York University AK/AS/SC MATH 1310 3.0AF
Integral Calculus with Applications

Midterm Examination II - *Solutions*
March 12, 2008

NAME & STUDENT NUMBER:

You have 50 minutes to complete this examination. There are 5 pages to the examination, consisting of a table of formulae and 6 questions, for a total score of 90 marks. You may not use a calculator, or any notes or books. Show all your work, and explain or justify your solutions to the extent possible. You may leave numerical answers unsimplified.

Trig formulae and error bounds:

$$1+\tan^2\theta = \sec^2\theta \qquad 1+\cot^2\theta = \csc^2\theta$$

$$\cos^2\theta+\sin^2\theta = 1 \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos^2\theta = (1+\cos 2\theta)/2 \qquad \sin^2\theta = (1-\cos 2\theta)/2$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$$

$$\cos(\pi/3) = \sin(\pi/6) = 1/2$$

$$\cos(\pi/6) = \sin(\pi/3) = \sqrt{3}/2$$

midpoint rule: $(b-a)^3K/24n^2$

trapezoidal rule: $(b-a)^3K/12n^2$

Simpson's rule: $(b-a)^5M/180n^4 \quad (n \text{ even})$

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1. [15] Evaluate $\int 3x^2 \cdot \ln x\, dx$ using integration by parts.

Solution: Let $u = \ln x$ and $dv = 3x^2\, dx$. Then $v = x^3$ and $du = dx/x$. So

$$\int 3x^2 \ln x\, dx = \int u\, dv = uv - \int v\, du = x^3 \ln x - \int x^3 \cdot \frac{dx}{x}$$

$$= x^3 \ln x - \int x^2\, dx = x^3 \ln x - \frac{x^3}{3} + C$$

2. Evaluate the following:

(a) [10] $\int (\sec x)(\sec^7 x + \tan x)\, dx$

(b) [10] $\int_0^1 \frac{2}{(4-x^2)^{3/2}}\, dx$.

Solution:

(a) Since the derivative of $\sec x$ is $\sec x \tan x$, and $1+\tan^2 x = \sec^2 x$,

$$\begin{aligned}\int (\sec x)(\sec^7 x + \tan x)\, dx &= \int \sec^8 x\, dx + \int \sec x \tan x\, dx \\ &= \int (\sec^2 x)^3 \sec^2 x\, dx + \sec x = \int (1+\tan^2 x)^3 \sec^2 x\, dx + \sec x\end{aligned}$$

Substituting $u = \tan x$, we have $du = \sec^2 x\, dx$, so this

$$\begin{aligned}&= \int (1+u^2)^3\, du + \sec x = \int (1+3u^2+3u^4+u^6)\, du + \sec x \\ &= u + u^3 + \frac{3u^5}{5} + \frac{u^7}{7} + \sec x + C \\ &= \tan x + \tan^3 x + \frac{3\tan^5 x}{5} + \frac{\tan^7 x}{7} + \sec x + C\end{aligned}$$

(b) Let $x = 2\sin\theta$, so $dx = 2\cos\theta\, d\theta$.
The endpoint $x = 0$ corresponds to $\theta = 0$.
The endpoint $x = 1$ corresponds to $2\sin\theta = 1$, or $\sin\theta = 1/2$, or $\theta = \pi/6$.
So

$$\begin{aligned}\int_0^1 \frac{2}{(4-x^2)^{3/2}}\, dx &= \int_0^{\pi/6} \frac{2}{(4(1-\sin^2\theta))^{3/2}} \cdot 2\cos\theta\, d\theta \\ &= \int_0^{\pi/6} \frac{4\cos\theta}{(4\cos^2\theta)^{3/2}}\, d\theta = \int_0^{\pi/6} \frac{4\cos\theta}{8\cos^3\theta}\, d\theta \\ &= \int_0^{\pi/6} \frac{d\theta}{2\cos^2\theta} = \frac{1}{2}\int_0^{\pi/6} \sec^2\theta\, d\theta \\ &= \frac{1}{2}\Big[\tan\theta\Big]_0^{\pi/6} = \frac{1}{2}\Big(\frac{1}{\sqrt{3}} - 0\Big) = \frac{1}{2\sqrt{3}}.\end{aligned}$$

3. [15] Evaluate

$$\int \frac{x^3}{x^2+3x+2}\,dx$$

Solution: First we do long division to reduce the degree of the numerator:

$$\begin{array}{r|l} & \quad x-3 \\ \hline x^2+3x+2 & \quad x^3 \\ & \quad x^3+3x^2+2x \\ \hline & \qquad -3x^2-2x \\ & \qquad -3x^2-9x-6 \\ \hline & \qquad\qquad\quad 7x+6 \end{array}$$

In other words, $\dfrac{x^3}{x^2+3x+2} = x-3+\dfrac{7x+6}{x^2+3x+2}$.

We can factor x^2+3x+2 as $(x+2)(x+1)$, so our partial fractions decomposition is

$$\frac{7x+6}{x^2+3x+2} = \frac{A}{x+2}+\frac{B}{x+1}.$$

Or in other words, $7x+6 = A(x+1)+B(x+2)$.

Substituting $x=-1$ gives $B=-1$. Substituting $x=-2$ gives $A=8$. So

$$\begin{aligned}\int \frac{x^3}{x^2+3x+2}\,dx &= \int \left(x-3+\frac{8}{x+2}-\frac{1}{x+1}\right)dx \\ &= \frac{x^2}{2}-3x+8\ln(x+2)-\ln(x+1)+C.\end{aligned}$$

[The above was acceptable, even though it tacitly assumes that x is large enough for the ln's to make sense. But

$$\frac{x^2}{2}-3x+8\ln|x+2|-\ln|x+1|+C$$

is actually a better answer.]

4. [10] Find

$$\int_0^\infty \frac{x\,dx}{(1+x^2)^2}$$

You may use the formula for the following indefinite integral

$$\int \frac{x\,dx}{(1+x^2)^2} = -\frac{1}{2(1+x^2)} + C,$$

if you wish. You should explicitly identify all values of x at which the integral is improper. You should also explicitly identify all limits needed to compute the integral, and briefly explain how you evaluate them.

Solution:
The only improper endpoint is $x = \infty$. So

$$= \lim_{B\to\infty} \int_0^B \frac{x\,dx}{(1+x^2)^2} = \lim_{B\to\infty} \left[-\frac{1}{2(1+x^2)} \right]_0^B = \lim_{B\to\infty} \frac{1}{2}\left(1 - \frac{1}{1+B^2}\right) = \frac{1}{2}$$

since when $B \to \infty$, also $1 + B^2 \to \infty$, so $1/(1 + B^2) \to 0$.

5. Consider the differential equation $y' = -x^2(y + 2)$.
   (a) [10] Find the general solution.
   (b) [5] If $y = 4$ when $x = 0$, find an expression for y when $x = 2$. It should be the kind of thing you could easily work out with a scientific calculator, if you had one.

Solution:
Rewrite the differential eq'n $\dfrac{dy}{dx} = -x^2(y + 2)$ as $\dfrac{dy}{y+2} = -x^2\,dx$.

Then integrate both sides, to give $\ln|y + 2| = -\dfrac{x^3}{3} + C$.
Taking exponentials gives $y + 2 = \pm e^C e^{-x^3/3}$, or $y = Ae^{-x^3/3} - 2$, where A is an arbitrary constant. This is the general solution.

To solve the initial value problem, substitute in the initial condition to give $4 = Ae^0 - 2$, or $A = 6$. So when $x = 2$ we have $y = 6e^{-8/3} - 2$.

6. [15] Consider the sequence $\left\{\frac{n!+(n+3)!}{(n+3)!}\right\}$.
Decide if it is increasing or decreasing.
Decide if it is bounded above or bounded below, and if it is, give explicit bounds.
Decide if it is convergent, and if it is, find its limit.

Solution: Note that $\frac{n!+(n+3)!}{(n+3)!} = \frac{1}{(n+3)(n+2)(n+1)} + 1$.
The sequence is decreasing, because each term in the denominator grows with n.
The sequence is bounded above, eg. by 2 (since the first term is ≤ 1).
The sequence is bounded below, eg. by 0 (since all terms are positive).
The sequence is convergent, with limit 1, since the denominator in the above fraction $\to \infty$.