

York University  
MATH 4430/6602 3.0AF (Stochastic Processes)  
Midterm Exam I – SOLUTIONS

October 17, 2008

NAME:

STUDENT NUMBER:

You have 50 minutes to complete the examination. There are 4 pages, containing 4 questions. You may bring one letter-sized two-sided formula sheet to the exam. No other books or notes may be used. You may use a calculator. Show all your work, and explain or justify your answers. You may leave numerical answers unsimplified, including binomial coefficients. Use the back of your page if you run out of room.

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1. [20] A Markov chain with state space  $\{0, 1, 2, 3, 4\}$  has transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}$$

Find all communicating classes and determine their periods. Explain your reasoning.

*Solution:* The communicating classes are  $\{0, 4\}$  and  $\{1, 2, 3\}$ . To see the former note that 0 and 4 do communicate (in 1 step), but that 0 is not accessible from any state other than 4, and vice versa. To see that  $\{1, 2, 3\}$  is a class, note that a circuit  $1 \rightarrow 2 \rightarrow 3 \rightarrow 2 \rightarrow 1$  is possible.

Both classes have period  $d = 2$ . To see this observe that in  $\{0, 4\}$  we alternate from 0 to 4 and back (until leaving the class). And in  $\{1, 2, 3\}$  we alternate between being in state 2 and being in states 1 or 3.

2. Define a Markov chain with state space  $E = \{0, 1, 2\}$  as follows:

- If  $X_0 = i$ , choose a potential next state  $j$  to move to, uniformly from  $E$ .
- If  $i \leq j$  then accept this move and let  $X_1 = j$ .
- If  $i > j$ , flip a fair coin. If it lands head, accept the move (ie take  $X_1 = j$ ). If it lands tails, reject the move (ie take  $X_1 = i$ ).

- (a) [10] Find the transition matrix  $P$ .  
 (b) [5] Find  $P(X_1 = 2, X_2 = 2, X_3 = 0 \mid X_0 = 0)$ .  
 (c) [5] Find  $P(X_2 = 1 \mid X_0 = 1)$ .

*Solution:*

- (a) For  $i < j$  we'll have  $P_{ij} = 1/3$  since that move, if proposed, will always be accepted. For  $i > j$  we'll have  $P_{ij} = 1/6$  since that move, if proposed, will be accepted half the time. We can then work out  $P_{ii}$  by using the fact that the row sums equal 1. In other words,

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}$$

- (b)  $P(X_1 = 2, X_2 = 2, X_3 = 0 \mid X_0 = 0) = P_{02}P_{22}P_{20} = \frac{1}{3} \times \frac{2}{3} \times \frac{1}{6} = \frac{1}{27}$ .  
 (c)  $P(X_2 = 1 \mid X_0 = 1) = \sum_j P_{1j}P_{j1} = \frac{1}{6} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{6} = \frac{13}{36}$ .

3. [20] A Markov chain with state space  $\{0, 1, 2, 3\}$  has transition matrix

$$P = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0 & 0.3 & 0.7 & 0 \\ 0.1 & 0.7 & 0 & 0.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let  $T$  be the total time spent in the even-numbered states (ie in states 0 or 2). Use first step analysis to set up an explicit family of equations satisfied by  $h_i = E[T \mid X_0 = i]$ ,  $i = 0, 1, 2, 3$ .

Justify your derivation, but **DO NOT SOLVE THESE EQUATIONS**.

*Solution:*

$\{0, 1, 2\}$  communicate, and 3 is absorbing. Since a transition  $2 \rightarrow 3$  is possible, eventually the process is absorbed in state 3. Thus the total time  $T$  spent in the even states (0 and 2) is finite. Let  $h_i = E[T \mid X_0 = i]$ . Then  $h_3 = 0$ . If  $i = 0, 2$  then  $h_i = 1 + \sum_j P_{ij}h_j$  since the first step contributes time 1 to  $T$ . But if  $i = 1$  then  $h_i = \sum_j P_{ij}h_j$ . Writing this out, we have

$$\begin{aligned} h_0 &= 1 + 0.4h_0 + 0.6h_1 \\ h_1 &= 0.3h_1 + 0.7h_2 \\ h_2 &= 1 + 0.1h_0 + 0.7h_1 + 0.2h_3 \\ h_3 &= 0 \end{aligned}$$

4. [20] Solve (a) or (b), depending on which class you are taking.
- (a) MATH 4430: Let  $Q$  be uniform on  $[0, 1]$ . Given  $Q = q$ , let  $X_1, X_2, \dots$  be independent, taking value 1 with probability  $q$  and 0 with probability  $1 - q$ . Find  $E[X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5 \cdot X_6 \cdot X_7 \cdot X_8 \cdot X_9 \cdot X_{10}]$
- (b) MATH 6602: Let  $X$  be a bounded random variable with distribution symmetric about 0. In other words,  $E[F(X)] = E[F(-X)]$  for every bounded Borel measurable function  $F$ . Prove that  $E[X | X^2] = 0$ . Be sure to define the meaning of this equality..

*Solution:*

- (a)  $E[X_1 \cdots X_{10}] = E[E[X_1 \cdots X_{10} | Q]] = E[E[X_1 | Q] \cdots E[X_{10} | Q]]$   
 $= E[Q^{10}] = \int_0^1 q^{10} dq = \frac{1}{11}$ .
- (b) Let  $Z = E[X | X^2]$ . By definition,  $Z$  is any random variable such that  $Z = h(X^2)$  for some function  $h$ , and  $E[Z1_A(X^2)] = E[X1_A(X^2)]$  for every Borel  $A$ .

We want to show that  $Z = 0$  satisfies the above two conditions. The first is obvious (take  $h(z) = 0$ ). The second is simply the statement that  $E[X1_A(X^2)] = 0$  for every Borel  $A$ .

By symmetry,  $E[X1_A(X^2)] = E[-X1_A((-X)^2)] = -E[X1_A(X^2)]$ . Collecting terms, this shows that  $E[X1_A(X^2)] = 0$ , which is what we needed to show.