York University AK/AS/SC MATH 1310 3.0AF
Integral Calculus with Applications

Midterm Examination I - *Solutions*
February 6, 2008

NAME & STUDENT NUMBER:

You have 50 minutes to complete this examination. There are 4 pages to the examination, consisting of a table of formulae and 6 questions, for a total score of 85 marks. You may not use a calculator, or any notes or books. Show all your work, and explain or justify your solutions to the extent possible. You may leave numerical answers unsimplified.

Trig formulae and error bounds:

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\cos^2\theta + \sin^2\theta = 1 \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos^2\theta = (1 + \cos 2\theta)/2 \qquad \sin^2\theta = (1 - \cos 2\theta)/2$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\text{midpoint rule: } (b-a)^3 K/24n^2$$

$$\text{trapezoidal rule: } (b-a)^3 K/12n^2$$

$$\text{Simpson's rule: } (b-a)^5 M/180n^4 \quad (n \text{ even})$$

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1. [10] Evaluate the definite integral $\displaystyle\int_1^2 \left(x - \frac{1}{x}\right)^2 dx$

Solution:

$$= \int_1^2 \left(x^2 - 2 + \frac{1}{x^2}\right) dx = \left[\frac{x^3}{3} - 2x - \frac{1}{x}\right]_1^2 = \left(\frac{8}{3} - 4 - \frac{1}{2}\right) - \left(\frac{1}{3} - 2 - 1\right) = \frac{5}{6}$$

2. Evaluate the following indefinite integrals:

(a) [10] $\displaystyle\int \cos^4\theta \sin^3\theta \, d\theta$

(b) [10] $\displaystyle\int \frac{1+t}{1+t^2}\, dt$

Solution:

(a) Let $u = \cos\theta$. So $du = -\sin\theta \, d\theta$ and

$$\begin{aligned}\int \cos^4\theta \sin^3\theta \, d\theta &= \int \cos^4\theta(\sin^2\theta)\sin\theta \, d\theta \\ &= \int \cos^4\theta(1-\cos^2\theta)\sin\theta \, d\theta \\ &= \int u^4(1-u^2)(-du) = \int (u^6 - u^4)\, du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C = \frac{\cos^7\theta}{7} - \frac{\cos^5\theta}{5} + C.\end{aligned}$$

(b) Let $u = 1 + t^2$. So $du = 2t\, dt$ and

$$\begin{aligned}\int \frac{1+t}{1+t^2}\, dt &= \int \frac{1}{1+t^2}\, dt + \int \frac{2t\, dt}{2(1+t^2)} \\ &= \arctan t + \int \frac{du}{2u} = \arctan t + \frac{1}{2}\ln u + C \\ &= \arctan t + \frac{1}{2}\ln(1+t^2) + C.\end{aligned}$$

3. [8] Rewrite the definite integral $\displaystyle\int_1^3 x^3 e^{1+x^2}\, dx$ as a definite integral in the variable $y = x^2$. DO NOT evaluate this integral.

Solution:

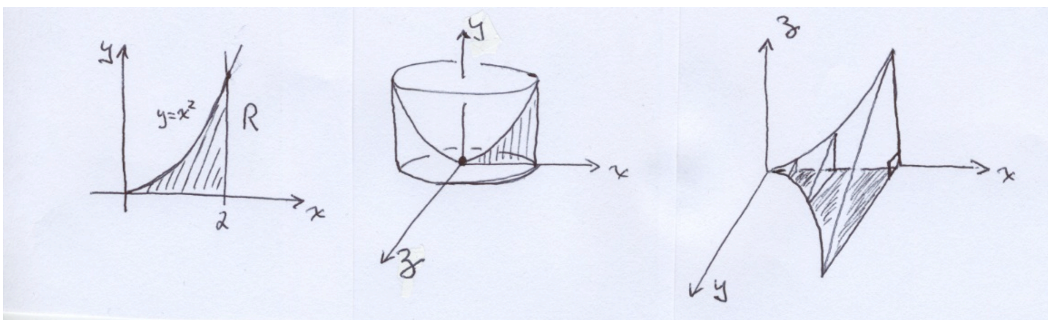
If $y = x^2$ then $dy = 2x\, dx$, so $\int x^3 e^{1+x^2}\, dx = \int \frac{1}{2} x^2 e^{1+x^2} 2x\, dx = \int \frac{1}{2} y e^{1+y}\, dy$. Also, when $x = 1$ then $y = 1^2 = 1$, and when $x = 3$ then $y = 3^2 = 9$. So

$$\int_1^3 x^3 e^{1+x^2}\, dx = \int_1^9 \frac{1}{2} y e^{1+y}\, dy.$$

4. Let R be the region in the xy plane defined by the inequalities $0 \leq y \leq x^2$ and $0 \leq x \leq 2$. In each part below, set up a definite integral (or sum or difference of definite integrals) whose value equals the given volume V.
DO NOT evaluate these integrals.

(a) [8] The volume of the solid S obtained by revolving R around the y-axis.

(b) [10] The volume of the solid S whose sections perpendicular to the x axis are all isosceles right triangles, with right angle sitting on the x-axis, and with base lying in R.



Solution:

(a) This could be done either with disks or shells. With disks we are slicing perpendicular to the y-axis, and the annular cross sections have outer radius $x = 2$ and inner radius $x = \sqrt{y}$ (since $y = x^2$). Moreover y runs from $0^2 = 0$ to $2^2 = 4$. So the volume is

$$\int_0^4 \pi[2^2 - (\sqrt{y})^2]\, dy = \int_0^4 \pi[4 - y]\, dy.$$

Alternatively we could use shells. The shell of radius x has circumference $2\pi x$ and height x^2, so the volume is

$$\int_0^2 2\pi x \cdot x^2\, dx = \int_0^2 2\pi x^3\, dx.$$

(b) The volume is $\int_0^2 A(x)\, dx$. Here $A(x)$ is the area of a right angled triangle, with equal base and height. The base sits in the xy-plane, and runs from $y = 0$ to $y = x^2$. In other words, it has length x^2. Thus
$A(x) = \frac{1}{2}(x^2)^2 = \frac{1}{2}x^4$. So the volume is

$$\int_0^2 \frac{1}{2}x^4\, dx.$$

5. Evaluate

(a) [7] $\dfrac{d}{dx}2^{\sin x}$

(b) [10] $\lim\limits_{x\to 0+} x^{2x}$

Solution:

(a) $= \dfrac{d}{dx}e^{\sin x \cdot \ln 2} = 2^{\sin x} \cdot \ln 2 \cdot \cos x$

(b) $= \lim\limits_{x\to 0+} e^{2x \ln x} = e^{\lim_{t\to 0+} \frac{2\ln x}{1/x}}$.

This involves an indeterminate form of type ∞/∞.

So by l'Hospital's rule, it $= e^{\lim_{x\to 0+} \frac{2/x}{-1/x^2}} = e^0 = 1$.

6. Consider the integral $\displaystyle\int_1^5 \frac{dx}{x}$.

(a) [4] Use Simpson's rule and the regular partition of $[1,5]$ into 4 intervals to estimate this integral.

(b) [8] Find a reasonable bound on the error of this estimate.

Solution:

(a) $= \dfrac{1}{3}\Big[1\cdot f(1) + 4\cdot f(2) + 2\cdot f(3) + 4\cdot f(4) + 1\cdot f(5)\Big]$

$= \dfrac{1}{3}\Big[1\cdot 1 + 4\cdot \dfrac{1}{2} + 2\cdot\dfrac{1}{3} + 4\cdot\dfrac{1}{4} + 1\cdot\dfrac{1}{5}\Big] = \dfrac{73}{45}.$

(b) $|\text{error}| \le \dfrac{4^5 M}{180\cdot 4^4} = \dfrac{M}{45}$, where M is a bound on $|f^{(4)}|$.

We know that $f(x) = 1/x$, $f'(x) = -1/x^2$, $f''(x) = 2/x^3$, $f^{(3)}(x) = -6/x^4$, and $f^{(4)} = 24/x^5$. This is decreasing on $[1,5]$ so takes its maximum value at $x = 1$, and hence $M = 24$. In other words, $|\text{error}| \le 24/45 \approx 0.5333$.

[In fact, this isn't such a great error estimate in this case. The true value of the integral is $\ln 5 \approx 1.6094$. Simpson's rule gives $73/45 \approx 1.6222$. So the real error is the difference, or ≈ 0.0128. Which is much smaller than the estimate.]