MATH 6910 3.0AF (Stochastic Calculus in Finance)
Practice problems – Solutions

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1. Let $f(x) = \log x$, for $x > 0$. Then $f'(x) = 1/x$ and $f''(x) = -1/x^2$. From the fact that $dS_t = \alpha_t S_t\,dt + \sigma_t S_t\,dW_t$ we get that $d[S,S]_t = \sigma_t^2 S_t^2\,dt$. So

$$\begin{aligned} d\log S_t &= f'(S_t)\,dS_t + \frac{1}{2}f''(S_t)d[S,S]_t \\ &= \frac{\alpha_t S_t\,dt + \sigma_t S_t\,dW_t}{S_t} - \frac{\sigma_t^2 S_t^2\,dt}{2S_t^2} \\ &= (\alpha_t - \frac{\sigma_t^2}{2})\,dt + \sigma_t\,dW_t. \end{aligned}$$

In particular, $\log S_t = \log S_0 + \int_0^t (\alpha_s - \frac{1}{2}\sigma_s^2)\,ds + \int_0^t \sigma_s\,dW_s$ so

$$S_t = S_0 e^{\int_0^t (\alpha_s - \frac{1}{2}\sigma_s^2)\,dt + \int_0^t \sigma_s\,dW_s}.$$

Thus this is the only possible solution, and solutions are therefore unique.

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2. Recall that $X_t = \int_0^t s\,dB_s$, so $dX_t = t\,dB_t$.

(a) It follows that $d[X,X]_t = t^2\,dt$ so $[X,X]_t = \int_0^t s^2\,ds = t^3/3$. We know that $E[X_t] = 0$, since X_t is a stochastic integral of Brownian motion (with a square integrable integrand). And by the Ito isometry, $\mathrm{Var}(X_t) = E[X_t^2] = E\Big[[X,X]_t\Big] = E[t^3/3] = t^3/3$.

(b) $Y_t = f(X_t)$ where $f(x) = e^x$. So $f'(x) = e^x = f''(x)$ too. In particular,

$$dY_t = f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,d[X,X]_t = e^{X_t}t\,dB_t + \frac{1}{2}e^{X_t}t^2\,dt.$$

Or in other words, $dY_t = \frac{1}{2}t^2 Y_t\,dt + tY_t\,dB_t$.

(c) Likewise, $Z_t = f(t, X_t)$ where $f(t,x) = e^{at}x$. In particular, $f_t = ae^{at}x$, $f_x = e^{at}$, and $f_{xx} = 0$. So

$$\begin{aligned} dZ_t &= f_t(t,X_t)\,dt + f_x(t,X_t)\,dX_t + \frac{1}{2}f_{xx}(t,X_t)\,d[X,X]_t \\ &= ae^{at}X_t\,dt + e^{at}t\,dB_t. \end{aligned}$$

In particular, $dZ_t = aZ_t\,dt + te^{at}\,dB_t$.

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3. $X_t = \int_0^t B_s\,dB_s$ so $[X,X]_t = \int_0^t B_s^2\,ds$. Because it is a stochastic integral (with a square integrable integrand), $E[X_t] = 0$. So by the Ito isometry, $\mathrm{Var}(X_t) = \int_0^t E[B_s^2]\,ds = \int_0^t s\,ds = t^2/2$.

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4. Here W_t is a Brownian motion.
   (a) By Ito's lemma, $dW_t^4 = 4W_t^3\,dW_t + 6W_t^2\,dt$. Since $W_0 = 0$,
   $$W_t^4 = \int_0^t 6W_s^2\,ds + \int_0^t 4W_s^3\,dB_s.$$
   (b) Taking expectations, $E[W_t^4] = \int_0^t E[6W_s^2]\,ds$. Here we're using linearity to interchange the $\int$ and E, and we're using the fact that the second term has mean 0. Thus $E[W_t^4] = \int_0^t 6s\,ds = 3t^2$.
   (c) Likewise $dW_t^6 = 6W_t^5\,dW_t + 15W_t^4\,dt$, so $W_t^6 = \int_0^t 6W_s^5\,dW_s + \int_0^t 15W_s^4\,ds$. So using the result of part (b), $E[W_t^6] = \int_0^t 15E[W_s^4]\,ds = \int_0^t 45s^2\,ds = 15t^3$.

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5. Let $B^1 = W^1$, which is therefore automatically a Brownian motion. Let
$$B_t^2 = \int_0^t \rho_s\,dW_s^1 + \int_0^t \sqrt{1-\rho_s^2}\,dW_s^2.$$
Because W^1 and W^2 are independent Brownian motions, $d[B^2,B^2]_t = \rho_t^2\,dt + (1-\rho_t^2)\,dt = dt$. Moreover, being a sum of stochastic integrals (with square integrable integrands), B_t^2 is a continuous martingale. So by Theorem 4.6.4 it is actually a Brownian motion.

Finally, we compute $d[B^1,B^2]_t$. This'll be $K\,dt$ where we get K by multiplying the respective coefficients of dW_t^1 and dW_t^2 and then adding up. In other words, $d[B^1,B^2]_t = (1\cdot\rho_t + 0\cdot\sqrt{1-\rho_t^2}\,dt = \rho_t\,dt$. Which is what I asked you to show.

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6. Let $f(t,x) = e^x \cos(x+at)$. Then $f_t = -ae^x \sin(x+at)$, $f - x = e^x[\cos(x+at) - \sin(x+at)]$, and $f_{xx} = -2e^x \sin(x+at)$. Then

$$\begin{aligned} df(t,B_t) &= f_t(t,B_t)\,dt + f_x(t,B_t)\,dB_t + \frac{1}{2} f_{xx}(t,B_t)\,dt \\ &= -e^{B_t}(a+1)\sin(B_t+at)\,dt + e^{B_t}[\cos(B_t+at) - \sin(B_t+at)]\,dB_t. \end{aligned}$$

The only way this can be a martingale is if the dt term vanishes. That'll happen only when $a = -1$.

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7. $Y_t = e^{X_t}$, where $X_t = at + \sigma B_t$.

(a) $f(x) = e^x$ has $f'(x) = f''(x) = e^x$. So $dY_t = e^{X_t}\,dX_t + \frac{1}{2}e^{X_t}\,d[X,X]_t = Y_t(a\,dt + \sigma\,dB_t) + \frac{1}{2}Y_t\sigma^2\,dt$. In other words,

$$dY_t = (a + \frac{1}{2}\sigma^2)Y_t\,dt + \sigma Y_t\,dB_t.$$

(b) $Z_t = 1/Y_t = e^{-at-\sigma B_t}$, which is also a GBM (just with new coefficients). So by part (a),

$$dZ_t = (-a + \frac{1}{2}\sigma^2)Y_t\,dt - \sigma Y_t\,dB_t.$$

(c) Now let $g(y) = 1/y$, so $g'(y) = -1/y^2$ and $g''(y) = 2/y^3$. Therefore

$$\begin{aligned} dZ_t &= g'(Y_t)\,dY_t + \frac{1}{2}g''(Y_t)\,d[Y,Y]_t \\ &= -\frac{(a+\frac{1}{2}\sigma^2)Y_t\,dt + \sigma Y_t\,dB_t}{Y_t^2} + \frac{2\sigma^2 Y_t^2\,dt}{2Y_t^3} \\ &= \frac{-a - \frac{1}{2}\sigma^2 + \sigma^2}{Y_t}\,dt - \frac{\sigma}{Y_t}\,dB_t \\ &= (-a + \frac{1}{2}\sigma^2)Z_t\,dt - \sigma Z_t\,dB_t. \end{aligned}$$

Which agrees with the result of part (b).