York University
MATH 4430 3.0AF (Stochastic Processes)
Assignment 3 – Solutions

November 2008 (postponed to February 2009) – Salisbury

In problems 1.1 and 1.2 below, I asked you to do additional work in one of the two problems. The solutions give full solutions to both, but you did not have to do the extra work on both problems in order to receive full credit

§IV exercise 1.4

The long run cost per period is the integral of the cost function against the stationary probability. In other words, $\sum c_k \pi_k$ where c_k is the cost for state k. So we need to find π.

We have that $\pi_j = \sum_i \pi_i P_{ij}$ and $\sum_i \pi_i = 1$. In particular,
$\pi_0 = 0.3\pi_0 + 0.5\pi_1 + 0.5\pi_2$, or $-0.7\pi_0 + 0.5\pi_1 + 0.5\pi_2 = 0$
$\pi_1 = 0.2\pi_0 + 0.1\pi_1 + 0.2\pi_2$, or $0.2\pi_0 - 0.9\pi_1 + 0.2\pi_2 = 0$
$\pi_2 = 0.5\pi_0 + 0.4\pi_1 + 0.3\pi_2$, or $0.5\pi_0 + 0.4\pi_1 - 0.7\pi_2 = 0$
$\pi_0 + \pi_1 + \pi_2 = 1$.

Adding multiples of the 4th equation to the 1st and 2nd (to eliminate π_0) gives $1.2\pi_1 + 1.2\pi_2 = 0.7$ and $-1.1\pi_1 = -0.2$, so $\pi_1 = 0.2/1.1 = 2/11 \approx 0.181818$, $\pi_2 = 7/12 - \pi_1 = 53/132 \approx 0.401515$, and $\pi_0 = 1 - \pi_1 - \pi_2 = 1 - 7/12 = 5/12 \approx 0.416667$

So the cost per period is $2\pi_0 + 5\pi_1 + 3\pi_2 = 389/132 \approx 2.94697$

§IV problem 1.1

Let X_n denote the number or balls in urn A at time n. The states are $\{0, 1, 2, 3, 4, 5\}$ and the transition matrix is

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

This is doubly stochastic, so the stationary probability has $\pi_k = 1/6$, $k = 0, \ldots, 5$ and the long run fraction of time the urn is empty (in state 0) is $\pi_0 = 1/6$.

If this is the problem you elected to go further with, let $h_i = E[T \mid X_0 = i]$, where T is the first time $n \geq 0$ that $X_n = 0$. Then by first step analysis, $h_0 = 0$ and for $i \neq 0$, $h_i = 1 + \sum_j P_{ij} h_j$. In particular,

$h_1 = 1 + \frac{1}{2}h_0 + \frac{1}{2}h_2 = 1 + \frac{1}{2}h_2$ (and therefore $h_2 = 2h_1 - 2$)
$h_2 = 1 + \frac{1}{2}h_1 + \frac{1}{2}h_3$ (and therefore $h_3 = 2h_2 - 2 - h_1 = 3h_1 - 6$)
$h_3 = 1 + \frac{1}{2}h_2 + \frac{1}{2}h_4$ (and therefore $h_4 = 2h_3 - 2 - h_2 = 4h_1 - 12$)
$h_4 = 1 + \frac{1}{2}h_3 + \frac{1}{2}h_5$ (and therefore $h_5 = 2h_4 - 2 - h_3 = 5h_1 - 20$)
$h_5 = 1 + \frac{1}{2}h_4 + \frac{1}{2}h_5$ (and therefore $h_5 = 2 + h_4$, or $h_1 = 10$). In other words, $h_0 = 0, h_1 = 10, h_2 = 18, h_3 = 24, h_4 = 28, h_5 = 30$.

We can therefore calculate the mean return time m_0 to state 0, either as

$$m_0 = 1 + \sum_j P_{0j}h_j = 1 + \frac{1}{2}h_0 + \frac{1}{2}h_1 = 1 + \frac{10}{2} = 6$$

or as

$$m_0 = \frac{1}{\pi_0} = \frac{1}{1/6} = 6.$$

§IV problem 1.2

Let X_n denote the number or balls in urn A at time n. The states are $\{0, 1, 2, 3, 4, 5\}$ and the transition matrix is

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Note that though this chain is irreducible, it is also periodic of period 2. There is still a unique stationary probability π, and if we also know that we'd be able to write $\pi = \frac{1}{2}\pi^o + \frac{1}{2}\pi^e$, where π^o and π^e are probabilities concentrated on the odd and even states respectively, satisfying $\pi^e P = \pi^e$ and $\pi^0 P = \pi^e$. But in fact, we don't need to make use of all this structure in this problem.

To find π we solve the equations $\pi_j = \sum_i \pi_i P_{ij}$ and $\sum_i \pi_i = 1$. In other words,
$\pi_0 = \frac{1}{5}\pi_1$ (so $\pi_1 = 5\pi_0$)
$\pi_1 = \pi_0 + \frac{2}{5}\pi_2$ (so $\pi_2 = 5(\pi_1 - \pi_0)/2 = 10\pi_0$)
$\pi_2 = \frac{4}{5}\pi_1 + \frac{3}{5}\pi_3$ (so $\pi_3 = \frac{5}{3}(\pi_2 - \frac{4}{5}\pi_1) = 10\pi_0$)
$\pi_3 = \frac{3}{5}\pi_2 + \frac{4}{5}\pi_4$ (so $\pi_4 = \frac{5}{4}(\pi_3 - \frac{3}{5}\pi_2) = 5\pi_0$)
$\pi_4 = \frac{2}{5}\pi_3 + \pi_5$ (so $\pi_5 = \pi_4 - \frac{2}{5}\pi_3 = \pi_0$)
$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 = 32\pi_0$ (so $\pi_0 = 1/32$).
Note that there is also a redundant equation for π_5, that I haven't written down. In any case, $\pi_0 = 1/32, \pi_1 = 5/32, \pi_2 = 10/32, \pi_3 = 10/32, \pi_4 = 5/32, \pi_5 = 1/32$.

If this is the problem you elected to go further with, let $h_i = E[T \mid X_0 = i]$, where T is the first time $n \geq 0$ that $X_n = 0$. Then by first step analysis, $h_0 = 0$ and for $i \neq 0$, $h_i = 1 + \sum_j P_{ij} h_j$. In particular,
$h_1 = 1 + \frac{1}{5}h_0 + \frac{4}{5}h_2 = 1 + \frac{4}{5}h_2$
$h_2 = 1 + \frac{2}{5}h_1 + \frac{3}{5}h_3$ (and therefore $h_3 = \frac{5}{3}(h_2 - \frac{2}{5}h_1 - 1) = \frac{17}{15}h_2 - \frac{7}{3}$)
$h_3 = 1 + \frac{3}{5}h_2 + \frac{2}{5}h_4$ (and therefore $h_4 = \frac{5}{2}(h_3 - \frac{3}{5}h_2 - 1) = \frac{4}{3}h_2 - \frac{25}{3}$)
$h_4 = 1 + \frac{4}{5}h_3 + \frac{1}{5}h_5$ (and therefore $h_5 = 5(h_4 - \frac{4}{5}h_3 - 1) = \frac{32}{15}h_2 - \frac{112}{3}$)
$h_5 = 1 + h_4$ (and therefore $\frac{32}{15}h_2 - \frac{112}{3} = \frac{4}{3}h_2 - \frac{22}{3}$, or $h_2 = \frac{75}{2}$). In other words, $h_0 = 0, h_1 = 31, h_2 = 75/2, h_3 = 241/6, h_4 = 125/3, h_5 = 128/3$.

We can therefore calculate the mean return time m_0 to state 0, either as

$$m_0 = 1 + \sum_j P_{0j} h_j = 1 + h_1 = 32$$

or as

$$m_0 = \frac{1}{\pi_0} = \frac{1}{1/32} = 32.$$

§IV problem 2.4

(a) The states are $\{0, 1, 2, 3\}$ and the transitions that happen with probability 1 are $3 \to 2, 2 \to 1$, and $1 \to 0$. While a new component with $\xi = k$ corresponds to a transition $0 \to k - 1$. In other words, the transition matrix is

$$P = \begin{pmatrix} 0.1 & 0.3 & 0.2 & 0.4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(b) The chain is irreducible and aperiodic, with finitely many states, so is automatically regular. The conditions satisfied by the stationary probability π are that $\pi_j = \sum_i \pi_i P_{ij}$ and $\sum_i \pi_i = 1$. In other words,
$\pi_0 = 0.1\pi_0 + \pi_1$ (or $\pi_1 = 0.9\pi_0$)
$\pi_1 = 0.3\pi_0 + \pi_2$ (or $\pi_2 = \pi_1 - 0.3\pi_0 = 0.6\pi_0$)
$\pi_3 = 0.4\pi_0$.
$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ (so $\pi_0 = 1/2.9 = 10/29$). Note that there is also a redundant equation for π_3, that I haven't written down. In any case, the limiting distribution is $\pi_0 = 10/29, \pi_1 = 9/29, \pi_2 = 6/29, \pi_3 = 4/29$. And to answer the question asked, the long run probability that the component in service has no remaining life is $\pi_0 = 10/29$.

(c) The mean component lifetime is
$E[\xi] = 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.2 + 4 \times 0.4 = 2.9$
To see the relationship with π, think of an excursion from 0. If $\xi = k$ then

we jump from state 0 to state $k-1$, and then it takes $k-1$ more steps to descend to state 0. In other words, the return time is $1+(k-1)=k=\xi$. Therefore we should have $m_0 = E[\xi]$. And since $m_0 = 1/\pi_0 = 29/10 = 2.9$ this is indeed the case.

§IV problem 2.8

State $0=(1,0)$ corresponds to the computer working. State $1=(0,0)$ corresponds to the first day of a computer being broken down, so we have a transition $0 \to 1$ with probability p. State $2=(1,0)$ corresponds to the second day of a computer being broken down. So we have transitions $1 \to 2$ and $2 \to 0$ with probability 1. The latter is because the computer will be fixed after 2 days of repairs. Therefore if $q = 1-p$, the transition matrix is

$$P = \begin{pmatrix} q & p & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

The conditions for the stationary probability are that $\pi_0 = q\pi_0 + \pi_2$, $\pi_1 = p\pi_0$, $\pi_2 = \pi_1$, and $\pi_0 + \pi_1 + \pi_2 = 1$. The first is redundant, but the others give that $\pi_0(1+2p) = 1$, or $\pi_0 = 1/(2p+1)$. In particular, π_0 is the fraction of time the chain is in state 0. That is, the fraction of time that the computer is working, and the system is available for business. The numbers asked for are $\pi_0 = 0.9804$ (when $p = 0.01$), 0.9615 (when $p = 0.02$), 0.9091 (when $p = 0.05$), and 0.8333 (when $p = 0.1$).

§IV problem 4.4

Starting at state 0 we jump somewhere, and then work our way down to 0, one step at a time. In other words, we return to 0 with probability 1, so the chain is recurrent and cannot ever be transient. (To be more precise, 0 cannot be a transient state. It is possible that only finitely many of the α_k are non-zero, in which case state i would be transient for large enough i. But 0 would still belong to a recurrent and aperiodic class.)

We know from class that the chain will be positive recurrent if there is a stationary probability measure, and that it will be null recurrent otherwise. So let's see under what conditions we can solve the equations $\pi_j = \sum_i \pi_i P_{ij}$, $\sum_i \pi_i = 1$, all $\pi_i \geq 0$.

Assume we have a solution. Then $\pi_j = \alpha_{j+1}\pi_0 + \pi_{j+1}$ for $j \geq 0$. In other words, $\pi_j - \pi_{j+1} = \alpha_{j+1}\pi_0$, so

$$\pi_0 \sum_{i=j}^{\infty} \alpha_{i+1} = \pi_0 \lim_{N\to\infty} \sum_{i=j}^{N} \alpha_{i+1} = \lim_{N\to\infty} \sum_{i=j}^{N} (\pi_i - \pi_{i+1}) = \lim_{N\to\infty} \pi_j - \pi_{N+1} = \pi_j$$

since if the π_i sum to 1, then certainly they must $\pi_{N+1} \to 0$ as $N \to \infty$. Of course these sums are finite; If ξ is a random variable equal to 1+ the site we jump to starting from 0, then ξ takes value k with probability α_k, and

$$\pi_j = \pi_0 \sum_{i=j}^{\infty} \alpha_{i+1} = \pi_0 \sum_{i>j} \alpha_i = \pi_0 P(\xi > j).$$

Moreover, by formula (5.2) of chapter 1,

$$1 = \sum_{k=0}^{\infty} \pi_k = \pi_0 \sum_{k=0}^{\infty} P(\xi > k) = \pi_0 E[\xi].$$

So $E[\xi] < \infty$. In particular, what we've shown is that if the chain is positive recurrent then there is a solution to our equations, and in turn $E[\xi] < \infty$.

Conversely, if $E[\xi] < \infty$ then we can set $\pi_0 = 1/E[\xi]$ and then $\pi_j = P(\xi > j)\pi_0$, and produce a solution to our equations. From this we conclude that the chain is positive recurrent. In other words, a necessary and sufficient condition for the chain to be positive recurrent is that $E[\xi] < \infty$, or equivalently, that

$$\sum_{k=1}^{\infty} k\alpha_k < \infty.$$

If this sum is infinite, the chain will be null recurrent. For example, if $\alpha_k = c/k^2$ for a constant c chosen to make the α_k sum to 1, then this produces a null recurrent chain.

I suggested that you work out the following two additional problems as practice for the midterm. But they were not required for credit

§IV problem 4.2

We start by solving the eigenvalue equations, but we then have to dig a bit deeper if we're going to understand what happens to P^n as $n \to \infty$. In particular, recall that there are two subclasses here, which I'll label as $A = \{0, 1\}$ and $B = \{2, 3\}$. The chain alternates between being in A and being in B. And we can write the stationary measure as $\pi = \frac{1}{2}\pi^A + \frac{1}{2}\pi^B$, where π^A lives on A, π^B lives on B, and $\pi^A P = \pi^B$, $\pi^B P = \pi^A$.

If $\pi_j = \sum_i \pi_i P_{ij}$ and $\sum_i \pi_i = 1$ then
$\pi_0 = \frac{1}{4}\pi_2 + \frac{1}{3}\pi_3$
$\pi_1 = \frac{3}{4}\pi_2 + \frac{2}{3}\pi_3$
$\pi_2 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1$
$\pi_3 = \frac{1}{2}\pi_0 + \frac{2}{3}\pi_1$
$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1.$

Adding the first two equations shows that $\pi_0 + \pi_1 = \pi_2 + \pi_3$, so both sums $= \frac{1}{2}$.
So $\pi_0 = \frac{1}{4}\pi_2 + \frac{1}{3}\pi_3 = \frac{1}{4}\pi_2 + \frac{1}{3}(\frac{1}{2} - \pi_2) = \frac{1}{6} - \frac{1}{12}\pi_2$.
Likewise, $\pi_2 = \frac{1}{2}\pi_0 + \frac{1}{3}\pi_1 = \frac{1}{2}\pi_0 + \frac{1}{3}(\frac{1}{2} - \pi_0) = \frac{1}{6} + \frac{1}{6}\pi_0 = \frac{1}{6} + \frac{1}{6}(\frac{1}{6} - \frac{1}{12}\pi_2)$.
In other words, $72\pi_2 = 12 + 2 - \pi_2$, or $\pi_2 = 14/73$. Thus
$\pi_0 = 11/73 \approx 0.1507, \pi_1 = 51/146 \approx 0.3496, \pi_2 = 14/73 \approx 0.1918$, and
$\pi_3 = 45/146 \approx 0.3082$.

It follows that $\pi^A = (2\pi_0, 2\pi_1, 0, 0) = (\frac{22}{73}, \frac{51}{73}, 0, 0)$ and
$\pi^B = (0, 0, 2\pi_2, 2\pi_3) = (0, 0, \frac{28}{73}, \frac{45}{73})$. And keeping track of which subclass the process is in after n steps, we see that P^n is asymptotically of the following form, when n is even:

$$P^n \sim \begin{pmatrix} \pi^A \\ \pi^A \\ \pi^B \\ \pi^B \end{pmatrix} = \begin{pmatrix} \frac{22}{73} & \frac{51}{73} & 0 & 0 \\ \frac{22}{73} & \frac{51}{73} & 0 & 0 \\ 0 & 0 & \frac{28}{73} & \frac{45}{73} \\ 0 & 0 & \frac{28}{73} & \frac{45}{73} \end{pmatrix}$$

And when n is odd, P^n is asymptotically of the form

$$P^n \sim \begin{pmatrix} \pi^B \\ \pi^B \\ \pi^A \\ \pi^A \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{28}{73} & \frac{45}{73} \\ 0 & 0 & \frac{28}{73} & \frac{45}{73} \\ \frac{22}{73} & \frac{51}{73} & 0 & 0 \\ \frac{22}{73} & \frac{51}{73} & 0 & 0 \end{pmatrix}$$

Basically, P^n asymptotically flips back and forth between these two matrices.

§IV problem 5.2

There are 2 communicating classes – a transient class $\{0, 1, 2, 3, 4\}$ and a recurrent class $\{5, 6, 7\}$. Because there is only one recurrent class, there will be a unique stationary measure, and it will be concentrated on the recurrent class. In other words, we know before we start that $\pi_0 = \pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$. So the remaining equations are that
$\pi_5 = 0.3\pi_5 + 0.2\pi_6 + 0.9\pi_7$
$\pi_6 = 0.4\pi_5 + 0.2\pi_6 + 0.1\pi_7$
$\pi_7 = 0.3\pi_5 + 0.6\pi_6$
$\pi_5 + \pi_6 + \pi_7 = 1$.

Substituting the 3rd equation into the first gives that $0.43\pi_5 = 0.76\pi_6$, or
$\pi_5 = 74\pi_6/43$. The third equation then gives that $\pi_7 = 48\pi_6/43$. Then the 4th

gives that $\pi_6 = 43/165$. So $\pi_5 = 74/165$ and $\pi_7 = 48/165$. Thus

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \\ 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \\ 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \\ 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \\ 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \\ 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \\ 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \\ 0 & 0 & 0 & 0 & 0 & \frac{74}{165} & \frac{43}{165} & \frac{48}{165} \end{pmatrix}$$

This would be a much better problem if P were modified slightly. Indeed, I think this is what the authors intended, but they wrote it down wrong. Let's shift the 3rd and 4th rows of P over one column. In other words, take

$$P = \begin{pmatrix} 0.1 & 0.2 & 0.1 & 0.1 & 0.2 & 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.2 & 0.1 & 0 & 0.3 & 0.1 & 0.2 \\ 0.5 & 0 & 0 & 0.2 & 0.1 & 0.1 & 0.1 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0.9 & 0.1 & 0 \end{pmatrix}$$

Now there are two recurrent classes, $A = \{3, 4\}$ and $B = \{5, 6, 7\}$ and one transient class $C = \{0, 1, 2\}$. Above we've found the "extreme" stationary probability $\pi^B = (0, 0, 0, 0, 0, \frac{74}{165}, \frac{43}{165}, \frac{48}{165}) = (0, 0, 0, 0, 0, 0.4485, 0.2606, 0.2909)$ corresponding to $\{5, 6, 7\}$. There is a second "extreme" stationary probability π^A living on $\{3, 4\}$, ie in which the only non-zero entries are π_3 and π_4. The general stationary probability then has the form $\lambda\pi^A + (1-\lambda)\pi^B$ where $0 \le \lambda \le 1$. Solving for π^A as above we get $\pi_3 = 0.3\pi_3 + 0.6\pi_4$ and $\pi_3 + \pi_4 = 1$. In other words, $\pi_3 = 6/13$ and $\pi_4 = 7/13$, or

$\pi^A = (0,0,0,\frac{6}{13},\frac{7}{13},0,0,0) = (0,0,0,0.4615,0.5385,0,0,0)$. It follows that

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \lambda_0\pi^A + (1-\lambda_0)\pi^B \\ \lambda_1\pi^A + (1-\lambda_1)\pi^B \\ \lambda_2\pi^A + (1-\lambda_2)\pi^B \\ \pi^A \\ \pi^A \\ \pi^B \\ \pi^B \\ \pi^B \end{pmatrix}$$

where λ_i is the probability of reaching A starting from state i.

We calculate λ_i by a first step analysis. It $= 1$ for $i = 3, 4$ and $= 0$ for $i = 5, 6, 7$. And for $i = 0, 1$, we have $\lambda_i = \sum_j P_{ij}\lambda_j$. In other words,
$\lambda_0 = 0.1\lambda_0 + 0.2\lambda_1 + 0.1\lambda_2 + 0.1 + 0.2$
$\lambda_1 = 0.1\lambda_1 + 0.2\lambda_2 + 0.1$
$\lambda_2 = 0.5\lambda_0 + 0.2 + 0.1 = 0.5\lambda_0 + 0.3$

Thus $0.9\lambda_1 = 0.2(0.5\lambda_0 + 0.3) + 0.1 = 0.1\lambda_0 + 0.16$, or $\lambda_1 = 0.1111\lambda_0 + 0.1778$. Likewise, $0.3 = 0.9\lambda_0 - 0.2\lambda_1 - 0.1\lambda_2 = \lambda_0(0.9 - 0.0222 - 0.05) - 0.0356 - 0.03 = 0.8278\lambda_0 - 0.0656$, so $\lambda_0 = 0.3656/0.8278 = 0.4416$, $\lambda_2 = 0.5208$, and $\lambda_1 = 0.2268$; Substituting in, we have

$$\lim_{n\to\infty} P^n = \begin{pmatrix} 0 & 0 & 0 & 0.2038 & 0.2378 & 0.2504 & 0.1455 & 0.1624 \\ 0 & 0 & 0 & 0.1047 & 0.1221 & 0.3467 & 0.2015 & 0.2249 \\ 0 & 0 & 0 & 0.2404 & 0.2804 & 0.2149 & 0.1249 & 0.1394 \\ 0 & 0 & 0 & 0.4615 & 0.5385 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4615 & 0.5385 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4485 & 0.2606 & 0.2909 \\ 0 & 0 & 0 & 0 & 0 & 0.4485 & 0.2606 & 0.2909 \\ 0 & 0 & 0 & 0 & 0 & 0.4485 & 0.2606 & 0.2909 \end{pmatrix}$$