3.2 Polynomial Functions and Their Graphs
Objectives

► Graphing Basic Polynomial Functions
► End Behavior and the Leading Term
► Using Zeros to Graph Polynomials
► Shape of the Graph Near a Zero
► Local Maxima and Minima of Polynomials
In this section we study polynomial functions of any degree. But before we work with polynomial functions, we must agree on some terminology.

**POLYNOMIAL FUNCTIONS**

A polynomial function of degree $n$ is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n$ is a nonnegative integer and $a_n \neq 0$.

The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.

The number $a_0$ is the **constant coefficient** or **constant term**.

The number $a_n$, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.
We often refer to polynomial functions simply as *polynomials*. The following polynomial has degree 5, leading coefficient 3, and constant term –6.

\[ 3x^5 + 6x^4 - 2x^3 + x^2 + 7x - 6 \]

- **Leading term**: $3x^5$
- **Leading coefficient**: 3
- **Degree**: 5
- **Constant term**: –6
- **Coefficients**: 3, 6, –2, 1, 7, and –6
Here are some more examples of polynomials.

\[ P(x) = 3 \quad \text{Degree 0} \]
\[ Q(x) = 4x - 7 \quad \text{Degree 1} \]
\[ R(x) = x^2 + x \quad \text{Degree 2} \]
\[ S(x) = 2x^3 - 6x^2 - 10 \quad \text{Degree 3} \]

If a polynomial consists of just a single term, then it is called a \textbf{monomial}. For example, \( P(x) = x^3 \) and \( Q(x) = -6x^5 \) are monomials.
Graphing Basic Polynomial Functions
Graphing Basic Polynomial Functions

The graphs of polynomials of degree 0 or 1 are lines, and the graphs of polynomials of degree 2 are parabolas. The greater the degree of a polynomial, the more complicated its graph can be.

However, the graph of a polynomial function is **continuous**. This means that the graph has no breaks or holes (see Figure 1).
Moreover, the graph of a polynomial function is a smooth curve; that is, it has no corners or sharp points (cusps) as shown in Figure 1.

The simplest polynomial functions are the monomials $P(x) = x^n$, whose graphs are shown in Figure 2.
Graphing Basic Polynomial Functions

As the figure suggests, the graph of $P(x) = x^n$ has the same general shape as the graph of $y = x^2$ when $n$ is even and the same general shape as the graph of $y = x^2$ when $n$ is odd.

(c) $y = x^3$

(d) $y = x^4$

(e) $y = x^5$

Graphs of monomials

Figure 2
However, as the degree $n$ becomes larger, the graphs become flatter around the origin and steeper elsewhere.
Example 1 – *Transformations of Monomials*

Sketch the graphs of the following functions.

(a) $P(x) = -x^3$  
(b) $Q(x) = (x - 2)^4$  
(c) $R(x) = -2x^5 + 4$

Solution:

We use the graphs in Figure 2.

(a) The graph of $P(x) = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis, as shown in Figure 3(a).
(b) The graph of \( Q(x) = (x - 2)^4 \) is the graph of \( y = x^4 \) shifted to the right 2 units, as shown in Figure 3(b).
(c) We begin with the graph of \( y = x^5 \). The graph of 
\( y = -2x^5 \) is obtained by stretching the graph vertically
and reflecting it in the x-axis (see the dashed blue graph
in Figure 3(c)).

Finally, the graph of \( R(x) = -2x^5 + 4 \) is obtained by
shifting upward 4 units (see the red graph in
Figure 3(c)).
End Behavior and the Leading Term
The **end behavior** of a polynomial is a description of what happens as \( x \) becomes large in the positive or negative direction. To describe end behavior, we use the following notation:

\[
x \to \infty \text{ means “} x \text{ becomes large in the positive direction”}
\]

\[
x \to -\infty \text{ means “} x \text{ becomes large in the negative direction”}
\]

For any polynomial *the end behavior is determined by the term that contains the highest power of* \( x \), *because when* \( x \) *is large, the other terms are relatively insignificant in size.*
End Behavior and the Leading Term

The following box shows the four possible types of end behavior, based on the highest power and the sign of its coefficient.

**END BEHAVIOR OF POLYNOMIALS**

The end behavior of the polynomial \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \) is determined by the degree \( n \) and the sign of the leading coefficient \( a_n \), as indicated in the following graphs.

- **\( P \) has odd degree**
  - Leading coefficient positive
  - \( y \to \infty \) as \( x \to \infty \)
  - \( y \to -\infty \) as \( x \to -\infty \)

- **\( P \) has even degree**
  - Leading coefficient negative
  - \( y \to \infty \) as \( x \to -\infty \)
  - \( y \to -\infty \) as \( x \to \infty \)
Determine the end behavior of the polynomial

\[ P(x) = -2x^4 + 5x^3 + 4x - 7 \]

Solution:
The polynomial \( P \) has degree 4 and leading coefficient \(-2\). Thus, \( P \) has even degree and negative leading coefficient, so it has the following end behavior:

\[ y \to -\infty \text{ as } x \to \infty \text{ and } y \to -\infty \text{ as } x \to -\infty \]
The graph in Figure 4 illustrates the end behavior of $P$. 

\[ P(x) = -2x^4 + 5x^3 + 4x - 7 \]
Using Zeros to Graph Polynomials
Using Zeros to Graph Polynomials

If \( P \) is a polynomial function, then \( c \) is called a **zero (a root)** of \( P \) if \( P(c) = 0 \). In other words, the zeros of \( P \) are the solutions of the polynomial equation \( P(x) = 0 \). Note that if \( P(c) = 0 \), then the graph of \( P \) has an \( x \)-intercept at \( x = c \), so the \( x \)-intercepts of the graph are the zeros of the function.

**REAL ZEROS OF POLYNOMIALS**

If \( P \) is a polynomial and \( c \) is a real number, then the following are equivalent:

1. \( c \) is a zero of \( P \).
2. \( x = c \) is a solution of the equation \( P(x) = 0 \).
3. \( x - c \) is a factor of \( P(x) \).
4. \( c \) is an \( x \)-intercept of the graph of \( P \).
To find the zeros of a polynomial \( P \), we factor and then use the Zero-Product Property.

For example, to find the zeros of \( P(x) = x^2 + x - 6 \), we factor \( P \) to get

\[
P(x) = (x - 2)(x + 3)
\]

From this factored form we easily see that

1. 2 is a zero of \( P \).
2. \( x = 2 \) is a solution of the equation \( x^2 + x - 6 = 0 \).
3. \( x - 2 \) is a factor of \( x^2 + x - 6 \).
4. 2 is an \( x \)-intercept of the graph of \( P \).

The same facts are true for the other zero, \( -3 \).
The following theorem has many important consequences. Here we use it to help us graph polynomial functions.

**INTERMEDIATE VALUE THEOREM FOR POLYNOMIALS**

If $P$ is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value $c$ between $a$ and $b$ for which $P(c) = 0$. 
Figure 6 shows why it is intuitively plausible.
Using Zeros to Graph Polynomials

GUIDELINES FOR GRAPHING POLYNOMIAL FUNCTIONS

1. Zeros. Factor the polynomial to find all its real zeros; these are the $x$-intercepts of the graph.

2. Test Points. Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the $x$-axis on the intervals determined by the zeros. Include the $y$-intercept in the table.

3. End Behavior. Determine the end behavior of the polynomial.

4. Graph. Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.
Example 5 – Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - x^2 - 3x$.

(a) Find the zeros of $P$.

(b) Sketch a graph of $P$.

Solution:

(a) To find the zeros, we factor completely.

$$P(x) = x^3 - x^2 - 3x$$
$$= x(x^2 - 2x - 3)$$
$$= x(x - 3)(x + 1)$$

Thus, the zeros are $x = 0$, $x = 3$, and $x = -1$. 
(b) The $x$-intercepts are $x = 0$, $x = 3$, and $x = -1$. The $y$-intercept is $P(0) = 0$. We make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros.

Since $P$ is of odd degree and its leading coefficient is positive, it has the following end behavior:

\[ y \to \infty \text{ as } x \to \infty \text{ and } y \to -\infty \text{ as } x \to -\infty \]
We plot the points in the table and connect them by a smooth curve to complete the graph, as shown in Figure 8.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-10</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$\frac{7}{8}$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-6</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
</tbody>
</table>

$P(x) = x^3 - 2x^2 - 3x$

Figure 8
Shape of the Graph Near a Zero
Shape of the Graph Near a Zero

In general, if $c$ is a zero of $P$, and the corresponding factor $x - c$ occurs exactly $m$ times in the factorization of $P$, then we say that $c$ is a **zero of multiplicity** $m$.

By considering test points on either side of the $x$-intercept $c$, we conclude that the graph crosses the $x$-axis at $c$ if the multiplicity $m$ is odd and does not cross the $x$-axis if $m$ is even.

Moreover, it can be shown by using calculus that near $x = c$ the graph has the same general shape as the graph of $y = A(x - c)^m$. 
SHAPE OF THE GRAPH NEAR A ZERO OF MULTIPLICITY $m$

If $c$ is a zero of $P$ of multiplicity $m$, then the shape of the graph of $P$ near $c$ is as follows.

**Multiplicity of $c$**  

<table>
<thead>
<tr>
<th>Multiplicity of $c$</th>
<th>Shape of the graph of $P$ near the $x$-intercept $c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m$ odd, $m &gt; 1$</td>
<td>![Graph for odd multiplicity] (OR) ![Graph for odd multiplicity]</td>
</tr>
<tr>
<td>$m$ even, $m &gt; 1$</td>
<td>![Graph for even multiplicity] (OR) ![Graph for even multiplicity]</td>
</tr>
</tbody>
</table>
Graph the polynomial $P(x) = x^4(x - 2)^3(x + 1)^2$.

Solution:
The zeros of $P$ are $-1$, $0$, and $2$ with multiplicities $2$, $4$, and $3$, respectively.

The zero $2$ has *odd* multiplicity, so the graph crosses the $x$-axis at the $x$-intercept $2$. 
Example 8 – Solution cont’d

But the zeros 0 and −1 have even multiplicity, so the graph does not cross the x-axis at the x-intercepts 0 and −1.

Since $P$ is a polynomial of degree 9 and has positive leading coefficient, it has the following end behavior:

\[ y \to \infty \text{ as } x \to \infty \quad \text{and} \quad y \to -\infty \text{ as } x \to -\infty \]
Example 8 – Solution

With this information and a table of values we sketch the graph in Figure 11. Note: If \( x = -0.5 \), then \( P(x) \approx -0.24414 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( P(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.3</td>
<td>-9.2</td>
</tr>
<tr>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>-0.5</td>
<td>-3.9</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>2.3</td>
<td>8.2</td>
</tr>
</tbody>
</table>

\[ P(x) = x^4(x - 2)^3(x + 1)^2 \]
Local Maxima and Minima of Polynomials
Local Maxima and Minima of Polynomials

If the point \((a, f(a))\) is the highest point on the graph of \(f\) within some viewing rectangle, then \(f(a)\) is a local maximum value of \(f\), and if \((b, f(b))\) is the lowest point on the graph of \(f\) within a viewing rectangle, then \(f(b)\) is a local minimum value (see Figure 12).

![Figure 12](image-url)
Local Maxima and Minima of Polynomials

We say that such a point \((a, f(a))\) is a **local maximum point** on the graph and that \((b, f(b))\) is a **local minimum point**.

The local maximum and minimum points on the graph of a function are called its **local extrema**.

For a polynomial function the number of local extrema must be less than the degree, as the following principle indicates.

**LOCAL EXTREMA OF POLYNOMIALS**

If \(P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0\) is a polynomial of degree \(n\), then the graph of \(P\) has at most \(n - 1\) local extrema.
A polynomial of degree $n$ may in fact have less than $n – 1$ local extrema. For example, $P(x) = x^5$ has no local extrema, even though it is of degree 5.

The preceding principle tells us only that a polynomial of degree $n$ can have no more than $n – 1$ local extrema.
Example 9 – The Number of Local Extrema

Determine how many local extrema at most each of the following polynomials may have:

(a) \( P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48; \)

(b) \( P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15; \)

(c) \( P_3(x) = 7x^4 + 3x^2 - 10x. \)
Example 9 – Solution

The graphs are shown in Figure 13.

\[ P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48 \]

\[ P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15 \]

(a) \( P_1 \) has two local minimum points and one local maximum point, for a total of three local extrema.

(b) \( P_2 \) has two local minimum points and two local maximum points, for a total of four local extrema.
(c) $P_3$ has just one local extremum, a local minimum.

$$P_3(x) = 7x^4 + 3x^2 - 10x$$

Figure 13