1.2 Exponents and Radicals
Objectives

- Integer Exponents
- Rules for Working with Exponents
- Scientific Notation
- Radicals
- Rational Exponents
- Rationalizing the Denominator
Integer Exponents
A product of identical numbers is usually written in exponential notation. For example, $5 \cdot 5 \cdot 5$ is written as $5^3$. In general, we have the following definition.

**EXPONENTIAL NOTATION**

If $a$ is any real number and $n$ is a positive integer, then the *nth power* of $a$ is

$$a^n = a \cdot a \cdot \ldots \cdot a$$

$n$ factors

The number $a$ is called the *base*, and $n$ is called the *exponent*. 
Example 1 – Exponential Notation

(a) \( \left( \frac{1}{2} \right)^5 = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) = \frac{1}{32} \)

(b) \((-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3)\)

\[= 81\]

(c) \(-3^4 = -(3 \cdot 3 \cdot 3 \cdot 3)\)

\[= -81\]
ZERO AND NEGATIVE EXPONENTS

If $a \neq 0$ is any real number and $n$ is a positive integer, then

$$a^0 = 1 \quad \text{and} \quad a^{-n} = \frac{1}{a^n}$$
Example 2 – Zero and Negative Exponents

(a) \( \left( \frac{4}{7} \right)^0 = 1 \)

(b) \( x^{-1} = \frac{1}{x^1} = \frac{1}{x} \)

(c) \( (-2)^{-3} = \frac{1}{(-2)^3} = \frac{1}{-8} = -\frac{1}{8} \)
Rules for Working with Exponents
Familiarity with the following rules is essential for our work with exponents and bases. In the table the bases $a$ and $b$ are real numbers, and the exponents $m$ and $n$ are integers.

### LAWS OF EXPONENTS

<table>
<thead>
<tr>
<th>Law</th>
<th>Example</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $a^m a^n = a^{m+n}$</td>
<td>$3^2 \cdot 3^5 = 3^{2+5} = 3^7$</td>
<td>To multiply two powers of the same number, add the exponents.</td>
</tr>
<tr>
<td>2. $\frac{a^m}{a^n} = a^{m-n}$</td>
<td>$\frac{3^5}{3^2} = 3^{5-2} = 3^3$</td>
<td>To divide two powers of the same number, subtract the exponents.</td>
</tr>
<tr>
<td>3. $(a^m)^n = a^{mn}$</td>
<td>$(3^2)^5 = 3^{2 \cdot 5} = 3^{10}$</td>
<td>To raise a power to a new power, multiply the exponents.</td>
</tr>
<tr>
<td>4. $(ab)^n = a^n b^n$</td>
<td>$(3 \cdot 4)^2 = 3^2 \cdot 4^2$</td>
<td>To raise a product to a power, raise each factor to the power.</td>
</tr>
<tr>
<td>5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$</td>
<td>$\left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2}$</td>
<td>To raise a quotient to a power, raise both numerator and denominator to the power.</td>
</tr>
</tbody>
</table>
Example 4 – Simplifying Expressions with Exponents

Simplify:

(a) \((2a^3b^2)(3ab^4)^3\)

(b) \(\left(\frac{x}{y}\right)^3\left(\frac{y^2x}{z}\right)^4\)

Solution:

(a) \((2a^3b^2)(3ab^4)^3 = (2a^3b^2)[3^3a^3(b^4)^3]\)

\[= (2a^3b^2)(27a^3b^{12})\]

\[= (2)(27)a^3a^3b^2b^{12}\]

Law 4: \((ab)^n = a^nb^n\)

Law 3: \((a^m)^n = a^{mn}\)

Group factors with the same base
Example 4 – **Solution**

(b) 

\[
\left( \frac{x}{y} \right)^3 \left( \frac{y^2x}{z} \right)^4 = \frac{x^3}{y^3} \frac{(y^2)^4x^4}{z^4}
\]

\[
= \frac{x^3}{y^3} \frac{y^8x^4}{z^4}
\]

\[
= (x^3x^4) \left( \frac{y^8}{y^3} \right) \frac{1}{z^4}
\]

\[
= \frac{x^7y^5}{z^4}
\]

= 54a^6b^{14}

Law 1: \(a^mb^n = a^{m+n}\)

Laws 5 and 4

Law 3

Group factors with the same base

Laws 1 and 2
We now give two additional laws that are useful in simplifying expressions with negative exponents.

<table>
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<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>6. ( \left( \frac{a}{b} \right)^{-n} = \left( \frac{b}{a} \right)^n )</td>
<td>( \left( \frac{3}{4} \right)^{-2} = \left( \frac{4}{3} \right)^2 )</td>
<td>To raise a fraction to a negative power, invert the fraction and change the sign of the exponent.</td>
</tr>
<tr>
<td>7. ( \frac{a^{-n}}{b^{-m}} = \frac{b^m}{a^n} )</td>
<td>( \frac{3^{-2}}{4^{-5}} = \frac{4^5}{3^2} )</td>
<td>To move a number raised to a power from numerator to denominator or from denominator to numerator, change the sign of the exponent.</td>
</tr>
</tbody>
</table>
Example 5 – *Simplifying Expressions with Negative Exponents*

Eliminate negative exponents and simplify each expression.

(a) \[ \frac{6st^{-4}}{2s^{-2}t^2} \]

(b) \[ \left( \frac{y}{3z^3} \right)^{-2} \]
Example 5 – Solution

(a) We use Law 7, which allows us to move a number raised to a power from the numerator to the denominator (or vice versa) by changing the sign of the exponent.

\[
\frac{6st^{-4}}{2s^{-2}t^{2}} = \frac{6ss^{2}}{2t^{2}t^{4}}
\]

\[
= \frac{3s^{3}}{t^{6}}
\]

Law 7

\[t^{-4} \text{ moves to denominator and becomes } t^{4}\]

\[s^{-2} \text{ moves to numerator and becomes } s^{2}\]
Example 5 – Solution

(b) We use Law 6, which allows us to change the sign of the exponent of a fraction by inverting the fraction.

\[
\left( \frac{y}{3z^3} \right)^{-2} = \left( \frac{3z^3}{y} \right)^2
\]

\[
= \frac{9z^6}{y^2}
\]

Law 6

Laws 5 and 4
Scientific Notation
Scientific Notation

**SCIENTIFIC NOTATION**

A positive number $x$ is said to be written in scientific notation if it is expressed as follows:

$$ x = a \times 10^n $$

where $1 \leq a < 10$ and $n$ is an integer

For instance, when we state that the distance to the star Proxima Centauri is $4 \times 10^{13}$ km, the positive exponent 13 indicates that the decimal point should be moved 13 places to the right:

$$ 4 \times 10^{13} = 40,000,000,000,000 $$

Move decimal point 13 places to the right
Scientific Notation

When we state that the mass of a hydrogen atom is $1.66 \times 10^{-24}$ g, the exponent $-24$ indicates that the decimal point should be moved 24 places to the left:

$$1.66 \times 10^{-24} = 0.00000000000000000000000166$$

Move decimal point 24 places to the left
Example 6 – Changing from Decimal to Scientific Notation

Write each number in scientific notation.

(a) 56,920
(b) 0.000093

Solution:

(a) $56,920 = 5.692 \times 10^4$

(b) $0.000093 = 9.3 \times 10^{-5}$
Radicals
Radicals

We know what $2^n$ means whenever $n$ is an integer. To give meaning to a power, such as $2^{4/5}$, whose exponent is a rational number, we need to discuss radicals.

The symbol $\sqrt{}$ means “the positive square root of.” Thus

$$\sqrt{a} = b \text{ means } b^2 = a \text{ and } b \geq 0$$

Since $a = b^2 \geq 0$, the symbol $\sqrt{a}$ makes sense only when $a \geq 0$. For instance,

$$\sqrt{9} = 3 \text{ because } 3^2 = 9 \text{ and } 3 \geq 0$$
Radicals

Square roots are special cases of $n$th roots. The $n$th root of $x$ is the number that, when raised to the $n$th power, gives $x$.

**DEFINITION OF $n$th ROOT**

If $n$ is any positive integer, then the **principal $n$th root** of $a$ is defined as follows:

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a$$

If $n$ is even, we must have $a \geq 0$ and $b \geq 0$. 
## PROPERTIES OF nth ROOTS

<table>
<thead>
<tr>
<th>Property</th>
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<tbody>
<tr>
<td>1. $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$</td>
<td>$\sqrt[3]{-8 \cdot 27} = \sqrt[3]{-8} \cdot \sqrt[3]{27} = (-2)(3) = -6$</td>
</tr>
<tr>
<td>2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$</td>
<td>$\sqrt[4]{\frac{16}{81}} = \frac{\sqrt[4]{16}}{\sqrt[4]{81}} = \frac{2}{3}$</td>
</tr>
<tr>
<td>3. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$</td>
<td>$\sqrt[3]{\sqrt[6]{729}} = \sqrt[6]{729} = 3$</td>
</tr>
<tr>
<td>4. $\sqrt[n]{a^n} = a$ if $n$ is odd</td>
<td>$\sqrt[3]{(-5)^3} = -5$, $\sqrt[5]{2^5} = 2$</td>
</tr>
<tr>
<td>5. $\sqrt[n]{</td>
<td>a</td>
</tr>
</tbody>
</table>
Example 8 – Simplifying Expressions Involving nth Roots

(a) \[ \sqrt[3]{x^4} = \sqrt[3]{x^3 x} \]

\[ = \sqrt[3]{x^3} \sqrt[3]{x} \]

\[ = x \sqrt[3]{x} \]

Factor out the largest cube

Property 1: \[ \sqrt[3]{ab} = \sqrt[3]{a} \sqrt[3]{b} \]

Property 4: \[ \sqrt[3]{a^3} = a \]

(b) \[ \sqrt[4]{81x^8y^4} = \sqrt[4]{81} \sqrt[4]{x^8} \sqrt[4]{y^4} \]

\[ = 3 \sqrt[4]{(x^2)^4} \mid y \mid \]

\[ = 3x^2 \mid y \mid \]

Property 1: \[ \sqrt[4]{abc} = \sqrt[4]{a} \sqrt[4]{b} \sqrt[4]{c} \]

Property 5, \[ \sqrt[4]{a^4} = \mid a \mid \]

Property 5: \[ \sqrt[4]{a^4} = \mid a \mid, \mid x^2 \mid = x^2 \]
Example 9 – **Combining Radicals**

(a) \( \sqrt{32} + \sqrt{200} = \sqrt{16 \cdot 2} + \sqrt{100 \cdot 2} \)  

\[ = \sqrt{16} \sqrt{2} + \sqrt{100} \sqrt{2} \]  

\[ = 4 \sqrt{2} + 10 \sqrt{2} = 14 \sqrt{2} \]  

**Factor out the largest squares**  

Property 1: \( \sqrt{ab} = \sqrt{a} \sqrt{b} \)

(b) If \( b > 0 \), then

\[ \sqrt{25b} - \sqrt{b^3} = \sqrt{25} \sqrt{b} - \sqrt{b^2} \sqrt{b} \]  

\[ = 5 \sqrt{b} - b \sqrt{b} \]  

\[ = (5 - b) \sqrt{b} \]  

**Property 1:** \( \sqrt{ab} = \sqrt{a} \sqrt{b} \)  

**Property 5, \( b > 0 \)**  

**Distributive property**
Rational Exponents
To define what is meant by a *rational exponent* or, equivalently, a *fractional exponent* such as $a^{1/3}$, we need to use radicals. To give meaning to the symbol $a^{1/n}$ in a way that is consistent with the Laws of Exponents, we would have to have

$$(a^{1/n})^n = a^{(1/n)n} = a^1 = a$$

So by the definition of $n$th root,

$$a^{1/n} = \sqrt[n]{a}$$
In general, we define rational exponents as follows.

**DEFINITION OF RATIONAL EXPONENTS**

For any rational exponent $m/n$ in lowest terms, where $m$ and $n$ are integers and $n > 0$, we define

$$a^{m/n} = (\sqrt[n]{a})^m$$  or equivalently  $$a^{m/n} = \sqrt[n]{a^m}$$

If $n$ is even, then we require that $a \geq 0$. 
Example 11 – *Using the Laws of Exponents with Rational Exponents*

(a) $a^{1/3}a^{7/3} = a^{8/3}$

(b) $\frac{a^{2/5}a^{7/5}}{a^{3/5}} = a^{2/5 + 7/5 - 3/5} = a^{6/5}$

(c) $(2a^3b^4)^{3/2} = 2^{3/2}(a^3)^{3/2}(b^4)^{3/2}$

\[ = (\sqrt{2})^3a^{3(3/2)}b^{4(3/2)} \]

\[ = 2\sqrt{2}a^{9/2}b^6 \]

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Law 1: $a^m b^n = a^{m+n}$

Law 1, Law 2: $\frac{a^m}{a^n} = a^{m-n}$

Law 4: $(abc)^n = a^n b^n c^n$

Law 3: $(a^m)^n = a^{mn}$
Example 11 – Using the Laws of Exponents with Rational Exponents

\[(d) \left( \frac{2x^{3/4}}{y^{1/3}} \right)^3 \left( \frac{y^4}{x^{-1/2}} \right) = \frac{2^3(x^{3/4})^3}{(y^{1/3})^3} \cdot (y^4x^{1/2}) \]

\[
= \frac{8x^{9/4}}{y^{1/3}} \cdot y^4x^{1/2} \\
= 8x^{11/4}y^{3/2} \]

Laws 5, 4, and 7
Law 3
Law 1, Law 2
Rationalizing the Denominator
Rationalizing the Denominator

It is often useful to eliminate the radical in a denominator by multiplying both numerator and denominator by an appropriate expression. This procedure is called **rationalizing the denominator**.

If the denominator is of the form $\sqrt{a}$, we multiply numerator and denominator by $\sqrt{a}$. In doing this we multiply the given quantity by 1, so we do not change its value. For instance,

$$
\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot 1 = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{a}
$$
Rationalizing the Denominator

Note that the denominator in the last fraction contains no radical. In general, if the denominator is of the form $\sqrt[n]{a^m}$ with $m < n$, then multiplying the numerator and denominator $\sqrt[n]{a^{n-m}}$ by will rationalize the denominator, because (for $a > 0$)

$$\sqrt[n]{a^m} \cdot \sqrt[n]{a^{n-m}} = \sqrt[n]{a^{m+n-m}} = \sqrt[n]{a^n} = a$$
Example 13 – **Rationalizing Denominators**

(a) \[
\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}
\]

(b) \[
\frac{1}{\sqrt[3]{x^2}} = \frac{1}{\sqrt[3]{x^2}} \cdot \frac{\sqrt[3]{x}}{\sqrt[3]{x}} = \frac{\sqrt[3]{x}}{x}
\]

(c) \[
\sqrt[7]{\frac{1}{a^2}} = \frac{1}{\sqrt[7]{a^2}} = \frac{1}{\sqrt[7]{a^2}} \cdot \frac{\sqrt[7]{a^5}}{\sqrt[7]{a^5}} = \frac{\sqrt[7]{a^5}}{\sqrt[7]{a^7}} = \frac{\sqrt[7]{a^5}}{a}
\]