York University
MATH 6602 3.0AF (Stochastic Processes)
Assignment 2 – Solutions

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(1) **§2, elementary problem 4**

(a) The equivalence classes are $\{0,1,2\}$ and $\{3\}$. This is because if we start at 3, we leave right away and never come back. But also $0 \to 2 \to 1 \to 0$ is possible. According to the convention on p. 61, $d_3 = 0$. And because both $0 \to 2 \to 1 \to 0$ and $0 \to 3 \to 0$ are possible, the period of $\{0,1,2\}$ is $\gcd(3,2) = 1$.

(b) The transition $0 \to 1 \to 3 \to 0$ is possible, as is the transition $2 \to 1 \to 3 \to 2$ so interweaving these circuits, all states communicate and there is a single equivalence class. The period is 3. Because we cycle between being in $\{0,2\}$, to being at 1, to being at 3, and then back to being in $\{0,2\}$.

(2) **§2, problem 7**

(a) Let $i_0, i_1, \dots, i_N$ be the smallest possible sequence of states for which $i_0 = j$, $i_N = k$, and each transition $i_n \to i_{n+1}$ happens with positive probability. By hypothesis, there is such a sequence, so there is certainly a shortest. I claim there will be no repetitions in the sequence. For if i_n appeared twice we could get an even shorter such sequence by deleting all the terms in between, along with one of the i_n's.

Since each terms appears at most once, and there are only r states, it follows that $N \le r - 1$.

(b) Since j is recurrent, we may assume that the whole chain consists of a single equivalence class. Let β_k be the probability that it takes more than r steps to get from k to j. By part (a), all the $\beta_k < 1$. Let $\beta < 1$ be the maximum of the β_k. I claim that

$$P(\text{it takes more than } n = mr \text{ steps to return to } j) \le \beta^m.$$

Let R_m denote the above event. For $m = 1$ we have $P(R_1) \le \beta$ by definition of β. If we fix some sequence $x_0, x_1, \dots, x_{mr}$ which doesn't return to j, then

$$\begin{aligned} &P(R_{m+1} \mid X_0 = x_0, \dots, X_{mr} = x_{mr}) \\ &\qquad = P(X_{mr+1} \neq j, \dots, X_{(m+1)r} \neq j \mid X_{mr} = x_{mr}) \le \beta_{x_{mr}} \le \beta. \end{aligned}$$

Thus

$$P(R_{m+1}, X_0 = x_0, \dots, X_{mr} = x_{mr}) \le \beta P(X_0 = x_0, \dots, X_{mr} = x_{mr}).$$

Summing over all such sequences $x_0, \dots, x_{mr}$ we get that $P(R_{m+1}) \le \beta P(R_m)$. So by induction, we conclude that $P(R_m) \le \beta^m$ for every m.

Now for arbitrary n, set $p_n = P(\text{it takes more than } n \text{ steps to return to } j)$. Then $p_n \le \beta^m$ where m is any number with $mr \le n$ (because it also takes more than mr steps to return). We may choose m so that $m \le n/r < m+1$. That is, $m > \frac{n}{r} - 1$. In particular,

$$p_n \le \beta^m < \beta^{\frac{n}{r}-1}.$$

Having this $\le \alpha^n$ just means that

$$\beta^{\frac{1}{r}-\frac{1}{n}} \le \alpha$$

for every $n > r$. But the largest this expression can be is when $n = r+1$, so we just take

$$\alpha = \beta^{\frac{1}{r}-\frac{1}{r+1}} = \beta^{\frac{1}{r(r+1)}} < 1.$$

(3) **§3, problem 1**

We have here a random walk on $\{0, 1, \dots, r\}$ with absorption at 0 and r. Let d_k be the mean absorption time, starting from k. Then $d_0 = d_r = 0$, and for $0 < i < r$ we have $d_i = 1 + \sum_j P_{ij} d_j = 1 + p d_{i+1} + q d_{i-1}$.

We know from class that $d_i = i(r-i)$ if $p = \frac{1}{2}$. So suppose $p \ne \frac{1}{2}$. I stated in class that these difference equations have unique solutions (this follows easily by linear algebra), so we just need to find one. We will search for a solution of the form $d_i = A + Bb^i + ci$. Substituting in, we get that $A + Bb^i + ci = 1 + A + Bb^{i-1}(pb^2+q) + c(i+p-q)$. This will work, for arbitrary A and B, provided $pb^2 - b + q = 0$ and $1 + c(p-q) = 0$. Since $p+q = 1$ the first factors as $(pb-q)(b-1) = 0$. In other words, anything of the form $A + B(q/p)^i + i/(q-p)$ solves the difference equation. To also solve the boundary conditions, we need that $A + B = 0$ and $A + B(q/p)^r + r/(q-p) = 0$. Therefore $B = -A$ and $A = r/[(q-p)(1-(q/p)^r)]$. In other words,

$$d_i = \frac{i}{q-p} - \frac{r}{(q-p)(1-(q/p)^r)}\Big(1 - (q/p)^i\Big).$$

Alternatively, we could get the same answer without having to guess the form of the solution. From the equation $(p+q)d_i = d_i = 1 + pd_{i+1} + qd_{i-1}$

we get recursively that for $1 \le i \le r-1$

$$\begin{aligned}
d_{i+1} - d_i &= \frac{q}{p}(d_i - d_{i-1}) - \frac{1}{p} \\
&= \Big(\frac{q}{p}\Big)^2 (d_{i-1} - d_{i-2}) - \frac{1}{p}\Big(1 + \frac{q}{p}\Big) \\
&= \Big(\frac{q}{p}\Big)^3 (d_{i-2} - d_{i-3}) - \frac{1}{p}\Big(1 + \frac{q}{p} + \big(\frac{q}{p}\big)^2\Big) \\
&= \dots \\
&= \Big(\frac{q}{p}\Big)^i (d_1 - d_0) - \frac{1}{p}\Big(1 + \frac{q}{p} + \cdots + \big(\frac{q}{p}\big)^{i-1}\Big).
\end{aligned}$$

Let $b = q/p$ and $c = 1/(q-p)$. Since $d_0 = 0$ we get that

$$d_{i+1} - d_i = b^i d_1 - \frac{1}{p} \cdot \frac{1-b^i}{1-b} = (d_1 - c)b^i + c.$$

Summing up, for $2 \le i \le r$ we have

$$\begin{aligned}
d_i = d_i - d_0 &= (d_i - d_{i-1}) + \cdots + (d_2 - d_1) + d_1 \\
&= \sum_{j=1}^{i-1} ((d_1 - c)b^i + c) + d_1 \\
&= d_1 + (i-1)c + (d_1 - c)\Big(\frac{1-b^i}{1-b} - 1\Big) \\
&= ci + (d_1 - c)\frac{1-b^i}{1-b}.
\end{aligned}$$

This formula is clearly true for $i = 1$ as well. Substituting $i = r$ and using that $d_r = 0$, we get that

$$\frac{d_1 - c}{1-b} = -\frac{cr}{1-b^r},$$

so

$$d_i = ci - \frac{cr}{1-b^r}(1 - b^i),$$

which is exactly the formula we had before.

(4) **§3, problem 4**

Let h_i be the probability of absorption at 0, starting from $X_0 = i$. Then $h_0 = 1$, $h_N = 0$, and for $1 \le i < N$,
$h_i = \sum_j P_{ij} h_j = \mu_i h_{i-1} + \lambda_i h_{i+1} + (1 - \lambda_i - \mu_i)h_i$.
In other words, $(\lambda_i + \mu_i)h_i = \lambda_i h_{i+1} + \mu_i h_{i-1}$, or $\lambda_i(h_{i+1} - h_i) = \mu_i(h_i - h_{i-1})$.
Let

$$\rho_0 = 1, \qquad \rho_i = \frac{\mu_1 \mu_2 \cdots \mu_i}{\lambda_1 \lambda_2 \cdots \lambda_i}.$$

It follows that $h_{i+1} - h_i = \rho_i(h_1 - h_0) = \rho_i(h_1 - 1)$.
Adding up, we get that $h_{i+1} - 1 = h_{i+1} - h_0 = (h_1 - 1)\sum_{j=0}^{i} \rho_j$.
Therefore $-1 = h_N - h_0 = (h_1 - 1)\sum_{j=0}^{N-1} \rho_j$.
In other words, $h_1 - 1 = -1/\sum_{j=0}^{N-1} \rho_j$ so

$$h_k = 1 - \frac{\sum_{j=0}^{k-1} \rho_j}{\sum_{j=0}^{N-1} \rho_j} = \frac{\sum_{j=k}^{N-1} \rho_j}{\sum_{j=0}^{N-1} \rho_j}.$$

Of course, the absorption probability at N is $1 - h_k$.

(5) If you ever toss just 1 coin, then $X = 1$ (because you will keep tossing that coin until it comes up heads). So consider a Markov chain X_n defined as follows: $X_n = k$ means we toss k coins on the $n+1$'st toss, and then (conditional on this) $X_{n+1} \sim \mathrm{Bin}(k, \frac{1}{2})$ counts the number of tails.
Let $h_i = P(\text{we reach state } 1 \mid X_0 = i)$. We use an initial distribution in which $X_0 = 5$ so what we want is to find h_5.

The transition matrix is

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} & 0 & 0 \\ \frac{1}{16} & \frac{1}{4} & \frac{3}{8} & \frac{1}{4} & \frac{1}{16} & 0 \\ \frac{1}{32} & \frac{5}{32} & \frac{5}{16} & \frac{5}{16} & \frac{5}{32} & \frac{1}{32} \end{pmatrix}$$

and the equation we need to solve is that $h_i = \sum_j P_{ij} h_j$ for $i \neq 1$. Clearly $h_0 = 0$ and $h_1 = 1$. The other equations therefore become
$h_2 = \frac{1}{4}h_0 + \frac{1}{2}h_1 + \frac{1}{4}h_2 = \frac{1}{2} + \frac{1}{4}h_2$, so $\frac{3}{4}h_2 = \frac{1}{2}$ or $h_2 = \frac{2}{3}$.
$h_3 = \frac{1}{8}h_0 + \frac{3}{8}h_1 + \frac{3}{8}h_2 + \frac{1}{8}h_3 = \frac{3}{8} + \frac{1}{4} + \frac{1}{8}h_3$, so $\frac{7}{8}h_3 = \frac{5}{8}$ or $h_3 = \frac{5}{7}$.
$h_4 = \frac{1}{16}h_0 + \frac{1}{4}h_1 + \frac{3}{8}h_2 + \frac{1}{4}h_3 + \frac{1}{16}h_4 = \frac{1}{4} + \frac{1}{4} + \frac{5}{28} + \frac{1}{16}h_4$, so $\frac{15}{16}h_4 = \frac{19}{28}$ or $h_4 = \frac{76}{105}$.
$h_5 = \frac{1}{32}h_0 + \frac{5}{32}h_1 + \frac{5}{16}h_2 + \frac{5}{16}h_3 + \frac{5}{32}h_4 + \frac{1}{32}h_5 = \frac{5}{32} + \frac{5}{24} + \frac{25}{112} + + \frac{19}{168} + \frac{1}{32}h_5$, so $\frac{31}{32}h_5 = \frac{157}{224}$ or $h_5 = \frac{157}{217} = 0.7235$

(6) Changing variables to $m = n - 1$,

$$\begin{aligned} h_i &= E[\sum_{n\geq 0} \beta^n c(X_n) \mid X_0 = i] = \beta^0 c(i) + E[\sum_{n\geq 1} \beta^n c(X_n) \mid X_0 = i] \\ &= c(i) + \sum_j P_{ij} E[\sum_{n\geq 1} \beta^n c(X_n) \mid X_0 = i, X_1 = j] \\ &= c(i) + \sum_j P_{ij} E[\sum_{m\geq 0} \beta^{m+1} c(X_m) \mid X_0 = j] \\ &= c(i) + \beta \sum_j P_{ij} E[\sum_{m\geq 0} \beta^m c(X_m) \mid X_0 = j] = c(i) + \beta \sum_j P_{ij} h_j. \end{aligned}$$

(7) There are several ways of approaching this problem. Let S_0 be the number of even times n that $X_n = 0$, and let S_1 be the number of even times n that $X_n = 1$. Observe that

$$\begin{aligned} P(T = 2n+1) &= P(X_{2n} = 0, X_{2n+1} = 2) + P(X_{2n} = 1, X_{2n+1} = 2) \\ &= 0.5P(X_{2n} = 0) + 0.4P(X_{2n} = 1). \end{aligned}$$

Summing over n we get that $P(T \text{ is odd}) = 0.5E[S_0] + 0.4E[S_1]$. If we look at X_n only at even times n, we have a Markov chain with transition matrix P^2, and a standard first step analysis for this chain will let us compute $E[S_0]$ and $E[S_1]$.

A related way of solving the problem is to let $f_i = P(T \text{ is odd} \mid X_0 = i)$ and to take two steps instead of one. In other words, $f_2 = 0$ and for $i = 0, 1$ $f_i = \sum_{j,k} P_{ij} P_{jk} P(T \text{ is odd} \mid X_0 = i, X_1 = j, X_2 = k)$. The above conditional probability is 1 if $j = 2$, 0 if $j \neq 2$ and $k = 2$, and f_k if $k = 0, 1$. So the equations to solve are actually that $f_i = P_{i2} + P^2_{i0} f_0 + P^2_{i1} f_1$, $i = 0, 1$.

Another way of solving the problem is to change to a Markov chain Y with 6 states $0, 1, 2, 0', 1', 2'$. States $0, 1$ for Y correspond to X being in states $0, 1$ at even times, and states $0', 1'$ for Y correspond to X being in states $0, 1$ at odd times. In other words, the transition matrix for Y is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0.3 & 0.2 & 0.5 \\ 0 & 0 & 0 & 0.5 & 0.1 & 0.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.3 & 0.2 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.1 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

If Y ends up absorbed in state 2, this means the original process X reached state 2 first at an even time. If Y ends up absorbed in state $2'$, this means the original process X reached state 2 first at an odd time. Now apply first step analysis to find the probability of Y being absorbed in state $2'$.

Yet another way of doing this is to let $f_i = P(T \text{ is odd} \mid X_0 = i)$ and $g_i = P(T \text{ is even} \mid X_0 = i)$. Then $g_i = 1 - f_i$. First step analysis gives that $f_2 = 0$ and for $i \neq 2$,

$$\begin{aligned} f_i &= \sum_j P_{ij} P(T \text{ is odd} \mid X_0 = i, X_1 = j) \\ &= \sum_j P_{ij} P(T - 1 \text{ is even} \mid X_1 = j) \\ &= \sum_j P_{ij} P(T \text{ is even} \mid X_0 = j) \\ &= \sum_j P_{ij} g_j = \sum_j P_{ij}(1 - f_j). \end{aligned}$$

I'll solve this version. In particular,
$f_0 = 0.3(1 - f_0) + 0.2(1 - f_1) + 0.5 = 1 - 0.3f_0 - 0.2f_1$, and
$f_1 = 0.5(1 - f_0) + 0.1(1 - f_1) + 0.4 = 1 - 0.5f_0 - 0.1f_1$.
Therefore $1.3f_0 + 0.2f_1 = 1 = 0.5f_0 + 1.1f_1$, so
$0.8 = 1.3 - 0.5 = 1.3(0.5f_0 + 1.1f_1) - 0.5(1.3f_0 + 0.2f_1) = (1.43 - 0.1)f_1 = 1.33f_1$. Thus $f_1 = 0.8/1.33 = 0.6015$ and $0.5f_0 = 1 - 1.1f_1$ implies that the answer to the question asked is $f_0 = 0.3383/0.5 = 0.6767$