

MATH 6910 3.0AF (Stochastic Calculus in Finance)
Assignment 2 – Solutions

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1. Recall that if

$$dU_t = A_t dt + H_t dB_t$$

$$dV_t = C_t dt + K_t dB_t$$

then $d[U, V]_t = H_t K_t dt$.

(a) Since

$$df'(B_t) = f''(B_t) dB_t + \frac{1}{2} f'''(B_t) dt,$$

it follows that $d[f'(B), B]_t = f''(B_t) dt$. Thus

$$\begin{aligned} \int_0^t f'(B_s) \circ dB_s &= \int_0^t f'(B_s) dB_s + \frac{1}{2} [f'(B), B]_t \\ &\quad \text{(by definition)} \\ &= \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds \\ &= f(B_t) - f(B_0) \\ &\quad \text{(by It\hat{o}'s lemma).} \end{aligned}$$

(b) By definition,

$$\begin{aligned} J_n - I_n &= \sum_{i=1}^n \left(\frac{H_{t_{i-1}} + H_{t_i}}{2} - H_{t_{i-1}} \right) (B_{t_i} - B_{t_{i-1}}) \\ &= \frac{1}{2} \sum (\Delta_i H)(\Delta_i B) \rightarrow \frac{1}{2} [H, B]_t. \end{aligned}$$

We had in class that $I_n \rightarrow \int_0^t H_s dB_s$. Thus

$$J_n = I_n + (J_n - I_n) \rightarrow \int_0^t H_s dB_s + \frac{1}{2} [H, B]_t = \int_0^t H_s \circ dB_s.$$

2. We try $X_t = \alpha(t) \left[x_0 + \int_0^t \beta(s) dB_s \right]$. Then

$$\begin{aligned} dX_t &= \alpha'(t) \left[x_0 + \int_0^t \beta(s) dB_s \right] dt + \alpha(t) \beta(t) dB_t \\ &= \frac{\alpha'(t)}{\alpha(t)} X_t dt + \alpha(t) \beta(t) dB_t. \end{aligned}$$

So, matching coefficients, we want $\alpha(0) = 1$ and $(\log \alpha(t))' = \alpha'(t)/\alpha(t) = t$. In other words, $\log \alpha(t) = t^2/2$, so $\alpha(t) = e^{t^2/2}$. We also want $\alpha(t)\beta(t) = 3e^{t^2/2}$. Thus $\beta(t) = 3$, and our tentative solution is

$$X_t = e^{t^2/2}(x_0 + 3B_t) = e^{t^2/2}(1 + 3B_t).$$

Conversely an appeal to Itô's lemma will verify that this solution really does work.

3.

- (a) Set $Y_t = x_0 + (\sigma/2) \int_0^t e^{\beta s/2} dB_s$. Then $X_t = e^{-\beta t/2} Y_t$, and $dY_t = (\sigma/2) e^{\beta t/2} dB_t$, so

$$\begin{aligned} dX_t &= -\frac{\beta}{2} e^{-\beta t/2} Y_t dt + e^{-\beta t/2} dY_t \\ &= -\frac{\beta}{2} X_t dt + e^{-\beta t/2} \frac{\sigma}{2} e^{\beta t/2} dB_t \\ &= -\frac{\beta}{2} X_t dt + \frac{\sigma}{2} dB_t \end{aligned}$$

as required.

- (b) Note that $x = |x| \cdot \text{Sign}(x)$. By Ito's lemma,

$$dR_t = 2X_t dX_t + d[X, X]_t.$$

From the expression for dX_t we infer that $d[X, X]_t = (\sigma/2)^2 dt$. Thus

$$\begin{aligned} dR_t &= 2X_t \left(\frac{\sigma}{2} dB_t - \frac{\beta}{2} X_t dt \right) + \frac{\sigma^2}{4} dt \\ &= \left(\frac{\sigma^2}{4} - \beta X_t^2 \right) dt + \sigma X_t dB_t \\ &= \left(\frac{\sigma^2}{4} - \beta R_t \right) dt + \sigma |X_t| \cdot \text{Sign}(X_t) dB_t \\ &= \left(\frac{\sigma^2}{4} - \beta R_t \right) dt + \sigma \sqrt{R_t} dW_t \end{aligned}$$

since $\sqrt{R_t} = |X_t|$, and $dW_t = \text{Sign}(X_t) dB_t$. Thus R_t is a CIR type process, with a special choice of α , namely $\sigma^2/4$.

To show that W_t is a Brownian motion we would use Levy's theorem, since it is clearly a martingale, and has $[W, W]_t = t$. To see the latter note that $[W, W]_t = \int_0^t \text{Sign}(X_s)^2 ds = \int_0^t ds = t$.

[Note: The point of this exercise is that we already know that X_t has a normal distribution. Thus R_t , being the square of X_t , has a non-central χ^2 distribution. In other words, we can find the risk neutral distribution explicitly, at least for this particular choice of α . This can then be used to price options in the CIR short rate model of interest rates. Problem 6.6

carries this technique further, and finds the distribution for other choices of α .]

4. Our CEO wants to construct a portfolio whose value at time T will equal $-A \log(S_T/K)$, as this portfolio will exactly neutralize her contract. She will have eliminated the uncertainty present in her contract, and will have no net cash flow at time T .

- (a) The market is complete (assuming she can trade in the company stock), so the cost of implementing the hedge must be

$$e^{-rT} \tilde{E}[-A \log(S_T/K)].$$

- (b) Since $S_T = S_0 e^{(r-\sigma^2/2)T + \sigma \tilde{B}_T}$, we have that

$$\begin{aligned} e^{-rT} \tilde{E}[-A \log(S_T/K)] &= -A e^{-rT} \tilde{E}[\log(S_0/K) + (r - \frac{\sigma^2}{2})T + \sigma \tilde{B}_T] \\ &= -A e^{-rT} [\log(S_0/K) + (r - \frac{\sigma^2}{2})T] \end{aligned}$$

- (c) Substituting the parameters, the above expression equals $-100,000 \times e^{-2 \times .03} [\log(12/10) + 2(.03 - .3 \times .3/2)] = -14,345$. Note that the hedging cost is negative, which means that the net result of hedging is that she receives an immediate payment of \$14,345. She has basically traded a certain immediate payment for an uncertain (though possibly larger) future payment/debt.
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5. Let A be the event that $X_T > y$. Then the payoff on the option is $K1_A$.

- (a) Because the market is complete, the option price is

$$e^{-rT} \tilde{E}[K1_A] = K e^{-rT} \tilde{P}(A).$$

We may write $X_T = m + vZ$ where m and v are the mean and variance of X_T , and Z has a standard Normal distribution. In particular, the price becomes $K e^{-rT} (1 - N((y - m)/\sqrt{v}))$.

- (b) Substituting, we get $m = 100e^{-3} = 134.99$, $v = .25(e^{-6} - 1)/.12 = 1.7127$, and $(y - m)/\sqrt{v} = 2.303$, so the price equals

$$30e^{-3}(1 - N(2.303)) = 30 \times .7408 \times .0106 = .2364$$

(c) Let $u(t, X_t)$ be the option price, so $V_t^0 = e^{-rt}u(t, X_t)$ is a risk neutral martingale.

$$\begin{aligned} dV_t^0 &= -re^{-rt}u(t, X_t) dt + e^{-rt}u_t(t, X_t) dt + \\ &\quad + e^{-rt}u_x(t, X_t) dX_t + \frac{1}{2}u_{xx}(t, X_t) d[X, X]_t \\ &= e^{-rt} \left[-ru(t, X_t) + u_t(t, X_t) + ru_x(t, X_t) + \right. \\ &\quad \left. + \frac{1}{2}\sigma^2u_{xx}(t, X_t) \right] dt + \sigma e^{-rt}u_x(t, X_t) d\tilde{B}_t. \end{aligned}$$

So for V_t^0 to be a martingale, we must have

$$-ru(t, x) + u_t(t, x) + ru_x(t, x) + \frac{1}{2}\sigma^2u_{xx}(t, x) = 0.$$

The terminal condition is that $u(T, x) = K1_{(y, \infty)}(x)$

6. $Y = X + \theta$ where $X \sim N(0, 1)$ under P and θ is deterministic. $Z = e^{-\theta X - \frac{1}{2}\theta^2}$ is positive and has mean 1, so we can define an equivalent probability \tilde{P} whose Radon Nikodym density with respect to P is Z . In other words, $\tilde{E}[W] = E[ZW]$ for every square integrable random variable W .

We need to show that $Y \sim N(0, 1)$ under \tilde{P} . To do that we compute Y 's moment generating function:

$$\begin{aligned} \tilde{E}[e^{uY}] &= E[Ze^{uY}] = E[e^{-\theta X - \frac{1}{2}\theta^2} e^{u(X+\theta)}] = e^{u\theta - \frac{1}{2}\theta^2} E[e^{(u-\theta)X}] \\ &= e^{u\theta - \frac{1}{2}\theta^2} e^{\frac{1}{2}(u-\theta)^2} = e^{u\theta - \frac{1}{2}\theta^2} e^{\frac{1}{2}u^2 - u\theta + \frac{1}{2}\theta^2} = e^{\frac{1}{2}u^2}. \end{aligned}$$

Since this IS the standard normal moment generating function, Y must be standard normal under \tilde{P} .

7. The values for B_0, B_1, B_2 and for $\tilde{B}_0, \tilde{B}_1, \tilde{B}_2$ are

$$\begin{array}{ccc} & 2 & 2 - 4\theta = 4q \\ 1 & & 1 - 2\theta = 2q \\ 0 & 0 & 0 \quad -4\theta = 2 - 4p = 4q - 2 = 2q - 2p \\ & -1 & -1 - 2\theta = -2p \\ & -2 & -2 - 4\theta = -4p \end{array}$$

$B_0 \quad B_1 \quad B_2$

$\tilde{B}_0 \quad \tilde{B}_1 \quad \tilde{B}_2$

We can use a model $\Omega = \{uu, ud, du, dd\}$ corresponding to the two up or down moves in the binomial tree. There are four atoms in \mathcal{F}_2 , namely $\{uu\}, \{ud\}, \{du\}, \{dd\}$. There are two atoms in \mathcal{F}_1 , namely $U = \{uu, ud\}$ and $D = \{du, dd\}$. And one atom in \mathcal{F}_0 , namely Ω itself.

- (a) To show that B_n is a martingale we need to show that when $i < j$ we have $E[B_j|\mathcal{F}_i] = B_i$. Technically there are three cases: $E[B_1|\mathcal{F}_0] = B_0$, $E[B_2|\mathcal{F}_1] = B_1$, and $E[B_2|\mathcal{F}_0] = B_0$. But the latter follows from the first two using the tower property of conditional expectations, so we'll only worry about the first two statements. If we calculate $E[B_{n+1}|\mathcal{F}_n]$ for $n = 0, 1$ we get the following:

$$E[B_{n+1}|\mathcal{F}_n]:$$

$$\begin{array}{r} \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) \qquad \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 \\ \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot (-2) \\ \hline n = 0 \qquad \qquad \qquad n = 1 \end{array}$$

Simplifying, this works out to be

$$\begin{array}{r} 1 \\ 0 \qquad \qquad \qquad -1 \\ \hline n = 0 \quad n = 1 \end{array}$$

which agrees with the values computed earlier for B_0 and B_1 . In other words,

$$\text{For any } \omega, E[B_1|\mathcal{F}_0](\omega) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0 = B_0(\omega).$$

$$\text{For } \omega \in U, E[B_2|\mathcal{F}_1](\omega) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot 0 = 1 = B_1(\omega).$$

$$\text{For } \omega \in D, E[B_2|\mathcal{F}_1](\omega) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot (-2) = -1 = B_1(\omega).$$

- (b) The same computation for \tilde{B}_n under Q gives

$$E_Q[\tilde{B}_{n+1}|\mathcal{F}_n]:$$

$$\begin{array}{r} p(4q) + q(2 - 4p) = 2q \\ p(2q) + q(-2p) = 0 \qquad \qquad \qquad p(4q - 2) + q(-4p) = -2p \\ \hline n = 0 \qquad \qquad \qquad n = 1 \end{array}$$

which agrees with the previously computed values of \tilde{B}_0 and \tilde{B}_1 .

- (c) Computing Z_n according to the given formula, we get

Z_n :

$$\begin{array}{ccc} & & 4p^2 \\ & 2p & \\ 1 & & 4pq \\ & 2q & \\ & & 4q^2 \\ \hline n = 0 & n = 1 & n = 2 \end{array}$$

Then we compute the density of Q with respect to P , which on each atom a will be the factor we multiply $P(a)$ by to get $Q(a)$. In other words,

$\frac{dQ}{dP}$:

$$\begin{array}{ccc} & & \frac{Q(\{uu\})}{P(\{uu\})} = \frac{p^2}{1/4} \\ & \frac{Q(U)}{P(U)} = \frac{p}{1/2} & \\ \frac{Q(\Omega)}{P(\Omega)} = 1 & & \frac{Q(\{ud\})}{P(\{ud\})} = \frac{Q(\{du\})}{P(\{du\})} = \frac{pq}{1/4} \\ & \frac{Q(D)}{P(D)} = \frac{q}{1/2} & \\ & & \frac{Q(\{dd\})}{P(\{dd\})} = \frac{q^2}{1/4} \\ \hline n = 0 & n = 1 & n = 2 \end{array}$$

Clearly these agree, so $Z_n = dQ/dP$ on \mathcal{F}_n . It is actually enough to have verified that they agree on atoms, because the statement for general events follows using the fact that every event is a union of atoms.

Finally, we must verify that Z_n is a P -martingale. As before, we must compute $E[Z_{n+1}|\mathcal{F}_n]$ for $n = 0$ and $n = 1$. Using the computed values of Z_n we get

$E[Z_{n+1}|\mathcal{F}_n]$:

$$\begin{array}{ccc} & & \frac{1}{2} \cdot 4p^2 + \frac{1}{2} \cdot 4pq = 2p(p+q) = 2p \\ \frac{1}{2} \cdot 2p + \frac{1}{2} \cdot 2q = p+q = 1 & & \\ & & \frac{1}{2} \cdot 4pq + \frac{1}{2} \cdot 4q^2 = 2q(p+q) = 2q \\ \hline n = 0 & & n = 1 \end{array}$$

which agree with the previously computed values of Z_0 and Z_1 .

(d) Finally, we want $Z\tilde{B}$ to be a P -martingale. First we record its values.

$Z_n \tilde{B}_n$:

$$\begin{array}{rcc}
 & & 4p^2(4q) = 16p^2q \\
 & 2p(2q) = 4pq & \\
 1 \cdot 0 = 0 & & 4pq(2q - 2p) = 8pq(q - p) \\
 & 2q(-2p) = -4pq & \\
 & & 4q^2(-4p) = -16pq^2 \\
 \hline
 n = 0 & n = 1 & n = 2
 \end{array}$$

Then we compute the following conditional expectations under P

$E[Z_{n+1} \tilde{B}_{n+1} | \mathcal{F}_n]$:

$$\begin{array}{rcc}
 & \frac{1}{2} \cdot 16p^2q + \frac{1}{2} \cdot 8pq(q - p) = 4pq(2p + q - p) = 4pq & \\
 \frac{1}{2} \cdot 4pq + \frac{1}{2} \cdot (-4pq) = 0 & & \\
 & \frac{1}{2} \cdot 8pq(q - p) + \frac{1}{2} \cdot (-16pq^2) = 4pq(q - p - 2q) = -4pq & \\
 \hline
 n = 0 & & n = 1
 \end{array}$$

which does agree with the computed values of $Z_0 \tilde{B}_0$ and $Z_1 \tilde{B}_1$.