$\uparrow\uparrow$ NAME $\uparrow\uparrow$ $\uparrow\uparrow$ STUDENT # $\uparrow\uparrow$

York University

MATH 1300 3.00MW – Differential Calculus with Applications

Midterm Examination II – SOLUTIONS

March 11, 2010

You have 80 minutes to complete this examination. There are 6 pages to the examination, consisting of 8 questions and a table of formulae, for a total score of 100 marks. You may not use a calculator, or any notes or books. Show all your work, and explain or justify your solutions to the extent possible. You may leave numerical answers unsimplified.

---

1. [10] Find $\dfrac{d}{dx}\arcsin\left(\sqrt{5x}\right)$, for $0 < x < \frac{1}{5}$.

Solution:

$$\frac{d}{dx}\arcsin\left(\sqrt{5x}\right) = \frac{1}{\sqrt{1-(\sqrt{5x})^2}}\frac{d}{dx}\sqrt{5x}$$
$$= \frac{1}{\sqrt{1-5x}} \cdot \frac{1}{2\sqrt{5x}}\frac{d}{dx}5x = \frac{5}{2\sqrt{5x(1-5x)}}.$$

Note that $0 < x < 1/5$ implies $0 < \sqrt{5x} < 1$, in which case $\arcsin\sqrt{5x}$ is defined.

2. [10] If $f(0) = 8$ and $f'(0) = 2$, find $\dfrac{d}{dx}\Big[f(x) + \arctan(7x)\Big]^{2/3}$ when $x = 0$.

Solution:

$$\frac{d}{dx}\Big[f(x)+\arctan(7x)\Big]^{2/3} = \frac{2}{3}\Big[f(x)+\arctan(7x)\Big]^{-1/3}\frac{d}{dx}\Big[f(x)+\arctan(7x)\Big]$$
$$= \frac{2}{3}\Big[f(x)+\arctan(7x)\Big]^{-1/3}\Big[f'(x)+\frac{7}{1+(7x)^2}\Big].$$

Substituting $x = 0$ gives

$$\frac{2}{3}\Big[f(0)+\arctan(0)\Big]^{-1/3}\Big[f'(0)+\frac{7}{1}\Big] = \frac{2}{3}[8+0]^{-1/3}[2+7] = \frac{2}{3}\cdot\frac{1}{2}\cdot 9 = 3.$$

3. [10] Find the equation of the tangent line to the curve

$$9 + x^5y + x^2 + xy^3 = 0$$

at the point $(-1, 2)$.

Solution: Differentiating implicitly with respect to x,

$$5x^4y + 2x + y^3 + \Big[x^5 + 3xy^2\Big]\frac{dy}{dx} = 0.$$

Substituting $x=-1$ and $y=2$, $10-2+8-[1+12]dy/dx=0$, so $dy/dx=16/13$. Therefore

$$(y-2)=\frac{16}{13}(x+1) \quad \Rightarrow \quad y=\frac{16x+42}{13}.$$

4. [10] Let $f=g^{-1}$, where g satisfies $g(0)=3$ and $g'(x)=5+\cos(x^2)$. Find $f'(3)$.

Solution: We know that $f'(y)=\dfrac{1}{g'(g^{-1}(y))}=\dfrac{1}{5+\cos(x^2)}$, where $x=g^{-1}(y)$. In other words,

$$f'(3)=\frac{1}{g'(0)}=\frac{1}{5+\cos(0^2)}=\frac{1}{5+1}=\frac{1}{6}.$$

5. Find

(a) [10] $\displaystyle\lim_{x\to 0}\frac{1-\cos(3x)}{x\sin(2x)}$

(b) [10] $\displaystyle\lim_{x\to\infty} x-\sqrt{x^2+3x+5}$

using l'Hospital's at least once in each part.
Remember to say why the rule applies.

Solution:

(a) L'Hospital's rule applies, since this limit has the form $\frac{0}{0}$. So

$$\lim_{x\to 0}\frac{1-\cos(3x)}{x\sin(2x)}=\lim_{x\to 0}\frac{3\sin(3x)}{\sin(2x)+2x\cos(2x)}.$$

This limit also has the form $\frac{0}{0}$, so we apply l'Hospital again, and get

$$=\lim_{x\to 0}\frac{9\cos(3x)}{2\cos(2x)+2\cos(2x)-4x\sin(2x)}=\frac{9}{2+2-0}=\frac{9}{4}.$$

(b) $x-\sqrt{x^2+3x+5}=x\left(1-\sqrt{1+\frac{3}{x}+\frac{5}{x^2}}\right)$. Setting $y=1/x$, this equals $\dfrac{1-\sqrt{1+3y+5y^2}}{y}$. Sending $x\to\infty$ makes $y\to 0$, which turns this expression into one of the form $\frac{0}{0}$. Applying l'Hospital makes

$$\lim_{x\to\infty} x-\sqrt{x^2+3x+5}=\lim_{y\to 0}\frac{1-\sqrt{1+3y+5y^2}}{y}=\lim_{y\to 0}\frac{-\frac{3+10y}{2\sqrt{1+3y+5y^2}}}{1}=-\frac{3}{2}.$$

6. Using the fact that $\frac{d}{dx}(x^4 + x^5) = 4x^3 + 5x^4$, find

(a) [5] The general solution to $\frac{dy}{dx} = 4x^3 + 5x^4$.

(b) [5] The solution to this differential equation, with the initial condition that $y = 12$ when $x = 1$.

Solution:

(a) $y = x^4 + x^5 + c$.

(b) $12 = 1^4 + 1^5 + c$, so $c = 10$. Therefore $y = x^4 + x^5 + 10$.

7. [10] Let $g(x) = 7 - \frac{x^2+1}{x^3-8}$. Find all horizontal or vertical asymptotes of g.

Sketch the behaviour of $g(x)$ **only** for x near any vertical asymptote.

Solution: g is continuous at x unless the denominator $x^3 - 8 = 0$. So $x = 2$ is the only possible vertical asymptote. Looking at the signs of $x^2 + 1$ and $x^3 - 8$, we get

$$\lim_{x\to 2+} 7 - \frac{x^2+1}{x^3-8} = -\infty, \quad \lim_{x\to 2-} 7 - \frac{x^2+1}{x^3-8} = +\infty$$

(see Figure 1 for a sketch).
The horizontal asymptote is 7, since

$$\lim_{x\to\pm\infty} 7 - \frac{x^2+1}{x^3-8} = \lim_{x\to\pm\infty} 7 - \frac{\frac{1}{x}+\frac{1}{x^3}}{1-\frac{8}{x^3}} = 7 - \frac{0}{1} = 7.$$

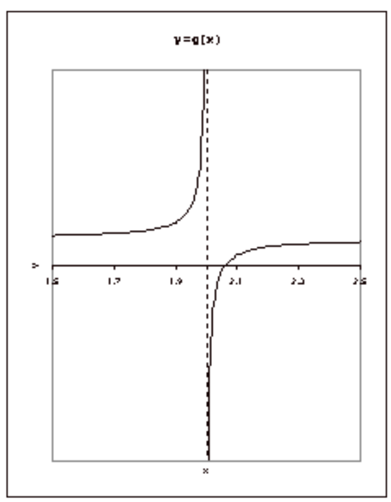

Figure 1. Sketch for problem 7

8. Let $f(x) = \dfrac{x^3}{3} - x^2 - 3x + 4$.

(a) [5] Find all critical points of f.
(b) [5] Find the intervals where f is increasing and decreasing, and use this to identify the x where f has relative maxima or minima.
(c) [5] Find all inflection points and identify the intervals where f is concave up or concave down.
(d) [5] Sketch f. Do not worry about the precise value of intercepts, or the precise value of f at critical or inflection points.

Solution:

(a) f is differentiable everywhere, so the only critical points are where $f' = 0$. So $0 = f'(x) = x^2 - 2x - 3 = (x-3)(x+1)$ implies the critical points are at $x = 3$ and $x = -1$.
(b) $f \uparrow$ on the intervals where $f' > 0$, that is, on $(-\infty, -1)$ and $3, \infty)$.
$f \downarrow$ on the intervals where $f' < 0$, that is, on $(-1, 3)$.
This implies a local maximum at $x = -1$ and a local minimum at $x = 3$.
(c) $0 = f''(x) = 2x - 2 = 2(x-1)$ implies that $x = 1$ is an inflection point, with f concave down (ie $f'' < 0$) on $(-\infty, 1)$ and f concave up (ie $f'' > 0$) on $(1, \infty)$.
(d) See Figure 2 for a sketch.

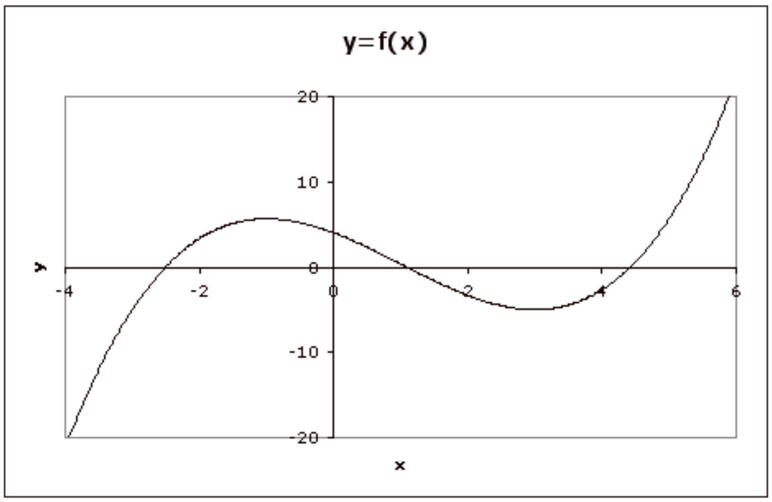

FIGURE 2. Sketch for problem 8

Trig formulae:

$$\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$$

$$\cos(\pi/3) = \sin(\pi/6) = 1/2 \qquad \cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos^2\theta = (1 + \cos 2\theta)/2 \qquad \sin^2\theta = (1 - \cos 2\theta)/2$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta \qquad \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Ranges:

The range of
arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, arccos is $[0, \pi]$, arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$, arccot is $(0, \pi)$,
arccsc is $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$, arcsec is $[0, \pi] \setminus \{\frac{\pi}{2}\}$.