Minicourse: Mathematical Finance

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Outline

Part I: Derivative Pricing

1. Basic ideas: Hedging and Arbitrage
2. Mathematical tools:
   Brownian Motion, Stochastic Calculus, and SDEs
3. Risk-Neutral valuation in continuous time
4. Binomial pricing: risk-neutral valuation in discrete time
5. Next steps: Incomplete markets, stochastic volatility
6. Term structure and the yield curve
7. American Options
Part II: Portfolio Optimization
1. Utility functions
2. Asset allocation, consumption, and the Merton problem
3. Stochastic Control Theory and Dynamic programming
4. Hamilton-Jacobi-Bellman equations
5. BSDEs
6. Optimal execution

Part III: Other Aspects of Risk Management
1. Actuarial Finance
2. Credit Risk: structural/reduced form models, CVA
3. Systemic Risk: random networks
Options: a first look at hedging

- The Black-Scholes-Merton formula (1973) revolutionized financial markets in the 1980’s.
- Buying a (European) call option on a stock gives you the right, but not the obligation, to buy a share from the seller of the option at a given price $K$ (the *strike price*) on a given date $T$ (the *expiry date*).
- If $S_t$ is the stock price at time $t$, this $\iff$ an obligation for a payment of $f(S_T)$ at time $T$, where $f(x) = (x - K)_+$. 
- In other words, a call option is a *derivative security*: its value derives from the price of some other security.
- Conversely a put option gives you the right to sell. $\iff f(x) = (K - x)_+$. 

Calls and Puts

\[ x = \text{stock price at time } T \]

Call payoff \( f(x) = (x - K)_+ \)

Put payoff \( f(x) = (K - x)_+ \)
Model: GBM

- Assume the stock follows a *Geometric Brownian motion*. ie it solves the *Stochastic Differential Equation* (SDE):

\[
dS_t = \mu S_t \, dt + \sigma S_t \, dB_t
\]

where the noise $B_t$ is a *Brownian motion* under the true $P$.

- Assume a constant *risk free interest rate* $r$. ie. Money market account follows $dR_t = rR_t \, dt$

- The option can then be *hedged*: $\exists$ self-financing portfolio $V_t$ (ie. take positions in both the stock and in a money market account – you can rebalance, but no money goes in or out) such that $V_T = f(S_T)$. 
Brownian motion $B_t$ (Wiener process)  
(a basic model for random noise)

GBM: $S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$ (P. Samuelson)  
(SDE sol’n – a model for stock prices)
Hedging

- Red curve = intrinsic value $f(S_t)$ of a call option, starting out of the money and moving into the money over time.
- Blue curve = hedging portfolio. They meet at time $T$.

- The option price must be $V_0$, ie. where the blue curve STARTS.
- In fact this yields a formula $V_t = v(t, S_t)$ for the derivative price at every time $t \leq T$. 
Arbitrage

Arbitrage means an opportunity guaranteed to make money, without having to invest anything up front.

If you see the option selling for any $V \neq V_0$, there is an arbitrage opportunity:

- If $V > V_0$ then sell the option and use the funds to buy the hedging portfolio. You pay nothing up front, and are guaranteed to have the returns on $V - V_0$ left at the end.
- If $V < V_0$ then buy the option and finance it by selling the hedging portfolio. At the end you have the returns on $V_0 - V$.

In the absence of arbitrage, price = $V_0 = \text{cost of setting up the hedging portfolio (no-arbitrage pricing)}$. 
Caveats

- Option price is a function $v(t, S_t)$
- But so is the number of stocks needed, $\Delta(t, S_t)$
- Both fluctuate wildly – a result of continuously rebalancing the portfolio. Transaction costs renders this impractical for small investors.
- Also, volatility and interest rates aren't actually constant.
- So BSM is only a good approximation – a starting point for our models. Can be made more sophisticated to correct (partially) for stochastic volatility, term structure, monthly or weekly rebalancing, transaction costs, price jumps, etc.
Brownian motion is the basic stochastic process people use in continuous time.

Its law is completely characterized by the following:

- $t \mapsto B_t$ is a *random continuous function*.
- If $I_i = [s_i, t_i]$ are a disjoint collection of intervals, then the increments $B_{t_i} - B_{s_i}$ are *independent Gaussian* r.v.’s with zero mean and variances $t_i - s_i$.

Observed in nature by the botanist Robert Brown in 1827 (pollen grains suspended in water).

Albert Einstein (1905) obtained a quantitative model, and used it to estimate Avogadro’s number.

Norbert Wiener (1927) gave a rigorous derivation. Properties then studied by Paul Lévy and others.
Applications in finance are based on stochastic calculus, invented by Kyosi Itô in 1944: tools for calculating with and manipulating Brownian paths.

Ordinary calculus deals with smooth functions, but the paths of Brownian motion are wild and jagged; eg. non-differentiable everywhere. New mathematics was needed to treat stochastic differential equations.

Itô’s ideas revolutionized probability theory in the 1950’s and 1960’s. Electrical engineers picked it up in the 1960’s for filtering random noise out of signals. Entered Economics and Finance in the 1970’s and 1980’s, after options traders in Chicago started basing decisions on the BSM formula, a major factor in the explosive growth of options markets. See Black & Scholes: *The Pricing of Options and Corporate Liabilities* (1973)
Kyoshi Itô, 1915–2008

For his work, Itô won the 1st Gauss Prize in 2006
“For new mathematics having remarkable applications to other fields.”
Stochastic Differential Equations

- eg SDE for GBM: $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$
- Give it meaning via its integrated form

$$S_t = S_0 + \int_0^t \mu S_q \, dq + \int_0^t \sigma S_q \, dB_q.$$  
- First integral is standard. Second is not, because Br.M. has infinite variation.
- Itô obtained this *stochastic integral* based on having finite *quadratic variation*, and a non-anticipating hypothesis on the integrand that implies near-cancellation of the oscillations.
- From that, can build a whole theory, eg $\exists!$ of sol’ns to SDE’s.
- Latest chapter in this story: Martin Hairer’s 2014 Fields medal for integration over *rough paths* & applications to SPDE’s.
Risk Neutral valuation (continuous time)

- Rewrite SDE as \( dS_t = rS_t \, dt + \sigma S_t \, d\tilde{B}_t \) (ie set \( \tilde{B}_t = B_t + (\mu - r)t/\sigma \))

- Fundamental principle: Can’t measure \( \mu \).
  So \( \exists \) mutually absolutely continuous measure \( Q \) (called the Risk Neutral probability measure), under which \( \tilde{B}_t \) is a Brownian motion.

- Growth rates of \( S_t \) (stock) and \( R_t \) (money market) are now equal. And can show that ANY self-financing\(^1\) portfolio \( V_t = \Delta_t S_t + \psi_t R_t \) also has growth rate \( r \).

- In fact, a simple calculation shows that
  \[
  dV_t = e^{-rt} V_t \, dt + e^{-rt} \, dV_t = \cdots \quad d\tilde{B}_t
  \]

- Conclusion: Discounted portfolio values are always stochastic integrals, and therefore Risk neutral martingales.
  ie. don’t drift/trend up or down.

\(^1\)\(dV_t = \Delta_t \, dS_t + \psi_t \, dR_t\)
Conversely, the *Martingale representation theorem* shows that every risk-neutral martingale (in this model) can be written as a stochastic integral with respect to $\tilde{B}_t$.

Then one reverses the previous calculation to represent it as the discounted value of a self-financing portfolio.

i.e. discounted portfolios $\leftrightarrow$ risk-neutral martingales.

But there is a recipe for finding such a martingale, with the correct terminal condition:

$$V_t = e^{-r(T-t)} E_Q[f(S_T) \mid \mathcal{F}_t] = v(t, S_t).$$

**CONCLUSION:** This $V_t$ is the value of a portfolio that hedges the option, and the option price is the hedging cost $V_0 = e^{-rT} E_Q[f(S_T)]$. 
For the European call, there is a formula for this expectation:

\[ \nu(t, s) = s \Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) \]

\[ d_1 = \frac{\log(s/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \]

\[ d_2 = d_1 - \sigma \sqrt{T-t} \]

Note: Doesn’t depend on \( \mu \)
And can solve numerically for the expectation (eg Monte Carlo) if we want more complicated \( f \) (using wrong – ie risk neutral rather than physical – parameters).
BSM European Call Option Prices

Call values, for multiple $t$
Fischer Black / Myron Scholes
Robert Merton
(Nobel prize, 1997)
PDE approach

- We know $e^{-rt}v(t, S_t)$ is a risk-neutral martingale.
- Apply Itô’s Lemma (change of variables formula for stochastic integrals) to write $de^{-rt}v(t, S_t)$ in the form
  
  $$
  \cdots \, dt + \cdots \, d\tilde{B}_t.
  $$

- Martingale property (no trend up/down) $\Rightarrow$ $dt$ terms $= 0$.
- A simple calculation expresses these terms in terms of derivatives of $v$. ie
  
  $$
  v_t + rxv_x + \frac{1}{2} \sigma^2 x^2 v_{xx} - rv = 0.
  $$

- ie obtain option prices as solutions to a PDE (maybe numerically).
Binomial Model – discrete time

- There’s an alternate approach to risk-neutral valuation using discrete time, that is more elementary.
- Current price $s$
- After 1 period, stock price goes up to $us$ or down to $ds$. Option pays $f(us)$ or $f(ds)$.
- Cash grows by factor $R$ over the same period.
Hedge option by holding $\Delta$ units of stock, and $B$ in cash, at the beginning of the period. If we can do this, then initial value $s\Delta + B = \text{option price.}$

At the end of the period, hedge position is worth either $us\Delta + BR$ or $ds\Delta + BR$. Set $= f(us), f(ds).$

Solve for $\Delta, B$:

$$\Delta = \frac{f(us) - f(ds)}{s(u - d)} \quad B = \frac{uf(ds) - df(us)}{R(u - d)}.$$
Risk Neutral valuation

- So initial price is

\[ s\Delta + B = \frac{1}{R} \left[ f(us) \frac{R - d}{u - d} p + f(ds) \frac{u - R}{u - d} (1-p) \right]. \]

- Set \( Q(\text{up move} = p) \). Provided \( d < R < u \), option value is

\[ \frac{1}{R} E^Q[f(S_1)] \]

where \( Q \) is the risk neutral measure \( \neq P \), the physical measure
- $p$ is **NOT** the real probability of an up move. It is an artificial value, designed to let us apply the technology for evaluating expectations (PDEs, Monte Carlo) to option pricing.

- RN growth rate is $E^Q[S_1] = [pu + (1 - p)d]s = Rs$. In other words, under $Q$, both stock and bond grow by the same factor per period, on average. If $Q$ were the real measure, there would be no premium for risk, and only people indifferent to risk (ie risk-neutral) would buy stocks. In the real world, we demand a higher return for stocks, before we’ll buy them.
Binomial Tree

- $s$
- $u^4s$
- $u^3ds$
- $u^2d^2s$
- $ud^3s$
- $d^4s$
With $n$ periods, possible option values are $f(u^k d^{n-k}s)$, $k = 0, 1, \ldots, n$

Can price by walking backwards through the tree (although it is not actually a tree) – dynamic programming

But conceptually, initial option value is

$$
\frac{1}{R^n} \sum_{k=0}^{n} f(u^k d^{n-k}s) \binom{n}{k} p^k (1 - p)^{n-k} = \frac{1}{R^n} E^Q[f(u^K d^{n-K}s)]
$$

where the number of up-moves is $K \sim \text{Binomial}(n, p)$. Risk Neutral expectation, discounted at the risk free rate.

with suitable parameters, when $n \to \infty$ get Binomial $\to$ Gaussian, and recover BSM in limit.

Binomial tree $\leftrightarrow$ finite difference scheme for PDE.
Incomplete Markets

- What about other market models, besides GBM?
- Call a market *Complete* if every option can be perfectly hedged. *Incomplete* otherwise.
- In a complete market much of the above still goes through, and no-arbitrage prices are well defined.
- But in an incomplete market, there may be a range of possible prices that avoid arbitrage.

**Fundamental Theorem of Asset Pricing:**
- Every arbitrage-free way to assign prices comes from some risk-neutral probability measure (or *equivalent martingale measure*). ie absence of arbitrage \( \Leftrightarrow \exists \) a RNM.
- Market is complete \( \Leftrightarrow \) there is a unique RNM.
- Worth noting that a trinomial discrete model is incomplete.
Stochastic Volatility

- For example, the major shortfall of GBM as a model is that realistic volatilities are NOT constant.
- An improvement: Heston model

\[
\begin{align*}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{\Sigma_t} dB_t \\
\frac{d\Sigma_t}{\Sigma_t} &= \kappa (\theta - \Sigma_t) dt + \sigma \sqrt{\Sigma_t} dW_t.
\end{align*}
\]

Now squared volatility is mean reverting to level \( \theta \) at rate \( \kappa \) (\( \Sigma_t \) follows a CIR model).
- Models are absolutely continuous when we vary \( \mu, \kappa, \theta \) (locally anyway), and being risk-neutral forces \( \mu = r \). But there is still freedom to vary \( \kappa \) and \( \theta \), so model is incomplete.
  ie CAN’T HEDGE just by trading in \( S_t \) and the money market account.
So what questions can one now answer?

One is to look for hedges that aren’t perfect, but are optimal according to some criterion. eg variance minimizing hedge, quantile hedging. This points in the direction of stochastic control theory, using techniques that we’ll cover later today.

Another is to add hedging instruments. For example, there is a volatility index VIX, on which one can purchase volatility swaps – derivative securities whose cash flows are determined by VIX. If we allow our portfolios to trade in both the stock and a volatility derivative, that constrains the possible RNM’s. Can be done in such a way that there is only one, ie that we complete the market.
Bonds and Interest

- Another defect with GBM is that realistic interest rates aren’t constant (as assumed in BSM formula)
- A more sophisticated approach is to model the short rate \( r_t \) (risk-free rate for instantaneous borrowing) as a stochastic process. For example, one might assume that risk-neutrally \( r_t \) follows a CIR model driven by yet another Brownian motion. This is an example of an affine short-rate model, in which it is possible to derive nice formulas for bond prices as \( E_Q[e^{-\int_0^T r_s ds}] \), in the case of a (zero-coupon) bond paying $1 at time \( T \).
- In practice such 1-factor models explain some – but not all – of the behaviour of bond prices.
More sophisticated models try to capture the behaviour for bonds of all maturities $T$, simultaneously.

This can be done in ways that involve multiple factors, eg HJM models for forward interest rates (the rate you promise today for lending at a future time).

Each bond has a *yield*: collapse all features of the bond to a single constant effective interest rate. Today’s *yield curve* (at time $t$) represents this yield as a random function of time-from-the-present $s$. HJM lets you model the evolution of this curve over time, ie as a function of $s$ and $t$. This can get pretty fancy, eg there are even SPDE versions of this.

Increasingly banks use computationally simpler approaches based on directly modelling LIBOR, eg BGM model.
American Options

- Before, European options had a fixed maturity $T$.
- Now the option holder decides (on the fly) when to exercise.
- eg. American put has intrinsic value $f(x) = (K - x)_+$. Hedge $V_t$ satisfies $V_t \geq f(S_t)$ for $t \leq$ expiration $T$, and $\exists$ stopping time $\tau \leq T$ with $V_\tau = f(S_\tau)$.
- Arbitrage, if option trades for $V \neq V_0$:
  - If $V > V_0$, sell option, implement hedge, pocket difference.
  - If $V < V_0$, buy option, sell hedge, exercise at time $\tau$.
- Risk-Neutral valuation: $V_0 = \sup_{\text{st. time } \tau \leq T} \mathbb{E}_Q[e^{-r\tau}f(S_\tau)]$.
- PDE for option price is now a free boundary value problem, as must find not just price, but also the optimal exercise region, with a smooth pasting condition across the boundary.
American put price $v(t, x)$

Option price $v(t, x)$, where $x =$ stock price at a fixed time $t$
Exercise boundary

Horizontal axis is time $t$. Blue = exercise boundary. Orange = stock price
Part II: Portfolio Optimization

- Let $X_t$ be our wealth at time $t$. Our objective is to “optimize” wealth at some time $T$ (e.g., retirement). But wealth is random, so how do we make this well-posed?

- Economists answer: let that person’s utility function be $u(x)$ and maximize $E[u(X_T)]$.

- Utility properties: $u$ is continuous, ↑, concave (as $u' =$ marginal impact of a new $\$, will ↓). Often also $u'(0+) = \infty$.

- A common choice is $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$ represents risk aversion. Called constant relative risk (CRRA) utility. When $\gamma = 1$ use $u(x) = \log x$ instead.

- High $\gamma$ means we’re very risk averse, low $\gamma$ means we’re not. $\gamma \to 0$ corresponds to risk neutrality (care only about average returns). Concavity $\Rightarrow$ given a choice of two investments with the same mean return, we prefer the less risky of the two.
Utility

Translated CRRA utilities

\[ u(x) = \begin{cases} 
  \frac{x^{1-\gamma}-1}{1-\gamma}, & \gamma \neq 1 \\
  \log x, & \gamma = 1 
\end{cases} \]
Merton problem

- Suppose wealth can be invested in a stock $S_t$ or a money market fund $R_t$, where (as before)

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t \quad dR_t = r R_t \, dt.$$ 

Note that everything is under the physical measure $P$.

- Self financing portfolio is $X_t = \phi_t S_t + \psi_t R_t$, and we get to control the amounts.

- Asset allocation is $\pi_t = \frac{\phi_t S_t}{X_t}$. Fraction of wealth in the stock. This is our control. $X_0 = x_0$ is given.

- Merton problem: find admissible $\pi_t$ that maximizes $E[u(X_T)]$.

- Simple calculation $\Rightarrow$ following dynamics:

$$dX_t = r X_t \, dt + (\mu - r) \pi_t X_t \, dt + \sigma \pi_t X_t \, dB_t.$$
Caveats

- Different people will have different $\gamma$’s, so different optima.
- Could restrict $0 < \pi_t < 1$, but for simplicity will allow $\pi_t < 0$ (short selling) or $\pi_t > 1$ (leverage).
- Some care is needed in specifying what $\pi_t$ are admissible. Obviously must be adapted (can’t peek into the future). But also need to be aware of “doubling” strategies or (their opposite) “suicide” strategies. Can usually ignore latter, as won’t be optimal.
- We’ll assume $\pi_t$ satisfies $X_t \geq 0 \forall t$. This turns out to rule out doubling.
- But for simplicity: also assume $\pi_t X_t$ is square integrable.
- $\exists$ many variants to the problem, including optimizing utility of consumption over time (rather than just terminal wealth).
Merton’s answer

\[
\pi_t = \frac{\mu - r}{\gamma \sigma^2}
\]

- Myopic solution: does not depend on \( t, T \), or wealth \( x \). So doesn’t change as horizon \( T \) gets closer.
- \( \pi_t > 0 \) so no short-selling. But may have \( \pi_t > 1 \) (leverage), for example if \( \gamma \) is small, or if \( \mu \) is big.
- A static asset allocation doesn’t mean you don’t trade. In fact, the portfolio must be continuously rebalanced to maintain the allocation. But the effect is to force you to sell stocks when they go up in price, and to buy them when they go down. In practice, financial advisors do often tell people to maintain a roughly constant asset allocation, with rebalancing taking place a few times per year.
Dynamic Programming

- Interpolate $T \sim 0$ by

$$V_t = \sup_{\text{admissable } \pi} E[u(X^\pi_T)|\mathcal{F}_t]$$

- Find $V_t$ (and therefore what we want, ie $V_0$) using *Hamilton-Jacobi-Bellman* (HJB) equations for the value function $v(t, x)$:

$$\sup_{\pi} L_\pi v(t, x) = 0 \quad \forall t, x$$

where $L_\pi v = v_t + v_x[r + (\mu - r)\pi]x + \frac{1}{2}v_{xx}\sigma^2\pi^2x^2$
HJB verification theorem

Suppose that we can find
- a smooth function \( v(t, x) \) s.t.
- \( v(T, x) = u(x) \) and
- \( L_\pi v \leq 0 \) for every \( t, x, \pi \).

Suppose we can also find a control \( \pi^*(t, x) \) s.t.
- \( L_{\pi^*} v = 0 \) for every \( t, x, \) and
- \( \pi^* \) is admissible.

Then
- \( v \) solves HJB
- \( V_0 = v(t, x_0) \) solves our optimization problem,
- \( \pi^* \) is optimal and \( V_t = v(t, X_t^{\pi^*}) \).
Proof

- Let $\pi_t$ be admissible, $X_t$ the corresponding wealth process.
- By Itô’s lemma, $dv(s, X_s) =$ 
  \[ v_s + v_x[r + (\mu - r)\pi]X_s + \frac{1}{2}v_{xx}\sigma^2\pi_s^2X_s^2 \] 
  $ds + v_x\sigma\pi_sX_s dB_s$
- The $ds$ term is negative, so
  $u(X_T) = v(T, X_T) \leq v(t, X_t) + \int_t^T v_x\sigma\pi_sX_s dB_s$
- By admissibility, the stochastic integral has mean $= 0$. 
  So $E[u(X_T)|\mathcal{F}_t] \leq v(t, X_t)$ and $\therefore V_t \leq v(t, X_t)$.
- Same argument with $\pi_s = \pi^*(s, X_s^\pi^*)$
  \[ \Rightarrow E[u(X_T)|\mathcal{F}_t] = v(t, X_t). \]
  So $V_t = v(t, X_t)$ and $\pi_s = \pi^*(s, X_s)$ is optimal.
Merton’s solution to HJB.

- Scaling argument suggest we try \( v(t, x) = f(t)x^{1-\gamma}/(1 - \gamma) \) with \( f(t) > 0 \) and \( f(T) = 1 \).
- Substituting, get that \( \pi \) maximizes a quadratic, and its claimed formula is elementary.
- Substituting back, PDE just becomes the ODE
  \[
  f'(t) + f(t)r(1 - \gamma) + f(t)\frac{(1-\gamma)(\mu-r)^2}{2\gamma\sigma^2} = 0,
  \]
  which has an explicit solution.
- Admissibility holds: optimal portfolio is just a GBM.
- So verification argument applies.
Optimality principle

- How do we obtain HJB in the first place?
- Heuristic: pretend we know ∃ optimal π*. Then $V_t = E[u(X_t^{\pi^*})|\mathcal{F}_t]$ is a martingale of form some $\nu(t, X_t^{\pi^*)}$.
- Under a sub-optimal control $\pi$, can interpret $\nu(t, X_t)$ as the utility from following $\pi$ on $[0, t]$ and then switching to $\pi^*$. From that get that for $s < t$, $E[\nu(t, X_t)|\mathcal{F}_s] \leq \nu(s, X_s)$. So drift is $\leq 0$.
- In other words, $\nu(t, X_t)$ is a supermartingale in general, and a martingale under optimal behaviour.
- Applying Itô’s lemma now yields HJB.
Other methods

- When an explicit solution does not exist, can sometimes solve HJB numerically, or approx. by a discrete control problem.
- To prove properties of the solution, can often obtain value function as a viscosity solution of HJB.
- Alternate approach: Focus on terminal wealth $Y = X_T$, and choose it to maximize $E[u(Y)]$ subject to a constraint. Namely that the risk neutral expectation $E_Q[e^{-rT}Y] = x_0$, the available initial wealth. This optimization is usually done using duality methods (basically Lagrange multipliers).
- Then extract the asset allocation $\pi_t$ that realizes the risk-neutral martingale $E_Q[e^{-rT}Y|\mathcal{F}_t]$ as a discounted portfolio $e^{-rt}X_t$. 
BSDE’s

- If \( Y \) is the optimal terminal wealth, can reformulate the risk-neutral martingale condition for the portfolio process as

\[
dV_t = rV_t \, dt + Z_t \, dB_t
\]

\[
V_T = Y
\]

for some process \( Z_t \), where \( B_t \) is a RN-Brownian motion.

- Sometimes constraints on how the portfolio can be managed get expressed via \( Z_t \), and the BSDE approach is particularly useful.

  Eg. El Karoui-Quenez-Peng (1997)
Optimal Execution

- Stochastic control problems arise throughout finance.
- For example: optimal execution of orders, drawn from the limit order book, in the context of high-frequency trading.
- See, for example the recent book by Jaimungal (in references)
Part III: Actuarial Finance

- Insurance industry blends actuarial risks with market risks
- eg. through retirement savings products such as **Variable Annuities**.
- Basically a mutual fund, but with a complicated collection of guarantees (aka options) layered on top.
- Can ask questions about pricing, hedging, or how consumers should optimally manage these guarantees.
- eg **Guaranteed lifetime withdrawal benefit** – converts to a life annuity if specified withdrawals ever bankrupt your account. Level of the annuity depends on when you opt to turn on income stream (a control problem).
Huang-Milevsky-Salisbury, IME 2014

Figure: Green region: wait to initiate income
Credit Risk

▶ A different problem arises if I’ve made a bunch of loans, and I want to understand the risk I’m carrying.
▶ Each borrower (obligor) has a certain probability of default (PD) and an associated loss given default (LGD).
▶ A bank’s risk management protocols require estimating these, tracking them, and having capital reserves in place to manage defaults when they happen. Both for the individual loans and for the whole portfolio of loans. A typical credit risk problem.
▶ But derivatives also come back into the picture: May be able to insure against default by buying a credit derivative. So have another credit risk problem: how to price credit derivatives?
For example, a Credit Default Swap (CDS) pays only at the random time $T$ that firm (obligor) defaults. Could pay a fixed amount upon default (standard CDS). Or the amount paid could vary with LGD. eg if company debt is $D$ & firm value at default is $V_T$, then CDS is a put option:
Incomplete Markets

- Unlike equity options, you can’t hedge these risks: even if you trade in the obligor’s stock, if they go bankrupt, the equity piece of your hedge won’t be there when you need it.
- That is, market is incomplete. Pricing amounts to identifying the RN measure implied by observed prices.
- Such as spread (difference) between corporate loans and government (risk free) loans.
- Risk Neutral PD is only useful for pricing. For risk management, need true PD (under physical measure). For that, can try to use historical data on defaults for obligors with similar credit ratings.
Prior to the financial crisis, there were two main classes of models . . . both of which go back to Merton
Credit Risk Models

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- Structural models

- Reduced form models
Credit Risk Models

Prior to the financial crisis, there were two main classes of models ... both of which go back to Merton

▶ Structural models
  ▶ Merton (1974) – the *Merton model* for credit risk
  ▶ Black–Cox (1976)
  ▶ Leland (1994)
  ▶ KMV (Moody’s – now CreditEdge)
  ▶ CreditMetrics (MSCI – now CreditManager)

▶ Reduced form models
  ▶ Merton (1976)
  ▶ Duffie-Singleton (1999)
  ▶ Hull–White (2000)
  ▶ Gaussian Copula (Li, 2000)
  ▶ CreditRiskPlus (Credit Suisse)
Structural models

- These try understand the root causes of default. Analogous (if default = death) to studying mortality medically, person by person.
- For example, Merton model starts by viewing a firm’s equity as assets minus recovery of debt:

\[
\begin{align*}
\text{Stock Valuation} &= \text{Call Option on Assets} \\
&= \text{max}(0, \text{Assets} - \text{Debt})
\end{align*}
\]

- i.e. stock valuation is a call option on the assets of the firm.
- Now seek to apply option pricing methodology to get risk neutral PD and other quantities of interest. One issue, of several, is that we can seldom actually observe the value of the firm’s assets.
Reduced form models

- Ignore causes, treat defaults as random, focus on hazard rates. Analogous to studying mortality as an actuary, not an MD.
- Because it is hard to model what is going on inside a firm, reduced form models are usually easier to develop, and often perform better in applications.
- But data is sparse (ie few companies actually default). And particularly hard to get for default correlations.
Correlations and the crisis

- Eg. the Gaussian copula model postulates a way of inferring correlations in defaults from marginal distributions of default times and correlations between prices of the assets. Easy to apply, and was widely adopted pre-crisis.

- This model failed in the crisis. Seriously underestimates correlations when they’re high. In addition, correlations changed rapidly in the crisis, which models didn’t account for.

- Caused problems with CDO’s: Collateralized Debt Obligations, and CDS’s written on them.

- Issue: vendors viewed the pieces of CDO’s as reasonably independent (and so likewise the CDS’s that went with them). Relied on law of large numbers to keep risk low. But once housing prices dropped, the independence disappeared, making CDO’s worthless (driving Lehman to bankruptcy), and triggering piles of the CDS’s AIG had written.
The secret formula that destroyed Wall Street

How one simple equation made billions for bankers – and nuked your 401(k)

– Wired magazine
Post-crisis research

▶ Prior to the crisis, people were using simple models to study complex and exotic credit derivatives (eg CDO’s)
▶ Post-crisis, these exotic products no longer trade. People now study detailed models for simple derivatives (eg CDS’s).
▶ One new research area is CVA’s: Regulations now require banks to calculate Credit Value Adjustments for significant transactions. ie to adjust values to account for the possibility of counterparty default. Collateral rules are also more explicit, and incorporated into capital reserve requirements.
▶ Another new research topic: Systemic Risk. Model networks of banks, and how default at one can spread through the banking network (contagion).
Arrows show obligations (say in units of $10M) ie cash flows when all loans are repaid.

Banks also have specified capital reserves (not shown).
References

- J. Hull, *Options, futures and other derivatives* 10th ed, Pearson 2017
- D. Lando, *Credit risk modelling*, Princeton 2004