/2} - 1] \), and \( M \) is a number such that \( |f''(x)| \leq M \), then we can use Taylor’s Inequality to write

\[
| R_1(x) | \leq \frac{M}{2!} x^2
\]
We have \( f''(x) = \frac{3}{4} m_0 c^2 (1 + x)^{-5/2} \) and we are given that \( |v| \leq 100 \text{ m/s} \), so

\[
|f''(x)| = \frac{3m_0 c^2}{4(1 - v^2/c^2)^{5/2}} \leq \frac{3m_0 c^2}{4(1 - 100^2/c^2)^{5/2}} \quad (= M)
\]

Thus, with \( c = 3 \times 10^8 \text{ m/s} \),

\[
|R_1(x)| \leq \frac{1}{2} \cdot \frac{3m_0 c^2}{4(1 - 100^2/c^2)^{5/2}} \cdot \frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0
\]

So when \( |v| \leq 100 \text{ m/s} \), the magnitude of the error in using the Newtonian expression for kinetic energy is at most \((4.2 \times 10^{-10})m_0\).