York University AK/AS/SC MATH 1310 3.0AF
Integral Calculus with Applications

Quiz 6 – Solution

March 26, 2008
Worth: 10 marks. Time allowed: 15 minutes. No calculators allowed.

(a) For the function $f(x) = e^{1-\sqrt{1+x}}$ find the second order Taylor polynomial $P_2(x)$ about $x = 0$.

(b) Suppose I give you the bound $|f'''(x)| \leq 7/8$, for $x > 0$. If you were to use the above polynomial to approximate $e^{1-\sqrt{1.1}}$, what can you say about the accuracy of this approximation? Be sure to explain the reasons for your answer.
(Your answer should be numerical, but may be left unsimplified. In other words, it need not be in decimal form.)

Solution: We calculate that

$$f'(x) = -\frac{1}{2\sqrt{1+x}}e^{1-\sqrt{1+x}}, \qquad f''(x) = \frac{1}{4(1+x)^{3/2}}e^{1-\sqrt{1+x}} + \frac{1}{4(1+x)}e^{1-\sqrt{1+x}},$$

and substituting $x = 0$ gives

$$f(0) = 1, \qquad f'(0) = -\frac{1}{2}, \qquad f''(0) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

So

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 = 1 - \frac{x}{2} + \frac{x^2}{4}.$$

If we approximate $f(0.1) = e^{1-\sqrt{1.1}}$ by $P_2(0.1)$, the error is $f'''(t)(0.1)^3/3!$ for some t between 0 and 0.1; In other words, the absolute error is $\leq \frac{7}{48} \times 0.001$;

This is a perfectly valid answer, but if you actually do the division you get 0.00015 as a concrete error estimate. In fact, the true error is about -0.00014