NONCOMMUTATIVE CANTOR-BENDIXSON DERIVATIVES AND SCATTERED C^*-ALGEBRAS

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Abstract. We analyze the sequence obtained by consecutive applications of the Cantor-Bendixson derivative for a noncommutative scattered C^*-algebra $\mathcal{A}$, using the ideal $\mathcal{I}^{At}(\mathcal{A})$ generated by the minimal projections of $\mathcal{A}$. With its help, we present some fundamental results concerning scattered C^*-algebras, in a manner parallel to the commutative case of scattered compact or locally compact Hausdorff spaces and superatomic Boolean algebras. It also allows us to formulate problems which have motivated the "cardinal sequences" programme in the classical topology, in the noncommutative context. This leads to some new constructions of noncommutative scattered C^*-algebras and new open problems. In particular, we construct a type I C^*-algebra which is the inductive limit of stable AF-ideals $\mathcal{A}_\alpha$, along an uncountable limit ordinal λ, such that $\mathcal{A}_{\alpha+1}/\mathcal{A}_\alpha$ is $*$-isomorphic to the algebra of all compact operators on a separable Hilbert space and $\mathcal{A}_{\alpha+1}$ is σ-unital and stable for each $\alpha < \lambda$, but $\mathcal{A}$ is not stable and where all ideals of $\mathcal{A}$ are of the form $\mathcal{A}_\alpha$. In particular, $\mathcal{A}$ is a nonseparable C^*-algebra with no stable ideal which is maximal among the stable ideals. This answers a question of M. Rørdam in the nonseparable case. Two more complex constructions based on the language developed in this paper are presented in separate papers [23, 24].

1. Introduction

1.1. **Background and goals.** The first modern paper on superatomic Boolean algebras (the algebras where every subalgebra has an atom) was written by Mostowski and Tarski in 1939 ([47]) while a serious research on dispersed (or scattered) compact Hausdorff spaces (the spaces where every subspace has a relative isolated point) can be traced back to Cantor with a substantial abstract result already in 1920 in the paper of Mazurkiewicz and Sierpiński ([46]). Since the papers of Day ([17]), Rudin ([59]), Pełczyński and Semadeni ([53]), it was realized that the topics of superatomic Boolean algebras and of scattered compact spaces are the same topics in two different languages (formally, by the Stone duality, a compact K is scattered if and only if the Boolean algebra $Clop(K)$, of all clopen subsets of K forms a basis for K and is superatomic) and that they produce an interesting class of Banach spaces of the form $C(K)$ with many peculiar features. In fact, already in 1930 J. Schreier used the compact scattered space $K = [0, \omega^\omega]$ to provide an ingenious negative answer to a problem of Banach, if nonisomorphic Banach spaces may have isomorphic dual spaces ([60]). Later two important classes of Banach spaces emerged by generalizing these $C(K)$s, namely, Asplund spaces (those Banach spaces where separable subspaces have separable duals) and Asplund generated spaces (those which admit

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a dense range linear bounded operator from an Asplund space). These classes assumed fundamental roles in the theory of vector valued measures and renorming theory of Banach spaces ([19, 18]), due to the Radon-Nikodym property in the dual and the Frechet differentiability properties.

At the same time the research related to the dual pair of superatomic Boolean algebras and scattered compact spaces underwent dramatic set-theoretic developments, especially related to the use of additional set-theoretic assumptions. For example, Ostaszewski constructed his hereditary separable non hereditarily Lindelöf space assuming Jensen's $\diamondsuit$ principle ([51]), Tall obtaind a separable nonmetrizable normal Moore space assuming $\mathfrak{p} > \omega_1$ ([63]). The interaction with Banach space theory very much included these developments. For example, under the continuum hypothesis Kunen's scattered compact space K, in the form of $C(K)$, provided an example of a nonseparable Banach space with no uncountable biorthogonal system ([50]). Pol used $C(K)$ for K equal to the ladder system space to answer a question of Corson, whether all weakly Lindelöf Banach spaces are weakly compactly generated ([54]). Many other equally fundamental constructions followed ([61], [26], [10]).

It was Tomiyama in 1963 who first addressed the question which now would be expressed as "what are the C^*-algebras which are Asplund Banach spaces?" ([62]). This was done just in the separable case, and later developed in the general case by Jensen ([31, 32, 33]) and Huraya ([28]). Further understanding was obtainen by Wojtaszczyk in [66], Chu in [13], where the Radon-Nikodym property of the duals of scattered C^*-algebras was proved. In [15], where the weak Banach-Saks property was investigated in the context of scattered C^*-algebras and in [14] where crossed products of scattered C^*-algebras were investigated. Morover Lin proved that scattered C^*-algebras are approximately finite in [42]. Recent papers [40], [39] of Kusuda look at scattered C^*-algebras in the context of AF-algebras and crossed products.

As often happens, the noncommutative theory was developed based on the noncommutative analogues of the phenomena taking place in the dual or the bidual rather than the algebra itself. This is because of the lack of a well-behaved version of the K for $C(K)$, in the general case. After all, K naturally can be considered as the subspace of the the dual Banach space of $C(K)$, the space of Radon measures on K. The purpose of this paper is to present an alternative approach which focuses on the phenomena taking place in the noncommutative analogues of Boolean algebra of clopen sets of K, in other words the projections of $C(K)$. Through the Stone duality, they are quite topological, referring to the level of K. We aim at three goals: a) to present the usefulness of a natural notion of Cantor-Bendixson derivative of a C^*-algebra by obtaining elementary and parallel (to the commutative case) new proofs of many results concerning scattered C^*-algebras; b) to motivate a collection of problems and constructions of C^*-algebras parallel to the commutative well established topic of cardinal sequences of superatomic Boolean algebras or scattered (locally) compact Hausdorff spaces (more constructions can be found in our two subsequent papers [23] and [24]); c) to illustrate the accessibility of the noncommutative realm to the reasonings of set-theoretic topological nature.

1.2. **Cantor-Bendixson derivatives.** In Section 2 we review fundamental facts and some proofs concerning the commutative case, i.e., the superatomic Boolean algebras and scattered compact spaces, to which we will be appealing for intuitions in the following sections.

Section 3 is devoted to developing elementary apparatus allowing us to work with scattered C^*-algebras in a manner parallel to the commutative case of superatomic Boolean algebras. The first surprising difference between the literatures concerning the commutative and the noncommutative case is the way one decomposes the commutative and noncommutative scattered objects. In the Boolean algebras and compact spaces the main operations leading to the decomposition is the Cantor-Bendixson derivative. In general, these kind of decompositions are usually called the "composition series" for a C^*-algebra (see IV.1.1.10 of [7]). On one hand, since scattered C^*-algebras are type I (or GCR) (see [31]), they are subject to a unique composition series of their ideals such that each consecutive quotient is the largest type I_0 (or CCR) subalgebra (see e.g, IV.1.1.12 of [7] and Theorem 1.5.5 of [2]). On the other hand, a generally "finer" composition series for scattered C^*-algebras is presented in Theorem 2 of [32], where each consecutive quotient is elementary (see also Theorem 1.4 (3)). However, these decompositions are in general quite unrelated to the Cantor-Bendixson sequence (Theorem 1.4 (2)). Recall that in the case of the Banach space theory manifestation of scattered objects, a version of the Cantor-Bendixson derivative (in the form of the Szlenk index) turned out to be a fundamental combinatorial tool as well (see e.g. [41]). So, we believe that one should investigate the C^*-algebra version of the Cantor-Bendixson decomposition. In our approach the role of Boolean atoms (corresponding to isolated points in topology) in C^*-algebras is played by the following well-known notion of a minimal projection:

Definition 1.1. *A projection p in a C^*-algebra $\mathcal{A}$ is called minimal if $p\mathcal{A}p = \mathbb{C}p$. The set of minimal projections of $\mathcal{A}$ will be denoted by $At(\mathcal{A})$. The $*$-subalgebra of $\mathcal{A}$ generated by the minimal projections of $\mathcal{A}$ will be denoted $\mathcal{I}^{At}(\mathcal{A})$.*

The origin of the notation $\mathcal{I}^{At}(\mathcal{A})$ is the Boolean algebra notation for the notion of an atom (Definition 2.1). For example, we have $\chi_{\{x\}}C(K)\chi_{\{x\}} = \mathbb{C}\chi_{\{x\}}$, for any isolated point of a compact space K, and in fact these are the only minimal projections in $C(K)$. The right candidate for the Cantor-Bendixson derivative turns out to be the mapping from a C^*-algebra $\mathcal{A}$ to its quotient $\mathcal{A}/\mathcal{I}^{At}(\mathcal{A})$ by the ideal $\mathcal{I}^{At}(\mathcal{A})$. The main observations from Section 3 concerning $\mathcal{I}^{At}(\mathcal{A})$ can be summarized as follows:

Theorem 1.2. *Suppose that $\mathcal{A}$ is a C^*-algebra.*

1. *$\mathcal{I}^{At}(\mathcal{A})$ is an ideal of $\mathcal{A}$,*
2. *$\mathcal{I}^{At}(\mathcal{A})$ is isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ of compact operators on a Hilbert space,*
3. *$\mathcal{I}^{At}(\mathcal{A})$ contains all ideals of $\mathcal{A}$ which are isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$,*
4. *if an ideal $\mathcal{I} \subseteq A$ is essential and isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$, then $\mathcal{I} = \mathcal{I}^{At}(\mathcal{A})$.*

Proof. See Propositions 3.14, 3.15, 3.20. □

This corresponds to the commutative case, when A is a Boolean ring, $\mathcal{K}(\mathcal{H})$ for $\mathcal{H} = \ell_2(\kappa)$ for some cardinal κ, is replaced by the Boolean ring $Fin(\kappa)$ of finite subsets of κ and essential ideals are replaced by dense Boolean ideals: the Boolean ring generated by atoms is an ideal, it is isomorphic to the ring $Fin(\kappa)$ for some κ, it contains all ideals of A isomorphic to $Fin(\lambda)$ for any λ, and if an ideal is dense and isomorphic to $Fin(\kappa)$, it is the ideal generated by all atoms.

It should be noted that one can consider the usual Cantor-Bendixson derivatives (i.e., removing all isolated points) in the spectrum space $\hat{\mathcal{A}}$ of a C^*-algebra $\mathcal{A}$. Moreover as proved by Jensen [32] and exploited by Lin in [42], this derivative makes sense for scattered C^*-algebras and, for example, the Cantor-Bendixson height of the spectrum is the same as the Cantor-Bendixson height of the algebra in our sense. However, the spectrum of $\mathcal{A}$ in nontrivial noncommutative cases is not a Hausdorff space, e.g., often (quite often in the scattered case) given any two points one is in the closure of the other (cf. Proposition 6.3). Moreover the spectrum does not determine the algebra, unlike the commutative case via the Stone duality, for example $\mathcal{K}(\ell_2) \oplus \mathcal{K}(\ell_2)$ and $C(\{1,2\})$ have the same spectra - two point Hausdorff space. So in a sense, our approach is to transport these derivatives to the algebra level, where it corresponds to intuitive Boolean notions and more readily becomes a tool for investigating the C^*-algebra. The techniques of section 3 are standard and similar to the development of the basic theory of CCR and GCR (or type I) algebras (see e.g. [2]).

1.3. **Scattered C^*-algebras.** We may define scattered C^*-algebras in a manner parallel to the superatomic Boolean algebras:

Definition 1.3. *A C^*-algebra $\mathcal{A}$ is called scattered if for every nonzero C^*-subalgebra $\mathcal{B} \subseteq \mathcal{A}$, the ideal $\mathcal{I}^{At}(\mathcal{B})$ is nonzero.*

It should be added that already in the paper [62] of Tomiyama the role of minimal projections in scattered C^*-algebras is exploited. In section 4 we prove the following six equivalent conditions including, (3) - (6) known in the C^*-algebras literature to be equivalent to the traditional definition.

Theorem 1.4 ([31, 32, 66])**.** *Suppose that $\mathcal{A}$ is a C^*-algebra. The following conditions are equivalent:*

(1) Every non-zero $*$-homomorphic image of $\mathcal{A}$ has a minimal projection.
(2) There is an ordinal $ht(\mathcal{A})$ and a continuous increasing sequence of closed ideals $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$ such that $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_{ht(\mathcal{A})} = \mathcal{A}$ and

$$\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha) = \{[a]_{\mathcal{I}_\alpha} : a \in \mathcal{I}_{\alpha+1}\},$$

for every $\alpha < ht(\mathcal{A})$.
(3) There is an ordinal $m(\mathcal{A})$ and a continuous increasing sequence of ideals $(\mathcal{J}_\alpha)_{\alpha \leq m(A)}$ such that $\mathcal{J}_0 = \{0\}$, $\mathcal{J}_{m(\mathcal{A})} = \mathcal{A}$ and $\mathcal{J}_{\alpha+1}/\mathcal{J}_\alpha$ is an elementary C^*-algebra (i.e., *-isomorphic to the algebra of all compact operators on a Hilbert space) for every $\alpha < m(\mathcal{A})$.
(4) Every subalgebra of $\mathcal{A}$ has a minimal projection.
(5) $\mathcal{A}$ does not contain a copy of the C^*-algebra $C([0,1])$.
(6) The spectrum of every self-adjoint element is countable.

The sequence from (2) will be called the Cantor-Bendixson sequence. Note that unlike the sequence from (3), it is unique. The ordinal $ht(A)$ is called the Cantor Bendixson height of $\mathcal{A}$.

These conditions have very precise analogs in the commutative setting (Theorem 2.7), which become additionally combinatorial due to the fact that all scattered compact spaces are totally disconnected, i.e., are precisely represented by the Boolean algebras of their clopen subsets (cf. Theorem 2.2). One can also note that scattered C^*-algebras possess the noncommutative version of the zero-dimensionality, i.e., they are of real rank zero (Proposition 4.5).

In section 4, after the proof of Theorem 1.4 we present several proofs of known properties of scattered C^*-algebras using our equivalences, for example, Kusuda's result that being scattered is determined by separable commutative subalgebras or that a C^*-algebra is scattered if and only if all of its subalgebras are AF ([40]).

In section 5, we calculate the Cantor-Bendixson derivatives of $\mathcal{A} \otimes \mathcal{K}(\ell_2)$ which is used extensively in the last section.

Having developed a language analogous to the Cantor-Bendixson derivatives for locally compact Hausdorff spaces the next natural step is to ask if analogous phenomena happen in the noncommutative context. The phenomena we focus on are related to the interaction between the height and the width of locally compact spaces:

Definition 1.5. *Suppose that $\mathcal{A}$ is a scattered C^*-algebra with the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$. The width of $\mathcal{A}$ is the supremum of κ, where $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is isomorphic to a nondegenerate subalgebra of $\mathcal{K}(\ell_2(\kappa))$, and it is denoted by $wd(\mathcal{A})$. An algebra $\mathcal{A}$ is called κ-thin-tall (κ-thin-very tall) if $ht(\mathcal{A}) = \kappa^+$ ($ht(\mathcal{A}) = \kappa^{++}$) and $wd(\mathcal{A}) = \kappa$. An ω-thin-tall (ω-thin-very tall) algebra is called thin-tall (thin-very tall).*

A scattered C^*-algebra $\mathcal{A}$ is called κ-short-wide if and only if $\mathcal{I}^{At}(\mathcal{A}) \cong \mathcal{K}(\ell_2(\kappa))$, $\mathcal{I}_2(\mathcal{A})/\mathcal{I}^{At}(\mathcal{A}) \cong \mathcal{K}(\ell_2(\kappa^+))$ and $\mathcal{I}_3(\mathcal{A})/\mathcal{I}_2(\mathcal{A}) = \{0\}$. An ω-short-wide algebra is called a Ψ-algebra.

The investigations of thin-tall, thin-very tall commutative algebras as well as Ψ-algebras led to many fundamental discoveries in topology and Banach space theory mentioned in the first part of the introduction (e.g., Ostaszewski's and Kunen's spaces are thin-tall and Tall's or Simon's are Ψ-spaces). We focus on the case of $\kappa = \omega$ because it is the most interesting and also because we would like to avoid discussing here constructions that require additional set-theoretic assumptions. For example, the only known constructions of commutative κ-thin-tall algebras for $\kappa > \omega$ use such assumptions ([36]) and for κ-short-wide algebras, it is known that such assumptions are necessary (Theorem 3.4 of [3]). Also it is known that already the existence of a thin-very tall commutative C^*-algebra is independent of the usual axioms ([35, 4]).

As we investigate the noncommutative constructions, we need to impose strong noncommutativity conditions. We consider such two conditions, the stability of a C^*-algebra $\mathcal{A}$, i.e., the condition that $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{K}(\ell_2)$ (see e.g. [58]) and the condition that all the quotients $\mathcal{I}_{\alpha+1}(\mathcal{A})/\mathcal{I}_\alpha(\mathcal{A})$ for $\alpha < ht(\mathcal{A})$ are isomorphic to $\mathcal{K}(\ell_2(\kappa_\alpha))$ for some infinite cardinal κ_α. The latter condition we call the "full noncommutativity" (Definition 6.1). Section 6 is devoted to proving some simple observations about this notion, which is equivalent to the fact that all ideals of the algebra are among the ideals $\mathcal{I}_\alpha(\mathcal{A})$ for $\alpha < ht(\mathcal{A})$ and that the centers of the multiplier algebras of all quotients of $\mathcal{A}$ (in particular of $\mathcal{A}$ itself) are trivial (6.3). Stability does not imply being fully noncommutative (take e.g. $c_0 \otimes \mathcal{K}(\ell_2)$) but for separable C^*-algebras the converse implication holds (7.3).

In the last section 7 we construct fully noncommutative and stable examples of thin-tall and Ψ-algebras as well as algebras of width κ and height θ for any ordinal $\theta < \kappa^+$ and any regular cardinal κ (7.4, 7.6, 7.8), showing that one can obtain in the noncommutative context all the constructions of scattered C^*-algebras analogous to the commutative examples that do not require additional set-theoretic assumptions.

The most interesting construction is of the thin-tall scattered C^*-algebra, which as in the commutative case (cf. [34], [55]) requires some kind of decomposition which is nontrvial at countable limit stages of the construction. As an interesting side-product of this technique we obtain a nonstable type I AF-algebra which is an uncountable inductive limit of stable AF-algebras (Theorem 7.7). However, a result of Hjelmborg and Rørdam (Corollary 4.1. [27]), shows that countable inductive limits of stable separable (or σ-unital, more generally) C^*-algebras, is again stable. This also follows from the model theory fact that being stable is $\forall\exists$-axiomatizable for separable C*-algebras (Proposition 2.7.7 of [22]), and that $\forall\exists$-axiomatizability is preserved under taking (countable) inductive limits (see Proposition 2.4.4 of [22]). Therefore, our example shows that these results can not be strengthened to the inductive limits along uncountable ordinals. Moreover this example shows that there may not be a maximal stable ideal among the ideals of a C^*-algebra. This answers the following question of Rørdam negatively for (only) nonseparable C^*-algebras.

Question 1.6. *(Question 6.5.* [58]*) Does every (separable) C*-algebra A have a greatest stable ideal (i.e., a stable ideal that contains all other stable ideals)?*

The argument here is based on the fact that we have the complete list of ideals in a fully noncommutative scattered C^*-algebra (Lemma 6.2).

Two further natural questions are whether the consistency results concerning superatomic Boolean algebras or locally compact scattered spaces can be obtained in the fully noncommutative and stable forms and weather the wider context of noncommutative C^*-algebras provides new possibilities. Perhaps the most interesting one is if it is consistent that there is a scattered C^*-algebra of countable width and height ω_3. Weather this is possible in the commutative case, is a very well known and old open problem. This leads to a more general question, whether "behind" any scattered C^*-algebra there is a commutative scattered C^*-algebra which "carries" similar combinatorics. Here one should mention that there are nonseparable scattered C^*-algebras with no nonseparable commutative subalgebras (see [5], [1], however there is a commutative algebra "behind" this example) and a very interesting result of Kusuda in [39], based on Dauns-Hofmann theorem, shows that a C^*-algebra is scattered if and only if it is GCR and its center is scattered (in most of our examples the center is null).

One can also wonder how interesting such constructions are from the point of view of C^*-algebras. We found two more examples, one of a thin-tall algebra and the other of a Ψ-algebra, which exhibit extraordinary behaviors, however, because of their complexity they are presented in separate papers [23, 24].

As the paper is intended for readers of diverse backgrounds including set theorists, classical topologists, Banach spaces theorists and C^*-algebraists, many arguments are explained in details not seen in the papers addressed to a monothematic group.

1.4. **Notation and terminology.** The notation and the terminology of the paper should be mainly standard. Initial parts of two introductory books [2] and [48] are completely sufficient as the C^*-algebras background. For topology we use the terminology of [21] and for Boolean algebras the ones of [38], [25]. In particular, we assume familiarity with the Stone duality. Usually compact spaces considered in this paper are Hausdorff. The only exception is the spectrum of a C^*-algebra. For a Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ denotes the C^*-algebra of all bounded operators on it. For a C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ and a subset $\mathcal{D}$ of $\mathcal{H}$, we use the notation $[\mathcal{A}\mathcal{D}]$ to denote the closed subspace of $\mathcal{H}$ spanned by the vectors $T\xi$ for $T \in \mathcal{A}$ and $\xi \in \mathcal{D}$. $\mathcal{A}|\mathcal{H}$ denotes the set of restrictions of all elements of $\mathcal{A}$ to an invariant subspace $\mathcal{H}$. When we talk about $\ell_2(X)$, then X is considered just as a set of indices regardless of the structure that can be carried by X. The (minimal) unitization of a C^*-algebra $\mathcal{A}$ will be denoted by $\widetilde{\mathcal{A}}$. Masa means a maximal self-adjoint abelian subalgebra and all the ideals considered in this paper are closed two-sided ideals. Also $\mathfrak{c}$ denotes the cardinal continuum.

2. Review of the commutative case - scattered compact spaces

We start reviewing the commutative situation by recalling basic facts concerning Boolean algebras:

Definition 2.1. *An atom of a Boolean algebra A is its nonzero element a such that $0 \leq b \leq a$ implies $b = 0$ or $b = a$ for $b \in A$. $I^{At}(A)$ denotes the Boolean ideal generated by the atoms of A.*

Theorem 2.2 ([47], [17])**.** *Suppose that A is a Boolean algebra. Then the following conditions are equivalent:*

(1) *Every subalgebra of A has an atom.*
(2) *A does not contain any free infinite subalgebra.*
(3) *Every homomorphic image of A is atomic, that is has an atom below every nonzero element.*
(4) *Every homomorphic image of A has an atom.*
(5) *There is an ordinal $ht(A)$ and a continuous increasing sequence of ideals $(I_\alpha)_{\alpha \leq ht(A)}$ whose union is A such that*

$$I^{At}(A/I_\alpha) = \{[a]_{I_\alpha} : a \in I_{\alpha+1}\}$$

holds for each $\alpha < ht(A)$.
(6) *There is an ordinal $k(A)$, and a continuous increasing sequence of ideals $(J_\alpha)_{\alpha<k(A)}$ whose union is A such that for every $\alpha < k(A)$ whenever $J \subseteq A$ is an ideal of A such that $J_\alpha \subseteq J \subseteq J_{\alpha+1}$, then $J = J_\alpha$ or $J = J_{\alpha+1}$.*

We do not include the proof of the theorem. However, in a sense it is included below, by the Stone duality, in the equivalences expressed in the topological language in Theorem 2.7. One can also find most of it in [55]. Item (6) is a version of Proposition 3.5 from [37].

Definition 2.3. *A Boolean algebra is called superatomic if and only if each of its homomorphic images is atomic, i.e., it has an atom below any nonzero element.*

Definition 2.4. *A compact Hausdorff space is called scattered if and only if each of its nonempty subsets has a relative isolated point.*

Lemma 2.5. *Suppose that K is a scattered compact Hausdorff space. Then K is totally disconnected and so 0-dimensional. Hence it is the Stone space of a Boolean algebra.*

Proof. As connected subspaces have no isolated points, the total disconnectedness is clear. The fact that for compact Hausdorff spaces the total disconnectedness implies 0-dimensionality follows from 6.1.23 of [21]. □

Proposition 2.6. *Suppose that K is a compact Hausdorff and totally disconnected space and A is a Boolean algebra. The Boolean algebra of clopen subsets of K is superatomic if and only if K is scattered. A is superatomic if and only if its Stone space is scattered.*

Proof. By taking closures, it is clear that every subset of K has an isolated point if and only if every closed subset of K has an isolated point. The rest follows from the Stone duality which gives the correspondence between the closed subsets of a compact space and homomorphic images of the dual Boolean algebra. □

If X is a topological space by the Cantor-Bendixson derivate X' of X we will denote the closed subspace of X consisting of all nonisolated points of X. Considering a compact Hausdorff scattered K, allows us to obtain more equivalent conditions for a compact space to be scattered, compared to the Boolean algebraic conditions from Theorem 2.2:

Theorem 2.7. *Suppose that K is a compact Hausdorff space. Then the following conditions are equivalent:*

(1) *Every nonempty (closed) subspace of K has an isolated point.*
(2) *There is an ordinal $ht(A)$ and a continuous decreasing sequence of closed subspaces $(K^{(\alpha)})_{\alpha\leq ht(K)}$ of K such that $K^{(0)}=K$, $K^{(ht(K))}=\emptyset$ and*
$$K^{(\alpha+1)}=(K^{(\alpha)})'$$
holds for each $\alpha<ht(K)$.
(3) *There is an ordinal $m(A)$ and a continuous decreasing sequence of closed subspaces $(L^{(\alpha)})_{\alpha\leq m(K)}$ of K such that $L^{(0)}=K$, $L^{(m(K)))}=\emptyset$ and $L^{(\alpha)}\setminus L^{(\alpha+1)}$ is one point for each $\alpha<m(K)$.*
(4) *Every continuous image of K has an isolated point.*
(5) *The range (spectrum) of every $f\in C(K)$ is countable.*
(6) *K does not map onto $[0,1]$.*

Proof. (1) ⇒ (2) By (1) the procedure described in (2) results in a strictly smaller subspace, hence at some point we end up with the empty subspace.

(2) ⇒ (3) The sets $K^{(\alpha)}\setminus K^{(\alpha+1)}$ are made of isolated points of $K^{(\alpha)}$, so any subspace of $K^{(\alpha)}\setminus K^{(\alpha+1)}$ is open in $K^{(\alpha)}$. In particular we can obtain $K^{(\alpha+1)}$ from $K^{(\alpha)}$ by producing a decreasing continuous sequence of closed sets each obtained from the previous by removing one point. This results in the decomposition from (3).

(3) ⇒ (4) Let $\phi: K\to M$ be a surjection, suppose that K satisfies (3) and that M has no isolated points. As in compact spaces infinite sets have (necessarily nonisolated) accumulation point, the preimage $\phi^{-1}(\{x\})$ of any point $x\in M$ must contain some nonisolated point of K, because otherwise it would be a finite set of isolated points, which would imply that x is isolated by the fact that ϕ is a closed mapping. This means that $\phi[L^{(\alpha)}]=M$ for every $\alpha\leq m(K)$ which contradicts the emptyness of $L^{(m(L))}$.

(4) ⇒ (5) If the range of some $f\in C(K)$ was uncountable, then it would contain a closed set with no isolated point by Cantor-Bendixson theorem and so it would

contain a copy of a Cantor set since subsets of $\mathbb{C}$ are metrizable. This Cantor set can be mapped onto $[0,1]$ and by the Tietze theorem this mapping can be extended to the entire range of f. The composition would give a continuous mapping from K onto $[0,1]$, contradicting (4).

(5) $\Rightarrow$ (6) clear.

(6) $\Rightarrow$ (1) If K had a subset with no isolated points, its closure L would have no isolated points as well. By the Tietze theorem it is enough to construct a continuous map from L onto $[0,1]$. As every open subset of L is infinite, one can construct a Cantor tree of open sets with inclusion of closures, i.e., open U_s for $s \in \{0,1\}^{<\mathbb{N}}$ such that $\overline{U_{s^\frown 1}}, \overline{U_{s^\frown 0}} \subseteq U_s$ and $\overline{U_{s^\frown 1}} \cap \overline{U_{s^\frown 0}} = \emptyset$ for each $s \in \{0,1\}^{<\mathbb{N}}$. Then one can define a continuous function ϕ from $M = \bigcap_{n\in\mathbb{N}} \bigcup_{|s|=n} \overline{U}_s$ onto $\{0,1\}^{\mathbb{N}}$ by puting $\phi(x) = y$ whenever $x \in \overline{U}_s$ if and only if $y|n = s$ for $n = |s|$. Then compose ϕ with a function from $\{0,1\}^{\mathbb{N}}$ onto $[0,1]$. Finally using the Tietze Theorem one can extend the composition from M to L and then to K.

□

For completeness we also add the following theorem (cf. Theorem 4.7) which shows the equivalent properties for K being compact scattered Hausdorff in terms of the Banach space of continuous functions, its Banach dual or bidual:

Theorem 2.8 ([53])**.** *Suppose that K is a compact Hausdorff space. All the following conditions are equivalent to K being scattered:*

(1) Every Radon measure on K is atomic, i.e., of the form $\mu = \Sigma_{n\in\mathbb{N}} t_n \delta_{x_n}$ where $x_n \in K$ and $\Sigma_{n\in\mathbb{N}} |t_n| < \infty$.
(2) The dual of $C(K)$, the space of Radon measures on K, is isometric to $\ell_1(K)$.
(3) The bidual of $C(K)$ is ℓ_∞ where the inclusion $C(K) \subseteq \ell_\infty(K)$ is the canonical embedding of $C(K)$ into the bidual $C(K)^{**}$.
(4) The Banach dual of every separable Banach subspace of $C(K)$ is separable.

There are at least two other long list of equivalent conditions for K to be scattered in terms of the Radon-Nikodym property in the vector valued integration ([19]) or in terms of strong differentiability in Banach spaces ([18]).

3. Atoms and the ideals they generate in noncommutative topology

3.1. **Minimal projections.**

Lemma 3.1. *Suppose that $\mathcal{H}$ is a Hilbert space and K is a compact Hausdorff space. Minimal projections in $\mathcal{B}(\mathcal{H})$ are orthogonal projections onto one-dimensional subspaces. Minimal projections of $C(K)$ are characteristic functions $\chi_{\{x\}}$ of isolated points $x \in K$.*

One easily checks that if $p \in \mathcal{A}$ is a minimal projection, then there is a linear functional λ_p such that $pap = \lambda_p(a)p$ for each $a \in \mathcal{A}$. For $p = \chi_{\{x\}}$ and $\mathcal{A} = C(K)$ the functional is δ_x, the evaluation at x. It can be proved that for any C^*-algebra λ_p is a pure state (see Chapter 5 of [48]). Recall that pure states are extreme points of the set of all positive linear functionals of norm ≤ 1. In the case of $C(K)$ they are exactly δ_x for all $x \in K$.

Lemma 3.2 (1.4.1 of [2])**.** *Suppose that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a C^*-algebra. An element $p \in \mathcal{K}(\mathcal{H})$ is a minimal projection in $\mathcal{A}$ if and only if p is a projection which is* *minimal with respect to the usual ordering of projections, i.e., there is no nonzero projection $q \in \mathcal{A}$ such that $q \leq p$ and $q \neq p$.*

In general the projections which are minimal with respect to the ordering do not need to be minimal in the sense of Definition 1.1. For example the unit of $C([0,1])$ is minimal with respect to the order but does not satisfy Definition 1.1. This example also shows that a minimal projection in a subalgebra may not be minimal in a bigger algebra. However we have the following:

Lemma 3.3. *Suppose that $\mathcal{A}$ is a C^*-algebra, $\mathcal{I} \subseteq \mathcal{A}$ its ideal and $\mathcal{B} \subseteq \mathcal{A}$ its masa. Then a minimal projection in $\mathcal{I}$ or in $\mathcal{B}$ is a minimal projection in $\mathcal{A}$.*

Proof. Suppose p is a minimal projection in $\mathcal{I}$. We have a functional $\phi \in \mathcal{I}^*$ such that $pcp = \phi(c)p$ for every $c \in \mathcal{I}$. Consider any $a \in \mathcal{A}$. Then $pap \in \mathcal{I}$, so $pap = p(pap)p = \phi(pap)p$ as required for a minimal projection in $\mathcal{A}$.

Let $\mathcal{B}$ be a masa of $\mathcal{A}$. The Gelfand transform $a \to \hat{a}$ is an $*$-isomorphism between $\mathcal{B}$ and $C_0(X)$ for some locally compact Hausdorff space X. Let the characteristic function $\chi_{\{x\}}$ of $\{x\}$ be a minimal projection in $C_0(X)$ and p be the unique minimal projection of $\mathcal{B}$ such that $\hat{p} = \chi$. We will show that p is also a minimal projection in $\mathcal{A}$. Take arbitrary elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$. There is $\lambda \in \mathbb{C}$ such that $pbp = \lambda p$ and

$$papb = papbp = \lambda pap = pbpap = bpap.$$

Therefore pap commutes with $\mathcal{B}$. By the maximality of $\mathcal{B}$, pap belongs to $\mathcal{B}$ and hence $p^2ap^2 = pap = \gamma p$ for some scalar γ, since p is a minimal projection in $\mathcal{B}$. □

Lemma 3.4. *For any C^*-algebra $\mathcal{A}$, $\mathcal{I}^{At}(\widetilde{\mathcal{A}}) = \mathcal{I}^{At}(\mathcal{A})$.*

Proof. The inclusion $\mathcal{I}^{At}(\mathcal{A}) \subseteq \mathcal{I}^{At}(\widetilde{\mathcal{A}})$ follows from the fact that $\mathcal{A}$ is an ideal of $\widetilde{\mathcal{A}}$ and Lemma 3.3.

For the proof of the other inclusion we may assume that $\mathcal{A}$ is not unital. Suppose that $(a, \alpha) \in \widetilde{\mathcal{A}}$ is a minimal projection for $\alpha \neq 0$. From this hypothesis we will derive a contradiction with $\mathcal{A}$ being nonunital. The fact that (a, α) must be an idempotent implies that $\alpha = 1$ and $a^2 = -a$, while the fact that $(a, 1)$ must be self-adjoint implies that a is selfadjoint. The minimality of $(a, 1)$ implies that $(a, 1)(b, 0)(a, 1) = \lambda(a, 1)$ for some $\lambda \in \mathbb{C}$ which must be 0. So $aba + ba + ab + b = 0$ for every $b \in \mathcal{A}$.

Let a net $(e_\xi)_{\xi \in \Xi}$ be an approximate unit for $\mathcal{A}$ (see Theorem 1.8.2 of [2]), that is in particular we have $\|xe_\xi - x\| \to 0$ and $\|e_\xi x - x\| \to 0$ for every $x \in \mathcal{A}$. Considering $b = e_\xi$ we get that $ae_\xi a + e_\xi a + ae_\xi = -e_\xi$ for all $\xi \in \Xi$ but the left hand side converges to $a^2 + 2a = a$, so the right hand side $(-e_\xi)$ must converge to a as well. This shows that $-a$ is the unit for $\mathcal{A}$, a contradiction.

□

We conclude this section by the next two elementary lemmas, exhibiting how the existence of certain projections in a C^*-algebra (or its subalgebras) can prevent it from being scattered, by imposing the existence of a copy of $C([0,1])$ in it.

Lemma 3.5. *Suppose that $\mathcal{A}$ is a C^*-algebra and $p \in \mathcal{A}$ is a projection which is not a minimal projection but such that there is no nonzero projection $q \subsetneq p$. Then $p\mathcal{A}p$ contains a copy of $C([0,1])$.*

Proof. Pick $q = prp$ for some $r \in \mathcal{A}$ which is not of the form λp for some $\lambda \in \mathbb{C}$. We have that $r = r_1 + ir_2$ where r_1 and r_2 are self adjoint, so we may assume that r is self adjoint. As $prp = p^2rp = prp^2$, we have that p and q commute. Since r is self adjoint, we have $q^* = (prp)^* = p^*r^*p^* = q$. So p and q are self adjoint commuting elements of $\mathcal{A}$, so the subalgebra $\mathcal{B} \subseteq \mathcal{A}$ which they generate is abelian and isomorphic to $C_0(X)$ for some locally compact X. Each projection p in such an algebra corresponds to the characteristic function χ_U for some clopen $U \subseteq X$. Note that U has more than one element, since otherwise it would be a minimal projection. As there is no projection smaller than p, it follows that U is connected and hence has $[0,1]$ as its continuous image, which yields $C([0,1])$ as a subalgebra of $C(U)$ and consequently a subalgebra of $p\mathcal{A}p$.

□

Lemma 3.6. *Suppose that $\mathcal{A}$ is a C^*-algebra with a nonzero projection such that every nonzero projection has a strictly smaller nonzero projection. Then $\mathcal{A}$ contains a copy of $C([0,1])$.*

Proof. Using the hypothesis we can construct projections $(p_s : s \in \{0,1\}^{<\mathbb{N}})$ such that $p_{s^\frown 0} + p_{s^\frown 1} = p_s$ for each $s \in \{0,1\}^{<\mathbb{N}}$ and all the projections are nonzero. If $s \subseteq t$ then $p_t \leq p_s$ and so $p_tp_s = p_tp_s = p_t$, that is all p_ss commute and so the algebra generated by them is $*$-isomorphic to a $C(K)$ for some compact Hausdorff K. The projections p_s correspond in $C(K)$ to a tree of nonempty clopen sets $(U_s : s \in \{0,1\}^{<\mathbb{N}})$ such that $U_{s^\frown 0} + U_{s^\frown 1} = U_s$ for each $s \in \{0,1\}^{<\mathbb{N}}$. Like in the proof of 2.7 (5) $\Rightarrow$ (1) this yields a surjective continuous map $\phi : K \to [0,1]$ and so $C([0,1]) \subseteq C(K) \subseteq \mathcal{A}$, as required. □

3.2. **Irreducible representations and the ideal $\mathcal{I}^{At}(\mathcal{A})$.** An irreducible representation π of a C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a *-homomorphisms from $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ such that its range $\pi[\mathcal{A}]$ is an irreducible subalgebra of $\mathcal{B}(\mathcal{H})$, i.e., it has no nontrivial $\pi[\mathcal{A}]$-invariant closed subspace in $\mathcal{H}$. Throughout this section we use different equivalent conditions for a representation to be irreducible (see e.g., [7, II.6.1.8.]). In the commutative case, irreducible representations correspond to the multiplicative functionals, i.e., functionals of the form $\pi_x : C_0(X) \to \mathbb{C}$, where $\pi_x(f) = f(x)$ for some $x \in X$. Here the Hilbert space $\mathcal{H}$ is the one-dimensional space $\mathbb{C}$.

Lemma 3.7. *Suppose that $\mathcal{A}$ is a C^*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation of $\mathcal{A}$. Let p be a minimal projection in $\mathcal{A}$, a unit vector v be in the range of $\pi(p)$ and $\mathcal{H}_0 = [\pi[\mathcal{A}]v]$. Then $\pi[\mathcal{A}]|_{\mathcal{H}_0}$ is an irreducible subalgebra of $\mathcal{B}(\mathcal{H}_0)$.*

Proof. Since $\mathcal{H}_0$ is $\pi[\mathcal{A}]$-invariant, the map $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ defined by $\pi_0(a) = \pi(a)|\mathcal{H}_0$ is a subrepresentation of π. Assume $T \in \mathcal{B}(\mathcal{H}_0)$ commutes with $\pi_0[\mathcal{A}]$. We will show that $T \in \mathbb{C}I$. By replacing T with $T - (Tv,v)I$, we may assume $(Tv,v) = 0$. Under this assumption we will show that $T = 0$. Now if $a, b \in \mathcal{A}$ we have $\pi_0(pb^*ap) = \lambda\pi_0(p)$. Therefore

$$(T\pi_0(a)v, \pi_0(b)v) = (T\pi_0(pb^*ap)v, v) = \lambda(T\pi_0(p)v, v) = \lambda(Tv, v) = 0.$$

Since $\mathcal{H}_0 = [\pi[\mathcal{A}]v]$, arbitrary two vectors of $\mathcal{H}_0$ can be approximated by vectors of the form $\pi_0(a)v$ and $\pi_0(b)v$ for some $a, b \in \mathcal{A}$, so it follows that $T = 0$.

□

Lemma 3.8. *Suppose that $\mathcal{A}$ is an irreducible $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. If p a minimal projection in $\mathcal{A}$, then p a minimal projection in $\mathcal{B}(\mathcal{H})$*

Proof. By Lemma 3.1 it is enough to show that p can not be an orthogonal projection onto a subspace with dimension more than one. If this was the case, take a nonzero vector v in the range of p and consider $\mathcal{X} = [\mathcal{A}v] \subseteq \mathcal{H}$. We claim that $\mathcal{X}$ would be a proper $\mathcal{A}$-invariant subspace of $\mathcal{H}$. It is clear that it is an invariant subspace of $\mathcal{H}$. As $p(v) = v$ and for some $\lambda \in \mathbb{C}$ by the minimality of p we have $pTp = \lambda p$, we obtain $pT(v) = (pTp)(v) = \lambda p(v) \in \mathbb{C}v$ for every $T \in \mathcal{A}$. By if the range of p is more than one-dimensional, there are vectors w in the range of p of it which are far from $\mathbb{C}v$. Since $p(w) = w$, the vector $T(v)$ cannot be close to w, which shows that $\mathcal{X}$ is a proper $\mathcal{A}$-invariant subspace of $\mathcal{H}$ which contradicts the irreducibility of $\mathcal{A}$.

□

Lemma 3.9. *Suppose that $\mathcal{A}$ is a C^*-algebra and p is its minimal projection. Let $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$ for $i = 1, 2$ be irreducible representations of $\mathcal{A}$. If $\pi_1(p) \neq 0 \neq \pi_2(p)$, then π_1 and π_2 are unitarily equivalent.*

Proof. Let $\phi \in \mathcal{A}^*$ be a functional such that $pap = \phi(a)p$ for every $a \in \mathcal{A}$. Pick vectors $v \in [\pi_1(p)\mathcal{H}_1] \subseteq \mathcal{H}_1$ and $w \in [\pi_2(p)\mathcal{H}_2] \subseteq \mathcal{H}_2$ of norm one. By Lemma 3.8 $\pi_1(p)$ and $\pi_2(p)$ are one-dimensional projections in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then for every $a \in \mathcal{A}$ we have

$$\begin{aligned}
\|\pi_1(a)v\|^2 &= \|\pi_1(ap)v\|^2 = (\pi_1(ap)v, \pi_1(ap)v) \\
&= (\pi_1(pa^*ap)v, v) = \phi(a^*a)(v, v) \\
&= \phi(a^*a)(w, w) = (\pi_2(pa^*ap)w, w) \\
&= \|\pi_2(a)w\|^2.
\end{aligned}$$

Note that this implies that $\pi_1(a) = \pi_1(a')$ is equivalent to $\pi_2(a) = \pi_2(a')$. Therefore the map U sending $\pi_1(a)v$ to $\pi_2(a)w$ extends to a well-defined linear isometry from $[\pi_1[\mathcal{A}]v]$ onto $[\pi_2[\mathcal{A}]w]$. By the irreducibility of π_1 and π_2 both v and w are cyclic vectors for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, i.e., $[\pi_1[\mathcal{A}]v] = \mathcal{H}_1$ and $[\pi_2[\mathcal{A}]w] = \mathcal{H}_2$. Therefore U is a unitary from $\mathcal{H}_1$ onto $\mathcal{H}_2$. Also if $a \in \mathcal{A}$ and $u \in \mathcal{H}_1$, there is $b \in \mathcal{A}$ such that $\pi_1(b)v = u$, then

$$(U^{-1}\pi_2(a)U)(u) = (U^{-1}\pi_2(a)U)(\pi_1(b)v) = U^{-1}\pi_2(a)\pi_2(b)w =$$
$$= U^{-1}\pi_2(ab)w = \pi_1(ab)v = \pi_1(a)u.$$

Thus π_1 and π_2 are unitarily equivalent. □

Definition 3.10. *Let $\mathcal{A}$ be a C^*-algebra and $(\pi_i, \mathcal{H}_i)_{i\in I}$ a maximal collection of pairwise unitarily inequivalent irreducible representations. Then $\pi:\mathcal{A} \to \prod_{i\in I} \mathcal{B}(\mathcal{H}_i)$ given by*

$$\pi(a)|\mathcal{H}_i = \pi_i(a)$$

is called the reduced atomic representation of $\mathcal{A}$.

Lemma 3.11. *The reduced atomic representation of a C^*-algebra is faithful.*

Proof. Suppose that $(\pi_i, \mathcal{H}_i)_{i\in I}$ a maximal collection of pairwise unitarily inequivalent irreducible representations, but there is $a \in \mathcal{A}$ such that $\pi_i(a) = 0$ for all $i \in I$ and $a \neq 0$. There is an irreducible representation σ such that $\sigma(a) \neq 0$ by Theorem

5.1.12. of [48]. Of course such a representation can not be unitarily equivalent to any π_is, which contradicts the maximality of $(\pi_i)_{i\in I}$. □

Proposition 3.12. *Suppose that $\mathcal{A}$ is a C^*-algebra, $\mathcal{I}$ is its ideal and $(\pi, \mathcal{H})$ is any faithful representation of $\mathcal{I}$. Then*

$$\pi[\mathcal{I}] \cap \mathcal{K}(\mathcal{H}) \subseteq \pi[\mathcal{I}^{At}(\mathcal{A})].$$

Proof. Let $T \in \pi[\mathcal{I}] \cap \mathcal{K}(\mathcal{H})$ be self-adjoint. Then by the spectral theorem $T = \Sigma_{n\in\mathbb{N}}\lambda_n P_n$ where P_ns are finite dimensional projections. Moreover, spectral theorem implies that $P_n \in \pi[\mathcal{I}]$. We can write each P_n as a finite sum of one dimensional (minimal) projections in $\pi[\mathcal{I}]$. This implies that T is in $\mathcal{I}^{At}(\pi[\mathcal{I}]) = \pi[\mathcal{I}^{At}(\mathcal{I})]$, the last equality follows from the faithfulness of π. However, $\mathcal{I}^{At}(\mathcal{I}) \subseteq \mathcal{I}^{At}(\mathcal{A})$, by Lemma 3.3 and therefore $\pi[\mathcal{I}] \cap \mathcal{K}(\mathcal{H}) \subseteq \pi[\mathcal{I}^{At}(\mathcal{A})]$. □

Proposition 3.13. *Suppose that $(\pi, \mathcal{H})$ is the reduced atomic representation of a C^*-algebra $\mathcal{A}$. Then*

$$\pi[\mathcal{A}] \cap \mathcal{K}(\mathcal{H}) = \pi[\mathcal{I}^{At}(\mathcal{A})].$$

Proof. Let $(\pi_i, \mathcal{H}_i)_{i\in I}$ be the maximal pairwise inequivalent irreducible representations of $\mathcal{A}$ which give rise to π. By Lemmas 3.9 for every minimal projection $p \in \mathcal{A}$, $\pi(p)$ is 0 on all but one $\mathcal{H}_i$. By Lemma 3.8 and Lemma 3.1 $\pi(p)$ is a one-dimensional projection on this $\mathcal{H}_i$. It follows that $\pi[\mathcal{I}^{At}(\mathcal{A})] \subseteq \mathcal{K}(\mathcal{H})$.

The other inclusion follows from Proposition 3.12 applied for $\mathcal{I} = \mathcal{A}$. □

Proposition 3.14. *Suppose $\mathcal{A}$ is a C^*-algebra. Then $\mathcal{I}^{At}(\mathcal{A})$ is isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

Proof. Apply Proposition 3.13 and Lemma 3.11. □

Proposition 3.15. *Suppose $\mathcal{A}$ is a C^*-algebra. Then $\mathcal{I}^{At}(\mathcal{A})$ is an ideal of $\mathcal{A}$. It contains all ideals isomorphic to a subalgebra of compact operators on a Hilbert space. Therefore there is no bigger ideal which can be faithfully mapped into an algebra of compact operators on a Hilbert space.*

Proof. Let π be the reduced atomic representation of $\mathcal{A}$ on $\mathcal{H}$. By Proposition 3.13 we have $\pi[\mathcal{A}] \cap \mathcal{K}(\mathcal{H}) = \pi[\mathcal{I}^{At}(\mathcal{A})]$. Since $\mathcal{K}(\mathcal{H})$ is an ideal of $\mathcal{B}(\mathcal{H})$, $\pi[\mathcal{I}^{At}(\mathcal{A})]$ is an ideal of $\pi[\mathcal{A}]$. Therefore $\mathcal{I}^{At}(\mathcal{A})$ is an ideal of $\mathcal{A}$. Any ideal $\mathcal{I}$ isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ is generated as an algebra by its minimal projections (by inspection in subalgebras of $\mathcal{K}(\mathcal{H})$). The minimal projections in $\mathcal{I}$ are minimal projections in $\mathcal{A}$ (Lemma 3.3). Thus $\mathcal{I} \subseteq \mathcal{I}^{At}(\mathcal{A})$. □

Proposition 3.16. *Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ is a subalgebra of $\mathcal{A}$. Then $\mathcal{I}^{At}(\mathcal{A}) \cap \mathcal{B} \subseteq \mathcal{I}^{At}(\mathcal{B})$. If $\mathcal{J}$ is an ideal if $\mathcal{A}$, then $\mathcal{I}^{At}(\mathcal{J}) = \mathcal{I}^{At}(\mathcal{A}) \cap \mathcal{J}$.*

Proof. By Proposition 3.15, $\mathcal{I}^{At}(\mathcal{A}) \cap \mathcal{B}$ is an ideal of $\mathcal{B}$. Since $\mathcal{I}^{At}(\mathcal{A})$ can be mapped faithfully into an algebra of compact operators (Proposition 3.14), so can $\mathcal{I}^{At}(\mathcal{A}) \cap \mathcal{B}$. However by Proposition 3.15 any such ideal of $\mathcal{B}$ is included in $\mathcal{I}^{At}(\mathcal{B})$. The second statement immediately follows from the first statement and Lemma 3.3. □

3.3. **Atoms and essential ideals.** Recall that an ideal $\mathcal{I}$ in a C^*-algebra $\mathcal{A}$ is called essential if and only if $\mathcal{I} \cap \mathcal{J} \neq \{0\}$ for any nonzero ideal of $\mathcal{A}$. Also $\mathcal{I}$ is essential if and only if $\mathcal{I}^\perp = \{a \in \mathcal{A} : a\mathcal{I} = 0\}$ is the zero ideal (see II.5.4.7 of [7]).

Lemma 3.17. *Suppose that $\mathcal{A}$ s a C^*-algebra, $\mathcal{I} \subseteq \mathcal{A}$ its essential ideal and $\mathcal{J} \subseteq \mathcal{I}$ an essential ideal of $\mathcal{I}$. Then $\mathcal{J}$ is an essential ideal of $\mathcal{A}$.*

Proof. To see that $\mathcal{J}$ is an ideal of $\mathcal{A}$, take $j \in \mathcal{J}$ and $a \in \mathcal{A}$. Let $(j_\xi)_{\xi\in\Xi}$ be an approximative unit for $\mathcal{J}$, in particular $\|jj_\xi - j\|$ converges to 0. We have $aj \in \mathcal{I}$ as $\mathcal{I}$ is an ideal of $\mathcal{A}$ and since $(ajj_\xi)_{\xi\in\Xi}$ converges to aj and each ajj_ξ is in $\mathcal{J}$, it follows that $aj \in \mathcal{J}$ as required. The proof for ja is analogous.

For the essentiality of $\mathcal{J}$ suppose that there is $a \in \mathcal{A}$ such that $a\mathcal{J} = 0$. However, the essentiality of $\mathcal{I}$ in $\mathcal{A}$ implies that there is $i \in \mathcal{I}$ such that $ia \neq 0$. Now the essentiality of $\mathcal{J}$ in $\mathcal{I}$ yields $j \in \mathcal{J}$ such that $iaj \neq 0$. It follows that $aj \neq 0$, which is a contradiction. □

Proposition 3.18. *Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{I} \subseteq \mathcal{A}$ is an essential ideal such that there is an embedding $i : \mathcal{I} \to \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then there is an extension of this embedding to an embedding $\tilde{i} : \mathcal{A} \to \mathcal{B}(\mathcal{H})$.*

Proof. Let $\mathcal{M}(\mathcal{I})$ be the multiplier algebra of $\mathcal{I}$, that is, a unital C^*-algebra containing $\mathcal{I}$ as an essential ideal such that it is universal in the sense that for any C^*-algebra $\mathcal{B}$ containing $\mathcal{I}$ as an ideal there is a unique homomorphism $h : \mathcal{B} \to \mathcal{M}(\mathcal{I})$ extending the identity map on $\mathcal{I}$ (see [7, I.7.3.1]). Since i is a faithful representation of $\mathcal{I}$, we can identify $\mathcal{M}(\mathcal{I})$ as the idealizer of the image of $\mathcal{I}$ inside $\mathcal{B}(\mathcal{H})$ (see [7, II.7.3.5]). Let $\tilde{i} : \mathcal{A} \to \mathcal{M}(\mathcal{I}) \subseteq \mathcal{B}(\mathcal{H})$ be the homomorphism obtained from the universality of the multiplier algebra, which extends i. We have that $\mathcal{I}^\perp = \{0\}$. Thus for every nonzero $a \in \mathcal{A}$ there is $b \in \mathcal{I}$ such that $ab \neq 0$. However ab belongs to $\mathcal{I}$ since it is an ideal, so $\tilde{i}(ab) = i(ab) \neq 0$ which means that $\tilde{i}(a) \neq 0$. Therefore $\tilde{i}$ is an embedding into $\mathcal{B}(\mathcal{H})$. □

Proposition 3.19. *Suppose that $\mathcal{A}$ a C^*-algebra and $\mathcal{I} \subseteq \mathcal{I}^{At}(\mathcal{A})$ is an essential ideal. Then $\mathcal{I} = \mathcal{I}^{At}(\mathcal{A})$.*

Proof. We know that $\mathcal{I}^{At}(\mathcal{A})$ is isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Proposition 3.14). By Proposition 3.18 we may assume that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{I} \subseteq \mathcal{K}(\mathcal{H})$. Therefore $\mathcal{I}^{At}(\mathcal{A})$ is equal to $\bigoplus_{i\in I} \mathcal{K}(\mathcal{H}_i)$ for a family of Hilbert spaces $\mathcal{H}_i \subseteq \mathcal{H}$ for $i \in I$ ([2, Theorem 1.4.5.]). Hence $\mathcal{I}$ is an essential ideal of $\bigoplus_{i\in I} \mathcal{K}(\mathcal{H}_i)$, which means that it must contain nonzero elements of each $\mathcal{K}(\mathcal{H}_i)$. By the fact that each $\mathcal{K}(\mathcal{H}_i)$ is simple, i.e., has no nontrivial ideals, it follows that $\mathcal{I}$ is the entire $\bigoplus_{i\in I} \mathcal{K}(\mathcal{H}_i)$. That is $\mathcal{I} = \mathcal{I}^{At}(\mathcal{A})$, as required. □

Proposition 3.20. *Suppose that $\mathcal{I}$ is an essential ideal of a C^*-algebra $\mathcal{A}$ which is isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then $\mathcal{I} = \mathcal{I}^{At}(\mathcal{A})$.*

Proof. Since $\mathcal{I}^{At}(\mathcal{A})$ is the largest ideal of $\mathcal{A}$ which is isomorphic to a subalgebra of $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ (Proposition 3.15), $\mathcal{I}$ is contained in $\mathcal{I}^{At}(\mathcal{A})$. Therefore Proposition 3.19 implies that $\mathcal{I} = \mathcal{I}^{At}(\mathcal{A})$.

□

4. The noncommutative case - scattered C^*-algebras

Theorem 1.4. [31, 32, 66] *Suppose that $\mathcal{A}$ is a C^*-algebra. The following conditions are equivalent:*

(1) Every non-zero $*$-homomorphic image of $\mathcal{A}$ has a minimal projection.
(2) There is an ordinal $ht(\mathcal{A})$ and a continuous increasing sequence of closed ideals $(\mathcal{I}_\alpha)_{\alpha\leq ht(\mathcal{A})}$ such that $\mathcal{I}_0=\{0\}$, $\mathcal{I}_{ht(\mathcal{A})}=\mathcal{A}$ and

$$\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)=\{[a]_{\mathcal{I}_\alpha}:a\in\mathcal{I}_{\alpha+1}\},$$

for every $\alpha<ht(\mathcal{A})$.
(3) There is an ordinal $m(\mathcal{A})$ and a continuous increasing sequence of ideals $(\mathcal{J}_\alpha)_{\alpha\leq m(A)}$ such that $\mathcal{J}_0=\{0\}$, $\mathcal{J}_{m(\mathcal{A})}=\mathcal{A}$ and $\mathcal{J}_{\alpha+1}/\mathcal{J}_\alpha$ is an elementary C^*-algebra (i.e., *-isomorphic to the algebra of all compact operators on a Hilbert space) for every $\alpha<m(\mathcal{A})$.
(4) Every subalgebra of $\mathcal{A}$ has a minimal projection.
(5) $\mathcal{A}$ does not contain a copy of the C^*-algebra $C([0,1])$.
(6) The spectrum of every self-adjoint element is countable.

Proof. In the proof we can assume that $\mathcal{A}$ is unital since it is easy to check that $\mathcal{A}$ satisfies any of the above conditions if and only if $\widetilde{\mathcal{A}}$, the unitization of $\mathcal{A}$, does.

(1) $\Rightarrow$ (2) We define the sequence $(\mathcal{I}_\alpha)_{\alpha<ht(\mathcal{A})}$ by induction. Put $\mathcal{I}_0=\{0\}$. At each successor stage $\alpha+1$, if $\mathcal{A}/\mathcal{I}_\alpha$ is non-zero, then $\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$ is non-zero by (1). Let $\mathcal{I}_{\alpha+1}=\sigma_\alpha^{-1}(\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha))$, where $\sigma_\alpha:\mathcal{A}\to\mathcal{A}/\mathcal{I}_\alpha$ is the quotient map. If γ is a limit ordinal let $\mathcal{I}_\alpha=\overline{\bigcup_{\alpha<\gamma}\mathcal{I}_\alpha}$. The induction has to terminate at some ordinal $ht(\mathcal{A})$, whose cardinality does not exceed $|\mathcal{A}|$.

(2 $\Rightarrow$ (3) For each ordinal $\alpha<ht(\mathcal{A})$ we have $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha=\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$ and hence by Proposition 3.14 the quotient $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is isomorphic to a subalgebra of the algebra of compact operators. Such algebras are isomorphic to algebras of the form $\bigoplus_{i\in\Lambda_\alpha}\mathcal{K}(\mathcal{H}_{\alpha,i})$ for some Hilbert spaces $\mathcal{H}_{\alpha,i}$ and a totally ordered set Λ_α (1.4.5. of [2]). Let $\nu_\alpha:\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha\to\bigoplus_{i\in\Lambda_\alpha}\mathcal{K}(\mathcal{H}_{\alpha,i})$ be such an isomorphism and define $\mathcal{J}_{\alpha,i}=\sigma_\alpha^{-1}\circ\nu_\alpha^{-1}(\bigoplus_{j\leq i}\mathcal{K}(\mathcal{H}_{\alpha,j}))$. Order $\Gamma=\{(\alpha,i):i\in\Lambda_\alpha,\alpha\leq ht(\mathcal{A})\}$ lexicographically and let $m(\mathcal{A})$ be the order type of this set. Re-enumerate the set of $\{\mathcal{J}_{\alpha,i}:(\alpha,i)\in\Gamma\}$ by the ordinal $m(\mathcal{A})$ as $\{\mathcal{J}_\gamma:\gamma\leq m(\mathcal{A})\}$. It is easy to check that $(\mathcal{J}_\gamma)_{\gamma\leq m(\mathcal{A})}$ has the required properties.

(3) $\Rightarrow$ (4) Let $(\mathcal{J}_\alpha)_{\alpha\leq\beta}$ be a composition series as in (3). Assume $\mathcal{C}$ is a unital subalgebra of $\mathcal{A}$. By passing to a masa of $\mathcal{C}$ and using Lemma 3.3, we may assume that $\mathcal{C}$ is commutative and hence isomorphic to an algebra of the form $C(K)$ for some compact Hausdorff K. Consider the sequence $(\mathcal{J}'_\alpha)_{\alpha<m(\mathcal{A})}$ of ideals of $C(K)$ defined by $\mathcal{J}'_\alpha=\mathcal{J}_\alpha\cap\mathcal{C}$. Then $\mathcal{J}'_{\alpha+1}/\mathcal{J}'_\alpha$ is isomorphic to a subalgebra of $\mathcal{J}_{\alpha+1}/\mathcal{J}_\alpha$, but being a commutative subalgebra of the algebra of compact operators $\mathcal{J}'_{\alpha+1}/\mathcal{J}'_\alpha$ is isomorphic to $c_0(\Gamma_\alpha)$ for some discrete space Γ_α. As all ideals in $C(K)$ are of the form $\{f\in C(K):f|F=0\}$ for some closed $F\subseteq K$, we obtain a corresponding decreasing continuous sequence $(F_\alpha)_{\alpha<m(\mathcal{A})}$. Since $C(K)/\{f\in C(K):f|F=0\}$ is canonically isomorphic to $C(F)$, we conclude that $\{f\in C(F_\alpha):f|F_{\alpha+1}=0\}$ is isomorphic to $c_0(\Gamma_\alpha)$, and therefore $F_\alpha\setminus F_{\alpha+1}$ is discrete, i.e. consists only of points isolated in F_α. Therefore, as in the proof of (2) to (3) of Theorem 2.7, we easily can obtain that K satisfies (3) of this theorem, so K is scattered and hence by Lemma 3.1, the algebra $C(K)$ and consequently $\mathcal{C}$ has a minimal projection.

(4) $\Rightarrow$ (5) It is clear as $C([0,1])$ has no minimal projections.

(5) $\Rightarrow$ (6) Assume not and take a self-adjoint element h of $\mathcal{A}$ with uncountable spectrum. Then the subalgebra generated by h and the unit of $\mathcal{A}$ is isomorphic to some $C(K)$, where K is metrizable and uncountable. A standard argument shows

that there is a continuous surjection from K onto $[0,1]$. Therefore $C([0,1])$ embeds into $\mathcal{A}$.

(6) $\Rightarrow$ (1) Let $\pi:\mathcal{A}\to\mathcal{B}$ be a surjective *-homomorphism. If $a-\lambda 1_{\mathcal{A}}$ is invertible, then $\pi(a)-\lambda 1_{\mathcal{B}}$ is invertible as well, so the spectrum of $\pi(a)$ in $\mathcal{B}$ is not bigger than the spectrum of a in $\mathcal{A}$, hence the spectra of all elements of $\mathcal{B}$ are countable and so $\mathcal{B}$ does not contain a subalgebra isomorphic to $C([0,1])$. It follows from Lemmas 3.5 and 3.6 that $\mathcal{B}$ has a minimal projection. $\square$

Before the next lemma recall Theorem 2.7.

Lemma 4.1. *Suppose that K is a scattered compact Hausdorff space with Cantor-Bendixson sequence $(K^{(\alpha)})_{\alpha\leq ht(K)}$. Then $C(K)$ is a commutative scattered C^*-algebra with the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha\leq ht(C(K))}$ satisfying $ht(C(K))=ht(K)$ and*

$$\mathcal{I}_\alpha = C^{(\alpha)}(K)=\{f\in C(K): f|K^{(\alpha)}=0\},$$
$$\mathcal{I}^{At}(C(K)/C^{(\alpha)}(K)\equiv c_0(K^{(\alpha)}\setminus K^{(\alpha+1)}).$$

Definition 4.2. *We call a C^*-algebra $\mathcal{A}$ atomic if and only if $\mathcal{I}^{At}(\mathcal{A})$ is an essential ideal.*

The following corresponds to the commutative fact that atoms are dense in superatomic Boolean algebras:

Proposition 4.3. *Every scattered C^*-algebra is atomic.*

Proof. Let $\mathcal{A}$ be a scattered C^*-algebra. It is easy to check that $\mathcal{I}^\perp$ is a (closed) ideal of $\mathcal{A}$ for any ideal $\mathcal{I}\subseteq\mathcal{A}$. Applying Proposition 3.16 for $\mathcal{J}=\mathcal{I}^{At}(\mathcal{A})^\perp$ we conclude that $\mathcal{I}^{At}(\mathcal{J})=\{0\}$, which by Theorem 1.4 (4) means that $\mathcal{J}=0$. Therefore $\mathcal{I}^{At}(\mathcal{A})$ is essential. $\square$

Proposition 4.4. *Every subalgebra and every $*$-homomorphic image of a scattered C^*-algebra is a scattered C^*-algebra.*

Proof. Condition (1) of Theorem 1.4 is hereditary with respect to $*$-homomorphic images and condition (4) is hereditary with respect to subalgebras. $\square$

The following corresponds to the fact (Proposition 2.5) that scattered compact Hausdorff spaces are zero-dimensional. Of course since every scattered C^*-algebra is an approximately finite C^*-algebra (Lemma 5.1 of [42]) which have real rank zero (V. 7. 2 of [16]), so do scattered C^*-algebras. However, this can be also observed independently.

Proposition 4.5. [42, 40] *Suppose that $\mathcal{A}$ is a scattered C^*-algebra. Then*

(1) $\mathcal{A}$ has real rank zero,
(2) $\mathcal{A}$ is approximately finite.

Proof. For (1) will use one of the equivalent conditions to the real rank zero property, which is self-adjoint elements of the algebra can be approximated by self-adjoint elements which have finite spectrum (V. 7. 3 of [16]). Suppose a is a self-adjoint element of $\mathcal{A}$. Let K be compact Hausdorff such that the unital C^*-algebra generated in $\mathcal{A}$ by a is isomorphic to $C(K)$. By Theorem 1.4 (4) K must be scattered and so the element corresponding to a in $C(K)$ must be approximated by a linear combination of projections which have spectrum $\{0,1\}$ and are self adjoint.

To prove (2) we need to use, as in [42], the fact due to Effros, that a separable C^*-algebra $\mathcal{A}$ is approximately finite if for an ideal $\mathcal{J} \subseteq \mathcal{A}$, both $\mathcal{J}$ and the quotient $\mathcal{A}/\mathcal{J}$ are approximately finite (III. 6. 3. [16]) and the fact that a C^*-algebra is approximately finite if all separable subalgebras are approximately finite, which follows directly from the definition. By Proposition 4.4 any separable subalgebra of $\mathcal{A}$ is scattered, so one can apply the above fact and the transfinite induction using either (2) or (3) of Theorem 1.4 and obtain condition (2). □

Proposition 4.6 ([40])**.** *The following are equivalent for a C^*-algebra $\mathcal{A}$:*

(1) $\mathcal{A}$ is scattered,
(2) Every subalgebra of $\mathcal{A}$ is approximately finite,
(3) Every commutative (and separable) subalgebra of $\mathcal{A}$ is scattered,

Proof. (1) $\to$ (2), (3). This follows from Propositions 4.4 and 4.5 (2). The fact that (2) or (3) imply (1) follows from Theorem 1.4 (5), as $C([0,1])$ is neither scattered nor AF. □

One should note that being a scattered C^*-algebra is equivalent to many other conditions which as in the commutative version, that is Theorem 2.8, are concerned with the dual and the bidual of the algebra:

Theorem 4.7 ([31], [32])**.** *Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{A}''$ is its enveloping von Neumann algebra. The following conditions are equivalent for $\mathcal{A}$ to be scattered:*

(1) $\mathcal{A}'' = \prod_{\alpha<\kappa} \mathcal{B}(\mathcal{H}_\alpha)$ for some cardinal κ.
(2) Each non-zero projection in $\mathcal{A}''$ majorizes a minimal projection in $\mathcal{A}$.
(3) Every non-degenerate representation of $\mathcal{A}$ is a sum of irreducible representations.
(4) Every positive functional μ on $\mathcal{A}$ is of the form $\Sigma_{n\in\mathbb{N}} t_n\mu_n$ where μ_ns are pure states and $t_n \in \mathbb{R}^+ \cup \{0\}$ are such that $\Sigma_{n\in\mathbb{N}} t_n < \infty$.
(5) The dual of $\mathcal{A}$ is isomorphic to the ℓ_1-sum $(\bigoplus_{\alpha<\beta} \mathcal{T}C(\mathcal{H}_\alpha))_{\ell_1}$ where $TC(\mathcal{H})$ denotes the trace class operators on a Hilbert space $\mathcal{H}$.
(6) The dual spaces $\mathcal{C}^*$ of separable subalgebras $\mathcal{C} \subseteq \mathcal{A}$ are separable.
(7) The spectrum spaces $\hat{\mathcal{C}}$ of separable subalgebras $\mathcal{C} \subseteq \mathcal{A}$ are countable.

5. Cantor-Bendixson derivatives of tensor products

Despite the following theorem of C. Chu it is unclear how to calculate the Cantor-Bendixson sequence of the tensor product of scattered C^*-algebras. Note that even in the commutative case $(K \times L)'$ is not $K' \times L'$, for K, L compact, however, $K \times L \setminus (K \times L)' = (K \setminus K') \times (L \setminus L')$.

Theorem 5.1 ([14])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. The maximal tensor product $\mathcal{A} \otimes_{\max} \mathcal{B}$ is scattered if and only if $\mathcal{A}$ and $\mathcal{B}$ are scattered.*

Recall that by (3) of Theorem 1.4 every scattered C^*-algebra is GCR (or type I or postliminar; see [2, Theorem 1.5.5] and page 169 of [48]). Therefore every scattered C^*-algebra $\mathcal{A}$ is nuclear (e.g., [7, IV.3.1.3]), i.e., for every C^*-algebra $\mathcal{B}$ there is a unique C^*-tensor product $\mathcal{A} \otimes \mathcal{B}$.

Lemma 5.2. *Assume $\mathcal{A}_1$ and $\mathcal{A}_2$ are two scattered C^*-algebras. Then*

$$\mathcal{I}^{At}(\mathcal{A}_1 \otimes \mathcal{A}_2) = \mathcal{I}^{At}(\mathcal{A}_1) \otimes \mathcal{I}^{At}(\mathcal{A}_2).$$

Proof. Let p_i be a minimal projection in $\mathcal{A}_i$ for $i = 1, 2$. Then

$$p_1 \otimes p_2(\mathcal{A}_1 \otimes \mathcal{A}_2)p_1 \otimes p_2 = p_1\mathcal{A}_1 p_1 \otimes p_2\mathcal{A}_2 p_2 = \mathbb{C}(p_1 \otimes p_2).$$

Therefore $p_1 \otimes p_2$ is a minimal projection in $\mathcal{A}_1 \otimes \mathcal{A}_2$ and hence $\mathcal{I}^{At}(\mathcal{A}_1) \otimes \mathcal{I}^{At}(\mathcal{A}_2) \subseteq \mathcal{I}^{At}(\mathcal{A}_1 \otimes \mathcal{A}_2)$.

Since every scattered C^*-algebra is atomic (Proposition 4.3) we know that $\mathcal{I}^{At}(\mathcal{A}_1)$ and $\mathcal{I}^{At}(\mathcal{A}_2)$ are essential ideals of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. If we show that $\mathcal{I}^{At}(\mathcal{A}_1) \otimes \mathcal{I}^{At}(\mathcal{A}_2)$ is an essential ideal of $\mathcal{A}_1 \otimes \mathcal{A}_2$ then by the above and Proposition 3.19 we have $\mathcal{I}^{At}(\mathcal{A}_1) \otimes \mathcal{I}^{At}(\mathcal{A}_2) = \mathcal{I}^{At}(\mathcal{A}_1 \otimes \mathcal{A}_2)$. Assume $\mathcal{J}$ is a nonzero ideal of $\mathcal{A}_1 \otimes \mathcal{A}_2$. It is known that $\mathcal{J}$ contains a nonzero elementary tensor product $x_1 \otimes x_2$, where $x_1 \in \mathcal{A}_1$ and $x_2 \in \mathcal{A}_2$ (this is true in general for nonzero closed ideals of the minimal tensor products of C^*-algebras; see for example [8, Lemma 2.12]). For $i = 1, 2$, since $\mathcal{I}^{At}(\mathcal{A}_i)$ is an essential ideal of $\mathcal{A}_i$, there is $a_i \in \mathcal{I}^{At}(\mathcal{A}_i)$ such that $a_i x_i \neq 0$. Then

$$a_1x_1 \otimes a_2x_2 = (a_1 \otimes a_2)(x_1 \otimes x_2) \neq 0$$

belongs to $\mathcal{J} \cap \mathcal{I}^{At}(\mathcal{A}_1) \otimes \mathcal{I}^{At}(\mathcal{A}_2)$. Thus $\mathcal{I}^{At}(\mathcal{A}_1) \otimes \mathcal{I}^{At}(\mathcal{A}_2)$ is an essential ideal of $\mathcal{A}_1 \otimes \mathcal{A}_2$.

□

Proposition 5.3. *Suppose that $\mathcal{A}$ is a scattered C^*-algebra with the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$. Then $\mathcal{A} \otimes \mathcal{K}(\ell_2)$ is a scattered C^*-algebra whose Cantor-Bendixson sequence $(\mathcal{J}_\alpha)_{\alpha \leq ht(\mathcal{A})}$ satisfies $\mathcal{J}_\alpha = \mathcal{I}_\alpha \otimes \mathcal{K}(\ell_2)$ for every $\alpha \leq ht(\mathcal{A} \otimes \mathcal{K}(\ell_2)) = ht(\mathcal{A})$. In particular $\mathcal{J}_{\alpha+1}/\mathcal{J}_\alpha$ is $*$-isomorphic to $(\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha) \otimes \mathcal{K}(\ell_2)$.*

Proof. We prove $\mathcal{J}_\alpha = \mathcal{I}_\alpha \otimes \mathcal{K}(\ell_2)$ by induction on $\alpha \leq ht(\mathcal{A})$. For $\alpha = 1$ this follows from Lemma 5.2. At a successor ordinal by Lemma 5.2 and the inductive assumption we have $\mathcal{J}_{\alpha+1}/\mathcal{J}_\alpha = \mathcal{I}^{At}\big((\mathcal{A} \otimes \mathcal{K}(\ell_2))/\mathcal{J}_\alpha\big) = \mathcal{I}^{At}\big(\big(\mathcal{A} \otimes \mathcal{K}(\ell_2)\big)/(\mathcal{I}_\alpha \otimes \mathcal{K}(\ell_2))\big) = \mathcal{I}^{At}\big((\mathcal{A}/\mathcal{I}_\alpha) \otimes \mathcal{K}(\ell_2)\big) = \mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha) \otimes \mathcal{K}(\ell_2) = (\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha) \otimes \mathcal{K}(\ell_2)$ and so $\mathcal{J}_{\alpha+1} = \mathcal{I}_{\alpha+1} \otimes \mathcal{K}(\ell_2)$. The limit ordinal case is immediate. □

6. Fully noncommutative scattered C^*-algebras

Definition 6.1. *Let $\mathcal{A}$ be a scattered C^*-algebra with the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$. $\mathcal{A}$ is called fully noncommutative if $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is isomorphic to $\mathcal{K}(\mathcal{H}_\alpha)$ for infinite dimensional Hilbert spaces $\mathcal{H}_\alpha$, for each $\alpha < ht(\mathcal{A})$.*

Note that if $\mathcal{A}$ is stable or $ht(\mathcal{A})$ is a limit ordinal, then in the definition above each $\mathcal{H}_\alpha$, $\alpha < ht(\mathcal{A})$ must be already infinite dimensional, since otherwise $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha = \mathcal{A}/\mathcal{I}_\alpha \cong M_n$ for some $n \in \mathbb{N}$, which is impossible in both cases.

Lemma 6.2. *Suppose that $\mathcal{A}$ is a fully noncommutative scattered C^*-algebra with the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$ and $\mathcal{J} \subseteq \mathcal{A}$ is an ideal of $\mathcal{A}$. Then $\mathcal{J} = \mathcal{I}_\alpha$ for some $\alpha \leq ht(\mathcal{A})$.*

Proof. Let $\beta \leq ht(\mathcal{A})$ be the minimal ordinal such that $\mathcal{I}_\beta \not\subseteq \mathcal{J}$. If there is no such β we have $\mathcal{I}_{ht(\mathcal{A})} = \mathcal{A} = \mathcal{J}$. It follows that $\beta = \alpha + 1$ for some $\alpha < ht(\mathcal{A})$. So $\sigma_\alpha[\mathcal{I}_{\alpha+1} \cap \mathcal{J}]$ is a proper ideal of $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$, where σ_α is the quotient map. By Definition 6.1 the quotient $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is isomorphic to the algebra of all compact operators on a Hilbert space, which has no nonzero proper ideals. So $(\mathcal{J} \cap \mathcal{I}_{\alpha+1})/\mathcal{I}_\alpha$ is the zero ideal. Moreover by Theorem 1.4 (2) $\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha) = \mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$, which is an essential ideal by Theorem 1.4 (1) and Proposition 4.3. Therefore $\mathcal{J}/\mathcal{I}_\alpha = \{0\}$ which implies that $\mathcal{J} = \mathcal{I}_\alpha$, as required. □

Proposition 6.3. *Suppose that $\mathcal{A}$ is a scattered C^*-algebra with the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha \leq ht(\mathcal{A})}$. If $\mathcal{A}$ is fully noncommutative, then*

(1) the ideals of $\mathcal{A}$ form a chain,
(2) the centers of the multipler algebras of any quotient of $\mathcal{A}$ are all trivial.

If $\mathcal{A}$ is stable or $ht(\mathcal{A})$ is a limit ordinal, then (1) *and* (2) *are equivalent to $\mathcal{A}$ being fully noncommutative.*

Proof. Assume $\mathcal{A}$ is fully noncommutative. The statement (1) is Lemma 6.2.

For (2), by Lemma 6.2 a nonzero quotient of $\mathcal{A}$ must be of the form $\mathcal{A}/\mathcal{I}_\alpha$ for some $\alpha < ht(\mathcal{A})$. By the Dauns-Hofmann theorem (4.4.8 of [52]) the center of the multiplier algebra of $\mathcal{A}/\mathcal{I}_\alpha$ is isomorphic to the C^*-algebra of the bounded continuous functions on the spectrum $\widehat{\mathcal{A}/\mathcal{I}_\alpha}$ of $\mathcal{A}/\mathcal{I}_\alpha$. The quotient $\mathcal{A}/\mathcal{I}_\alpha$ is again scattered and therefore $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is an essential ideal of $\mathcal{A}/\mathcal{I}_\alpha$ (Proposition 4.3). This implies that $\mathcal{A}/\mathcal{I}_\alpha$ can be embedded into the multiplier algebra of $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha \cong \mathcal{K}(\ell_2(\kappa))$, for some cardinal κ so that the embedding is the identity on $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$. It follows that $\mathcal{A}/\mathcal{I}_\alpha$ has a faithful representation in $\mathcal{B}(\ell_2(\kappa))$, where $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is mapped onto $\mathcal{K}(\ell_2(\kappa))$. So this representation is irreducible and its kernel zero is included in the kernel of any other irreducible representation of $\mathcal{A}/\mathcal{I}_\alpha$. Therefore all points of the spectrum of $\mathcal{A}/\mathcal{I}_\alpha$ are in the closure (with the Jacobson topology) of this one point. Hence the only continuous functions on the spectrum of $\mathcal{A}/\mathcal{I}_\alpha$ are constant maps. It follows from the Dauns-Hofmann theorem that the center of the multipliers of $\mathcal{A}/\mathcal{I}_\alpha$ is trivial.

Now assume either $\mathcal{A}$ is stable or $ht(\mathcal{A})$ is a limit ordinal.

Assume (1) holds. Suppose that one of the quotients $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is not isomorphic to the algebra of all compact operators on an infinite dimensional Hilbert space. Since $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha = \mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$ and by Proposition 3.14, $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ it is isomorphic to some subalgebra of all compact operators on a Hilbert space ([2, Theorem 1.4.5.]). Hence, by the comment after Definition 6.1 we may assume that $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is isomorphic to a subalgebra of an algebra of the form $\mathcal{K}(\mathcal{H}_1) \oplus \mathcal{K}(\mathcal{H}_2)$ for some Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Now $\mathcal{I}_\alpha + \mathcal{K}(\mathcal{H}_1) \oplus \{0\}$ and $\mathcal{I}_\alpha + \{0\} \oplus \mathcal{K}(\mathcal{H}_2)$ are two incomparable (with respect to inclusion) ideals of $\mathcal{A}$.

Assume (2) holds. If $\mathcal{A}$ is not fully commutative, then it has a quotient $\mathcal{A}/\mathcal{I}_\alpha$ with an essential ideal $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ which is isomorphic to $\bigoplus_{i\in I} \mathcal{K}(\mathcal{H}_i)$ for some nonzero Hilbert spaces $\mathcal{H}_i$ and some set I with at least two elements. By the essentiality of the ideal $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ in $\mathcal{A}/\mathcal{I}_\alpha$, there is an embedding of $\mathcal{A}/\mathcal{I}_\alpha$ into $\bigoplus_{i\in I} \mathcal{B}(\mathcal{H}_i)$, which sends $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ onto $\bigoplus_{i\in I} \mathcal{K}(\mathcal{H}_i)$ and $\bigoplus_{i\in I} \mathcal{B}(\mathcal{H}_i)$ can be identified with the algebra of multipliers of $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$. However since $|I| \geq 2$, the projections on the factors $\mathcal{B}(\mathcal{H}_i)$ witness the fact that the center of the algebra of multipliers of $\mathcal{A}/\mathcal{I}_\alpha$ is nontrivial. □

7. Some constructions of scattered C^*-algebras

In this section we give examples of constructions of scattered C^*-algebras with some prescribed sequences of Cantor-Bendixson derivatives. All these examples correspond to classical classes of scattered locally compact spaces. As we want to produce "genuinely" noncommutative examples, we will focus on two notions of being fully noncommutative (Definition 6.1) and the stability of C^*-algebras (cf. [58]):

Definition 7.1. *A C^*-algebra $\mathcal{A}$ is called stable if and only if $\mathcal{A}$ is $*$-isomorphic to $\mathcal{A} \otimes \mathcal{K}(\ell_2)$.*

The first immediate group of constructions is obtained by tensoring the commutative examples by $\mathcal{K}(\ell_2)$ and applying Proposition 5.3. They are not fully noncommutative, but are stable. Before the following theorem recall the definitions of width and height of a Scattered C^*-algebras (Definition 1.5).

Theorem 7.2. *Let κ be a regular cardinal.*

- *There are stable scattered C^*-algebras of countable width and height α for any $\alpha < \omega_2$.*
- *It is consistent that there are stable scattered C^*-algebras of countable width and height α for any $\alpha < \omega_3$.*
- *It is consistent that there are stable scattered C^*-algebras of width κ and height κ^+.*
- *There are stable scattered C^*-algebras $\mathcal{A}$ of height 2 with $\mathcal{I}^{At}(\mathcal{A}) \cong c_0 \otimes \mathcal{K}(\ell_2)$ and $\mathcal{A}/\mathcal{I}^{At}(\mathcal{A}) \cong c_0(\mathfrak{c}) \otimes \mathcal{K}(\ell_2)$ where $\mathfrak{c}$ denotes the cardinality of the continuum.*

Proof. In all these cases we apply Proposition 5.3 for tensor products $C(K) \otimes \mathcal{K}(\ell_2)$ for appropriate compact scattered Hausdorff spaces K from [34], [4], [45], [36] and the Ψ-spaces surveyed e.g. in [30]. See also [55]. □

In the rest of this section we focus on obtaining fully noncommutative examples which do not require any special set-theoretic assumptions. Before doing so let us inquire about the relationship between the stability and being fully noncommutative. As shown in the above examples the former does not imply the latter, but being fully noncommutative implies the stability for separable scattered C^*-algebras.

Lemma 7.3. *Suppose that $\mathcal{A}$ is a separable nonunital scattered C^*-algebra with the Cantor-Bendixson sequence $(\mathcal{I}_\alpha)_{\alpha < ht(\mathcal{A})}$. Then $\mathcal{A}$ is stable if and only if $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is stable for each $\alpha < ht(\mathcal{A})$. In particular, if $\mathcal{A}$ is fully noncommutative, then $\mathcal{A}$ is stable.*

Proof. For the forward implication uses the fact that ideals and quotients of stable C^*-algebras are stable (2.3. (ii) of [58]).

The proof of the backward implication is by induction on the height of the algebra. Suppose that the height is a successor ordinal $\alpha + 1$. We have that $\mathcal{A} = \mathcal{I}_{\alpha+1}$ and that both $\mathcal{I}_\alpha$ and that $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ are stable, by the inductive assumption. Consider the short exact sequence,

$$0 \to \mathcal{I}_\alpha \to \mathcal{A} \to \mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha \to 0.$$

Now we use the fact that scattered C^*-algebras are approximately finite (Lemma 5.1 of [42]) and Blackadar's characterization of separable stable AF-algebras as AF-algebras with no nontrivial bounded trace ([6], [58]). A nontrivial bounded trace on $\mathcal{A}$ would need to be zero on $\mathcal{I}_\alpha$, by the inductive assumption. Hence it would define a nontrival bounded trace on $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$, which is impossible since it is stable (see also Proposition 6.12 of [58]). If the height of $\mathcal{A}$ is a limit ordinal, then the result follows from the fact that countable inductive limits of separable stable C^*-algebras are also stable (Corollary 2.3 of [58]). □

Examples from papers [23, 24] show that nonseparable fully noncommutative scattered C^*-algebras do not have to be stable. However by tensoring a fully-noncommutative C^*-algebra by $\mathcal{K}(\ell_2)$ we obtain a fully noncommutative C^*-algebra of the same height and width which is additionally stable.

Recall that for a cardinal κ, κ^+ denotes the minimal cardinal which is strictly bigger than the κ. The following proposition shows that one can start from $\mathcal{K}(\ell_2(\kappa))$, where κ is a regular cardinal and increase the height of the algebra up to any ordinal $\theta < \kappa^+$. In these constructions the algebras of lower height are not essential ideals of those previously constructed, and so we can not get to the height κ^+ without increasing the width. These constructions correspond to consecutive application of one-point compactification and the Cartesian product by κ with the discrete topology, that is taking the disjoint union of κ-many copies of one point compactification of the previous spaces.

Proposition 7.4. *Suppose that κ is a regular cardinal. For any $\theta < \kappa^+$ there is a scattered C^*-algebra $\mathcal{A}_\theta$ of height θ such that if $(\mathcal{I}^\theta_\alpha)_{\alpha \le ht(\mathcal{A}_\theta)}$ is the Cantor-Bendixson sequence of $\mathcal{A}_\theta$, the algebra $\mathcal{I}^\theta_{\alpha+1}/\mathcal{I}^\theta_\alpha$ is isomorphic to $\mathcal{K}(\ell_2(\kappa))$ for every $\alpha < ht(\mathcal{A}_\theta)$. In particular, each $\mathcal{A}_\theta$ is fully noncommutative, with $wd(\mathcal{A}_\theta) = \kappa$. There are stable examples with this property.*

Proof. Let $\{1_{\beta,\alpha} : \alpha, \beta < \kappa\}$ be a system of matrix units for $\mathcal{K}(\ell_2(\kappa))$. Define $\mathcal{A}_\theta$ for $\theta < \kappa^+$ by recursion together with embeddings $\pi_{\theta,\eta} : \mathcal{A}_\eta \to \mathcal{A}_\theta$ for $\eta \le \theta < \kappa^+$. The embeddings satisfy the condition that $\pi_{\gamma,\theta} \circ \pi_{\theta,\eta} = \pi_{\gamma,\eta}$ for $\eta \le \theta \le \gamma < \kappa^+$.

Let $\mathcal{A}_0 = \{0\}$ and $\mathcal{A}_1 = \mathcal{K}(\ell_2(\kappa))$. Fix $\theta < \kappa^+$ and suppose we are given $\mathcal{A}_\eta$ and $\pi_{\gamma,\eta}$ for every $\eta \le \gamma < \theta$. First consider the case where $\theta = \eta + 1$ for some $\eta < \kappa^+$. Let $\mathcal{A}_\theta = \widetilde{A_\eta} \otimes \mathcal{K}(\ell_2(\kappa))$. By the assumption we have $ht(\mathcal{A}_\eta) = \eta$ (i.e., $\mathcal{I}^\eta_\eta = \mathcal{A}_\eta$). Define the embedding $\pi_{\theta,\eta} : \mathcal{A}_\eta \to \mathcal{A}_\theta$ by mapping $a \to (a, 0) \otimes 1_{0,0}$ (the top left entry of the $\kappa \times \kappa$ matrix over $\widetilde{A_\eta}$) and let $\pi_{\theta,\alpha} = \pi_{\theta,\eta} \circ \pi_{\eta,\alpha}$ for $\alpha < \eta$.

We claim that $\mathcal{I}^\theta_\beta = \mathcal{I}^\eta_\beta \otimes \mathcal{K}(\ell_2(\kappa))$ for all $\beta < ht(\mathcal{A}_\eta) = \eta$. We show this by induction on β. Assume this is true for some $\beta < \eta$. Then

$$
\begin{aligned}
\mathcal{I}^\theta_{\beta+1}/\mathcal{I}^\theta_\beta = \mathcal{I}^{At}(\mathcal{A}_\theta/\mathcal{I}^\theta_\beta) &= \mathcal{I}^{At}\big(\frac{\widetilde{A_\eta} \otimes \mathcal{K}(\ell_2(\kappa))}{\mathcal{I}^\eta_\beta \otimes \mathcal{K}(\ell_2(\kappa))}\big) \\
&\cong \mathcal{I}^{At}\big(\frac{\widetilde{A_\eta}}{\mathcal{I}^\eta_\beta} \otimes \mathcal{K}(\ell_2(\kappa))\big) \\
\text{(Lemma 5.2)} \qquad &= \mathcal{I}^{At}\big(\frac{\widetilde{A_\eta}}{\mathcal{I}^\eta_\beta}\big) \otimes \mathcal{I}^{At}\big(\mathcal{K}(\ell_2(\kappa))\big) \\
\text{(Lemma 3.4)} \qquad &= \frac{\mathcal{I}^\eta_{\beta+1}}{\mathcal{I}^\eta_\beta} \otimes \mathcal{K}(\ell_2(\kappa)) \\
&\cong \mathcal{K}(\ell_2(\kappa) \otimes \ell_2(\kappa)) \cong \mathcal{K}(\ell_2(\kappa)).
\end{aligned}
$$

Let $\sigma_{\theta,\beta}$ (resp. $\sigma_{\eta,\beta}$) denote the canonical quotient maps from $\mathcal{A}_\theta$ (resp. $\mathcal{A}_\eta$) onto $\mathcal{A}_\theta/\mathcal{I}^\theta_\beta$ (resp. $\mathcal{A}_\eta/\mathcal{I}^\eta_\beta$). Also let

$$
\psi : \frac{\mathcal{A}_\theta}{\mathcal{I}^\theta_\beta} = \frac{\widetilde{A_\eta} \otimes \mathcal{K}(\ell_2(\kappa))}{\mathcal{I}^\eta_\beta \otimes \mathcal{K}(\ell_2(\kappa))} \to \frac{\widetilde{A_\eta}}{\mathcal{I}^\eta_\beta} \otimes \mathcal{K}(\ell_2(\kappa))
$$

be the natural isomorphism. Then the composition $\psi \circ \sigma_{\theta,\beta}$ is the map which sends $(a, \lambda) \otimes T \in \mathcal{A}_\theta$ to $((a, \lambda) + \mathcal{I}^\eta_\beta) \otimes T$. Therefore $\psi \circ \sigma_{\theta,\beta} = \widetilde{\sigma}_{\eta,\beta} \otimes id$ (where $\widetilde{\sigma}_{\eta,\beta}$ is the natural extension of $\sigma_{\eta,\beta}$ to $\widetilde{\mathcal{A}_\eta}$ and id is the identity map on $\mathcal{K}(\ell_2(\kappa)))$. Thus

$$\begin{aligned}\mathcal{I}^\theta_{\beta+1} &= \sigma^{-1}_{\theta,\beta}(\mathcal{I}^{At}(\mathcal{A}_\theta/\mathcal{I}^\theta_\beta))\\ &= \sigma^{-1}_{\theta,\beta}\circ\psi^{-1}\big(\frac{\mathcal{I}^\eta_{\beta+1}}{\mathcal{I}^\eta_\beta}\otimes\mathcal{K}(\ell_2(\kappa))\big)\\ &= (\widetilde{\sigma}_{\eta,\beta}\otimes id)^{-1}\big(\frac{\mathcal{I}^\eta_{\beta+1}}{\mathcal{I}^\eta_\beta}\otimes\mathcal{K}(\ell_2(\kappa))\big)\\ &= \mathcal{I}^\eta_{\beta+1}\otimes\mathcal{K}(\ell_2(\kappa)).\end{aligned}$$

Assume $\beta\le\eta$ is a limit ordinal and the claim is true for all the ordinals below β.

$$\begin{aligned}\mathcal{I}^\theta_\beta &= \overline{\bigcup_{\alpha<\beta}\mathcal{I}^\theta_\alpha} = \overline{\bigcup_{\alpha<\beta}\mathcal{I}^\eta_\alpha\otimes\mathcal{K}(\ell_2(\kappa))}\\ &= \overline{\bigcup_{\alpha<\beta}\mathcal{I}^\eta_\alpha}\otimes\mathcal{K}(\ell_2(\kappa)) = \mathcal{I}^\eta_\beta\otimes\mathcal{K}(\ell_2(\kappa)).\end{aligned}$$

Next step is to show that $ht(\mathcal{A}_\theta)=\theta=\eta+1$. First notice that

$$\begin{aligned}\mathcal{I}^\theta_\theta/\mathcal{I}^\theta_\eta = \mathcal{I}^{At}\big(\mathcal{A}_\theta/\mathcal{I}^\theta_\eta\big) &\cong \mathcal{I}^{At}\big(\frac{\widetilde{A_\eta}\otimes\mathcal{K}(\ell_2(\kappa))}{\mathcal{A}_\eta\otimes\mathcal{K}(\ell_2(\kappa))}\big)\\ &\cong \mathcal{I}^{At}\big(\frac{\widetilde{A_\eta}}{\mathcal{A}_\eta}\otimes\mathcal{K}(\ell_2(\kappa))\big)\\ \text{(Lemma 5.2)}\qquad &= \mathcal{I}^{At}\big(\frac{\widetilde{A_\eta}}{\mathcal{A}_\eta}\big)\otimes\mathcal{I}^{At}\big(\mathcal{K}(\ell_2(\kappa))\big)\\ &\cong \mathbb{C}\otimes\mathcal{K}(\ell_2(\kappa))\cong\mathcal{K}(\ell_2(\kappa)).\end{aligned}$$

Similar to the above we have

$$\mathcal{I}^\theta_\theta = (\widetilde{\sigma}_{\eta,\eta}\otimes id)^{-1}\big(\frac{\widetilde{A_\eta}}{\mathcal{A}_\eta}\otimes\mathcal{K}(\ell_2(\kappa))\big) = \mathcal{A}_\theta.$$

Suppose θ is a limit ordinal. Let $\mathcal{A}_\theta$ be the inductive limit of the system $\{(\mathcal{A}_\eta,\pi_{\gamma,\eta}):\eta\le\gamma<\theta\}$. By repeatedly using the above claim we have

$$\mathcal{I}^\theta_\beta = \mathcal{I}^\beta_\beta\otimes\bigotimes_{\beta<\gamma<\theta}\mathcal{K}(\ell_2(\kappa))$$

for $\beta<\theta$. For every $\eta<\theta$, the map $\pi_{\theta,\eta}:\mathcal{A}_\eta\to\mathcal{A}_\theta$ defined by

$$\pi_{\theta,\eta}(a) = \varinjlim_{\gamma<\theta}\pi_{\gamma,\eta}(a)$$

is an embedding. Assume $a\in\mathcal{A}_\theta$ and $\epsilon>0$ are given. Then there are $\eta<\theta$ and $b\in\mathcal{A}_\eta$ such that

$$\|\pi_{\theta,\eta}(b)-a\|<\epsilon.$$

Since $\mathcal{A}_\eta=\mathcal{I}^\eta_\eta$, we have $\pi_{\theta,\eta}(b)\in\mathcal{I}^\theta_\eta$. Therefore $a\in\overline{\bigcup_{\eta<\theta}\mathcal{I}^\theta_\eta}=\mathcal{I}^\theta_\theta$. Thus $\mathcal{A}_\theta=\mathcal{I}^\theta_\theta$ and $ht(\mathcal{A}_\theta)=\theta$. Also

$$\mathcal{I}^\theta_{\beta+1}/\mathcal{I}^\theta_\beta \cong \frac{\mathcal{I}^{\beta+1}_{\beta+1}}{\mathcal{I}^{\beta+1}_\beta}\otimes\bigotimes_{\beta<\eta\le\theta}\mathcal{K}(\ell_2(\kappa))\cong\mathcal{K}(\ell_2(\kappa)),$$

if $\beta<\theta$. This completes the proof. □

It is more difficult to increase the height of the algebra further without increasing its width, i.e., keeping the previous algebra as an essential ideal. In the following we focus on the case $\kappa = \omega$, and try to increase the height of the algebra to uncountable ordinals while preserving the width ω. We will see that the stability of an algebra can be employed to guarantee that it would be an essential ideal in the next algebra, helping us to maintain the same width throughout the construction. The successor stage corresponds in the commutative case to dividing a locally compact scattered Hausdorff space into infinitely many clopen noncompact subspaces of the same height and one-point compactifying each part. This technique is behind the standard example of a thin-tall locally compact space due to Juhasz and Weiss ([34]). For a scattered C^*-algebra $\mathcal{A}$, let $(\mathcal{I}_\beta(\mathcal{A}))_{\beta\leq ht(\mathcal{A})}$ denote its Cantor-Bendixson sequence,

Lemma 7.5. *Suppose that $\mathcal{A}$ a scattered stable C^*-algebra of height β. Then there is a stable scattered C^*-algebra $\mathcal{B}$ of height $\beta+1$ containing $\mathcal{A}$ as and essential ideal such that $\mathcal{I}_\beta(\mathcal{B}) = \mathcal{A}$ and $\mathcal{B}/\mathcal{A} \cong \mathcal{K}(\ell_2)$.*

Proof. Since $\mathcal{A}$ is stable, we can identify $\mathcal{A}$ with $\mathcal{A} \otimes \mathcal{K}(\ell_2)$. Let

$$\mathcal{B} = \widetilde{\mathcal{A}} \otimes \mathcal{K}(\ell_2).$$

As $\mathcal{A}$ is an essential ideal of $\widetilde{\mathcal{A}}$, we have that $\mathcal{A} \otimes \mathcal{K}(\ell_2)$ is an essential ideal of $\mathcal{B}$ (see the proof of Proposition 5.2). It is also clear that $\mathcal{B}/\mathcal{A} \cong \mathbb{C} \otimes \mathcal{K}(\ell_2) \cong \mathcal{K}(\ell_2)$. To calculate the Cantor-Bendixson derivatives of $\mathcal{B}$ we use Proposition 5.3 and conclude that $\mathcal{I}_\alpha(\mathcal{B}) = \mathcal{I}_\alpha(\widetilde{\mathcal{A}}) \otimes \mathcal{K}(\ell_2)$ for $\alpha \leq \beta$. Morover the isomophism between $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{K}(\ell_2)$ sends $\mathcal{I}_\alpha(\mathcal{A})$ onto $\mathcal{I}_\alpha(\widetilde{\mathcal{A}}) \otimes \mathcal{K}(\ell_2)$, by Proposition 5.3 and the fact that an isomorphism must preserve the Cantor-Bendixson ideals. □

Theorem 7.6. *There is a thin-tall fully noncommutative C^*-algebra (which is a subalgebra of $\mathcal{B}(\ell_2)$).*

Proof. We construct an inductive limit $\mathcal{A}$ of stable scattered separable fully noncommutative algebras $(\mathcal{A}_\alpha)_{\alpha<\omega_1}$ such that

- $\mathcal{A}_0 = \mathcal{K}(\ell_2)$,
- $\mathcal{A}_\alpha$ is an essential ideal of $\mathcal{A}_{\alpha+1}$ for every $\alpha < \omega_1$,
- $\mathcal{A}_{\alpha+1}/\mathcal{A}_\alpha$ is $*$-isomorphic to $\mathcal{K}(\ell_2)$ for every $\alpha < \omega_1$,
- $\mathcal{A}_\lambda$ is an inductive limit of $\mathcal{A}_\alpha$s for $\alpha < \lambda$ if λ is a countable limit ordinal.

This is enough, since then by Lemma 3.17 we have that $\mathcal{A}_{\alpha+1}/\mathcal{A}_\alpha$ is an essential ideal in $\mathcal{A}/\mathcal{A}_\alpha$, and so by Proposition 3.20 we conclude that $\mathcal{I}^{At}(\mathcal{A}/\mathcal{A}_\alpha) = \mathcal{A}_{\alpha+1}/\mathcal{A}_\alpha$.

Given $\mathcal{A}_\alpha$ apply Lemma 7.5 to get $\mathcal{A}_{\alpha+1}$. The resulting algebra satisfies the induction requirements, it is stable, fully noncommutative, separable and scattered of height $\alpha + 1$.

At countable limit ordinals take the inductive limit of the previously constructed chain of ideals. It is clear that we obtain a fully noncommutative scattered algebra of appropriate height. To prove that it is stable we use Corollary 2.3. (i) of [58] implying that the countable inductive limit of separable stable algebras is stable.

The final algebra is the inductive limit of the entire uncountable sequence. □

Whether the algebra constructed in the theorem above is stable depends on the additional features of the construction. Clearly $\mathcal{A} \otimes \mathcal{K}(\ell_2)$ is stable, for $\mathcal{A}$ from the above theorem, so the construction can be carried out in such a way that this isomorphism is respected. However one can construct a fully noncommutative thin-tall scattered C^*-algebra which is not stable (see [24]). Note that the Cantor-Bendixson ideals of such C^*-algebras are always stable by Lemma 7.3. This provides examples of C^*-algebras which are nonstable and with no maximal stable ideal. This follows from the fact that in fully commutative scattered C^*-algebras all ideals are among the Cantor-Bendixson ideals by Lemma 6.2.

Theorem 7.7. *There is a fully noncommutative scattered algebra $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ which is an inductive limit of a sequence $(\mathcal{I}_\alpha)_{\alpha<\lambda}$ of its essential ideals for a limit ordinal $\lambda \leq \mathfrak{c}^+$ of uncountable cofinality such that $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is isomorphic to $\mathcal{K}(\ell_2)$ for each $\alpha < \lambda$, with each $\mathcal{I}_\alpha$ stable but $\mathcal{A}$ nonstable. In particular stability of C^*-algebras is not preserved by uncountable inductive limits and there are nonstable C^*-algebras without a stable ideal which is maximal among stable ideals.*

Proof. Perform the recursive construction as in the proof of Theorem 7.6 obtaining $\mathcal{A}_\alpha$s which are stable, fully noncommutative, scattered of height α and form an increasing chain of essential ideals. Let λ be the first ordinal at which the construction can not be continued anymore because $\mathcal{A}_\lambda$ is not stable.

First we prove that there is such $\lambda < \mathfrak{c}^+$. Suppose not, and let us derive a contradiction. Then we can produce $\mathcal{A}_{\mathfrak{c}^+}$ and consider $\mathcal{I}^{At}(\mathcal{A}_{\mathfrak{c}^+})$, which is *-isomorphic to $\mathcal{K}(\ell_2)$ and it is an essential ideal of $\mathcal{A}_{\mathfrak{c}^+}$. It follows that $\mathcal{A}_{\mathfrak{c}^+}$ embeds into the multiplier algebra of $\mathcal{K}(\ell_2)$, which is $\mathcal{B}(\ell_2)$. It follows that the density of $\mathcal{A}_{\mathfrak{c}^+}$ is at most continuum, and so the height $ht(\mathcal{A}_{\mathfrak{c}^+}) < (2^\omega)^+$ as successor cardinals are regular, a contradiction.

Now let us see that λ is a limit ordinal of uncountable cofinality. If λ is a successor, then the resulting algebra is stable by Lemma 7.5. At limit ordinals of countable cofinality we can use Corollary 4.1 of from [27], which states that an inductive limit of a sequence of σ-unital stable C^*-algebras is stable. Note that algebras of the form $\mathcal{B} = \widetilde{\mathcal{A}} \otimes \mathcal{K}(\ell_2)$ are σ-unital, because $(1 \otimes p)$ form a countable approximate unit of them, where p runs through finite dimensional projections onto spaces spanned by first $n \in \mathbb{N}$ vectors of some fixed orthonormal basis of ℓ_2. The countable cofinality of λ ensures that our inductive limit $\mathcal{A}_\lambda$ is also an inductive limit of a sequence of algebras $(\mathcal{A}_{\lambda_n})_{n\in\mathbb{N}}$, where $(\lambda_n)_{n\in\mathbb{N}}$ is cofinal in λ. This implies that $\mathcal{A}_\lambda$ is stable, which is a contradiction. Therefore λ is as required.

The absence of a maximal stable ideal in $\mathcal{A}_\lambda$ follows from the minimality of λ and the fact that all ideals in a fully noncommutative scattered C^*-algebras are among $\mathcal{A}_\alpha$s for $\alpha < \lambda$ (Lemma 6.2). □

Our final example is a noncommutative version of the Ψ-space (cf. [30]):

Theorem 7.8. *There is a stable and fully noncommutative scattered C^*-algebras $\mathcal{A}$ of height 2 with $\mathcal{I}^{At}(\mathcal{A}) \cong \mathcal{K}(\ell_2)$ and $\mathcal{A}/\mathcal{I}^{At}(\mathcal{A}) \cong \mathcal{K}(\ell_2(\mathfrak{c}))$.*

Proof. Let $(A_\xi : \xi < \mathfrak{c})$ be an almost disjoint family of subsets of $\mathbb{N}$, that is such a family of infinite subsets of $\mathbb{N}$ that $A_\xi \cap A_\eta$ is finite for any two distinct $\xi, \eta < \mathfrak{c}$. Define in $\mathcal{B}(\ell_2)$ orthogonal projections P_ξ onto the spaces $span(\{e_n : n \in A_\xi\})$ where $(e_n)_{n\in\mathbb{N}}$ is a fixed orthogonal basis of ℓ_2. Let $\sigma_\xi : A_0 \to A_\xi$ be bijections and $T_{\xi,0} \in \mathcal{B}(\ell_2)$ be corresponding partial isometries i.e., $T_{\xi,0}(e_n) = e_{\sigma_\xi(n)}$ if $n \in A_0$ and $T_{\xi,0}(e_n) = 0$ otherwise. Note that $T^*_{\xi,0}(e_n) = e_{\sigma_\xi^{-1}(n)}$ if $n \in A_\xi$ and $T_{\xi,0}(e_n) = 0$ otherwise. For all $\xi, \eta < \kappa$ define

$$T_{\eta,\xi} = T_{\eta,0}T^*_{\xi,0}.$$

That is $T_{\eta,\xi}(e_n) = e_{\sigma_\eta \circ \sigma_\xi^{-1}(n)}$ if $n \in A_\xi$ and $T_{\eta,\xi}(e_n) = 0$ otherwise. Note that $T_{\eta,\xi} = P_\eta T_{\eta,\xi} = T_{\eta,\xi} P_\xi = P_\eta T_{\eta,\xi} P_\xi$.

Let $\mathcal{A}$ be a C^*-algebra generated in $\mathcal{B}(\ell_2)$ by $\{T_{\eta,\xi} : \xi, \eta < \mathfrak{c}\}$ and the compact operators. As $\mathcal{K}(\ell_2)$ is essential ideal in $\mathcal{B}(\ell_2)$, it is essential in $\mathcal{A}$ and so by Proposition 3.20 we have $\mathcal{I}^{At}(\mathcal{A}) = \mathcal{K}(\ell_2)$. Now consider the quotient $\mathcal{A}/\mathcal{K}(\ell_2)$. We have

$$[T_{\eta,\xi}]^*_{\mathcal{K}(\ell_2)} = [T_{\xi,\eta}]_{\mathcal{K}(\ell_2)} \tag{1}$$

as $T^*_{\eta,\xi} = T_{\xi,\eta}$. Moreover

$$[T_{\beta,\alpha}]_{\mathcal{K}(\ell_2)} [T_{\xi,\eta}]_{\mathcal{K}(\ell_2)} = \delta_{\alpha,\xi} [T_{\beta,\eta}]_{\mathcal{K}(\ell_2)} \tag{2}$$

where $\delta_{\alpha,\xi} = 1$ if $\xi = \alpha$ and $\delta_{\alpha,\xi} = 0$ otherwise. This is checked directly if $\alpha = \xi$ and for $\alpha \neq \xi$ we use $T_{\beta,\alpha} T_{\xi,\eta} = T_{\beta,\alpha} P_\alpha P_\xi T_{\xi,\eta}$ and the fact that $P_\alpha P_\xi$ is the projection on a finite dimensional space $span(\{e_n : n \in A_\alpha \cap A_\xi\})$, since A_ξs are almost disjoint. It follows that $T_{\beta,\alpha} T_{\xi,\eta}$ is compact if $\alpha \neq \xi$. It is well-known that C^* algebras having nonzero generators satisfying (1) and (2) are isomorphic to the algebra of compact operators on the Hilbert space $\ell_2(\mathfrak{c})$. This is for example checked in Section 2.3. of [23] and this completes the proof.

□

As in the case of a thin-tall C^*-algebra the above Ψ-algebra may or may not be stable. An example exhibiting very strong nonstability (its multiplier algebra is isomorphic to the minimal unitization) is constructed in [23].

References

1. C. Akemann, J. Doner, *A nonseparable C^*-algebra with only separable abelian C^*-subalgebras.* Bull. London Math. Soc. 11 (1979), no. 3, 279–284.
2. W. Arveson, *An invitation to C^*-algebras*, Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg, 1976.
3. J. Baumgartner, *Almost-disjoint sets, the dense set problem and the partition calculus.* Ann. Math. Logic 9 (1976), no. 4, 401–439.
4. J.Baumgartner, S. Shelah, *Remarks on superatomic Boolean algebras.* Ann. Pure Appl. Logic 33 (1987), no. 2, 109–129.
5. T. Bice, P. Koszmider, *A note on the Akemann-Doner and Farah-Wofsey constructions*, `http://arxiv.org/abs/1602.02429`
6. B. Blackadar, *Traces on simple AF C^*-algebras.* J. Funct. Anal. 38 (1980), no. 2, 156–168.
7. B. Blackadar, *Operator algebras: theory of C*-algebras and von Neumann algebras*, Vol. 122. Springer Science & Business Media, 2006.
8. E. Blanchard, and E. Kirchberg, *Non-simple purely infinite C^*-algebras: the Hausdorff case.* Journal of Functional Analysis 207.2 (2004): 461-513.
9. R. Bonnet, M. Rubin, *A thin-tall Boolean algebra which is isomorphic to each of its uncountable subalgebras.* Topology Appl. 158 (2011), no. 13, 1503–1525.
10. C. Brech, P. Koszmider, *Thin-very tall compact scattered spaces which are hereditarily separable.* Trans. Amer. Math. Soc. 363 (2011), no. 1, 501–519.
11. L. Brown, R. Douglas, P. Fillmore, *Extensions of C^*-algebras and K-homology.* Ann. of Math. (2) 105 (1977), no. 2, 265–324.
12. L. Brown, G. Pedersen, *C^*-algebras of real rank zero.* J. Funct. Anal. 99 (1991), no. 1, 131–149.
13. C. Chu, *A note on scattered C^*-algebras and the Radon-Nikodym property*, J. London Math. Soc. 24 (1981), 533–536.
14. C. Chu, *Crossed products of scattered C^*-algebras*, J. London Math. Soc. (2) 26 (1982), no. 2, 317–324.
15. C. Chu, *The weak Banach-Saks property in C^*-algebras.* J. Funct. Anal. 121 (1994), no. 1, 1–14.

16. K. Davidson, *C^*-algebras by example.* Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.
17. G. Day, *Superatomic Boolean algebras.* Pacific J. Math. 23 1967 479–489.
18. R. Deville, G. Godefroy, V. Zizler, *Smoothness and renormings in Banach spaces.* Pitman Monographs and Surveys in Pure and Applied Mathematics, 64. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1993.
19. J. Diestel, J. Uhl, *Vector measures.* With a foreword by B. J. Pettis. Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.
20. A. Dow, P. Simon, *Thin-tall Boolean algebras and their automorphism groups.* Algebra Universalis 29 (1992), no. 2, 211–226.
21. R. Engelking, *General topology.* Translated from the Polish by the author. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.
22. I. Farah, B. Hart, M. Lupini, L. Robert, A. Tikuisis, A. Vignati, W. Winter, *Model theory of C^*-algebras.* arXiv:1602.08072.
23. S. Ghasemi, P. Koszmider, *An extension of compact operators by compact operators with no nontrivial multipliers not stable.* Preprint, 2016.
24. S. Ghasemi, P. Koszmider *On the stability of thin-tall scattered C^*-algebras.* Preprint, 2016.
25. S. Givant, P. Halmos, *Introduction to Boolean algebras.* Undergraduate Texts in Mathematics. Springer, New York, 2009.
26. P. Hajek, V. Montesinos Santalucia, J. Vanderwerff, V. Zizler, *Biorthogonal systems in Banach spaces.* CMS Books in Mathematics/Ouvrages de Mathematiques de la SMC, 26. Springer, New York, 2008.
27. J. Hjelmborg, M. Rørdam, *On stability of C^*-algebras.* J. Funct. Anal. 155 (1998), no. 1, 153–170.
28. T. Huruya, *A spectral characterization of a class of C^*-algebras.* Sci. Rep. Niigata Univ. Ser. A No. 15 (1978), 21–24.
29. T. Huruya, *The Second Dual Of A Tensor Product Of C^*-algebras, II.* Science Reports of Niigata University. Series A, Mathematics 11 (1974): 21-23.
30. M. Hrusak, *Almost disjoint families and topology.* Recent progress in general topology. III, 601–638, Atlantis Press, Paris, 2014.
31. H. Jensen, *Scattered C^*-algebras.* Math. Scand. 41 (1977), no. 2, 308–314.
32. H. Jensen, *Scattered C^*-algebras. II.* Math. Scand. 43 (1978), no. 2, 308–310 (1979).
33. H. Jensen, *Scattered C^*-algebras with almost finite spectrum.* J. Funct. Anal. 50 (1983), no. 2, 127–132.
34. I. Juhasz, W. Weiss, *On thin-tall scattered spaces.* Colloq. Math. 40 (1978/79), no. 1, 63–68.
35. W. Just, *Two consistency results concerning thin-tall Boolean algebras.* Algebra Universalis 20 (1985), no. 2, 135–142.
36. P. Koepke, J. Martinez, *Superatomic Boolean algebras constructed from morasses.* J. Symbolic Logic 60 (1995), no. 3, 940–951.
37. S. Koppelberg, *Minimally generated Boolean algebras.* Order 5 (1989), no. 4, 393–406.
38. S. Koppelberg, *Handbook of Boolean algebras.* Vol. 1. Edited by J. Donald Monk and Robert Bonnet. North-Holland Publishing Co., Amsterdam, 1989.
39. M. Kusuda, *A characterization of scattered C^*-algebras and its applications to C^*-crossed products*, J. Operator Theory 63 (2010), 417–424.
40. M. Kusuda, *C^*-algebras in which every C^*-subalgebra is AF* Q. J. Math. 63 (2012), no. 3, 675–680.
41. G. Lancien, *A survey on the Szlenk index and some of its applications.* RACSAM. Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 100 (2006), no. 1-2, 209–235.
42. H. X. Lin, *The structure of quasimultipliers of C^*-algebras.* Trans. Amer. Math. Soc. 315 (1989), no. 1, 147–172.
43. T. Loring, *Lifting solutions to perturbing problems in C^*-algebras.* Fields Institute Monographs, 8. American Mathematical Society, Providence, RI, 1997.
44. J. C. Martinez, *A consistency result on thin-tall superatomic Boolean algebras.* Proc. Amer. Math. Soc. 115 (1992), no. 2, 473–477.
45. J. C. Martinez, *A consistency result on thin-very tall Boolean algebras.* Israel J. Math. 123 (2001), 273–284.

46. S. Mazurkiewicz, W. Sierpinski, *Contribution a la topologie des ensembles denombrables*, Fund Math 1, 1920, 17–27.
47. A. Mostowski, A. Tarski, *Booleshe Ringe mit ordneter basis*. Fund. Math. 32, 69–86.
48. G. Murphy, *C^*-algebras and operator theory*. Academic Press, Inc., Boston, MA, 1990.
49. S. Mrowka. *Some set-theoretic constructions in topology*. Fund. Math., 94 (2), 83–92, 1977.
50. S. Negrepontis, *Banach spaces and topology*. Handbook of set-theoretic topology, 1045–1142, North-Holland, Amsterdam, 1984.
51. A. Ostaszewski, *On countably compact, perfectly normal spaces*. J. London Math. Soc. (2) 14 (1976), no. 3, 505–516.
52. G. K. Pedersen, *C^*-algebras and their automorphism groups*. London Mathematical Society Monographs No. 14, Academic Press, 1979.
53. A. Pełczyński, Z. Semadeni, *Spaces of continuous functions. III. Spaces $C(\Omega)$ for ω without perfect subsets*. Studia Math. 18 1959 211–222.
54. R. Pol, *A function space $C(X)$ which is weakly Lindelöf but not weakly compactly generated*. Studia Math. 64 (1979), no. 3, 279–285.
55. J. Roitman, *Superatomic Boolean algebras*. Handbook of Boolean algebras, Vol. 3, 719–740, North-Holland, Amsterdam, 1989.
56. J. Roitman, *$\omega!$ can be a nontrivial automorphism group*. Proc. Amer. Math. Soc. 112 (1991), no. 3, 623–628.
57. J. Roitman, *Uncountable autohomeomorphism groups of thin-tall locally compact scattered spaces*. Algebra Universalis 29 (1992), no. 3, 323–331.
58. M. Rørdam, *Stable C^*-algebras*. Operator algebras and applications, 177–199, Adv. Stud. Pure Math., 38, Math. Soc. Japan, Tokyo, 2004.
59. W. Rudin, *Continuous functions on compact spaces without perfect subsets*. Proc. Amer. Math. Soc. 8 (1957), 39–42.
60. J. Schreier, *Ein Gegenbeispiel zur Theorie der schwachen Convergenz*, Studia Math. 2, 1930, 58–62.
61. P. Simon, *A compact Frechet space whose square is not Frechet*. Comment. Math. Univ. Carolin. 21 (1980), no. 4, 749–753.
62. J. Tomiyama, *A characterization of C^*-algebras whose conjugate spaces are separable*. Tohoku Mathematical Journal, Second Series 15.1 (1963): 96-102.
63. F. D. Tall, *Set-theoretic consistency results and topological theorems concerning the normal Moore space conjecture and related problems*, Thesis (Ph.D.) -The University of Wisconsin - Madison. 1969.
64. D. P. Williams, *Crossed products of C^*-algebras*. No. 134. American Mathematical Soc., 2007.
65. E. Wofsey, *$\wp(\omega)/fin$ and projections in the Calkin algebra*, Proc. Amer. Math. Soc. 136 (2008), no. 2, 719–726.
66. P. Wojtaszczyk, *On linear properties of separable conjugate spaces of C^*-algebras*. Studia Math. 52 (1974), 143–147.

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