

↑↑ NAME ↑↑

↑↑ STUDENT # ↑↑

York University
MATH 1300 3.00MW – Differential Calculus with Applications

Midterm Examination I – Solutions

February 4, 2010

You have 80 minutes to complete this examination. There are 5 pages to the examination, consisting of a table of formulae and 8 questions, for a total score of 75 marks. You may not use a calculator, or any notes or books. Show all your work, and explain or justify your solutions to the extent possible. You may leave numerical answers unsimplified.

Trig formulae:

$$\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$$

$$\cos(\pi/3) = \sin(\pi/6) = 1/2$$

$$\cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$$

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$1 + \tan^2 \theta = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\cos^2 \theta + \sin^2 \theta = 1$$

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$

$$\cos^2 \theta = (1 + \cos 2\theta)/2$$

$$\sin^2 \theta = (1 - \cos 2\theta)/2$$

$$\sin(\theta + \phi) = \sin \theta \cos \phi + \sin \phi \cos \theta$$

$$\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

Ranges: The range of

arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, arccos is $[0, \pi]$, arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$, arccot is $(0, \pi)$,

arccsc is $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$, arcsec is $[0, \pi] \setminus \{\frac{\pi}{2}\}$.

1. Find

(a) $[5] \cos(\arctan(7))$

(b) $[3] \cos(\arctan(-7))$

Solution:

(a) $\arctan 7$ is the angle θ whose \tan is 7, so θ is the angle of a right triangle whose opposite side is 7 and whose adjacent side is 1. This makes the hypotenuse $\sqrt{50}$ by Pythagoras, so $\cos \theta = 1/\sqrt{50}$. (Or $1/5\sqrt{2}$ if you chose to simplify it).

(b) $\arctan(-7)$ is an angle in the 4th quadrant, because the range of \arctan is defined to be $(-\frac{\pi}{2}, \frac{\pi}{2})$. So if $\theta = \arctan 7$, then $\arctan(-7) = -\theta$. Therefore $\cos(-\theta) = \cos(\theta) = 1/\sqrt{50}$ also.

2. Let $f(x) = \sqrt{3 - \sqrt{x}}$.

- (a) [5] What is the (largest) domain of f ?
 (b) [5] Use the intermediate value theorem to identify the range of f .
You may assume that f is continuous and decreasing on its domain.

Solution:

- (a) We need $x \geq 0$ to have \sqrt{x} defined. Likewise we need $3 - \sqrt{x} \geq 0$. So the domain is $[0, 9]$.
 (b) I claim the range is $[0, \sqrt{3}]$. To see this, note that $f(0) = \sqrt{3}$ and $f(9) = 0$. Because f is decreasing, $0 \leq x \leq 9 \Rightarrow \sqrt{3} = f(0) \geq f(x) \geq f(9) = 0$. In other words, the range is $\subset [0, \sqrt{3}]$. Conversely, because f is continuous and takes the values 0 and $\sqrt{3}$, the intermediate value theorem tells us it takes every value in between. In other words, the range is $\supset [0, \sqrt{3}]$. Together, these two inclusions tell us that the range = $[0, \sqrt{3}]$.

3. Let g be the function with domain $[0, \infty)$, defined by $g(x) = \frac{3 + 2x}{1 + x}$.

You may assume that g is one-to-one, with range $(2, 3]$.

- (a) [4] Identify the domain and range of g^{-1} ?
 (b) [6] Find an explicit formula for g^{-1}

Solution:

- (a) $\text{Domain}(g^{-1}) = \text{Range}(g) = (2, 3]$ and $\text{Range}(g^{-1}) = \text{Domain}(g) = [0, \infty)$.
 (b) If $y = g(x)$ then we need to solve for x . Cross multiplying, $y + yx = y(1 + x) = 3 + 2x$. Grouping x terms, $x(y - 2) = yx - 2x = 3 - y$, so $x = (3 - y)/(y - 2)$. In other words,

$$g^{-1}(y) = \frac{3 - y}{y - 2}.$$

Note that you are free to use any variable name you like to write this down. The book's choice would be to write $g^{-1}(x) = (3 - x)/(x - 2)$, which means exactly the same thing.

4. [10] Find

$$\lim_{z \rightarrow 2} \frac{z^5 - 2z^4 + 5z^2 - 9z - 2}{z^2 - 4}.$$

Solution: Direct substitution doesn't work, since putting $z = 2$ gives $0/0$. But since the top and bottom are polynomials, that tells us that $z - 2$ must divide both. On the bottom, $z^2 - 4 = (z - 2)(z + 2)$. On the top we need to do a long division. Carrying that out, we get

$z^5 - 2z^4 + 5z^2 - 9z - 2 = (z - 2)(z^4 + 5z + 1)$. So

$$\begin{aligned} \lim_{z \rightarrow 2} \frac{z^5 - 2z^4 + 5z^2 - 9z - 2}{z^2 - 4} &= \lim_{z \rightarrow 2} \frac{(z - 2)(z^4 + 5z + 1)}{(z - 2)(z + 2)} \\ &= \lim_{z \rightarrow 2} \frac{z^4 + 5z + 1}{z + 2} = \frac{16 + 10 + 1}{2 + 2} = \frac{27}{4} \end{aligned}$$

5. [10] Let

$$h(x) = \begin{cases} \frac{2\sin^2 x}{(3x)^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

At what values of x is h continuous?

Explain why it is or is not continuous at $x = 0$.

Solution: $\sin x$ is continuous, as is x , and algebraic manipulations preserve continuity as long as we don't divide by 0. That tells us that $h(x)$ is continuous everywhere other than $x = 0$. To see what happens at $x = 0$ we compute

$$\lim_{x \rightarrow 0} \frac{2\sin^2 x}{(3x)^2} = \lim_{x \rightarrow 0} \frac{2}{9} \left(\frac{\sin x}{x} \right)^2 = \frac{2}{9} \cdot 1^2 \neq 1.$$

Since this limit $\neq h(0)$, we see that h is not continuous at $x = 0$. In other words, the set of values at which h is continuous is $\mathbb{R} \setminus \{0\}$.

6. [7] Let $g(2) = 3$ and $g'(2) = 5$. Find the equation of the tangent line to the curve $y = g(x)$ at the point on this curve where $x = 2$.

Solution: The line has slope 5 and passes through $(2, 3)$ so has equation $y - 3 = 5(x - 2)$. Or equivalently, $y = 5x - 7$. Either answer is acceptable.

7. [10] Let $y = \frac{5x+2}{x^3+7}$. Find $\frac{dy}{dx}\big|_{x=-1}$ basing your answer on differentiation formulae.

Solution: By the quotient rule,

$$\frac{dy}{dx} = \frac{5(x^3+7) - (5x+2)(3x^2)}{(x^3+7)^2}.$$

Substituting $x = -1$ we get

$$\frac{dy}{dx}\big|_{x=-1} = \frac{5(-1+7) - (-5+2)(3)}{(-1+7)^2} = \frac{30+9}{36} = \frac{13}{12}.$$

8. [10] Let $f(x) = \sqrt{3x}$. Find $f'(3)$ basing your answer on the definition of the derivative.

Solution:

$$\begin{aligned} f'(3) &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{3(3+h)} - \sqrt{3 \cdot 3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{9+3h} - \sqrt{9}}{h} \cdot \frac{\sqrt{9+3h} + \sqrt{9}}{\sqrt{9+3h} + \sqrt{9}} \\ &= \lim_{h \rightarrow 0} \frac{(9+3h) - 9}{h(\sqrt{9+3h} + \sqrt{9})} = \lim_{h \rightarrow 0} \frac{3h}{h(\sqrt{9+3h} + \sqrt{9})} \\ &= \lim_{h \rightarrow 0} \frac{3}{\sqrt{9+3h} + \sqrt{9}} = \frac{3}{2\sqrt{9}} = \frac{1}{2}. \end{aligned}$$