MATH 6910 3.0AF (Stochastic Calculus in Finance)
Assignment 2 – Salisbury

Due (changed date) April 10, 2007

1. Let X_t and Y_t be continuous semimartingales. Define the *Stratonovich Integral*

$$\int_0^t X_s \circ dY_s = \int_0^t X_s \, dY_s + \frac{1}{2}[X, Y]_t.$$

   (a) Show that Stratonovich integrals obey the change of variables formula from ordinary calculus. In other words, show that if B_t is a Brownian motion then

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) \circ dB_s$$

   (you should assume that f is three times continuously differentiable).

   [Note: This makes Stratonovich integrals more convenient than Ito integrals for some purposes, especially if one is trying to convert a result from ordinary calculus into stochastic calculus. This sounds appealing, but there are drawbacks which mean that Ito integrals are almost everyone's preferred choice, in practice. For example, the integrands have to be semimartingales, which is much more restrictive than simply being predictable. And unlike Ito integrals, Stratonovich integrals typically aren't martingales.]

   (b) In ordinary calculus there are many numerical schemes for computing Riemann integrals, all of which agree in the limit. But the same schemes generally don't agree for stochastic integrals. An example is the midpoint rule:

$$J_n = \sum_{i=1}^{n} \frac{H_{t_{i-1}} + H_{t_i}}{2} (B_{t_i} - B_{t_{i-1}}),$$

   where as usual, we divide $[0, t]$ into n equal pieces, and will then let $n \to \infty$. Let H_t be a continuous semimartingale. Show that J_n converges, not to $\int_0^T H_s \, dB_s$, but to $\int_0^T H_s \circ dB_s$. Hint: Show that $J_n - I_n \to \frac{1}{2}[H, B]_T$, where

$$I_n = \sum_{i=1}^{n} H_{t_{i-1}} (B_{t_i} - B_{t_{i-1}}).$$

   You may use the fact, stated in class, that $I_n \to \int_0^t H_s \, dB_s$.
   [Note: This part of the question is a variation on problem 4.4 from page 190 of the text, but bypassing the mean-variance argument mentioned there]

2. It turns out that most SDEs of the form

$$dX_t = a(t)X_t\,dt + b(t)\,dB_t$$

(where a and b are deterministic functions) can be solved by the following technique. Substitute $X_t = \alpha(t)[x_0 + \int_0^t \beta(s)\,dB_s]$, find its differential, and match coefficients to determine the right choice of deterministic functions α and β. Carry this out for the SDE

$$dX_t = tX_t\,dt + 3e^{t^2/2}\,dB_t.$$

[Note: This problem is drawn from Steele's book]

3. This problem concerns the Cox-Ingersoll-Ross term structure model, and is a simplified version of problem 6.6 on page 286 of the text.

(a) Let

$$X_t = e^{-\beta t/2}(x_0 + \frac{\sigma}{2}\int_0^t e^{\beta s/2} dB_s)$$

and verify that

$$dX_t = \frac{\sigma}{2} dB_t - \frac{\beta}{2} X_t\,dt$$

(this isn't really new - it is a special case of the Ornstein Uhlenbeck process from class.)

(b) Let $R_t = X_t^2$ and

$$W_t = \int_0^t \text{Sign}(X_s)\,dB_s$$

Show that

$$dR_t = (\frac{\sigma^2}{4} - \beta R_t)dt + \sigma\sqrt{R_t}dW_t.$$

Note that $\text{Sign}(x) = \pm 1$ depending on whether $x \geq 0$ or $x < 0$. Verify that $d[W,W]_t = dt$. From this it follows immediately from Lévy's theorem that W_t is a Brownian motion too. In other words, we have obtained a solution to a particular case of the Cox-Ingersoll-Ross SDE from page 151 of the text.

4. The price of the stock of ABC corporation satisfies the SDE

$$dS_t = \mu S_t dt + \sigma S_t\,dB_t$$

where B_t is a Brownian motion. The corporation enters into a contract with its CEO, worth

$$A \ln\left(\frac{S_T}{K}\right)$$

at time T. Note that if the stock price S_T is greater than K, the CEO receives a payment, but if $S_T < K$ then she has to pay the corporation. In other words, this is an incentive for her to see that the stock price goes up. In order to neutralize

the contract, she decides to hedge. Ignoring transaction costs, how much does it cost her at time $t = 0$ to implement a hedge that will exactly balance this contract at time $t = T$? You should obtain your answer by
(a) Expressing the hedging cost in terms of risk neutral expectations,
(b) evaluating these expectations.
(c) Finally, work out an actual cost, where T corresponds to 2 years, $r = 3\%$ per year, $\mu = 6\%$ per year, $\sigma = 30\%$ per year, $K = 10$, the intial price of the stock is $S_0 = 12$, and $A = 100,000$.

5. Let X_t denote the price of a stock, obeying the stochastic differential equation

$$dX_t = rX_t\,dt + \sigma\,d\tilde{B}_t$$

under the risk-neutral measure $\tilde{P}$, where r is the risk-free rate of return. As usual, $\tilde{B}_t$ is a Brownian motion under $\tilde{P}$. Note that this differs from the Black-Scholes-Merton model, as there is no factor of X_t in front of the $d\tilde{B}_t$. This is a somewhat strange model (eg. prices can go negative), but it can be shown (and you may assume this) that it is complete and that option values V_t still have the property that $e^{-rt}V_t$ is a martingale under $\tilde{P}$. Note that the above is an Ornstein-Uhlenbeck SDE, so that we know from class that the solution is

$$X_t = x_0e^{rt} + \sigma e^{rt}\int_0^t e^{-rs}\,d\tilde{B}_s,$$

and that X_t has a (risk neutral) normal distribution with mean x_0e^{rt} and variance $\frac{\sigma^2(e^{2rt}-1)}{2r}$.

(a) Find a formula for the price (at time 0) of a binary option in this model. In other words, find the no-arbitrage price of an option that pays an amount K if the stock value at time T exceeds a given value y, and pays nothing otherwise.
(b) In particular, what will this price be if $r = 0.06$, $T = 5$, $\sigma = 0.50$, $x_0 = 100$, $y = 138$, and $K = 30$?
(c) Assume that $u(t, x)$ is a smooth function so that the option price is realized as $u(t, X_t)$. Following the argument from class, derive the PDE satisfied by u.

6. Do problem 1.11 on page 45 of the text

7. The following concerns two steps of a binary tree model. Let $B_0 = 0$, and define B_1 and B_2 so that $B_{n+1} = B_n \pm 1$. In other words, the possible values for

B_0, B_1, B_2 are as follows.

$$
\begin{array}{ccc}
 & & 2 \\
 & 1 & \\
0 & & 0 \\
 & -1 & \\
 & & -2
\end{array}
$$

Let $\mathcal{F}_n$ be the filtration generated by B_n, so $\mathcal{F}_0$ is trivial and has one atom Ω, $\mathcal{F}_1$ has 2 atoms, and $\mathcal{F}_2$ has 4 atoms.

(a) Let P give probability 1/2 to each upward branch in the tree (ie $B_{n+1} = B_n \pm 1$ each have probability 1/2). Verify directly that B_n is a P-martingale. In this setting, this simply means computing that $E[B_1|\mathcal{F}_0] = B_0$ and $E[B_2|\mathcal{F}_1] = B_1$.

(b) Let Q be a different probability, giving probabilities p to the upward branches, and $q = 1 - p$ to the downward ones. Set $\theta = p - \frac{1}{2}$ and $\tilde{B}_n = B_n - 2n\theta$. Calculate its values at each node in the tree, and then show by direct computation that $\tilde{B}_n$ is a Q-martingale.

(c) Let $Z_n = (2\sqrt{pq})^n(\sqrt{p/q})^{B_n}$. Or equivalently (to make it look more like what we did in class in the Brownian context), $Z_n = e^{B_n \log \sqrt{p/q} + n \log(2\sqrt{pq})}$. Clearly Z_n is positive. Calculate its values at each node in the tree, and show that Z_n is actually the density of Q with respect to P on $\mathcal{F}_n$. Thus it should be a P-martingale. Verify this fact by direct calculation.

(d) Because $\tilde{B}_n$ is a Q-martingale it should follow that $Z_n\tilde{B}_n$ is a P-martingale. Again, calculate its values at each node of the tree and verify this property by direct calculation.