**Stochastic Calculus in Finance**
**MATH 6910 - Salisbury**

## Introduction to American Options

(1) An *American Option* is one that the owner can exercise at any time $t \leq T$, rather than only at time $T$ as is the case with a European option. If exercised at time $t$ the owner receives an amount $f(S_t)$ depending on the current value $S_t$ of the stock. A perfect hedge has to work regardless of when it is exercised. To hedge, we look for a portfolio $V_t$ such that

- $V_t \geq f(S_t)$ for every $t \leq T$;
- For every $s < T$ there exists a $t \in [s, T]$ with $V_t = f(S_t)$.

If we can find such a portfolio then the no-arbitrage price at any time $s$ will be $V_s$.

Why? If at time $s$ we observe a price $> V_s$, we sell the option, purchase the portfolio, and pocket the difference as arbitrage. Regardless of when the option is exercised, our risk is covered (and we may in fact reap a further profit if the option is not exercised optimally).

Conversely, if we observe a price $< V_s$ we buy the option, sell the portfolio, and pocket the difference as arbitrage. We then wait till a time $t$ so that $V_t = f(S_t)$ to exercise the option, at which point the portfolio and option positions exactly net out.

(2) More precisely, take our usual model for a stock and a money market fund:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t, \qquad dR_t = rR_t \, dt.$$

The market is complete, so let $\tilde{P}$ be the risk-neutral measure. Consider a continuous function $v(t, x) \geq f(x)$ and set $V_t = v(t, S_t)$. Let $\tau_s$ be the first time $t > s$ at which $V_t = f(S_t)$. Suppose we can choose $v$ so that for $s < T$

- $e^{-rt} V_t$ is a $\tilde{P}$-supermartingale, $t \in [0, T]$;
- $e^{-r(t \wedge \tau_s)} V_{t \wedge \tau_s}$ is a $\tilde{P}$-martingale, $t \in [s, T]$;
- $\tau_s \leq T$ almost surely.

In other words, suppose that discounting $V_t$ produces a supermartingale, which actually becomes a martingale when we stop it at the random times $\tau_s$. Then it follows that

$$\boxed{V_t \text{ is the desired hedge.}}$$

To justify this statement, note that this portfolio has the properties mentioned in the paragraph (1). But why do we want a supermartingale, when what we already know is that discounted self-financing portfolios are risk-neutral martingales? The reason is that this hedging portfolio does not actually have to be self-financing. We won't ever put new money in, but we

WILL be allowed to withdraw money from the portfolio, if the hedging cost drops as a result of sub-optimal behaviour on the part of the option holder. In other words, the portfolios we are interested in are martingales up to the first time the option holder SHOULD exercise the option. But if the option isn't exercised then, the hedger can start withdrawing any funds that are freed up by this failure to exercise.

Assuming we can actually find such a function $v$, it follows that American options should typically be exercised as soon as their price reaches the *intrinsic value* $f(S_t)$. Any other exercise policy will normally allow the hedger to earn excess profits by making withdrawals from the hedging portfolio. In other words,

$$\boxed{\tau_s = \text{the optimal exercise time}}$$

for an individual holding the option at time $s$.

**Exercise:** Show that there can only be one such function $v$. You'll need to use the *optional sampling* property, which implies that for the bounded stopping time $\tau_s \geq s$ we have that $\tilde{E}[M_{\tau_s}|\mathcal{F}_s] = M_s$ if $M$ is a martingale, and $\tilde{E}[M_{\tau_s}|\mathcal{F}_s] \leq M_s$ if $M$ is a supermartingale. See below for the definition of a stopping time.

(3) The *continuation region* is the set $\Gamma$ of $(s,x)$ such that $v(s,x) > f(x)$. The *stopping region* is the set $\Sigma$ of $(s,x)$ such that $v(t,x) = x$. We will seek for solutions $v$ which are smooth in the continuation region. Applying Ito's lemma to $V_t$ we get

$$de^{-rt}V_t = e^{-rt}[v_t - rv + rS_t v_x + \frac{1}{2}\sigma^2 S_t^2 v_{xx}]\,dt + \sigma S_t\,d\tilde{B}_t$$

For $(s,x) \in \Gamma$ we have $\tau_s > s$ almost surely, and stopping at $\tau_s$ can therefore only give us a martingale if the above $dt$ terms vanish. In other words, the BSM equation

$$v_t - rv + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0$$

should hold on $\Gamma$. Of course, $v(t,x) = f(x)$ on $\Sigma$, but these two equations aren't actually enough to determine $v$, because we haven't yet figured out how to position the boundary between $\Gamma$ and $\Sigma$.

(4) **The American Put**
At this point we'll focus on the American put option, with $f(x) = (K - x)_+$. With a bit of hindsight, it is reasonable to expect that we should exercise only when the option value is sufficiently high. In other words, that there will be an optimal exercise level $L_t < K$ such that we continue if $x > L_t$ and we exercise if $x \leq L_t$ (or $t = T$). This leads to the following *free boundary value problem*:

- $v_t - rv + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0$ for $x > L_t$
- $v(t,x) = K - x$ for $x < L_t$
- $v_x$ is continuous at $x = L_t$.

Typically there is no closed form solution, and the above problem is normally solved numerically. The boundary $L_t$ between the regions is free to move, and solving for it is part of the problem. There are both numerical techniques for doing so, and approximations that let one analyze the solution quantitatively near the exercise boundary. The third condition above goes by the name of the *smooth pasting* condition

But why will solving the free BVP give us a solution to our probability problem? The martingale property of $V_t$ follows from the first condition above. The fact that $\tau_s \le T$ follows from the second condition and the easily checked fact that $v(t,x) \to f(x)$ as $t \uparrow T$. So the real issue (and the reason for the smooth pasting condition) is the supermartingale property.

Here there are two issues. We need that the $dt$ terms of $dV_t$ are $\le 0$ when $S_t \ne L_t$. And we need to look carefully at what happens when $S_t = L_t$. The former is easy: the $dt$ terms equal 0 when $S_t > L_t$, and a direct computation shows that they equal $-rKdt \le 0$ when $S_t < L_t$.

(5) **The Perpetual Put**

It is easiest to understand the role of the smooth pasting condition in the special case of the *perpetual American put* option (first analyzed this way by Henry McKean), in which the option never expires ($T = \infty$). The simplification in this case is that $v(t,x)$ and $L_t$ should no longer depend on time $t$, and the free BVP becomes an differential equation in the single variable $x$:

$$\begin{aligned} -rv(x) + rxv'(x) + \frac{1}{2}\sigma^2 x^2 v''(x) &= 0 \qquad \text{for } x > L \\ v(x) &= K - x \qquad \text{for } x < L \\ v'(L) &= -1 \end{aligned}$$

The general solution to the first equation has the form $A_1 x^{a_1} + A_2 x^{a_2}$, and substituting in we get that

$$-r + ra_i + \frac{1}{2}\sigma^2 a_i(a_i - 1) = 0, \qquad i = 1, 2.$$

This is a quadratic relation in the $a$'s, whose solutions are $a_2 = 1$ and $a_1 = -2r/\sigma^2 < 0$. We want the solution which vanishes at $\infty$, so $A_2 = 0$. Matching values at $x = L$ gives a solution of the form

$$v(x) = \begin{cases} K - x, & x \le L \\ (K - L)(x/L)^{-2r/\sigma^2}, & x > L. \end{cases}$$

Then smooth pasting lets us solve for $L$, giving

$$L = \frac{K}{1 + \frac{\sigma^2}{2r}}. \tag{A}$$

The point I want to emphasize, in order to understand the smooth pasting condition, is why other values of $L$ are impossible.

Let $\bar{L}$ be the value from (A), and let $\nu$ be the slope of $v$ immediately to the right of $L$. If $L < \bar{L}$ then it is easy to see that $\nu < -1$. In other words, $v$ would dip down below the line $K - x$. This contradicts the condition that $v \geq f$, so is impossible.

On the other hand, if $L > L_0$ then we get that $\nu > -1$. In other words, $v'(x)$ makes a positive jump as we pass through $x = L$. But that contradicts the supermartingale property of $v(S_t)$. To see this, think of smoothing out that jump by making $v''(x)$ very large for $x$ close to $L$, But a large positive value of $v''(x)$ would make the $dt$ terms of $dv(S_t)$ positive, which isn't possible for a supermartingale. So the only possibility is to take $L = \bar{L}$, as given by the smooth pasting condition.

[Note: the way to make the last argument fully rigorous is to use an extension of Ito's lemma for non-$C^2$ functions, known as *Tanaka's formula*.

Note also that this has an economic interpretation: Say you maintain a hedge constructed from the solution $v$ with a $L > \bar{L}$. Funds can be withdrawn from this hedge when $S_t < L$. But it turns out that funds also need to be added whenever $S_t = L$. This is what the failure of the supermartingale property means.]

(6) **Risk-Neutral valuation**

Returning to the case $T < \infty$, there is an American version of our risk-neutral pricing formula. Namely that

$$V_t = \sup_{\tau \text{ a stopping time, } t \leq \tau \leq T} \tilde{E}[e^{-r(\tau - t)} f(S_\tau) | \mathcal{F}_t]. \tag{B}$$

This is an example of what is called an *optimal stopping problem*. Here a *stopping time* is a special type of random time. The idea is that at any time $t$ we make the decision whether to stop or not, based on information in $\mathcal{F}_t$. In other words, without peeking into the future. More formally, we require that $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t$. For example, the time at which $S_t$ assumes its maximum value on $[0, T]$ is NOT a stopping time, because we need all the information in $\mathcal{F}_T$ to decide whether $\tau \leq t$. On the other hand, if $V_t$ is an adapted portfolio process then $\tau =$ the first time $V_t = f(S_t)$ (ie our optimal exercise time) is easily be shown to satisfy the stopping time property.

Think of a stopping time as being a strategy for exercising the option. Each risk-neutral expectation is the cost of hedging against one such strategy, and the American option price is the largest of these.

The way to connect this to our other results is to show that the $V_t$ defined in (B) satisfies the appropriate martingale/supermartingale properties. This is done in almost exactly the same way as in the portfolio optimization problem. For example, if we postulate that there is a stopping time $\tau^*$ which is optimal for the above stopping problem, then

$$V_t = \tilde{E}[e^{-r(\tau^*-t)} f(S_{\tau^*}) | \mathcal{F}_t].$$

From which it follows that

$$e^{-r(t\wedge\tau^*)} V_{t\wedge\tau^*} = \tilde{E}[e^{-r\tau^*} f(S_{\tau^*}) | \mathcal{F}_t]$$

is a martingale.

(7) **The American Call**

The price of an American option must be at least as large as that of the corresponding European option. A European call written on a non-dividend-paying stock turns out to always be more expensive than its intrinsic value $f(x) = (x - K)_+$. Thus the American call price won't coincide with intrinsic value until time $T$, hence it will never be optimal to exercise early. So American calls should always be held to maturity $T$. Likewise the American and European prices will agree.

On the other hand, a European put is cheaper than intrinsic value $f(x) = (K - x)_+$ when the stock price is low. Thus the American put price is strictly larger than the European put price at those levels, making early exercise an event with positive probability.

For a dividend paying stock, whose risk-neutral evolution is

$$dS_t = (r - \delta) S_t \, dt + \sigma S_t \, d\tilde{B}_t,$$

the same is true. That is, calls have lower European prices than intrinsic value, when the stock price is high enough. In this case early exercise is sensible, and one also prices the American option by solving a free BVP.

(8) **Exercise: the perpetual butterfly spread.**

Consider a perpetual American option with intrinsic value $f(x) = \min\Big((K - x)_+, (x - K(1-2a))_+\Big)$. This payoff structure is called a butterfly spread. Assume that $a < \sigma^2/(\sigma^2 + 2r)$. Show that the option price is

$$v(x) = \begin{cases} \frac{ax}{1-a}, & 0 < x < K(1-a) \\ K - x, & K(1-a) \le x \le L \\ (K-L)(x/L)^{-2r/\sigma^2}, & L < x \end{cases}$$

where $L$ is as in (A). In other words, the stopping region is $[K(1-a), L]$.

One point of the exercise is to show that $v$ need not be differentiable. In this case $v'$ jumps downward at $x = K(1-a)$. So while an upwards jump in $v'$ is ruled out by the supermartingale property as in paragraph (5), downward jumps in $v'$ are OK. Smooth pasting gets rid of the "corners" at $K(1-2a)$ and $K$, but not the one at $K(1-a)$.

Taking this further, consider the same option written on a dividend paying stock. Can you find situations in which $K(1-a)$ is in the interior of the stopping region? Or when the stopping region consists only of the value $K(1-a)$?