$\uparrow\uparrow$ NAME $\uparrow\uparrow$ $\uparrow\uparrow$ STUDENT # $\uparrow\uparrow$

York University
MATH 1300 3.00MW – Differential Calculus with Applications

Quiz 3 – Solutions
March 25, 2010

You have 30 minutes to complete this quiz. There are two pages, including some formulae, but you may write on the back. You may not use a calculator, notes, or books. Show your work, and explain or justify your solutions if possible. You may leave numerical answers unsimplified.

Trig formulae:

$$\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$$

$$\cos(\pi/3) = \sin(\pi/6) = 1/2 \qquad \cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos^2\theta = (1 + \cos 2\theta)/2 \qquad \sin^2\theta = (1 - \cos 2\theta)/2$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta \qquad \cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

Ranges:
The range of
arcsin is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, arccos is $[0, \pi]$, arctan is $(-\frac{\pi}{2}, \frac{\pi}{2})$, arccot is $(0, \pi)$,
arccsc is $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$, arcsec is $[0, \pi] \setminus \{\frac{\pi}{2}\}$.

You have 20 metres of fencing, to fence in an area for your pet rabbit at the side of your house. The area will look like a rectangle with a triangle on the side, and you have decided that the triangle will be similar to a right angle'd triangle with sides 3:4:5, as in the diagram below. So the fencing goes along the hypotenuse of the triangle, and then along 2 sides of the rectangle. What should the dimensions of the rectangular and triangular pieces be, so as to maximize the area?

(I'll scan in the diagram for a subsequent version of the solutions)

Solution: The length of the fence is

$$20 = \frac{5y}{3} + x + y = \frac{8y}{3} + x.$$

The area of the rectangle is xy and the area of the triangle is $\frac{1}{2}y(4y/3)$, so the area of the region is

$$A = xy + \frac{1}{2}y\left(\frac{4y}{3}\right) = xy + \frac{2}{3}y^2.$$

Solving for x from the first equation, and substituting it into the second gives

$$x = 20 - \frac{8y}{3} \qquad \Rightarrow \qquad A = \left(20 - \frac{8y}{3}\right)y + \frac{2}{3}y^2 = 20y - 2y^2.$$

To find the critical point, we set

$$0 = \frac{dA}{dy} = 20 - 4y \quad \Rightarrow \quad y = 5 \quad \Rightarrow \quad x = 20 - \frac{8 \times 5}{3} = \frac{20}{3}.$$

We're not finished yet though – we still need to see why the critical point actually gives the maximum. This isn't just being pedantic, since there are in fact constraints on the allowed values of y. Namely that $x \geq 0$ and $y \geq 0$.

One way of seeing that we've found the maximum area is to observe first that the function $f(y) = 20y - y^2$ (without any constraint on y) has a global maximum at $y = 5$ (either by noting that f is a downward parabola, or by using the sign of f' to see that f is increasing/decreasing to the left/right of $y = 5$). So if the critical point satisfies the constraints, then it'll give the maximum for our problem as well. And indeed, $y = 5$ and $x = 20/3$ do satisfy $y \geq 0$ and $x \geq 0$.

Another way of seeing this is to note that $x \geq 0$ is equivalent to $8y/3 \leq 20$, or $y \leq 15/2 = 7.5$; In other words, we're maximizing $20y - 2y^2$ subject to $0 \leq y \leq 15/2$. $y = 5$ does lie in this interval, so by the max/min theorem we have three values to check: the endpoints $y = 0, 15/2$ and the critical point $y = 5$. Substituting in we get $A = 0$ (this corresponds to a microscopic triangle, with the fence actually right up against the wall), $A = 75/2 = 37.5$ (here the rectangle disappears, and our region is just a big triangle), and $A = 50$. The biggest of these is $A = 50$, so choosing $y = 5$ does indeed maximize the area.