1. A random variable $X$ has density function

$$f(x) = \begin{cases} 
0, & x < 0 \\
c + x, & 0 < x < 1 \\
0, & x > 1
\end{cases}$$

(a) [5] Find $c$.

(b) [10] Find the cumulative distribution function of $X$.

(c) [10] Find $E[X]$.

Solution:

(a) $1 = \int_{-\infty}^{\infty} f(x) \, dx = \int_0^1 (c + x) \, dx = \left[ cx + \frac{x^2}{2} \right]_0^1 = c + \frac{1}{2}$. So $c = \frac{1}{2}$. 
(b) \( F(x) = \int_{-\infty}^{x} f(t) \, dt = \begin{cases} \int_{-\infty}^{x} 0 \, dt = 0, & x < 0 \\ \int_{-\infty}^{0} 0 \, dt + \int_{0}^{x} (\frac{1}{2} + t) \, dt = \left[ \frac{t}{2} + \frac{t^2}{2} \right]_{0}^{x} = \frac{x^2 + x^2}{2}, & 0 \leq x < 1 \\ \int_{-\infty}^{0} 0 \, dt + \int_{0}^{1} (\frac{1}{2} + t) \, dt + \int_{1}^{x} 0 \, dt = 1, & 0 \leq x < 1. \end{cases} \)

(c) \( E[X] = \int_{-\infty}^{\infty} xf(x) \, dx = \int_{0}^{1} x(\frac{1}{2} + x) \, dx = \left[ \frac{x^2}{4} + \frac{x^3}{3} \right]_{0}^{1} = \frac{7}{12}. \)

2. A discrete random variable \( X \) has distribution

<table>
<thead>
<tr>
<th>( x )</th>
<th>-1</th>
<th>0</th>
<th>5</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X = x) )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
<td>( \frac{1}{6} )</td>
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Let \( Y \sim N(3, 7) \) and let \( A \) be the event that \( X \geq 2. \)

(a) [10] Find \( E[X + 2Y - 1_A + 5]. \)
(b) [10] Find \( E[X1_A]. \)

Solution:

(a) \( E[X] = -1 \times \frac{1}{2} + 0 \times \frac{1}{6} + 5 \times \frac{1}{6} + 7 \times \frac{1}{6} = \frac{9}{6}. \) Also \( E[Y] = 3 \) and \( E[1_A] = P(A) = P(X = 5 \text{ or } X = 7) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6}. \) So \( E[X + 2Y - 1_A + 5] = E[X] + 2E[Y] - E[1_A] + 5 = \frac{73}{6}. \)

(b) The possible values of \( X1_A \) are 0 (probability \( \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \)), 5 (probability \( \frac{1}{6} \)), and 7 (probability \( \frac{1}{6} \)). So \( E[X1_A] = 0 \times \frac{2}{3} + 5 \times \frac{1}{6} + 7 \times \frac{1}{6} = \frac{12}{6} = 2. \)

[Or, though we didn’t study the formula \( E[g(X)] = \sum g(x)P(X = x) \) till after the midterm cutoff, I would also have accepted \( E[X1_A] = -1 \times 0 \times \frac{1}{2} + 0 \times 0 \times \frac{1}{6} + 5 \times 1 \times \frac{1}{6} + 7 \times 1 \times \frac{1}{6} = 2. \)]

3. Let \( Z \sim N(0, 1) \), and \( Y = Z^{1/3}. \)

(a) [15] Find the density of \( Y. \)
(b) [10] Find \( P(Y \geq (\frac{1}{2})^{1/3}). \)

Solution:

(a) Let \( H(y) \) be the cdf of \( Y. \) Then since the cubed-root function is 1-1, \( H(y) = P(Y \leq y) = P(Z^{1/3} \leq y) = P(Z \leq y^3) = \Phi(y^3). \) Therefore the density of \( Y \) is \( H'(y) = 3y^2\Phi(y^3) = \frac{3y^2}{\sqrt{2\pi}}e^{-y^6/2}. \)

(b) \( P(Y \geq (1/2)^{1/3}) = 1 - H((1/2)^{1/3}) = 1 - \Phi(\frac{1}{2}) = 1 - 0.6915 = 0.3085 \)

[Or alternately, \( P(Y \geq (1/2)^{1/3}) = P(Z \geq \frac{1}{2}) = \ldots. \)]
4. Ralph enrolls in Calculus by mistake, and only discovers this after the end of classes. He decides to show up to the exam anyway. The exam consists of 50 multiple-choice questions, worth 1 mark each, with possible answers (a) (b) (c) (d) (e). He has no clue how to answer any of the questions, so fills in answers guessed at random.

(a) [5] What is his mean score? (You must justify your answer to receive credit)
(b) [25] What (approximately) is the probability that he gets 15 or more questions correct?

**Solution:**

(a) Let $X$ be his score. Then $X \sim \text{Bin}(50, \frac{1}{5})$, so $E[X] = np = 50/5 = 10$.

(b) $np(1-p) = 8$ so $P(X \geq 15) = P(X \geq 14.5) = P \left( \frac{X-10}{\sqrt{8}} \geq \frac{14.5-10}{\sqrt{8}} \right)$

$\approx P(Z \geq 1.5910) \approx P(Z \geq 1.59) = 1 - \Phi(1.59) = 1 - 0.9441 = 0.0559$