Stochastic Calculus in Finance
Assignment 2

Due Friday November 29, 2002 4:00 PM

Problem 1. Use Itô's lemma to determine a constant a so that

$$M_t = e^{B_t} \cos(B_t + at)$$

is a martingale.

Problem 2. Do problem 9.2 from p. 150 of Steele. You can use the trick from section 9.3

Problem 3. The price of the stock of ABC corporation satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t \, dB_t$$

where B_t is a Brownian motion. The corporation enters into a contract with its CEO, worth

$$A \ln \left(\frac{S_T}{K} \right)$$

at time T. Note that if the stock price S_T is greater than K, the CEO receives a payment, but if $S_T < K$ then she has to pay the corporation. In other words, this is an incentive for her to see that the stock price goes up. In order to neutralize the contract, she decides to hedge. Ignoring transaction costs, how much does it cost her at time $t = 0$ to implement a hedge that will exactly balance this contract at time $t = T$? You should obtain your answer by
(a) Expressing the hedging cost in terms of risk neutral expectations,
(b) evaluating these expectations.
(c) Finally, work out an actual cost, where T corresponds to 2 years, $r = 3\%$ per year, $\mu = 6\%$ per year, $\sigma = 30\%$ per year, $K = 10$, the intial price of the stock is $S_0 = 12$, and $A = 100,000$.

Problem 4. Let X_t denote the price of a stock, obeying the stochastic differential equation

$$dX_t = rX_t \, dt + \sigma \, d\tilde{B}_t$$

under the risk-neutral measure $\tilde{P}$, where r is the risk-free rate of return. As usual, $\tilde{B}_t$ is a Brownian motion under $\tilde{P}$. Note that this differs from the Black-Scholes-Merton model, as there is no factor of X_t in front of the $d\tilde{B}_t$. This is a somewhat strange model (eg. prices can go negative), but it can be shown (and you may assume this) that it is complete and that option values V_t still have the property that $e^{-rt}V_t$ is a martingale under $\tilde{P}$. Note that the above is an Ornstein-Uhlenbeck SDE, so that we know from class that the solution is

$$X_t = x_0 e^{rt} + \sigma e^{rt} \int_0^t e^{-rs}\, d\tilde{B}_s,$$

and that X_t has a normal distribution with mean $x_0 e^{rt}$ and variance $\frac{\sigma^2(e^{2rt}-1)}{2r}$.

(a) Find a formula for the price (at time 0) of a binary option in this model. In other words, find the no-arbitrage price of an option that pays an amount K if the stock value at time T exceeds a given value x, and pays nothing otherwise.

(b) In particular, what will this price be if $r = 0.06$, $T = 5$, $\sigma = 0.50$, $x_0 = 100$, $x = 138$, and $K = 30$?

Problem 5. (Cox-Ingersoll-Ross interest rate model).

(a) Let

$$dr_t = (\alpha - \beta r_t)dt + \sigma\sqrt{r_t}dB_t.$$

where α, β, σ, and the initial value r_0 are all positive constants. Write this SDE in integral form, take expectations, and then differentiate with respect to t. Express the result as an ordinary differential equation for $u(t) = E[r_t]$. In other words, an equation for u_t' in terms of u_t.

(b) Use Ito's lemma to obtain an SDE for r_t^2. Carry out the same procedure as above to obtain an ODE for $v_t = E[r_t^2]$. In other words, find an equation for v_t' in terms of v_t and u_t.

[Note: these ODEs can actually be solved, though I'm not asking you to do this. This allows one to calculate the mean and variance of r_t, which is useful, as the above is one of the standard models for spot interest rates.]

Problem 6. (CIR continued)

(a) Let

$$X_t = e^{-\beta t/2}(x_0 + \frac{\sigma}{2}\int_0^t e^{\beta s/2} dB_s)$$

and check that

$$dX_t = \frac{\sigma}{2}dB_t - \frac{\beta}{2}X_t\, dt$$

(this isn't new - it is a special case of the Ornstein Uhlenbeck process from class.)

(b) Let $R_t = X_t^2$ and

$$W_t = \int_0^t \text{Sign}(X_s)\, dB_s$$

Show that

$$dR_t = (\frac{\sigma^2}{4} - \beta R_t)dt + \sigma \sqrt{R_t} dW_t.$$

Note that $\text{Sign}(x) = \pm 1$ depending on whether $x \geq 0$ or $x < 0$. It can be shown that W_t is a Brownian motion too, so this gives us a solution to a particular case of the Cox-Ingersoll-Ross model.