Stochastic Calculus in Finance
MATH 6910 - Salisbury

Girsanov Transformations - A short introduction

(1) Two probability measures P and $\tilde{P}$ are equivalent on a sigma field $\mathcal{G}$ if

$$P(A) = 0 \iff \tilde{P}(A) = 0$$

for every $A \in \mathcal{G}$. The Radon-Nikodym theorem (from measure theory) states that this is equivalent to the existence of a density $M > 0$ for $\tilde{P}$ with respect to P. That is, that there exists an $\mathcal{G}$-measurable random variable $M > 0$ such that

$$\tilde{E}[X] = E[MX]$$

for every suitably integrable $\mathcal{G}$-measurable X. In particular,

$$\tilde{P}(A) = E[M1_A] \tag{A}$$

for every $A \in \mathcal{G}$. One thinks of $M(\omega)$ as the factor by which one reweights the probability $P(d\omega)$ to get $\tilde{P}(d\omega)$; $\tilde{P}(d\omega) = M(\omega)P(d\omega)$. In the discrete case this is exactly what happens (see (6) below).

Note also that it must be the case that $E[M] = 1$. Turning things around, let M be any $\mathcal{G}$-measurable random variable with $M > 0$ and $E[M] = 1$. Then the formula (A) can be taken to *define* a probability measure $\tilde{P}$, which is automatically equivalent to P.

(2) If P and $\tilde{P}$ are equivalent on $\mathcal{F}_t$, for every $t \geq 0$, then we get a family of densities M_t, one for each t. In other words, M_t is $\mathcal{F}_t$-measurable, and satisfies

$$\tilde{P}(A) = E[M_t 1_A] \tag{B}$$

for every $A \in \mathcal{F}_t$. In this case, we have that

$$\boxed{M_t \text{ is a martingale under } P.}$$

PROOF: for $s < t$ we need that $E[M_t \mid \mathcal{F}_s] = M_s$. According to the definition, this means that

$$E[M_s 1_A] = E[M_t 1_A]$$

for every $A \in \mathcal{F}_s$. But the LHS equals $\tilde{P}(A)$ by definition, and so does the RHS, because A also belongs to $\mathcal{F}_t$.

Conversely, let M_t be any P-martingale with $M_t > 0$ $P - a.s.$ and $E[M_t] = 1$, for every t. Then (B) defines a measure $\tilde{P}$ on $\mathcal{F}_t$. And the argument just given can be reversed to show that these measures are consistent (that is, if $A \in \mathcal{F}_{t_1}$ and if we compute $\tilde{P}(A)$ from (B) with $t = t_1$, then we get the same answer as if we compute it from (B) using some other $t = t_2 > t_1$). In other words, there is a precise correspondence between strictly positive P-martingales on the one hand, and probabilities equivalent to P on the other.

It is worth noting that none of this says that P and $\tilde{P}$ are equivalent on the whole σ-field $\mathcal{F} = \mathcal{F}_\infty$. In our financial models they typically aren't; we can't tell μ from r with certainty based on the information in $\mathcal{F}_t$, that is, up to any finite time t. But we *can* tell them apart if we let $t = \infty$.

(3) Let B_t be a Brownian motion under P, and let $\mathcal{F}_t$ be its filtration. Then for any predictable process γ_t for which the stochastic integral $\int_0^t \gamma_s \, dB_s$ makes sense, we know by Itô's lemma that

$$M_t = e^{-\int_0^t \gamma_s \, dB_s - \frac{1}{2}\int_0^t \gamma_s^2 \, ds} \tag{C}$$

defines a local martingale. If a suitable integrability condition is satisfied then M_t is also a martingale. For example, this is true if γ_t is bounded, or more generally, if the *Novikov condition* $E[e^{\frac{1}{2}\int_0^t \gamma_s^2 \, ds}] < \infty \; \forall t$ is satisfied. In this case, the passage from P to $\tilde{P}$ is known as a *Girsanov transformation*, and we have the following result:

$$\boxed{\tilde{B}_t = B_t + \int_0^t \gamma_s \, ds \text{ is a Brownian motion under } \tilde{P}.} \tag{D}$$

(4) Take $\gamma_t = \gamma = (\mu - r)/\sigma$. For our GBM model of stock prices, γ is the *market price of risk*, or *Sharpe ratio*, or *equity risk premium*. It is the risk premium per unit volatility that investors will demand before they buy the stock. The fact that it is strictly positive is a consequence of the capital asset pricing model (CAPM), and is based on the sub-linear utility of wealth.

Under P we have the equation

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t.$$

substituting $d\tilde{B}_t = dB_t + \gamma \, dt$ gives that

$$dS_t = r S_t \, dt + \sigma S_t \, d\tilde{B}_t.$$

Our pricing and hedging arguments are all based on the hypothesis that we can find an equivalent probability measure $\tilde{P}$ under which $\tilde{B}_t$ is a Brownian motion. The obvious candidate is the Girsanov transformation by (C), and the result of (D) tells us that this works.

(5) It remains to prove the result (D) above. First we prove the following:

$$\boxed{N_t \text{ is a } \tilde{P}\text{-martingale} \iff N_t M_t \text{ is a } P\text{-martingale.}}$$

PROOF: Let N_t be a $\tilde{P}$-martingale. Then $E[N_t \mid \mathcal{F}_s] = N_s$ for $s < t$. This means that

$$\tilde{E}[N_t 1_A] = \tilde{E}[N_s 1_A]$$

for every $A \in \mathcal{F}_s$. Turning the $\tilde{E}$'s into E's, we get that $E[M_t N_t 1_A] = E[M_s N_s 1_A]$ for every $A \in \mathcal{F}_s$, which means exactly that $E[M_t N_t \mid \mathcal{F}_s] = M_s N_s$. In other words, we do get that $M_t N_t$ is a P-martingale. The converse is obtained by simply reversing the argument.

PROOF OF (D): We use Itô's lemma on $\tilde{B}_t M_t$, to check that it is a P-martingale. First note that $dM_t = -\gamma_t M_t dB_t$, so $d[\tilde{B}, M]_t = d[B, M]_t = -\gamma_t M_t dt$. Thus

$$\begin{aligned} d\tilde{B}_t M_t &= M_t d\tilde{B}_t + \tilde{B}_t dM_t + d[\tilde{B}, M]_t \\ &= M_t(dB_t + \gamma_t dt) + \tilde{B}_t dM_t - \gamma_t M_t dt \\ &= M_t dB_t + \tilde{B}_t dM_t. \end{aligned}$$

Since both B_t and M_t are martingales under P, we see that $\tilde{B}_t M_t$ is one too (actually, this only shows that it's a local martingale, but it isn't hard to verify the appropriate integrability condition). So by the previous argument, $\tilde{B}_t$ is a martingale under $\tilde{P}$.

To finish the proof, we use Lévy's theorem (see Sharpe, p. 168), that a continuous martingale is a Brownian motion if and only if it has quadratic variation t. In our case, $\tilde{B}_t$ is clearly continous, and we have just verified that it is a $\tilde{P}$-martingale. Since it only differs from B_t by a term of bounded variation, also $[\tilde{B}, \tilde{B}]_t = [B, B]_t = t$, and we're done.