

BLOWUP AND CONDITIONINGS OF ψ -SUPER BROWNIAN EXIT MEASURES

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ABSTRACT. We extend earlier results on conditioning of super-Brownian motion to general branching rules. We obtain representations of the conditioned process, both as an h -transform, and as an unconditioned superprocess with immigration along a branching tree. Unlike the finite-variance branching setting, these trees are no longer binary, and strictly positive mass can be created at branch points. This construction is singular in the case of stable branching. We analyze this singularity first by approaching the stable branching function via analytic approximations. In this context the singularity of the stable case can be attributed to blowup of the mass created at the first branch of the tree. Other ways of approaching the stable case yield a branching tree that is different in law. To explain this anomaly we construct a family of martingales whose backbones have multiple limit laws.

1. INTRODUCTION

The ψ -super-Brownian motion X_t is a measure valued diffusion with branching mechanism given by

$$\psi(\lambda) = a_1\lambda + a_2\lambda^2 + \int_0^\infty [e^{-\lambda r} - 1 + \lambda r] \pi(dr),$$

where $\lambda \geq 0$, $a_1 \in \mathbb{R}$, $a_2 \geq 0$, and $\pi(\cdot)$ is the associated Lévy measure. More precisely, the log-laplace functional of X_t ,

$$u(t, x) = -\log E_{\delta_x} \exp\left(-\int \phi(y) dX_t(y)\right)$$

solves the initial value problem

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \psi(u), \quad u(0, \cdot) = \phi(\cdot),$$

whenever ϕ is a non-negative bounded continuous function. These diffusions can be realised as scaling limits of a system of particles that perform branching Brownian motions. The exit measure X^D of such a process from a bounded domain D is a random measure on the boundary of D , supported on the set of points where the particles first exit the

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domain. It is known that for $d \geq 3$ that the random measure does not see points (see Le Gall [20] and Dynkin [8]).

In Salisbury and Verzani [29] and [30] the exit measure of super Brownian motion, X^D with critical binary branching (that is, with branching mechanism $\psi(\lambda) = 2\lambda^2$) is conditioned to charge small balls $\Delta_\epsilon^{z_i} = B(z_i, \epsilon) \cap \partial D$, where $z_1, \dots, z_n \in \partial D$. Letting $\epsilon \rightarrow 0$ they obtain a conditioned process, which is a martingale transform of super Brownian motion by a “polynomial” martingale of degree n . They describe this process in terms of a “backbone” consisting of a binary tree that realizes the trajectory of the mass that reaches z_1, \dots, z_n . These results do not generalize to a stable branching mechanism $\psi(\lambda) = c\lambda^{1+\beta}$, since “polynomials” of the exit measure will not even be integrable, let alone give martingales, when $n \geq 2$. Understanding why formed the primary motivation for this paper.

Our aim is to understand conditioning based on general branching mechanisms well enough to analyze how these martingales blow up as we approach the stable case. Consider a bounded domain $D \subset \mathbb{R}^d$, when $d \geq 3$. Let $\epsilon > 0$ and fix $z_i \in \partial D$ for $i = 1, 2, \dots, n$. As above, define $\Delta_\epsilon^{z_i}$ to be a ball on ∂D of radius ϵ . We condition ψ -super Brownian motion to hit balls $\Delta_\epsilon^{z_i}$ and obtain both the martingale transform that represents this process, and the probabilistic representation of this process in terms of immigrating mass along a branching “backbone”. We explicitly describe the evolution of the backbone tree, the manner mass is generated along the backbone, and the way it evolves afterwards (See Theorem 3.1 in Section 3). Unlike the results of [29] and [30], we now have to handle the probabilities of multiple branches and the distribution of positive mass created at the branch points of the tree.

We first take the limit as $\epsilon \rightarrow 0$, when $d \geq 4$ and ψ is a real-analytic function. We establish that the limit exists and is a martingale change of measure (see Theorem 4.2 and Theorem 4.1 in Section 4.2). The proof here follows the road map laid out in [29] but the significant estimates appear to require more delicate arguments (Lemma 4.2 and Lemma 4.3). The limiting backbone produced is a tree with precisely n leaves (see Section 4.3), but is no longer binary. Next we let the branching mechanism approach the stable case ($\psi(\lambda) = c\lambda^{1+\beta}$). With this particular order of taking the two limits, the backbone remains well behaved, as does the mass creation at non-branch points of the backbone. However, the mass created at branch points gets arbitrarily large, resulting in an explosion that allows us to pinpoint the source of the blowup (See Section 4.4 and Theorem 4.4). We also consider other ways of approaching the stable case, for example, to simply let $\epsilon \rightarrow 0$ with branching mechanism $\psi(\lambda) = c\lambda^{1+\beta}$. In this case, the backbone remains well-behaved but has a different law than in the previous mechanism. To explain this anomaly we construct a simpler family of martingales whose backbones have multiple limit laws, interpolating between analogues of both types obtained above (See Section 4.5).

The decomposition of the conditioned superprocesses in terms of an “immortal backbone” has been considered in the literature, in other contexts. In particular, h -transforms of critical super-Brownian motion have been studied by Roelly-Coppoletta and Roualt [27],

Evans and Perkins [17], and Overbeck [24], [25], while the immortal particle representation was originally discovered in Evans [16]. Other studies include Serlet [31] and Etheridge [14]. Salisbury and Sezer [28] and Verzani [33] consider more general conditionings for binary branching. Moras [23] considers specific classes of unbounded domains D , again for binary branching.

Versions for non-binary branching were studied in Etheridge and Williams [15] and Kyprianou et al [19] (we learned of the latter after completing our research). In [15], the super Brownian motion on all of \mathbb{R}^d , with stable $(1 + \beta)$ branching, is conditioned on survival until some fixed time T . This backbone has a Poisson number of immortal trees (conditioned on there being at least one), along which mass (conditioned to die before time T) is immigrated. The rate of immigration is random and there is additional immigration whenever the immortal tree branches. In the limit as $T \rightarrow \infty$, the immortal trees degenerate to the Evans immortal particle and the immigration (of unconditioned mass) along the particle is dictated by a stable subordinator. In [19], to study the travelling wave equation associated to the parabolic semi-group equation of ψ -super-Brownian motion, the authors show a similar immortal backbone on all of \mathbb{R} (which they call a "spine decomposition").

1.1. Layout of the paper. The rest of the paper is organised as follows. In the next section we discuss preliminaries with regard to conditioned diffusions, ψ -super Brownian motion and potentials. The Palm formula in Lemma 2.3 and identity for potentials in Lemma 2.5, presented here, are used significantly in the proof of main results. In Section 3 we prove Theorem 3.1. In particular, we define a martingale change of measure representing various conditionings and for each provide a description of the associated branching backbone representation.

In Section 4, we consider a specific condition namely that of the exit measure charging finitely many points on the boundary. The work here (as explained earlier) is divided into three parts. First for ψ -analytic, we condition the exit measure to hit balls of radius ϵ and establish a limit as $\epsilon \rightarrow 0$. This is proved in Theorem 4.1 and Theorem 4.2. A limiting backbone is then described in Section 4.3. Secondly, in Section 4.4 we consider the second limiting procedure of ψ -analytic to approximate $\psi(\lambda) = c\lambda^{1+\beta}$ for $0 < \beta \leq 1$. We explain the explosion effect precisely. Finally in Section 4.5, we begin by explaining other possible limits (if the order of limits done earlier are interchanged). We conclude by constructing a family of related martingales whose backbones have multiple limit laws. These interpolate between analogues of the two types of limits obtained earlier.

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2. PRELIMINARIES

2.1. Notation. For each set A , let $|A|$ denote its cardinality, and let $\mathcal{P}(A)$ denote the collection of partitions of A . Impose an order on A . Then for any $\sigma \in \mathcal{P}(A)$, we may order the sets in σ by their smallest elements. Letting $\sigma(j)$ denote the j th element of σ in this order, we may switch at will between the following two notations:

$$\prod_{C \in \sigma} \langle X^D, v^C \rangle = \prod_{j=1}^{|\sigma|} \langle X^D, v^{\sigma(j)} \rangle.$$

The following elementary combinatorial result will be convenient

Lemma 2.1. *Let $A \subset B \subset C$. be subsets of $\{1, 2, \dots, n\}$. Then*

$$\sum_{A \subset B \subset C} (-1)^{|B|} = (-1)^{|C|} \mathbf{1}_{A=C}$$

Proof. Both sides equal $(-1)^{|A|}(1-1)^{|C \setminus A|}$. See Lemma 2.1 of [29] □

2.2. Facts about conditioned diffusions. First we recall some familiar formulae for conditioned Brownian motion.

Let B be d -dimensional Brownian motion started from x , under a probability measure P_x . Write $\tau_D = \tau_D(B)$ for the first exit time of B from D . Let $g : D \rightarrow [0, \infty)$ be bounded on compact subsets of D , and set

$$L_g = \frac{1}{2} \Delta - g.$$

Let ξ_t be a process which, under a probability law P_x^g , has the law of a diffusion with generator L_g started at x and killed upon leaving D . In other words, ξ is a Brownian motion in D , killed at rate g . Write ζ for the lifetime of ξ . Then

$$(2.1) \quad P_x^g(\xi_t \in A, \zeta > t) = P_x \left(\exp - \int_0^t g(B_s) ds, B_t \in A, \tau_D > t \right).$$

Let $U^g f(x) = \int_0^\infty P_x^g(f(\xi_t) \mathbf{1}_{\{\zeta > t\}}) dt$ be the potential operator for L_g . If $g = 0$ we write U for U^g . If $0 \leq u$ is L_g -superharmonic, then the law of the u -transform of ξ is determined by the formula

$$P_x^{g,u}(\Phi(\xi) \mathbf{1}_{\{\zeta > t\}}) = \frac{1}{u(x)} P_x^g(\Phi(\xi) u(\xi_t) \mathbf{1}_{\{\zeta > t\}})$$

for $\Phi(\xi) \in \sigma\{\xi_s; s \leq t\}$. Assuming that $0 < u < \infty$ on D , this defines a diffusion on D . If u is L_g -harmonic, then it dies only upon reaching ∂D . In fact, the generator of the u -transform is

$$L_{g,u} f = \frac{1}{u} L_g(u f) = \frac{1}{2} \Delta f + \frac{1}{u} \nabla u \cdot \nabla f.$$

If $u = U^g f$ for some $f \geq 0$ (that is, if u is a potential) then the u -transform dies in the interior of D , and $P_x^{g,u}$ satisfies

$$P_x^{g,u}(\Phi(\xi)) = \frac{1}{u(x)} \int_0^\infty P_x^g(\Phi(\xi_{\leq t})f(\xi_t)\mathbf{1}_{\{\zeta > t\}}) dt,$$

where $\xi_{\leq t}$ is the process ξ killed at time t . See [2].

More generally, if $L_g u = -f$ then by breaking u into a potential plus a harmonic function one has that

$$(2.2) \quad P_x^{g,u}(\Phi(\xi)\mathbf{1}_{\{\zeta < \tau_D\}}) = \frac{1}{u(x)} \int_0^\infty P_x^g(\Phi(\xi_{\leq t})f(\xi_t)\mathbf{1}_{\{\zeta > t\}}) dt.$$

2.3. ψ super Brownian-motion. Let

$$(2.3) \quad \psi(\lambda) = a_1\lambda + a_2\lambda^2 + \int_0^\infty [e^{-\lambda r} - 1 + \lambda r] \pi(dr),$$

where $\lambda \geq 0, a_2 \geq 0$, and $\pi(\cdot)$ is the associated Lévy measure. We will assume the following:

$$(2.4) \quad a_1 \geq 0; \quad \int_0^\infty \min(r, r^2) \pi(dr) < \infty.$$

Lévy process exist under the weaker condition $\int_0^\infty \min(1, r^2) \pi(dr) < \infty$, and indeed there is a well known construction of continuous state branching processes (CSBP) as time-changes of Lévy processes. The stronger moment condition assumed above can be thought of as a condition for this CSBP (or equivalently, this time change) not to blow up in finite time. See [18]. It is easily seen that $\psi^{(n)}(\lambda) = \frac{d^n}{d\lambda^n} \psi(\lambda)$ exists for all n and $\lambda > 0$.

Let D be a domain in \mathbb{R}^d . Take ξ_t to denote Brownian motion on D , and let X_t be the ψ -super Brownian motion on \mathbb{R}^d . Let X^D be the associated exit measure on ∂D . In [12], Dynkin constructs this object, assuming $a_1 = 0$. But by Dawson's Girsanov theorem (see section 10.1.2 of [7]), applying Dynkin's construction to Brownian motion killed at rate a_1 yields precisely the desired ψ -super Brownian exit measure. Note that we are using the condition $a_1 \geq 0$ at this point. For $a_1 < 0$ and $\pi = 0$ one could in fact produce a superprocess that survives forever with positive probability, by taking D large enough (see [13]). Thus the exit measure could be infinite in that case.

\mathbb{N}_x will denote the excursion measure. LeGall used this measure extensively in the case $\psi(\lambda) = 2\lambda^2$ (see [21]). For the general case see Dynkin [12]. A key benefit to working under \mathbb{N}_x is that genealogies simplify – all mass descends from a single massless initial individual. The price one pays for this simplification is that \mathbb{N}_x is an infinite measure. Only events that involve extinction at short times receive infinite mass, so $\mathbb{N}_x(X^D \neq 0) < \infty$. It can be realized by having the superprocess start with initial value $\gamma\delta_x$ under a probability measure $\mathbb{P}_{\gamma\delta_x}$, sending $\gamma \downarrow 0$, and renormalizing $\mathbb{P}_{\gamma\delta_x}$ to obtain a non-trivial limit. Alternatively,

the superprocess with initial value μ (under a probability \mathbb{P}_μ) can be realized in terms of a Poisson random measure having $\int \mathbb{N}_x \mu(dx)$ as intensity. See section 3

Let $e_\phi^D = \exp -\langle X^D, \phi \rangle$. When convenient we will also denote this $e^D(\phi)$ or just $e(\phi)$. Let D_k be a sequence of smooth subdomains increasing to D and denote X^{D_k} by X^k and $e_\phi^{D_k}$ by e_ϕ^k . Set

$$(2.5) \quad \mathcal{N}_t(f) = \exp \left(- \int_0^t \psi'(\mathbb{N}_{\xi_s}(1-f)) ds \right).$$

Since $a_1 \geq 0$ we have $\psi'(\lambda) \geq 0$ when $\lambda \geq 0$, so in particular $\mathcal{N}_t(f) \leq 1$ for $f \geq 0$.

We will need the following results.

Lemma 2.2. *Assume (2.4) and let D be a domain in \mathbb{R}^d .*

- (a) *Let $\Gamma \subset \partial D$. Then $g(y) = \mathbb{N}_y(X^D(\Gamma) > 0)$ satisfies $\frac{1}{2}\Delta g = \psi(g)$.*
- (b) *Let g be a non-negative solution to $\frac{1}{2}\Delta g = \psi(g)$, and let D_k be an increasing sequence of smooth subdomains of D . Then for each k ,*

$$\mathbb{N}_x(1 - e_g^k) = g(x).$$

- (c) $\mathbb{N}_x(\langle X^D, \phi \rangle e_f^D) = E_x(\phi(\xi_{\tau_D}) \mathcal{N}_{\tau_D}(e_f^D))$ for $f, \phi \geq 0$.

Proof. Follows from Theorem 4.2.1 in [10], Theorem 1.1. in Chapter 4 of [12] and Theorem 11.7.1 in [7]. □

For $m \geq 2$ an integer, $y \in \mathbb{R}^d$, and measurable ϕ we define

$$(2.6) \quad b(m, \phi, y) = (-1)^m \psi^{(m)}(\mathbb{N}_y(1 - e_\phi)).$$

Note that for $\lambda > 0$

$$(2.7) \quad \psi^{(m)}(\lambda) = \begin{cases} a_1 + 2a_2\lambda + \int_0^\infty r[1 - e^{-\lambda r}] \pi(dr), & m = 1 \\ 2a_2 + \int_0^\infty r^2 e^{-\lambda r} \pi(dr), & m = 2 \\ (-1)^m \int_0^\infty r^m e^{-\lambda r} \pi(dr), & m \geq 3, \end{cases}$$

so in particular, $b(m, \phi, y)$ is well defined and non-negative, for $m \geq 2$ and for any $\phi \geq 0$ such that $\mathbb{N}_y(\langle X^D, \phi \rangle > 0) > 0$. It is decreasing in ϕ .

We will need the following assumption on ψ for certain results, which, among other things, makes the latter qualification unnecessary.

$$(A1) \quad \exists \lambda_0 > 0 \text{ such that } \int_1^\infty e^{r\lambda_0} \pi(dr) < \infty.$$

Under (A1) we have that $\int_0^\infty r^n \pi(dr) < \infty$ for $n \geq 2$, and by dominated convergence,

$$(2.8) \quad \psi(\lambda) = a_1 \lambda + a_2 \lambda^2 + \sum_{j=2}^{\infty} (-1)^j \frac{\lambda^j}{j!} \int_0^\infty r^j \pi(dr)$$

for $\lambda \leq \lambda_0$. Thus in this case, (2.7) will hold as well for $\lambda = 0$.

Recursive Palm formulae for moments have a long history, and in this context are due to Dynkin – see Theorem 1.1 of [12] or Theorem 4.1 of [11]. We work instead with a formulation along the lines of Lemma 2.6 in [29].

Lemma 2.3. *Assume (2.4) and let $N = \{1, 2, 3, \dots, n\}$, $n \geq 2$. Let $D \subset \mathbb{R}^d$ be a domain, $\phi \geq 0$, and let ξ be a Brownian motion in D with exit time τ . Let $\{v_i\}$ be a family of positive measurable functions. Then*

$$\mathbb{N}_x(e_\phi \prod_{i \in N} \langle X^D, v_i \rangle) = E_x \left(\sum_{\substack{\beta \in \mathcal{P}(N) \\ |\beta| \geq 2}} \int_0^\tau \mathcal{N}_t(e_\phi) b(|\beta|, \phi, \xi_t) \prod_{A \in \beta} \mathbb{N}_{\xi_t}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) dt \right).$$

Proof. Let $N^* = \{2, 3, \dots, N\}$.

$$\begin{aligned} \mathbb{N}_x(e_\phi \prod_{i \in N} \langle X^D, v_i \rangle) &= \mathbb{N}_x(\langle X^D, v_1 \rangle e_\phi \prod_{i \in N^*} \langle X^D, v_i \rangle) \\ &= (-1)^{n-1} \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial \lambda_3} \dots \frac{\partial}{\partial \lambda_n} \Big|_{\lambda_2 = \dots = \lambda_n = 0} \mathbb{N}_x(\langle X^D, v_1 \rangle e(\phi + \sum_{i=2}^n \lambda_i v_i)), \end{aligned}$$

where we differentiate under the integral sign using monotone convergence. Using (c) of Lemma 2.2 i.e. the one-dimensional case of the Palm formula, the above expression equals

$$(-1)^{n-1} \frac{\partial}{\partial \lambda_2} \frac{\partial}{\partial \lambda_3} \dots \frac{\partial}{\partial \lambda_n} \Big|_{\lambda_2 = \dots = \lambda_n = 0} E_x(v_1(\xi_\tau) \mathcal{N}_\tau(e(\phi + \sum_{i=2}^n \lambda_i v_i)))$$

Differentiating this expression under the integral sign (given the conditions on π this can be easily justified) and using the definition of b , this equals

$$\begin{aligned}
& E_x \left(v_1(\xi_\tau) \mathcal{N}_\tau(e_\phi) \sum_{\sigma \in \mathcal{P}(N^*)} \prod_{j=1}^{|\sigma|} \int_0^\tau \sum_{\beta_j \in \mathcal{P}(\sigma(j))} (-1)^{|\beta_j|+1} \psi^{|\beta_j|+1} \mathbb{N}_{\xi_{t_j}}(1 - e_\phi) \times \right. \\
& \qquad \qquad \qquad \left. \times \prod_{A \in \beta_j} \mathbb{N}_{\xi_{t_j}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) dt_j \right) \\
(2.9) \quad & = E_x \left(v_1(\xi_\tau) \mathcal{N}_\tau(e_\phi) \sum_{\sigma \in \mathcal{P}(N^*)} \prod_{j=1}^{|\sigma|} \int_0^\tau \sum_{\beta_j \in \mathcal{P}(\sigma(j))} b(|\beta_j| + 1, \phi, \xi_{t_j}) \times \right. \\
& \qquad \qquad \qquad \left. \times \prod_{A \in \beta_j} \mathbb{N}_{\xi_{t_j}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) dt_j \right).
\end{aligned}$$

Standard integration manipulations then show that the above equals

$$\begin{aligned}
& E_x \left(v_1(\xi_\tau) \mathcal{N}_\tau(e_\phi) \sum_{\sigma \in \mathcal{P}(N^*)} \sum_{k=1}^{|\sigma|} \int_0^\tau \sum_{\beta_k \in \mathcal{P}(\sigma(k))} b(|\beta_k| + 1, \phi, \xi_{t_k}) \prod_{A \in \beta_k} \mathbb{N}_{\xi_{t_k}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) \times \right. \\
& \qquad \qquad \qquad \left. \times \prod_{j \neq k} \left(\int_{t_k}^\tau \sum_{\beta_j \in \mathcal{P}(\sigma(j))} b(|\beta_j| + 1, \phi, \xi_{t_j}) \prod_{A \in \beta_j} \mathbb{N}_{\xi_{t_j}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) dt_j \right) dt_k \right) \\
& = E_x \left(\sum_{\sigma \in \mathcal{P}(N^*)} \sum_{k=1}^{|\sigma|} \int_0^\tau \sum_{\beta_k \in \mathcal{P}(\sigma(k))} b(|\beta_k| + 1, \phi, \xi_{t_k}) \prod_{A \in \beta_k} \mathbb{N}_{\xi_{t_k}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) \times \right. \\
& \qquad \qquad \qquad \times E_x \left(v_1(\xi_\tau) \mathcal{N}_\tau(e_\phi) \prod_{j \neq k} \int_{t_k}^\tau \sum_{\beta_j \in \mathcal{P}(\sigma(j))} b(|\beta_j| + 1, \phi, \xi_{t_j}) \times \right. \\
& \qquad \qquad \qquad \left. \left. \times \prod_{A \in \beta_j} \mathbb{N}_{\xi_{t_j}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) \mid \mathcal{F}_k dt_j \right) dt_k \right)
\end{aligned}$$

where \mathcal{F}_k denote the filtration of ξ_{t_k} . Applying the Markov Property at time t_k , this equals

$$E_x \left(\sum_{\sigma \in \mathcal{P}(N^*)} \sum_{k=1}^{|\sigma|} \int_0^\tau \mathcal{N}_{t_k}(e_\phi) \sum_{\beta_k \in \mathcal{P}(\sigma(k))} b(|\beta_k| + 1, \phi, \xi_{t_k}) \prod_{A \in \beta_k} \mathbb{N}_{\xi_{t_k}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) \times \right. \\ \left. \times E_{\xi_{t_k}} \left(v_1(\xi_\tau) \mathcal{N}_\tau(e_\phi) \prod_{j \neq k} \int_0^\tau \sum_{\beta_j \in \mathcal{P}(\sigma(j))} b(|\beta_j| + 1, \phi, \xi_{t_j}) \times \right. \right. \\ \left. \left. \times \prod_{A \in \beta_j} \mathbb{N}_{\xi_{t_j}}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) dt_j \right) dt_k \right).$$

Setting $M = \sigma(k)$ and summing over N^* this becomes

$$E_x \left(\sum_{\emptyset \neq M \subset N^*} \int_0^\tau \mathcal{N}_t(e_\phi) \sum_{\gamma \in \mathcal{P}(M)} b(|\gamma| + 1, \phi, \xi_t) \prod_{A \in \gamma} \mathbb{N}_{\xi_t}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) \times \right. \\ \left. \times \sum_{\beta \in \mathcal{P}(N^* \setminus M)} E_{\xi_t} \left(v_1(\xi_\tau) \mathcal{N}_\tau(e_\phi) \prod_{j=1}^{|\beta|} \int_0^\tau \sum_{\beta_j \in \mathcal{P}(\sigma(j))} b(|\beta_j| + 1, \phi, \xi_s) \times \right. \right. \\ \left. \left. \times \prod_{A \in \beta_j} \mathbb{N}_{\xi_s}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) ds \right) dt \right).$$

Using the identity obtained in (2.9), this equals

$$E_x \left(\sum_{\emptyset \neq M \subset N^*} \int_0^\tau \mathcal{N}_t(e_\phi) \sum_{\beta \in \mathcal{P}(M)} b(|\beta| + 1, \phi, \xi_t) \prod_{A \in \beta} \mathbb{N}_{\xi_t}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) \times \right. \\ \left. \times \mathbb{N}_{\xi_t}(e_\phi \prod_{i \in N \setminus M} \langle X^D, v_i \rangle) dt \right).$$

For any $\beta \in \mathcal{P}(N)$ with $|\beta| \geq 2$ we can realize a term of the above expression, letting $N \setminus M$ be the element of β containing 1, and letting γ be the restriction of β to M . Therefore

$$\mathbb{N}_x(e_\phi \prod_{i \in N} \langle X^D, v_i \rangle) = E_x \left(\sum_{\substack{\beta \in \mathcal{P}(N) \\ |\beta| \geq 2}} \int_0^\tau \mathcal{N}_t(e_\phi) b(|\beta|, \phi, \xi_t) \prod_{A \in \beta} \mathbb{N}_{\xi_t}(e_\phi \prod_{i \in A} \langle X^D, v_i \rangle) dt \right).$$

□

Lemma 2.4. *Assume (A1). Let D be a domain in \mathbb{R}^d satisfying $\sup_{x \in D} E_x(\tau_D) < \infty$, where τ_D is the exit time from D for Brownian motion. Then there exists $\lambda > 0$ such that*

$$\sup_{x \in D} \mathbb{N}_x(\exp(\lambda \langle X^D, 1 \rangle) - 1) < \infty.$$

Proof. Let $c_n = \sup_{x \in D} \mathbb{N}_x(\langle X^D, 1 \rangle^n)$. As $a_1 \geq 0$, we have $c_1 \leq 1$ by (c) of Lemma 2.2. By Lemma 2.3, with $\phi = 0$ we have the following recursion relation, for $n \geq 2$,

$$(2.10) \quad c_n \leq E_x \left(\sum_{\substack{\beta \in \mathcal{P}(N) \\ |\beta| \geq 2}} \int_0^\tau b(|\beta|, 0, \xi_t) \prod_{A \in \beta} c_{|A|} dt \right)$$

$$(2.11) \quad \leq K \sum_{\substack{\beta \in \mathcal{P}(N) \\ |\beta| \geq 2}} m_{|\beta|} \prod_{A \in \beta} c_{|A|}$$

$$(2.12) \quad = K \sum_{j=2}^n \frac{m_j}{j!} \sum_{\substack{i_1, i_2, \dots, i_j \geq 1 \\ i_1 + i_2 + \dots + i_j = n}} \frac{n!}{i_1! i_2! \dots i_j!} \prod_{i=1}^j c_{i_k}$$

where $K > 0$, $m_2 = 2a_2 + \int_0^\infty r^2 \pi(dr)$ and $m_k = \int_0^\infty r^k \pi(dr)$ for $k \geq 3$. For $N \geq 1$, let $g_N : [0, \infty) \rightarrow [0, \infty)$ given by

$$(2.13) \quad g_N(\lambda) = \sum_{n=1}^N \frac{c_n \lambda^n}{n!}, \quad \lambda > 0.$$

Using (2.10), by an inductive argument it is easy to see that $c_n < \infty$ for all $n \geq 2$, so g_N is well defined for all $N \geq 1$. Set $\tilde{\psi}(u) = \sum_{j=2}^\infty \frac{m_j}{j!} u^j = \psi(u) - a_1 u - a_2 u^2$ for $0 \leq u < \lambda_0$. Then using (2.10) again, for $\lambda > 0$

$$(2.14) \quad \begin{aligned} g_N(\lambda) &\leq \lambda + K \sum_{n=2}^N \lambda^n \sum_{j=2}^n \frac{m_j}{j!} \sum_{\substack{i_1, i_2, \dots, i_j \geq 1 \\ i_1 + i_2 + \dots + i_j = n}} \prod_{i=1}^j \frac{c_{i_k}}{i_k!} \\ &\leq \lambda + K \sum_{j=2}^N \frac{m_j}{j!} \sum_{1 \leq i_1, i_2, \dots, i_j \leq N} \prod_{i=1}^j \frac{c_{i_k} \lambda^{i_k}}{i_k!} \\ &= \lambda + K \sum_{j=2}^N \frac{m_j}{j!} (g_N(\lambda))^j \\ &\leq \lambda + K \tilde{\psi}(g_N(\lambda)), \end{aligned}$$

provided $g_N(\lambda) < \lambda_0$. By the assumptions on ψ , $\tilde{\psi}$ is infinitely differentiable on $[0, \lambda_0)$ with $\psi'' \geq 0$ and $\tilde{\psi}(0) = 0 = \tilde{\psi}'(0)$. So we may find $x_0 < \lambda_0$ sufficiently small that the line through $(x_0, \tilde{\psi}(x_0))$ with slope $1/K$ is secant to the graph of g_N . That is,

$$(2.15) \quad \tilde{\psi}(x) > \tilde{\psi}(x_0) + \frac{x - x_0}{K} \text{ for } x \in [0, x_0).$$

Let $\lambda_1 = x_0 - K \tilde{\psi}(x_0) < x_0 < \lambda_0$. Since

$$0 = \tilde{\psi}(0) > \tilde{\psi}(x_0) - \frac{x_0}{K} = -\frac{\lambda_1}{K},$$

we also have $\lambda_1 > 0$.

We claim that

$$(2.16) \quad g_N(\lambda) \leq x_0 \quad \forall \lambda \in [0, \lambda_1].$$

To see this, observe that g_N is continuous and strictly increasing, with $g_N(0) = 0$. So if (2.16) fails to hold, there will be a unique $\lambda < \lambda_1$ with $g_N(\lambda) = x_0$. But by (2.14),

$$x_0 = g_N(\lambda) \leq \lambda + K\tilde{\psi}(g_N(\lambda)) < \lambda_1 + K\tilde{\psi}(x_0) = x_0,$$

which is impossible.

Now just let $N \rightarrow \infty$ in (2.16) to complete the proof. \square

The authors are grateful to Amram Meir, who showed them how to construct this type of argument. For example, see [22].

2.4. Potentials. We will need certain results concerning potentials of specific partial differential equations. These potentials will be used to describe the exit measure conditioned to hit certain points on the boundary of D .

Let $N = \{1, 2, \dots, n\}$ be as before. Then for every non-empty subset $A \subset N$ let us suppose we are given a solution $u^A > 0$ to the equation $\frac{1}{2}\Delta u = \psi(u)$ in D . For convenience we also set $u^A = 0$ for $A = \emptyset$. Define

$$(2.17) \quad v_A = \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} u^B \quad \text{and} \quad v^A = \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} u^B.$$

We shall assume that

$$(2.18) \quad v_A \geq 0 \quad \text{for all } \emptyset \neq A \subset N.$$

The example to keep in mind is as follows: for $\Gamma_1, \dots, \Gamma_n \subset \partial D$, let

$$u^A(x) = \mathbb{N}_x(X^D \text{ charges } \bigcup_{i \in A} \Gamma_i),$$

$$v^A(x) = \mathbb{N}_x(X^D \text{ charges } \Gamma_i \text{ for every } i \in A), \text{ and}$$

$$v_A(x) = \mathbb{N}_x(X^D \text{ charges } \Gamma_i \text{ but not } \Gamma_j, \text{ for every } i \in A \text{ and } j \notin A).$$

(2.17) holds in this case, by a simple inclusion-exclusion argument (see also Lemma 5.1 in [29] and Section 4 in [30]).

Lemma 2.5. *Assume (2.4) and (2.18). Then*

$$(a) \quad u^A = \sum_{\substack{B \subset N \\ A \cap B \neq \emptyset}} v_B = \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} v^B \quad \text{and} \quad v^A = \sum_{A \subset B \subset N} v_B.$$

(b) For $A \neq \emptyset$,

$$(2.19) \quad \frac{1}{2}\Delta v_A - \psi'(u^N)v_A = - \sum_{j=2}^{\infty} \frac{b(j, u^N, \cdot)}{j!} \sum_{\substack{C_1 \cup C_2 \dots \cup C_j = A \\ C_i \neq \emptyset}} \prod_{i=1}^j v_{C_i}.$$

(c) Assume also (A1). The following then holds at any point where $u^A < \lambda_0$:

$$(2.20) \quad \frac{1}{2}\Delta v^A - \phi(u^A, v^A)v^A = - \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\substack{C_1 \cup C_2 \dots \cup C_j = A \\ \emptyset \neq C_i \neq A}} (-1)^{|A|} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|}$$

where $\phi(u^A, v^A) = \frac{\psi(u^A + (-1)^{|A|}v^A) - \psi(u^A)}{(-1)^{|A|}v^A} \geq 0$.

In particular, if $\int_1^{\infty} e^{r\lambda} \pi(dr) < \infty$ for every $\lambda > 0$ then (2.20) holds without restriction.

Note that all terms in the sum from (2.19) have the same sign, whereas those from (2.20) vary in sign. The extra conditions in part (c) arise because of the possibility of conditional convergence. Part (b) describes the functions we're primarily interested in, but part (c) will be useful for asymptotics.

Proof. The proof of first equality in part (a) is the same as that of (a) of Lemma 4.1 in [30]. As indicated in Remark 4.2 of [30], the proof of the second and third equality follow similarly.

We will show part (b) first.

Proof of (2.19) :

$$\begin{aligned} \frac{1}{2}\Delta v_A &= \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} \frac{1}{2}\Delta u^B \\ &= \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} \psi(u^B) \\ &= \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} \left(a_1 u^B + a_2 (u^B)^2 + \int_0^{\infty} (e^{-ru^B} + ru^B - 1) \pi(dr) \right) \\ &= a_1 v_A + 2a_2 u^N v_A - a_2 \sum_{\substack{C \cup C' = A \\ C, C' \neq \emptyset}} v_C v_{C'} + \\ &\quad + \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} \int_0^{\infty} (e^{-ru^B} + ru^B - 1) \pi(dr) \end{aligned}$$

In the last line we have used the definition of v_A and (b) of Lemma 4.1 of [30] (which will be recognized as the current lemma in the case $\psi(u) = 2u^2$.) Using the expression for

$\psi'(\cdot)$ from (2.7), and that of v_A we get that

$$\begin{aligned}
\frac{1}{2}\Delta v_A - \psi'(u^N)v_A &= - \int_0^\infty (-rv_A e^{-u^N r} + rv_A) \pi(dr) - a_2 \sum_{\substack{C \cup C' = A \\ C, C' \neq \emptyset}} v_C v_{C'} + \\
&\quad + \sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \int_0^\infty (e^{-ru^B} + ru^B - 1) \pi(dr) \\
(2.21) \quad &= -a_2 \sum_{\substack{C \cup C' = A \\ C, C' \neq \emptyset}} v_C v_{C'} + \sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \int_0^\infty (e^{-ru^B} + ru^B e^{-ru^N} - 1) \pi(dr).
\end{aligned}$$

Consider the last term of this expression. $A \neq \emptyset$ so by Lemma 2.1,

$$\sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \int_0^\infty (1 - ru^N e^{-ru^N} - e^{-ru^N}) \pi(dr) = 0.$$

Therefore

$$\begin{aligned}
&\sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \int_0^\infty (e^{-ru^B} + ru^B e^{-ru^N} - 1) \pi(dr) \\
&= \sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \int_0^\infty (e^{-ru^B} + ru^B e^{-ru^N} - ru^N e^{-ru^N} - e^{-ru^N}) \pi(dr) \\
&= \sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \int_0^\infty e^{-ru^N} (e^{r(u^N - u^B)} - r(u^N - u^B) - 1) \pi(dr) \\
&= \sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \int_0^\infty e^{-ru^N} \sum_{j=2}^\infty \frac{r^j}{j!} (u^N - u^B)^j \pi(dr).
\end{aligned}$$

By part (a), $u^N - u^B = \sum_{\substack{\emptyset \neq C \subset N \\ C \cap B = \emptyset}} v_C$. So by monotone convergence we have that the above

is

$$\begin{aligned}
&= \sum_{N \setminus ACBCN} (-1)^{|A|+|B|+n+1} \sum_{j=2}^\infty \int_0^\infty e^{-ru^N} \frac{r^j}{j!} \left(\sum_{\substack{\emptyset \neq C \subset N \\ C \cap B = \emptyset}} v_C \right)^j \pi(dr) \\
(2.22) \quad &= \sum_{j=2}^\infty \int_0^\infty e^{-ru^N} \frac{r^j}{j!} \sum_{\substack{N \setminus ACBCN \\ B \neq \emptyset}} (-1)^{|A|+|B|+n+1} \left(\sum_{\substack{\emptyset \neq C \subset N \\ C \cap B = \emptyset}} v_C \right)^j \pi(dr).
\end{aligned}$$

Using standard multinomial expansions and Lemma 2.1 we observe that j -th summand is,

$$\begin{aligned}
&= \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} \left(\sum_{\substack{\emptyset \neq C \subset N \\ C \cap B = \emptyset}} v_C \right)^j \\
&= \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} \sum_{\substack{C_1, C_2, \dots, C_j \\ \emptyset \neq C_i \subset N \setminus B}} \prod_{i=1}^j v_{C_i} \\
&= \sum_{\substack{C_1, C_2, \dots, C_j \\ \emptyset \neq C_i \subset N}} \left(\prod_{i=1}^j v_{C_i} \right) \sum_{N \setminus A \subset B \subset N \setminus \cup_i^j C_i} (-1)^{|A|+|B|+n+1} \\
&= \sum_{\substack{C_1, C_2, \dots, C_j \\ \emptyset \neq C_i \subset N}} \left(\prod_{i=1}^j v_{C_i} \right) (-1)^{|A|+n+1} (-1)^{n-|A|} 1_{A = \cup_i^j C_i} \\
&= - \sum_{\substack{C_1 \cup C_2 \dots \cup C_j = A \\ C_i \neq \emptyset}} \prod_{i=1}^j v_{C_i}.
\end{aligned}$$

Together with (2.21) and (2.22) this implies (2.19). It is clear from the proof that the sum in (2.19) converges.

Proof of (2.20) :

$$\begin{aligned}
\frac{1}{2} \Delta v^A &= \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} \frac{1}{2} \Delta u^B \\
&= \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} \psi(u^B).
\end{aligned}$$

Observe that $u^B \leq u^A$ by part (a) and (2.18). So by (2.8), we can expand ψ in a series, at any point where $u^A < \lambda_0$. Therefore

$$\begin{aligned}
\frac{1}{2} \Delta v^A &= \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} \left(a_1 u^B + \sum_{j=2}^{\infty} \frac{\psi^{(j)}(0)}{j!} (u^B)^j \right) \\
(2.23) \quad &= a_1 v^A + \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} (-u^B)^j.
\end{aligned}$$

For $j \geq 2$, we have by part (a) that

$$\begin{aligned}
& \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} (-u^B)^j \\
&= \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} \left(\sum_{\emptyset \neq C \subset B} v^C (-1)^{|C|} \right)^j \\
&= \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} \sum_{\emptyset \neq C_1, C_2, \dots, C_j \subset B} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} \\
&= (-1)^{|A|+1} \sum_{\emptyset \neq C_1, C_2, \dots, C_j \subset A} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} \mathbf{1}(A = \cup_{i=1}^j C_i) \\
&\quad (\text{using Lemma 2.1}) \\
&= (-1)^{|A|+1} \sum_{\substack{A = \cup_{i=1}^j C_i \\ \emptyset \neq C_1, C_2, \dots, C_j \neq A}} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} \\
&\quad + (-1)^{|A|+1} \sum_{k=1}^j \binom{j}{k} ((-1)^{|A|} v^A)^k \left(\sum_{\emptyset \neq C_1, C_2, \dots, C_{j-k} \neq A} \prod_{i=1}^{j-k} v^{C_i} (-1)^{|C_i|} \right) \\
&= (-1)^{|A|+1} \sum_{\substack{A = \cup_{i=1}^j C_i \\ \emptyset \neq C_1, C_2, \dots, C_j \neq A}} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} \\
&\quad + (-1)^{|A|+1} \sum_{k=1}^j \binom{j}{k} ((-1)^{|A|} v^A)^k (-u^A + -(-1)^{|A|} v^A)^{j-k} \\
&= (-1)^{|A|+1} \sum_{\substack{A = \cup_{i=1}^j C_i \\ \emptyset \neq C_1, C_2, \dots, C_j \neq A}} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} + (-1)^{|A|+1} (-u^A)^j - (-u^A + -(-1)^{|A|} v^A)^j. \\
&= \sum_{\emptyset \neq B \subset A} (-1)^{|B|+1} \left(\sum_{\emptyset \neq C \subset B, C \neq B} v^C (-1)^{|C|} \right)^j + (-1)^{|A|+1} (-u^A)^j - (-u^A + -(-1)^{|A|} v^A)^j.
\end{aligned}
\tag{2.24}$$

Using (2.23) and (2.24) we have

$$\begin{aligned}
\frac{1}{2}\Delta v^A &= a_1 v^A + (-1)^{|A|+1} \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\substack{A=\cup_{i=1}^j C_i \\ \emptyset \neq C_1, C_2, \dots, C_j \neq A}} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} + \\
&\quad + (-1)^{|A|+1} \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} (-u^A)^j - (-u^A + -(-1)^{|A|} v^A)^j \\
&= a_1 v^A + (-1)^{|A|+1} \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\substack{A=\cup_{i=1}^j C_i \\ \emptyset \neq C_1, C_2, \dots, C_j \neq A}} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} + \\
&\quad + (-1)^{|A|+1} \int_0^{\infty} (e^{-ru^A} - e^{-r(u^A + (-1)^{|A|} v^A)}) \pi(dr) \\
&= -(-1)^{|A|} \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} (-1)^{|A|+1} \sum_{\substack{A=\cup_{i=1}^j C_i \\ \emptyset \neq C_1, C_2, \dots, C_j \neq A}} \prod_{i=1}^j v^{C_i} (-1)^{|C_i|} + \\
&\quad + (-1)^{|A|} (\psi(u^A + (-1)^{|A|} v^A) - \psi(u^A))
\end{aligned}$$

which gives (2.20). Positivity of ϕ follows from monotonicity of ψ' . \square

3. THE BRANCHING PARTICLE DESCRIPTION

In this section we shall define a martingale change of measure which will represent various conditionings of the exit measure of ψ super-Brownian motion. For each such conditioning we shall also present a branching backbone representation, in which the conditioned process is realized as an unconditioned superprocess with immigration of mass along a branching tree.

Let $\emptyset \neq A \subset N = \{1, 2, \dots, n\}$ and let $u^A \geq 0$ and $v_A \geq 0$ be as in Section 2.4. Specific examples will be described later, but even at this level of generality we can use these functions to define an associated martingale. Define

$$(3.1) \quad \check{M}_k = \exp(-\langle X^k, u^N \rangle) \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\substack{C_1 \cup \dots \cup C_m = N \\ C_i \neq \emptyset}} \prod_{i=1}^m \langle X^k, v_{C_i} \rangle \geq 0.$$

In Lemma 4.3 of [30], it is shown that this sum is finite, and

$$\check{M}_k = \sum_{A \subset N} (-1)^{|A|} \exp(-\langle X^k, u^A \rangle)$$

(note that the $A = \emptyset$ term equals 1, by our convention that $u^\emptyset = 0$).

From this or otherwise it can be easily checked that \check{M}_k is a martingale under \mathbb{N}_x , with respect to the filtration \mathcal{F}_k . So for Φ measurable with respect to \mathcal{F}_k let $\check{\mathbb{M}}_x = \frac{1}{v_N(x)} \mathbb{N}_x(\Phi \check{M}_k)$ be the associated Girsanov transformation – that is, the h -transform, for the “harmonic” function

$$h(\mu) = \sum_{A \subset N} (-1)^{|A|} \exp(-\langle \mu, u^A \rangle).$$

Dynkin has developed a general framework for such h -transforms of superprocesses, which he calls *X-harmonic functions*. See [11].

Branching backbone : We describe the direct construction of $\check{\mathbb{M}}_x$ in terms of a branching particle system tracing out a backbone along which mass gets created. Let $\check{L}f = L_{\psi' \circ u^N} f = \frac{1}{2} \Delta f - \psi'(u^N) f$. Essentially we will constructively generate a measure $\check{\mathbb{N}}_x$ and then show that it agrees with $\check{\mathbb{M}}_x$. $\check{\mathbb{N}}_x$ will have a branching backbone Υ equipped with mass creation/immigration both along the branches and at the nodes. Each branch will be labeled by a nonempty subset A of N . The mass created along the backbone will evolve as an unconditioned superprocess but with a modified killing rate and Lévy measure. The first order of business is to give a more precise description of the various components of this process.

Evolution of Mass: Once mass has been created, it evolves as an unconditioned super-Brownian motion but with a modified branching law. More formally, for Φ measurable with respect to \mathcal{F}_k , let $\check{\mathbb{N}}_y(\Phi) = \mathbb{N}_y(\Phi e_{u^N}^k)$. We will show that this measure indeed describes a superprocess.

For any $\lambda > 0$ and $y \in D$ define $\check{\psi}(y, \lambda) = \psi(u^N(y) + \lambda) - \psi(u^N(y))$. It is easily checked that

$$\check{\psi}(y, \lambda) = \check{a}_1(y) \lambda + a_2 \lambda^2 + \int_0^\infty e^{-u^N(y)} (e^{-\lambda r} - 1 + \lambda r) \pi(dr)$$

where $\check{a}_1(y) = a_1 + 2a_2 u^N(y) + \int_0^\infty r(1 - e^{-ru^N(y)}) \pi(dr)$. This is therefore a spatially varying branching law, of the form (2.3). But now in places where u^N is large, we will have that the killing rate \check{a}_1 becomes large too (provided $a_2 \neq 0$), and the Lévy branching measure $\pi(dr) e^{-ru^N}$ becomes small.

The following is a special case of Dawson’s Girsanov formula (see Theorem 7.2.2. in [7]),

Lemma 3.1. *Assume (2.4). Let $\check{\mathbb{N}}_x$ be the excursion law for super-Brownian motion in D with branching mechanism $\check{\psi}$. Then for every $\phi \geq 0$,*

$$\check{\mathbb{N}}_x(1 - e^{-\langle X^k, \phi \rangle}) = \check{\mathbb{N}}_x(1 - e^{-\langle X^k, \phi \rangle}).$$

Proof. Let g be the solution to

$$\begin{aligned}\frac{1}{2}\Delta g(y) &= \check{\psi}(y, g(y)) \text{ for } y \in D_k \\ g &= \phi \text{ on } \partial D_k.\end{aligned}$$

Then $f = g + u^N$ is the solution to

$$\begin{aligned}\frac{1}{2}\Delta f &= \psi(f) \text{ in } D_k \\ f &= \phi + u^N \text{ on } \partial D_k,\end{aligned}$$

so

$$\begin{aligned}\tilde{\mathbb{N}}_x(1 - e^{-\langle X^k, \phi \rangle}) &= \mathbb{N}_x\left((1 - e^{-\langle X^k, \phi \rangle})e^{-\langle X^k, u^N \rangle}\right) \\ &= \mathbb{N}_x(1 - e^{-\langle X^k, \phi + u^N \rangle}) - \mathbb{N}_x(1 - e^{-\langle X^k, u^N \rangle}) \\ &= \left(g(x) + u^N(x)\right) - u^N(x) = g(x) = \tilde{\mathbb{N}}_x(1 - \exp -\langle X^k, \phi \rangle).\end{aligned}$$

This suffices. \square

Υ Backbone: Under $\check{\mathbb{N}}_x$ we begin with one particle in the system that performs a v_N transform of the motion with generator \check{L} in D . Note that by Lemma 2.5, v_N is \check{L} -superharmonic (as indeed are all the v_A). Let $-V_A(\cdot)$ denote the right hand side of (b) of Lemma 2.5.

If the particle dies at some site y , a random number of particles are born in its place, and numbered $1, 2, \dots, j$. The first particle is then assigned a randomly generated tag A_1 , the second is tagged with A_2 , etc. For $j \geq 2$, the probability that $j \geq 2$ particles are born and that their tags are a specified sequence A_1, \dots, A_j is given by:

$$(3.2) \quad \frac{1}{V_N(y) \cdot j!} b(j, u^N(y), y) \prod_{i=1}^j v_{A_i}(y).$$

Here the A_i are chosen so that $A_i \neq \emptyset$ and $\cup_{i=1}^j A_i = N$. In full generality, particles need not die before exiting D , but in our main example below they will in fact always die in the interior of D (and the v_A will accordingly be \check{L} -potentials). Note that the above defines a probability measure, by definition of V_N .

Each of the particles so created now evolves as a v_A transform of the motion with generator \check{L} (where A is the particle's label), until it dies. Whereupon a new collection of random branches is created, each labeled by an $A'_i \neq \emptyset$ with $\cup_i A'_i = A$, etc.

It will be convenient to describe this backbone as follows. Let n_t denote the number of particles alive at time t . Label them using an index i , $1 \leq i \leq n_t$, and for each one let $x_i(s)$, $0 \leq s \leq t$ denote the location of the particle or its ancestor at time s . Let Υ_t^k be

the measure putting unit mass at each point $x_i(t)$ for which x_i has not exited D_k by time t . Then Υ^k represents the backbone killed upon leaving D_k , and we recover the whole backbone by letting $k \rightarrow \infty$. Without comment we will feel free to refer to Υ^k in terms of the underlying particles, though formally it is still a measure-valued process.

Immigration at nodes: The birth of j particles at y is accompanied by site-dependent creation of mass. Conditional on the location being y and the number of particles being $j \geq 2$, the mass created is a random variable $R_j \geq 0$ whose law $\mu_{j,y}(dr)$ is given for $r > 0$ by

$$\mu_{j,y}(dr) = \begin{cases} \frac{r^j e^{-ru^N(y)} \pi(dr)}{\int_0^\infty r^j e^{-ru^N(y)} \pi(dr)}, & j \geq 3 \\ \frac{r^2 e^{-ru^N(y)} \pi(dr)}{2a_2 + \int_0^\infty r^2 e^{-ru^N(y)} \pi(dr)}, & j = 2. \end{cases}$$

In the case $j = 2$, $\mu_{j,y}$ also has an atom at $r = 0$ of size $2a_2/[2a_2 + \int_0^\infty r^2 e^{-ru^N(y)} \pi(dr)]$. In other words, if $a_2 > 0$ then it is possible for no mass to be created at a branch with $j = 2$. In any case, all mass created at the node then evolves according to $\tilde{\mathbb{N}}_y$.

Temporarily writing $Y_{j,y}^k$ for the exit measure from D_k resulting from such a creation of mass, and $E_{j,y}$ for expectations under this conditional law, we can write $Y_{j,y}^k$ as $\int \nu N(d\nu)$ where N is a Poisson random measure with intensity $n(d\nu) = R_j \tilde{\mathbb{N}}_y(X^k \in d\nu)$. Thus

$$\begin{aligned} (3.3) \quad E_{j,y}[e^{-\langle Y_{j,y}^k, \phi \rangle}] &= E_{j,y}[e^{-\int \langle \nu, \phi \rangle N(d\nu)}] = E_{j,y}[e^{-\int (1 - e^{-\langle \nu, \phi \rangle}) n(d\nu)}] \\ &= E_{j,y}[e^{-R_j \tilde{\mathbb{N}}_y(1 - e^{-\langle X^k, \phi \rangle})}] = \int_{[0, \infty)} e^{-r \tilde{\mathbb{N}}_y(1 - e^{-\langle X^k, \phi \rangle})} \mu_{j,y}(dr) \\ &= \frac{b(j, u^N(y) + \tilde{\mathbb{N}}_y(1 - e^{-\langle X^k, \phi \rangle}), y)}{b(j, u^N(y), y)}. \end{aligned}$$

Denote this quantity by $M^k(j, \phi, y)$.

Immigration along branches:

For any $\lambda > 0$ and $y \in D$ define

$$\begin{aligned} \eta(y, \lambda) &= \psi'(u^N(y) + \lambda) - \psi'(u^N(y)) \\ &= 2a_2\lambda + \int_0^\infty r e^{-ru^N(y)} (1 - e^{-\lambda r}) \pi(dr). \end{aligned}$$

Notice that this has the same form as ψ' in (2.7) but with $a_1 = 0$ and a spatially varying $\pi_y(dr) = e^{-ru^N(y)} \pi(dr)$.

We create mass along the branches. The mass created to time t forms a spatially dependent Lévy process, with Lévy exponent η . In other words, if L_t is the mass created until

time t , then

$$\check{\mathbb{N}}_x(e^{-\lambda L_t}) = \check{\mathbb{N}}_x(e^{-\int_0^t \int \eta(y, \lambda) \Upsilon_s^k(dy) ds}).$$

This mass then evolves according to $\check{\mathbb{N}}_y$. In particular, if τ denotes the first branch time and ξ_t the position along the first branch to time t then

$$\check{\mathbb{N}}_x(e^{-\lambda L_\tau}) = \check{\mathbb{N}}_x(e^{-\int_0^\tau \eta(\xi_t, \lambda) dt}).$$

If Y_{Br}^k denotes the exit measure of the mass created along this branch till time τ then a calculation similar to (3.3) shows that

$$(3.4) \quad \begin{aligned} \check{\mathbb{N}}_x(e^{-\langle Y_{\text{Br}}^k, \phi \rangle}) &= \check{\mathbb{N}}_x(e^{-\int_0^\tau \check{\mathbb{N}}_{\xi_t}(1 - e^{-\langle X^k, \phi \rangle}) dL_t}) \\ &= \check{\mathbb{N}}_x(e^{-\int_0^\tau \eta(\xi_t, \check{\mathbb{N}}_{\xi_t}(1 - e^{-\langle X^k, \phi \rangle})) dt}). \end{aligned}$$

In other words, we obtain an expression similar to that of (2.5).

With this, we've finished describing the construction of an exit measure Y^k under a probability measure $\check{\mathbb{N}}_x$, and are ready for the following:

Theorem 3.1. *Assume conditions (2.4) and (2.18). Then*

$$\check{\mathbb{M}}_x(\exp -\langle X^k, \phi \rangle) = \check{\mathbb{N}}_x(\exp -\langle Y^k, \phi \rangle).$$

Note that this shows that, under the two measures $\check{\mathbb{M}}_x$ and $\check{\mathbb{N}}_x$, the exit measures from each D_k have distributions which agree. Using historical processes (as in [29]) one can show that the same result carries over to the process level. That is, that the laws of the full superprocesses agree under these two measures.

Proof. In the present context, it is useful to label all the particles of Υ^k that exit D_k , by placing an order on them. So let F_k be the set of such particles, and set $\gamma_k = |F_k|$. For $A \subset N$, let

$$\mathcal{S}_m(A) = \{(C_1, \dots, C_m) : C_1 \cup \dots \cup C_m = A, \emptyset \neq C_i \forall i\}.$$

If $\gamma_k = m$, choose at random an ordering of F_k , and for $\Lambda = (C_1, \dots, C_m) \in \mathcal{S}_m(N)$, write $\Upsilon^k \approx \Lambda$ for the event that the i th particle is tagged with the set C_i , $i = 1, \dots, m$. Thus for example,

$$(3.5) \quad \check{\mathbb{N}}_x(\gamma_k = m) = \sum_{\Lambda \in \mathcal{S}_m(N)} \check{\mathbb{N}}_x(\Upsilon^k \approx \Lambda).$$

Note that if M_1, \dots, M_j are disjoint ordered sets, with $|M_i| = k_i$ and $m = \sum k_i$ then there are $\frac{m!}{k_1!k_2!\dots k_j!}$ orderings of $S = \bigcup M_i$ which are compatible with the given orders on each M_i . In other words, if σ is any order on S , and if Σ is an order on S picked at random, then the conditional probability

$$(3.6) \quad P(\Sigma = \sigma \mid \Sigma_{M_i} = \sigma_{M_i}, i = 1, \dots, j) = \frac{k_1!k_2!\dots k_j!}{m!}$$

(writing σ_M etc... for the restriction of σ to M).

As described initially, the root particle of the tree is always a v^N -particle. It is convenient, for purposes of induction, to allow the same notation to cover the situation that we start with our root being a v^A -particle for some $A \subset N$. In this case, (3.5) still holds, but with $\Lambda \in \mathcal{S}_m(N)$ replaced by $\Lambda \in \mathcal{S}_m(A)$. With this in mind, we may define another restriction operation as follows. For $1 \leq i_1 < \dots < i_k \leq m$, set

$$(C_1, \dots, C_m)|_{\{i_1, \dots, i_k\}} = (C_{i_1}, \dots, C_{i_k}).$$

Thus, if $\Lambda = (C_1, \dots, C_m) \in \mathcal{S}_m(A)$ and $M \subset \{1, \dots, m\}$, we will have that $\Lambda|_M \in \mathcal{S}_m(B)$, for $B = \cup_{i \in M} C_i$. As a shorthand for the latter, we write $\Lambda(M) = \cup_{i \in M} C_i$.

The case of interest is that the first branch of Υ^k partitions $\{1, \dots, m\}$ via the descent relation. If β is this partition then there are $|\beta|$ particles born at this branch, and

$$(3.7) \quad |\beta|! \text{ ways of tagging the 1st particle, 2nd particle, etc.}$$

with distinct elements of β .

We will show, by induction on $m \geq 1$, that for $\emptyset \neq A \subset N$, and $(C_1, \dots, C_m) \in \mathcal{S}_m(A)$,

$$(3.8) \quad \begin{aligned} \check{\mathbb{N}}_x(\exp -\langle Y^k, \phi \rangle > \Upsilon^k &\approx (C_1, \dots, C_m)) \\ &= \frac{1}{m!v_A(x)} \mathbb{N}_x(e_{\phi+u^N}^k \prod_{i=1}^m \langle X^k, v_{C_i} \rangle). \end{aligned}$$

Taking $A = N$ and summing over $\mathcal{S}_m(N)$ will then establish the theorem.

We start with the case $m = 1$. Here Υ^k will have a single v_N - process with lifetime $\zeta \geq \tau_k$. Therefore,

$$\begin{aligned} &\check{\mathbb{N}}_x(\exp -\langle Y^k, \phi \rangle; \Upsilon^k \approx N) \\ &= E_x^{\psi' \circ u^N, v_N} \left[\mathbf{1}_{\zeta < \tau_k} e^{-\int_0^\zeta \eta(\xi_s, \check{\mathbb{N}}_{\xi_s}(1-e_\phi^k)) ds} \right] \\ &= \frac{1}{v_N(x)} E_x^{\psi' \circ u^N} \left[v_N(\xi_{\tau_k}) \mathbf{1}_{\zeta < \tau_k} e^{-\int_0^\zeta \eta(\xi_s, \check{\mathbb{N}}_{\xi_s}(1-e_\phi^k)) ds} \right] \\ &= \frac{1}{v_N(x)} E_x \left[v_N(\xi_{\tau_k}) e^{-\int_0^{\tau_k} \psi' \circ u^N(\xi_s) ds} e^{-\int_0^{\tau_k} \eta(\xi_s, \check{\mathbb{N}}_{\xi_s}(1-e_\phi^k)) ds} \right] \\ &= \frac{1}{v_N(x)} E_x \left[v_N(\xi_{\tau_k}) \mathcal{N}_{\tau_k}(e_{\phi+u^N}^k) \right] \\ &= \frac{1}{v_N(x)} \mathbb{N}_x(e_{\phi+u^N}^k \langle X^k, v_N \rangle), \end{aligned}$$

where the second last equality follows from definition of η and \mathcal{N} , and the last equality follows from Lemma 2.2 (c).

Turning to the inductive step, let $m > 1$ and assume the inductive hypothesis for all $A \subset N$, and for all values smaller than m . For simplicity, we will verify (3.8) in the case

$A = N$. For ζ the lifetime of the initial particle, and $\Lambda = (C_1, \dots, C_m)$, we have by (3.4), (3.3), (3.7), (3.2), (3.6), and the definition of Υ , that

$$\begin{aligned} & \check{N}_x(e^{-\langle Y^k, \phi \rangle}, \Upsilon^k \approx \Lambda) \\ &= \sum_{\substack{\beta \in \mathcal{P}(\{1, \dots, m\}) \\ |\beta| \geq 2}} E_x^{\psi' \circ u^N, v_N} \left[1_{\zeta < \tau_k} e^{-\int_0^\zeta \eta(\xi_s, \check{N}_{\xi_s}(1-e_\phi^k)) ds} \times \right. \\ & \quad \times M(|\beta|, \phi, \xi_\zeta) \times |\beta|! \times \frac{b(|\beta|, u^N, \xi_\zeta) \prod_{A \in \beta} v_{\Lambda(A)}(\xi_\zeta)}{V_N |\beta|!} \times \\ & \quad \left. \times \frac{\prod_{A \in \beta} |A|!}{m!} \times \prod_{A \in \beta} \check{N}_{\xi_\zeta}(e^{-\langle Y^k, \phi \rangle}; \Upsilon^k \approx \Lambda|_A) \right]. \end{aligned}$$

By (2.2) and (2.1) this equals

$$\begin{aligned} & \frac{1}{m! v_N(x)} E_x^{\psi' \circ u^N} \left(\sum_{\substack{\beta \in \mathcal{P}(\{1, \dots, m\}) \\ |\beta| \geq 2}} \int_0^{\tau_k} e^{-\int_0^t \eta(\xi_s, \check{N}_{\xi_s}(1-e_\phi^k)) ds} \times \right. \\ & \quad \left. \times b(|\beta|, \phi + u^N, \xi_t) \times \prod_{A \in \beta} |A|! v_{\Lambda(A)}(\xi_t) \check{N}_{\xi_t}(e^{-\langle Y^k, \phi \rangle}; \Upsilon^k \approx \Lambda|_A) \right) dt \\ &= \frac{1}{m! v_N(x)} E_x \left(\sum_{\substack{\beta \in \mathcal{P}(\{1, \dots, m\}) \\ |\beta| \geq 2}} \int_0^{\tau_k} e^{-\int_0^t [\psi'(u^N(\xi_s)) + \eta(\xi_s, \check{N}_{\xi_s}(1-e_\phi^k))] ds} \times \right. \\ & \quad \left. \times b(|\beta|, \phi + u^N, \xi_t) \prod_{A \in \beta} |A|! v_{\Lambda(A)}(\xi_t) \check{N}_{\xi_t}(e^{-\langle Y^k, \phi \rangle}; \Upsilon^k \approx \Lambda|_A) \right) dt. \end{aligned}$$

By definition of η this is

$$\begin{aligned} & \frac{1}{m! v_N(x)} E_x \left(\sum_{\substack{\beta \in \mathcal{P}(\{1, \dots, m\}) \\ |\beta| \geq 2}} \int_0^{\tau_k} \mathcal{N}_t(e_{\phi+u^N}^k) b(|\beta|, \phi + u^N, \xi_t) \times \right. \\ & \quad \left. \times \prod_{A \in \beta} |A|! v_{\Lambda(A)}(\xi_t) \check{N}_{\xi_t}(e^{-\langle Y^k, \phi \rangle}; \Upsilon^k \approx \Lambda|_A) \right) dt, \end{aligned}$$

which by induction equals

$$\frac{1}{m!v_N(x)} E_x \left(\sum_{\substack{\beta \in \mathcal{P}(\{1, \dots, m\}) \\ |\beta| \geq 2}} \int_0^{\tau_k} \mathcal{N}_t(e_{\phi+u^N}^k) b(|\beta|, \phi + u^N, \xi_t) \times \right. \\ \left. \times \prod_{A \in \beta} \mathbb{N}_{\xi_t}(e_{\phi+u^N}^k \prod_{i \in A} \langle X^k, v_{C_i} \rangle) \right) dt.$$

By Lemma 2.3 we conclude that

$$\check{\mathbb{N}}_x(e^{-\langle Y^k, \phi \rangle}, \Upsilon^k \approx \Lambda) = \frac{1}{m!v_N(x)} \mathbb{N}_x(e_{\phi+u^N}^k \prod_{i=1}^m \langle X^k, v_{C_i} \rangle).$$

□

4. CONDITIONING THE EXIT MEASURE TO HIT n POINTS

In this section we shall consider the exit measure when it is conditioned to give positive mass to n small balls on the boundary of D . We shall study the limit as the radius of these small balls tends to 0. In the first part, we set notation. In the second part, we discuss the case when ψ is analytic. Here the limit is given by a martingale change of measure as in the previous section. Then we describe an “explosion” phenomenon when $n = 2$ and $\psi(\lambda) = \lambda^{1+\beta}$, $0 \leq \beta \leq 1$. Finally we discuss instability in the limiting process by establishing a range of possible limits. We begin by fixing some notation for this section.

4.1. Conditioning to hit n small balls. Let D be a C^2 -bounded domain in \mathbb{R}^d , for $d \geq 4$, $N = \{1, 2, \dots, n\}$ for $n \in \mathbb{N}$, and let $\{z_1, z_2, \dots, z_n\}$ be distinct points on the boundary of D . Let U be the potential operator for Brownian motion killed upon hitting the boundary of D . For $x \in D$ let $K_x^D(\cdot, \cdot)$ be the Martin Kernel on D and let $G_D(\cdot, \cdot)$ be the Green function. For any $z \in \partial D$ and $\epsilon > 0$ we define $\Delta(z, \epsilon) = B(z, \epsilon) \cap \partial D$ where $B(z, \epsilon)$ is the Euclidean ball of radius ϵ in \mathbb{R}^d .

For any $A \subset N$, set

$$B_\epsilon^A = \cup_{i \in A} B_\epsilon^i \equiv \cup_{i \in A} B(z_i, \epsilon) \text{ and } \Delta_\epsilon^A = \cup_{i \in A} \Delta_\epsilon^i \equiv \cup_{i \in A} \Delta(z_i, \epsilon).$$

Fix $0 < \delta_0 < 1$ such that $B(z_i, \delta_0) \cap B(z_j, \delta_0) = \emptyset$ if $i \neq j \in N$ and for any $\epsilon > 0$, set

$$D_\epsilon^A = D \setminus \overline{\cup_{i \in A} B(z_i, \epsilon)} \text{ and } \tau_\epsilon = \tau_\epsilon^A = \tau_{D_\epsilon^A}.$$

For any $x \in D$, define the functions:

$$u_\epsilon^A(x) = \mathbb{N}_x \left(\sum_{i \in A} \langle X^D, 1_{\Delta(z_i, \epsilon)} \rangle > 0 \right),$$

$$v_\epsilon^A(x) = \mathbb{N}_x \left(\prod_{i \in A} \langle X^D, 1_{\Delta(z_i, \epsilon)} \rangle > 0 \right),$$

and

$$v_{A, \epsilon}(x) = \mathbb{N}_x(X^D \text{ charges all } \Delta_\epsilon^i \text{ for } i \in A \text{ and does not charge } \Delta_\epsilon^j \text{ for } j \notin A).$$

It is easy to see by an exclusion-inclusion argument that

$$(4.1) \quad \begin{aligned} v_{A, \epsilon} &= \sum_{N \setminus A \subset B \subset N} (-1)^{|A|+|B|+n+1} u_\epsilon^B, \\ v_\epsilon^A &= - \sum_{\emptyset \neq B \subset A} (-1)^{|B|} u_\epsilon^B, \\ u_\epsilon^A &= - \sum_{\emptyset \neq B \subset A} (-1)^{|B|} v_\epsilon^B. \end{aligned}$$

Thus (2.17) and (2.18) hold. By (a) of Lemma 2.2, $\frac{1}{2}\Delta u_\epsilon^A = \psi(u_\epsilon^A)$.

We will need the following two Lemmas in section 4.2.

Lemma 4.1. *Let D be a bounded C^2 domain in dimension $d \geq 4$. Let B be d -dimensional Brownian motion started from $x \in D$, under a probability measure P_x . For $y \in D$ let m_y^D be the harmonic measure starting from w . Let $z \in \partial D$ and $z_0 \in \partial D$. Then for $x \in D$ fixed, \exists constants $C_k, k = 1, 2, 3, 4$ such that*

$$(a) \quad P_y(B_{\tau_{2\epsilon}} \in \partial B(z, 2\epsilon)) \leq C_1 m_y^D(\Delta(z, \epsilon)) \text{ for } y \in D \setminus B(z, 4\epsilon),$$

$$(b) \quad \int_D G_D(x, y) m_y^D(\Delta(z, \epsilon))^2 dy \leq C_2 \epsilon^2 m_x^D(\Delta(z, \epsilon)),$$

$$(c) \quad K_x^D(y, z) \leq C_3 |y - z|^{-d} \text{dist}(y, \partial D), \text{ and}$$

$$(d) \quad m_x(\Delta(z, \epsilon)) \geq C_4 \epsilon^{d-1}.$$

Proof. This uses the proof of Lemma 5.4 in [29], which in turn uses an argument from [1]. More specifically, (a) follows from (5.6) in [29] and (b) follows from (5.8) in [29]. We note that in [29] the domain D is assumed to be bounded Lipschitz domain which includes the class of C^2 -bounded domains. Part (c) follows from Lemma 2.1 in [1]. Part (d) is just the fact that the density of harmonic measure with respect to surface area is bounded away from 0. \square

Set $\rho = (d - 3)/(d - 1)$, and note that $0 < \rho < 1$ since $d \geq 4$.

Lemma 4.2. *Let D be a bounded C^2 domain in dimension $d \geq 4$, $A \subset N = \{1, \dots, n\}$, and $\lambda > 0$. Then there exists $\theta > 0$ and $\epsilon_0 > 0$, such that whenever $0 < \epsilon < \epsilon_0$ and $y \in D_{\theta\epsilon^\rho}^A$, $\forall i \in A$ then*

$$v_\epsilon^i(y) \leq \lambda.$$

Proof. (a) By the maximum principle, it is enough to prove the lemma for $y \in D_{\theta\epsilon^\rho}^A$ and $\text{dist}(y, \partial D) \leq \theta\epsilon^\rho$.

Fix an $i \in A$, and denote v_ϵ^i by v_ϵ and z_i by z . Let $\theta > 2$, $\epsilon_0 < \min(\delta_0, 1)$, $0 < \epsilon < \epsilon_0$. Using the comparison principle (see 8.2.H in [10]), and (2.2.10) in [32] there exists $c_1 > 0$ such that

$$(4.2) \quad v_\epsilon(x) \leq c_1 \text{dist}(x, \partial D)^{-2}, \forall x \in D.$$

Now using the Feynman-Kac formula we have,

$$(4.3) \quad \begin{aligned} v_\epsilon(y) &= E_y \left(v_\epsilon(B(\tau_{2\epsilon})) \exp\left(-\int_0^{\tau_{2\epsilon}} \frac{\psi(v_\epsilon(B_s))}{v_\epsilon(B_s)} ds\right) 1(\tau_{2\epsilon} < \tau_D) \right) \\ &\leq E_y(v_\epsilon(B(\tau_{2\epsilon})); 1(\tau_{2\epsilon} < \tau_D)) \\ &\leq c_1 \epsilon^{-2} P_y(\tau_{2\epsilon} \leq \tau_D) \\ &\leq c_2 \epsilon^{-2} P_y(B_{\tau_D} \in \Delta(z, \epsilon)), \end{aligned}$$

where the last inequality follows from the proof of Theorem 3.1 in [1]. As $\rho < 1$ and $\theta > 2$, $0 < \epsilon < \frac{1}{2}|y - z|$. Using the above and Lemma 2.1 in [1], we have that

$$(4.4) \quad v_\epsilon(y) \leq c_3 |y - z|^{-d} \text{dist}(y, \partial D) \epsilon^{d-3}.$$

Since D is bounded we have for all $y \in D_{\theta\epsilon^\rho}^{\{i\}}$,

$$v_\epsilon(y) \leq c_3 (\theta\epsilon^\rho)^{-d+1} \epsilon^{d-3} \leq c_3 \theta^{-d+1} \epsilon^{d-3+(-d+1)\rho} = c_3 \theta^{-d+1}.$$

As $d \geq 4$, we can choose θ large enough to obtain the required upper bound. The argument is independent of i and the proof of the lemma is complete. \square

4.2. Asymptotics for analytic ψ . In this section we shall assume that ψ is a real analytic function. We also assume that $a_1 = 0$ (a version of the results should hold for $a_1 > 0$ as well, but for the Martin kernel of killed Brownian motion rather than of Brownian motion itself). Note that if $a_1 = 0$ then (A1) \Rightarrow (2.4). The results presented here and their proofs largely mirror those presented in [29] which considered the case $\psi(u) = 2u^2$. The principal difference lies in the proof of Lemma 4.3, where a more delicate argument is required in order to obtain convergence of the power series arising there.

Theorem 4.1. *Let D be a bounded C^2 domain in dimension $d \geq 4$. Assume (A1) and that $a_1 = 0$. Let $A \subset N = \{1, \dots, n\}$. Then for $x, y \in D$,*

$$\lim_{\epsilon \rightarrow 0} \frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} = K_x^A(y),$$

where

$$K_x^A(y) = \begin{cases} K_x^D(y, z_i), & A = \{i\}, \\ \sum_{\sigma \in \mathcal{P}(A)} (-1)^{|\sigma|} \psi^{(|\sigma|)}(0) U\left(\prod_{C \in \sigma} K_x^C\right)(y), & |A| \geq 2. \end{cases}$$

As part of the argument, it will emerge that K_x^A is actually finite. Note that if $|A| \geq 2$ we may also write

$$(4.5) \quad K_x^A(y) = \sum_{j=2}^{|A|} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\substack{\cup_{i=1}^j C_i = A, \\ C_i \neq \emptyset \text{ and disjoint}}} U\left(\prod_{i=1}^j K_x^{C_i}\right)(y).$$

It is the latter form that comes naturally out of the formulae of Sections 2 and 3.

We need several lemmas before starting the proof.

Lemma 4.3. *Assume the conditions of Theorem 4.1, and let ρ , ϵ_0 , and θ be as in Lemma 4.2 with $\lambda < \lambda_0/2^n$. Then there is a $C < \infty$ such that $\forall 0 < \epsilon < \epsilon_0$ and $y \in D_{2|A|\theta\epsilon}^A$*

$$\frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} \leq C \sum_{i \in A} K_x^D(y, z_i).$$

Proof. We prove the lemma by induction on the size of A .

Step 1: ($|A| = 1$) Let $A = \{z\}$ and $v_\epsilon = v_\epsilon^A$. As in the proof of Lemma 4.2, we use (4.2), the Feynman Kac formula, and (a) of Lemma 4.1, to get

$$(4.6) \quad \begin{aligned} v_\epsilon(y) &= E_y \left(v_\epsilon(B(\tau_{2\epsilon})) \exp\left(-\int_0^{\tau_{2\epsilon}} \frac{\psi(v_\epsilon(B_s))}{v_\epsilon(B_s)} ds\right) \right) \\ &\leq E_y(v_\epsilon(B(\tau_{2\epsilon})); B(\tau_{2\epsilon}) \in \partial B(z, 2\epsilon)) \\ &\leq c_1 \epsilon^{-2} P_y(B(\tau_{2\epsilon}) \in \partial B(z, 2\epsilon)) \\ &\leq c_2 \epsilon^{-2} m_y^D(\Delta(z, \epsilon)) \end{aligned}$$

for $y \in D_{4\epsilon}^A$. Using the Palm formula (Lemma 2.3) with $\phi = 0$ and $n = 2$ we then obtain that

$$(4.7) \quad \mathbb{N}_x(X^D(\Delta(z, \epsilon))^2) = \psi^{(2)}(0) E_x \int_0^{\tau_D} [\mathbb{N}_{B_s}(X^D(\Delta(z, \epsilon)))]^2 ds.$$

Using the above, the Cauchy-Schwartz inequality, Lemma 2.3, and Lemma 4.1 (b) we have

$$\begin{aligned}
v_\epsilon(x) &= \mathbb{N}_x(X^D(\Delta(z, \epsilon)) > 0) \\
&\geq \frac{\mathbb{N}_x(X^D(\Delta(z, \epsilon)))^2}{\mathbb{N}_x(X^D(\Delta(z, \epsilon))^2)} \\
&= \frac{m_x^D(\Delta(z, \epsilon))^2}{\psi^{(2)}(0) \int_D G_D(x, y) m_y^D(\Delta(z, \epsilon))^2 dy} \\
(4.8) \quad &\geq c_3 \epsilon^{-2} m_x^D(\Delta(z, \epsilon)).
\end{aligned}$$

Consequently, (4.6), (4.8), and the boundary Harnack principle (see [3]) yield

$$\frac{v_\epsilon(y)}{v_\epsilon(x)} \leq c_4 \frac{\epsilon^2 m_y^D(\Delta(z, \epsilon))}{\epsilon^2 m_x^D(\Delta(z, \epsilon))} \leq c_5 K_x(y, z)$$

for $y \in D_{4\epsilon}^A$. By decreasing ϵ_0 if necessary, we can (and will) assume that $D_{4\epsilon}^A \supset D_{2\theta\epsilon}^A$. This establishes the case $|A| = 1$.

Step 2: ($|A| > 1$) Assume the result for every proper subset of A . Set $q = |A|$ and $\alpha(k) = 2^k \theta \epsilon^\rho$ for $k = 1, \dots, q$. By hypothesis, each $v_\epsilon^i(y) < \lambda_0/2^n$ on $D_{\alpha(1)}^A$. The same is then true of v^B for $B \subset A$, so by Lemma 2.5 $u^A < \lambda_0$ on $D_{\alpha(1)}^A$. Thus (2.20) applies and by the Feynman-Kac formula, we obtain the following on $D_{\alpha(q)}^A$:

$$\begin{aligned}
v_\epsilon^A(y) &= E_y(e^{-\int_0^{\tau_{\alpha(q-1)}} \phi_\epsilon^A(B_r) dr} v_\epsilon^A(B_{\tau_{\alpha(q-1)}})) \\
&+ (-1)^{|A|} \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\substack{\cup_{i=1}^j C_i = A \\ \emptyset \neq C_i \neq A}} E_y \int_0^{\tau_{\alpha(q-1)}} \prod_{i=1}^j v_\epsilon^{C_i}(B_t) (-1)^{|C_i|} e^{-\int_0^t \phi_\epsilon^A(B_r) dr} dt \\
(4.9) \quad &= I + II.
\end{aligned}$$

Here $0 \leq \phi_\epsilon^A = \phi(v_\epsilon^A, u_\epsilon^A)$. Consider the first term in (4.9):

$$\begin{aligned}
I &\leq E_y(v_\epsilon^A(B_{\tau_{\alpha(q-1)}})) \\
&\leq \sum_{i \in A} E_y(v_\epsilon^A(B_{\tau_{\alpha(q-1)}}) 1(B_{\tau_{\alpha(q-1)}} \in \partial B(z_i, \alpha(q-1)))) \\
&\leq \sum_{i \in A} \sup_{\partial B(z_i, \alpha(q-1))} v_\epsilon^A(\cdot) P_y(B_{\tau_{\alpha(q-1)}} \in \partial B(z_i, \alpha(q-1))) \\
&\leq c_6 \sum_{i \in A} \sup_{\partial B(z_i, \alpha(q-1))} v_\epsilon^A(\cdot) m_y^D(\Delta(z_i, \alpha(q-2))),
\end{aligned}$$

where the last inequality follows from Lemma 4.1 (a) [with z_i for z and $\alpha(q-2)$ for ϵ]. We know that $v_\epsilon^A \leq v_\epsilon^{A \setminus \{i\}}$. Using this and (4.8) with $z = z_i$, we have

$$(4.10) \quad \begin{aligned} \frac{I}{\prod_{j \in A} v_\epsilon^j(x)} &\leq c_6 \sum_{i \in A} \frac{\sup_{\partial B(z_i, \alpha(q-1))} v_\epsilon^{A \setminus \{i\}}(\cdot) m_y^D(\Delta(z_i, \alpha(q-2)))}{\prod_{j \in A, j \neq i} v_\epsilon^j(x) v_\epsilon^i(x)} \\ &\leq c_7 \sum_{i \in A} \frac{\sup_{\partial B(z_i, \alpha(q-1))} v_\epsilon^{A \setminus \{i\}}(\cdot) m_y^D(\Delta(z_i, \alpha(q-2)))}{\prod_{j \in A, j \neq i} v_\epsilon^j(x) \epsilon^{-2} m_x^D(\Delta(z_i, \epsilon))}. \end{aligned}$$

By the boundary Harnack principle,

$$m_y^D(\Delta(z_i, \alpha(q-2))) \leq c_8 m_x^D(\Delta(z_i, \alpha(q-2))) K_x(y, z_i),$$

and as in (4.4), $m_x^D(\Delta(z_i, \alpha(q-2))) \leq c_9 \alpha(q-2)^{d-1} \leq c_{10} \epsilon^{d-3}$. Likewise $\epsilon^{-2} m_x^D(\Delta(z_i, \epsilon)) \geq c_{11} \epsilon^{d-3}$, by (d) of Lemma 4.1. By induction, $v_\epsilon^{A \setminus \{i\}}(\cdot) / \prod_{j \in A, j \neq i} v_\epsilon^j(x)$ is bounded, in particular, on $D_{\alpha(q-1)}^{A \setminus \{i\}}$ by $c_{12} \sum_{j \in A \setminus \{i\}} K(\cdot, z_j)$, which by Lemma 4.1 (c) is bounded by $c_{13} \alpha(q-1)$ on $\partial B(z_i, \alpha(q-1))$. Therefore

$$(4.11) \quad \frac{I}{\prod_{j \in A} v_\epsilon^j(x)} \leq c_{14} \alpha(q-1) \sum_{i \in A} K_x(y, z_i)$$

Let

$$E_j = \{ \{C_i\}_{i=1}^j : \emptyset \neq C_i \neq A, \cup_{i=1}^j C_i = A, \text{ and } |A| + \sum_{i=1}^j |C_i| \text{ is even} \}$$

and

$$O_j = \{ \{C_i\}_{i=1}^j : \emptyset \neq C_i \neq A, \cup_{i=1}^j C_i = A, \text{ and } |A| + \sum_{i=1}^j |C_i| \text{ is odd} \}.$$

Then the second term (II) in (4.9) is,

$$\begin{aligned} &= \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{E_j} E_y \int_0^{\tau_{\alpha(q-1)}} \prod_{i=1}^j v_\epsilon^{C_i}(B_t) e^{-\int_0^t \phi_\epsilon^A(B_r) dr} dt \\ &\quad - \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{O_j} E_y \int_0^{\tau_{\alpha(q-1)}} \prod_{i=1}^j v_\epsilon^{C_i}(B_t) e^{-\int_0^t \phi_\epsilon^A(B_r) dr} dt \\ &\leq \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{E_j} E_y \int_0^{\tau_{\alpha(q-1)}} \prod_{i=1}^j v_\epsilon^{C_i}(B_t) dt \end{aligned}$$

as all terms in the second summand are non-negative. Therefore

$$(4.12) \quad \frac{II}{\prod_{i \in A} v_\epsilon^i(x)} \leq \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{E_j} E_y \int_0^{\tau_{\alpha(q-1)}} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(B_t)}{\prod_{i \in A} v_\epsilon^i(x)} dt.$$

First consider the case $j \leq |A|$ and $\{C_i\}_{i=1}^j \in E_j$. We observe that for $k \neq l$, if $C_k \setminus C_l \neq \emptyset$ then

$$v_\epsilon^{C_k} v_\epsilon^{C_l} \leq v_\epsilon^{C_k \setminus C_l} v_\epsilon^{C_l}.$$

So every term in this sum with $j \leq |A|$ is dominated by another term in which $\{C_i\}_{i=1}^j \in E_j$ are such that $C_k \cap C_l = \emptyset$ for $k \neq l$. Thus it suffices to bound such terms. For disjoint $\{C_i\}_{i=1}^j \in E_j$,

$$\begin{aligned} E_y \int_0^{\tau_{\alpha(q-1)}} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(B_t)}{\prod_{i \in A} v_\epsilon^i(x)} dt &\leq C^j E_y \int_0^{\tau_{\alpha(q-1)}} \frac{\prod_{i=1}^j \sum_{k \in C_i} K_x(B_t, z_k) \prod_{l \in C_i} v_\epsilon^l(x)}{\prod_{i \in A} v_\epsilon^i(x)} dt \\ (4.13) \qquad \qquad \qquad &= C^j E_y \int_0^{\tau_{\alpha(q-1)}} \prod_{i=1}^j \sum_{k \in C_i} K_x(B_t, z_k) dt. \end{aligned}$$

Let k_1, \dots, k_j be distinct. Then

$$\begin{aligned} E_y \int_0^{\tau_{\alpha(q-1)}} \prod_{i=1}^j K_x(B_t, z_{k_i}) dt &\leq G_D\left(\prod_{i=1}^j K_x(\cdot, z_{k_i})\right)(y) \\ &\leq c_{15} \sum_{i=1}^j G_D(K_x(\cdot, z_{k_i}))(y) \leq c_{15} \sum_{i=1}^j K_x(y, z_{k_i}) \\ (4.14) \qquad \qquad \qquad &\leq c_{16} \sum_{k=1}^{|A|} K_x(y, z_k) \end{aligned}$$

where the second inequality is due to the fact that z_i 's are separated by δ_0 and the third inequality follows from the “3-G” theorem (see (5.17) [29]). It follows from (4.13) that

$$(4.15) \qquad E_y \int_0^{\tau_{\alpha(q-1)}} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(B_t)}{\prod_{i \in A} v_\epsilon^i(x)} dt \leq c_{17} \sum_{k=1}^{|A|} K_x(y, z_k).$$

Now consider the case $j > |A|$. For $\{C_i\}_{i=1}^j \in E_j$ we can select $|A|$ of the C_i whose union is A , and apply the above bound to them. For the other C_i we apply Lemma 4.2, which gives that $v_\epsilon^{C_i} \leq \max_k v_\epsilon^k \leq \lambda$, where $2^{|A|}\lambda < \lambda_0$. Using (4.12), and (4.15) we have

$$\begin{aligned} \frac{II}{\prod_{i \in A} v_\epsilon^i(x)} &\leq c_{17} \left[\sum_{j=2}^{|A|} \frac{(-1)^j \psi^{(j)}(0) |E_j|}{j!} + \sum_{j=|A|+1}^{\infty} \frac{(-1)^j \psi^{(j)}(0) |E_j| \lambda^{j-|A|}}{j!} \right] \sum_{k=1}^{|A|} K_x(y, z_k) \\ &\leq c_{17} \left[\sum_{j=2}^{|A|} \frac{(-1)^j \psi^{(j)}(0) 2^{j|A|}}{j!} + \sum_{j=|A|+1}^{\infty} \frac{(-1)^j \psi^{(j)}(0) 2^{j|A|} \lambda^{j-|A|}}{j!} \right] \sum_{k=1}^{|A|} K_x(y, z_k) \\ &\leq c_{18} \sum_{k=1}^{|A|} K_x(y, z_k). \end{aligned}$$

□

Lemma 4.4. *Assume the conditions of Theorem 4.1. Let $y \in D_\delta^A$, where $\delta < \delta_0$. Then uniformly in $z \in \partial D$,*

$$\lim_{\epsilon \rightarrow 0} P_{yz}(\exp(-\int_0^{\tau_\delta} \phi_\epsilon^A(B_t) dt)) = 1.$$

Proof. We have $v_\epsilon^i(\cdot) \rightarrow 0$ as $\epsilon \rightarrow 0$. So by the assumptions on ψ , and Lemma 4.3 and

$$\phi_\epsilon^A(\cdot) = \frac{\psi(u_\epsilon^A + (-1)^{|A|}v_\epsilon^A) - \psi(u_\epsilon^A)}{(-1)^{|A|}v_\epsilon^A}(\cdot) \rightarrow 0.$$

Consequently it suffices to prove that

$$\lim_{\lambda \rightarrow 0} P_{yz}(\exp(-\lambda\tau_\delta)) = 1,$$

uniformly in z . As D is Lipschitz, we have

$$\sup_{y \in D_\delta, z \in \partial D} P_{yz}(\tau_\delta) < \infty$$

by [5]. Then Lemma 3.7 in [4] implies the result. \square

Proof of Theorem 4.1: We will use induction on the size of A . First consider the case where our target is a single point. Let $A = \{i\}$. When convenient in the proof we will use D_δ for D_δ^A for some $\delta > 0$. Let $x, y \in D_{\delta_0}$. By the Feynman-Kac Formula, for each fixed $\delta < \delta_0$ we have

$$\begin{aligned} \frac{v_\epsilon^i(y)}{v_\epsilon^i(x)} &= \frac{E_y(v_\epsilon^i(B_{\tau_\delta}) \exp(\int_0^{\tau_\delta} \phi_\epsilon^A(B_r) dr))}{E_x(v_\epsilon^i(B_{\tau_\delta}) \exp(\int_0^{\tau_\delta} \phi_\epsilon^A(B_r) dr))} \\ &= \frac{\int_{\partial D_\delta} P_{yz}(\exp(-\int_0^{\tau_\delta} \phi_\epsilon^A(B_t) dt)) K_x^{D_\delta}(y, z) v_\epsilon^i(z) m_x^{D_\delta}(dz)}{\int_{\partial D_\delta} P_{xz}(\exp(-\int_0^{\tau_\delta} \phi_\epsilon^A(B_t) dt)) v_\epsilon^i(z) m_x^{D_\delta}(dz)} \\ &= \frac{\int_{\partial D_\delta} P_{yz}(\exp(-\int_0^{\tau_\delta} \phi_\epsilon^A(B_t) dt)) K_x^{D_\delta}(y, z) v_\epsilon^i(z) m_x^{D_\delta}(dz)}{\int_{D_\delta} v_\epsilon^i(z) m_x^{D_\delta}(dz)} \times \\ &\quad \times \frac{\int_{D_\delta} v_\epsilon^i(z) m_x^{D_\delta}(dz)}{\int_{\partial D_\delta} P_{xz}(\exp(-\int_0^{\tau_\delta} \phi_\epsilon^A(B_t) dt)) v_\epsilon^i(z) m_x^{D_\delta}(dz)}. \end{aligned}$$

The measure

$$\lambda_{\epsilon, \delta}(x, dz) = \frac{v_\epsilon^i(z) m_x^{D_\delta}(dz)}{\int_{D_\delta} v_\epsilon^i(z) m_x^{D_\delta}(dz)}$$

is a probability measure on ∂D_δ^A . Since the boundary of D_δ^A is compact, by Prohorov's theorem any sequence ϵ_j has a subsequence, again written ϵ_j , for which $\lambda_{\epsilon_j, \delta}(x, dz) \Rightarrow \lambda_\delta(x, dz)$ weakly in the space of probability measures. Also, $K_x^{D_\delta}(y, z)$ is continuous and

bounded in z , for $z \in D \cap \partial D_\delta^A$, when $x, y \in D_{\delta_0}^A$. Consequently, Lemma 4.4 implies for $x, y \in D_{\delta_0}^A$ and for all $\delta < \delta_0$

$$\lim_{j \rightarrow \infty} \frac{v_{\epsilon_j}^i(y)}{v_{\epsilon_j}^i(x)} = \int_{\partial D_\delta} K_x^{D_\delta}(y, z) \lambda_\delta(x, dz).$$

The limiting function is harmonic in y for $y \in D_{\delta_0}$. By a diagonalization argument, we can assume there exists a convergent subsequence of our sequence such that the convergence holds simultaneously for a sequence of δ_j 's which converge to 0. By Lemma 4.3 we see then that the limit is harmonic in y with boundary value 0 on $\partial D \cap \partial D_\delta^A$ for all $\delta > 0$, and is 1 at $y = x$. This implies that the limit is the Martin Kernel for Brownian motion in D . This all subsequences have a subsequence which converges to the Martin kernel, and so the limit itself exists.

To prove the induction step, fix A and assume that the result is true for all proper subsets of A . Therefore if $\cup_{i=1}^j C_i = A$ and $\emptyset \neq C_i \neq A$ we have

$$\lim_{\epsilon \rightarrow 0} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(y)}{\prod_{i \in A} v_\epsilon^i(x)} = \begin{cases} \prod_{i=1}^j K_x^{C_i}(y) 1_{\{C_k \cap C_l = \emptyset, 1 \leq k \neq l \leq n\}} & \text{if } 1 \leq j \leq |A| \\ 0 & \text{otherwise.} \end{cases}$$

Let $x, y \in D$. Let $\epsilon > 0, \eta = \theta \epsilon^\rho > 0$ be small enough so that $x, y \in D_\eta$ and Lemma 4.3, Lemma 4.4, and Lemma 4.2 apply. By the Feynmann Kac formula and (2.20), we have then

$$\begin{aligned} \frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} &\geq \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{E_j} E_y \int_0^{\tau_\eta} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(B_t)}{\prod_{i \in A} v_\epsilon^i(x)} e^{-\int_0^t \phi_\epsilon^A(B_r) dr} dt \\ &\quad - \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{O_j} E_y \int_0^{\tau_\eta} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(B_t)}{\prod_{i \in A} v_\epsilon^i(x)} e^{-\int_0^t \phi_\epsilon^A(B_r) dr} dt. \end{aligned}$$

By Lemma 4.3, Lemma 4.4, the dominated convergence theorem (which applies as in Lemma 4.3 by the bound provided by Lemma 4.2) and the induction hypothesis we have

$$(4.16) \quad \liminf_{\epsilon \rightarrow 0} \frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} \geq \sum_{j=2}^{|A|} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\substack{\cup_{i=1}^j C_i = A \\ C_i \neq \emptyset = C_k \cap C_l}} E_y \int_0^{\tau_D} \prod_{i=1}^j K_x^{C_i}(B_t) dt.$$

For the upper bound, as in the proof of Lemma 4.3 we have

$$\begin{aligned} \frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} &\leq \frac{E_y(v_\epsilon^A(B_{\tau_\eta}))}{\prod_{i \in A} v_\epsilon^i(x)} + \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{E_j} E_y \int_0^{\tau_\eta} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(B_t)}{\prod_{i \in A} v_\epsilon^i(x)} dt \\ &\leq c_4 2^{|A|-1} \eta \sum_{i \in A} K_x(y, z_i) + \sum_{j=2}^{\infty} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{E_j} E_y \int_0^{\tau_\eta} \frac{\prod_{i=1}^j v_\epsilon^{C_i}(B_t)}{\prod_{i \in A} v_\epsilon^i(x)} dt. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, the first term $\rightarrow 0$ (since $\eta \rightarrow 0$). Using the inductive hypothesis and dominated convergence theorem for the second term we obtain

$$(4.17) \quad \limsup_{\epsilon \rightarrow 0} \frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} \leq \sum_{j=2}^{|A|} \frac{(-1)^j \psi^{(j)}(0)}{j!} \sum_{\substack{\cup_{i=1}^j C_i = A \\ C_i \neq \emptyset = C_k \cap C_l}} E_y \int_0^{\tau_D} \prod_{i=1}^j K_x^{C_i}(B_t) dt.$$

(4.5), (4.16) and (4.17) now yield the result. \square

Theorem 4.2. *Assume the conditions of Theorem 4.1. Let $\Phi_k \in \mathcal{F}_k$ be bounded, and fix $x \in D$. Then for $y \in D$,*

$$\lim_{\epsilon \rightarrow 0} \mathbb{N}_y(\Phi_k | \prod_{i=1}^n \langle X^D, 1_{\Delta_\epsilon^i} \rangle > 0) = \frac{1}{K_x^N(y)} \mathbb{N}_y(\Phi_k M_k^N),$$

where

$$M_k^N = \sum_{\sigma \in \mathcal{P}(N)} \prod_{C \in \sigma} \langle X^k, K_x^C \rangle.$$

Proof. We will need two preliminary Lemmas.

Lemma 4.5. *Assume the conditions of Theorem 4.1. Set $W_\epsilon^C = \exp((-1)^{|C|} \langle X^k, v_\epsilon^C \rangle) - 1$. Then*

$$\mathbb{N}_y \left(\prod_{i=1}^n \langle X^D, 1_{\Delta_\epsilon^i} \rangle > 0 | \mathcal{F}_k \right) = \sum_{j=1}^{2^{|N|}-1} \frac{1}{j!} \sum_{\substack{C_1 \cup C_2 \dots C_j \\ C_1 \cup C_2 \dots \cup C_j = N \\ \emptyset \neq C_i \forall i}} \left(\prod_{i=1}^j |W_\epsilon^{C_i}| \right) (-1)^{n + \sum_{i=1}^j |C_i|}$$

Proof. Using the arguments presented in [9] or otherwise one can verify that

$$u_\epsilon^{A,\lambda}(y) = \mathbb{N}_y(1 - \exp(-\lambda \langle X^D, \sum_{i \in A} 1_{\Delta_\epsilon^i} \rangle))$$

increases to $u_\epsilon^A(y)$ for all $y \in D$ as $\lambda \rightarrow \infty$. From here on, the proof of this lemma is the same as the proof of Lemma 5.8 in [29]. We will not present it again here, other than to remark that it is based on the Markov property of exit measures. Note that a different indexing system is used in [29], which accounts for the $j!$ factor. \square

Lemma 4.6. *Assume the conditions of Theorem 4.1. Let $\Phi_k \in \mathcal{F}_k$ be bounded. Let C_1, \dots, C_j be distinct and nonempty, with $\cup_{i=1}^j C_i = N$. Then*

$$\lim_{\epsilon \rightarrow 0} \frac{\mathbb{N}_y(\Phi_k \prod_{i=1}^j |W_\epsilon^{C_i}|)}{\prod_{i=1}^n v_\epsilon^i(x)} = \mathbb{N}_x(\Phi_k \prod_{i=1}^j \langle X^k, K_x^{C_i} \rangle) 1_{\{C_i \text{ disjoint}\}}.$$

Proof. The proof of this lemma can be obtained by imitating the proof of Lemma 5.9 in [29]. The only changes one has to make are to use: Theorem 4.1 in place of Theorem 5.3; Lemma 4.3 instead of Lemma 5.4; Lemma 2.4 instead of Lemma 2.7. \square

To complete the proof of the Theorem, observe that

$$\begin{aligned} & \mathbb{N}_y(\Phi_k | \prod_{i=1}^n \langle X^D, 1_{\Delta_\epsilon^i} \rangle > 0) \\ &= \frac{\mathbb{N}_y(\Phi_k, \prod_{i=1}^n \langle X^D, 1_{\Delta_\epsilon^i} \rangle > 0)}{\mathbb{N}_y(\prod_{i=1}^n \langle X^D, 1_{\Delta_\epsilon^i} \rangle > 0)} \\ &= \frac{\mathbb{N}_y(\Phi_k \mathbb{N}_y(\prod_{i=1}^n \langle X^D, 1_{\Delta_\epsilon^i} \rangle > 0 | \mathcal{F}_k))}{\prod_{i=1}^n v_\epsilon^i(x)} \times \frac{\prod_{i=1}^n v_\epsilon^i(x)}{v_\epsilon^N(y)}. \end{aligned}$$

By Theorem 4.1,

$$\frac{\prod_{i=1}^n v_\epsilon^i(x)}{v_\epsilon^N(y)} \rightarrow \frac{1}{K_x^N(y)}.$$

By Lemma 4.5 and Lemma 4.6,

$$\begin{aligned} & \frac{\mathbb{N}_y(\Phi_k \mathbb{N}_y(\prod_{i=1}^n \langle X^D, 1_{\Delta_\epsilon^i} \rangle > 0 | \mathcal{F}_k))}{\prod_{i=1}^n v_\epsilon^i(x)} \\ &= \sum_{j=1}^{2^{|N|-1}} \frac{1}{j!} \sum_{\substack{C_1 \cup C_2 \dots C_j \\ C_1 \cup C_2 \dots \cup C_j = N \\ \emptyset \neq C_i \forall i}} \frac{\mathbb{N}_y(\Phi_k \prod_{i=1}^j |W_\epsilon^{C_i}| (-1)^{n+\sum_{i=1}^j |C_i|})}{\prod_{i=1}^n v_\epsilon^i(x)} \\ &\rightarrow \sum_{j=1}^{2^{|N|-1}} \frac{1}{j!} \sum_{\substack{C_1 \cup C_2 \dots C_j \\ C_1 \cup C_2 \dots \cup C_j = N \\ \emptyset \neq C_i \forall i}} \mathbb{N}_y(\Phi_k \prod_{i=1}^j \langle X^k, K_x^{C_i} \rangle 1_{\{C_i \text{ disjoint}\}}) \\ &= \sum_{\sigma \in \mathcal{P}(A)} \mathbb{N}_y \left(\Phi_k \prod_{C \in \sigma} \langle X^k, K_x^C \rangle \right) = \mathbb{N}_y(\Phi_k M_k^N). \end{aligned}$$

The statement of the Theorem then follows easily. \square

4.2.1. *Weakening Hypothesis (A1).* It should be possible to weaken the assumption (A1) in Theorem 4.1 to the following:

$$(A2) \quad \int_1^\infty r^n \pi(dr) < \infty,$$

when $N = \{1, 2, \dots, n\}$. We are able to prove Theorem 4.1 under this weakened assumption, though only in the case $n = 2$. We begin with a lemma.

Lemma 4.7. *Let D be a bounded C^2 domain in dimension $d \geq 4$. Assume (A2) and that $a_1 = 0$. Let $\epsilon > 0$. There exists $a, b \in \mathbb{R}, c_1 > 0, f_\epsilon, g_\epsilon, h_\epsilon : D_\epsilon^{1,2} \rightarrow [a, b]$ such that for all*

$y \in D_\epsilon^{1,2}$,

$$(4.18) \quad v_\epsilon^{1,2}(y) \geq E_y(e^{-\int_0^{\tau_\alpha} f_\epsilon(B_t) dt} v_\epsilon^{1,2}(B_{\tau_\alpha})) + E_y\left(\int_0^{\tau_\alpha} g_\epsilon(B_t) v_\epsilon^1(B_t) v_\epsilon^2(B_t) dt\right)$$

$$(4.19) \quad v_\epsilon^{1,2}(y) \leq E_y(e^{-\int_0^{\tau_\alpha} f_\epsilon(B_t) dt} v_\epsilon^{1,2}(B_{\tau_\alpha})) + E_y\left(\int_0^{\tau_\alpha} h_\epsilon(B_t) v_\epsilon^1(B_t) v_\epsilon^2(B_t) dt\right),$$

$$(4.20) \quad \lim_{\epsilon \rightarrow 0} g_\epsilon(y) = \lim_{\epsilon \rightarrow 0} h_\epsilon(y) = \psi^{(2)}(0)$$

Proof of Lemma: Observe that,

$$\begin{aligned} \frac{1}{2} \Delta v_\epsilon^{1,2} &= \sum_{\emptyset \neq B \subset \{1,2\}} (-1)^{|B|+1} \frac{1}{2} \Delta u_\epsilon^B \\ &= \sum_{\emptyset \neq B \subset \{1,2\}} (-1)^{|B|+1} \psi(u_\epsilon^B) \\ &= \psi(v^1) + \psi(v_\epsilon^2) - \psi(v_\epsilon^1 + v_\epsilon^2 - v_\epsilon^{1,2}) \\ &= -[\psi(v_\epsilon^1 + v_\epsilon^2) - \psi(v_\epsilon^1)] + [\psi(v_\epsilon^2) - \psi(0)] + [\psi(v_\epsilon^1 + v_\epsilon^2) - \psi(v_\epsilon^1 + v_\epsilon^2 - v_\epsilon^{1,2})] \end{aligned}$$

When $n = 1$, (A2) implies that ψ is twice continuously differentiable. Using Taylor's theorem on ψ , we have

$$\begin{aligned} \frac{1}{2} \Delta v_\epsilon^{1,2} &= -[\psi^{(1)}(v_\epsilon^1) v_\epsilon^2 + \psi^{(2)}(\alpha_1) \frac{(v_\epsilon^2)^2}{2}] + [\psi^{(1)}(0) v_\epsilon^2 + \psi^{(2)}(\alpha_2) \frac{(v_\epsilon^2)^2}{2}] + [\psi^{(1)}(\alpha_3) v_\epsilon^{1,2}] \\ &= -[\psi^{(1)}(v_\epsilon^1) - \psi^{(1)}(0)] v_\epsilon^2 - [\psi^{(2)}(\alpha_1) - \psi^{(2)}(\alpha_2)] \frac{(v_\epsilon^2)^2}{2} + \psi^{(1)}(\alpha_3) v_\epsilon^{1,2} \\ &= -\psi^{(2)}(\alpha_4) v_\epsilon^1 v_\epsilon^2 - [\psi^{(2)}(\alpha_1) - \psi^{(2)}(\alpha_2)] \frac{(v_\epsilon^2)^2}{2} + \psi^{(1)}(\alpha_3) v_\epsilon^{1,2} \end{aligned}$$

where $\alpha_i : D_\epsilon^{1,2} \rightarrow [0, \infty)$ are measurable functions such that $v_\epsilon^1 \leq \alpha_1 \leq v_\epsilon^2 + v_\epsilon^1$, $0 \leq \alpha_2 \leq v_\epsilon^2$, $v_\epsilon^1 + v_\epsilon^2 - v_\epsilon^{1,2} \leq \alpha_3 \leq v_\epsilon^2 + v_\epsilon^1$, and $0 \leq \alpha_4 \leq v_\epsilon^1$.

Repeating the above calculation with the roles of v_ϵ^1 and v_ϵ^2 reversed, we would obtain

$$\frac{1}{2} \Delta v_\epsilon^{1,2} = -\psi^{(2)}(\beta_4) v_\epsilon^1 v_\epsilon^2 - [\psi^{(2)}(\beta_1) - \psi^{(2)}(\beta_2)] \frac{(v_\epsilon^1)^2}{2} + \psi^{(1)}(\alpha_3) v_\epsilon^{1,2},$$

where $\beta_i : D_\epsilon^{1,2} \rightarrow [0, \infty)$ are measurable functions such that $v_\epsilon^2 \leq \beta_1 \leq v_\epsilon^2 + v_\epsilon^1$, $0 \leq \beta_2 \leq v_\epsilon^1$, and $0 \leq \beta_4 \leq v_\epsilon^2$.

Using the Feynman-Kac formula, we have

$$v_\epsilon^{1,2}(y) = E_y(e^{-\int_0^{\tau_\alpha} \psi^{(1)}(\alpha_3(B_r)) dr} v_\epsilon^{1,2}(B_{\tau_\alpha})) + E_y\left(\int_0^{\tau_\alpha} e^{-\int_0^t \psi^{(1)}(\alpha_3(B_r)) dr} J_\epsilon(B_t) dt\right),$$

where $J_\epsilon : D \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} J_\epsilon &= \psi^{(2)}(\alpha_4)v_\epsilon^1v_\epsilon^2 + (\psi^{(2)}(\alpha_1) - \psi^{(2)}(\alpha_2)) \frac{(v_\epsilon^2)^2}{2} \\ &= \psi^{(2)}(\beta_4)v_\epsilon^1v_\epsilon^2 + (\psi^{(2)}(\beta_1) - \psi^{(2)}(\beta_2)) \frac{(v_\epsilon^1)^2}{2}. \end{aligned}$$

Now, $\psi^{(2)}(\cdot)$ is a non-negative decreasing continuous function. If $v_\epsilon^1 \leq v_\epsilon^2$ then $\beta_2 \leq \beta_1$ and

$$\psi^{(2)}(\beta_4)v_\epsilon^1v_\epsilon^2 \geq J_\epsilon \geq \left(\psi^{(2)}(\beta_4) + \frac{\psi^{(2)}(\beta_1) - \psi^{(2)}(\beta_2)}{2} \right) v_\epsilon^1v_\epsilon^2.$$

Likewise, if $v_\epsilon^2 \leq v_\epsilon^1$ then $\alpha_2 \leq \alpha_1$ and

$$\psi^{(2)}(\alpha_4)v_\epsilon^1v_\epsilon^2 \geq J_\epsilon \geq \left(\psi^{(2)}(\alpha_4) + \frac{\psi^{(2)}(\alpha_1) - \psi^{(2)}(\alpha_2)}{2} \right) v_\epsilon^1v_\epsilon^2.$$

At any given point one of the above happens. Therefore (4.18) and (4.19) hold with

$$\begin{aligned} f_\epsilon &= \psi^{(1)}(\alpha_3), \\ g_\epsilon &= \min\left(\psi^{(2)}(\alpha_4) + \frac{\psi^{(2)}(\alpha_1) - \psi^{(2)}(\alpha_2)}{2}, \psi^{(2)}(\beta_4) + \frac{\psi^{(2)}(\beta_1) - \psi^{(2)}(\beta_2)}{2}\right) \\ &\quad \text{and,} \\ h_\epsilon &= \max(\psi^{(2)}(\alpha_4), \psi^{(2)}(\beta_4)). \end{aligned}$$

Note that for a given ϵ , f_ϵ, g_ϵ and h_ϵ are bounded functions. Also for all $i = 1, 2, 3, 4$, $\alpha_i \rightarrow 0$ as $\epsilon \rightarrow 0$ and for all $i = 1, 2, 3$, $\beta_i \rightarrow 0$ as $\epsilon \rightarrow 0$. So (4.20) holds. \square

The Lemma below, is the equivalent of Lemma 4.3.

Lemma 4.8. *Let D be a bounded C^2 domain in dimension $d \geq 4$. Assume (A2) and that $a_1 = 0$. For $x \in D, A \subset N, \exists C < \infty, \epsilon_0 > 0$, such that $\forall \epsilon < \epsilon_0$ and $y \in D_{8\epsilon}^A$*

$$(4.21) \quad \frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} \leq C \sum_{i \in A} K_x^D(y, z_i).$$

Proof of Lemma: As before, one proceeds in two steps. The Step 1 proof, in Lemma 4.3, follows verbatim. In Step 2, use (4.18) instead of (2.20) to get

$$(4.22) \quad v_\epsilon^{1,2}(y) \leq E_y(v_\epsilon^{1,2}(B_{\tau_\epsilon})) + E_y\left(\int_0^{\tau_\alpha} h_\epsilon(B_t)v_\epsilon^1(B_t)v_\epsilon^2(B_t) dt\right).$$

It is easy to see that the analysis used in obtaining (4.11), or from [29], will imply that for $y \in D_{8\epsilon}^{1,2}$,

$$(4.23) \quad \frac{E_y(v_\epsilon^{1,2}(B_{\tau_\epsilon}))}{v_\epsilon^1(x)v_\epsilon^2(x)} \leq c_1\epsilon^2 (K_x(y, z_1) + K_x(y, z_2)).$$

The induction hypothesis and the fact that h_ϵ is bounded will imply

$$(4.24) \quad \frac{E_y(\int_0^{\tau_\alpha} h_\epsilon(B_t)v_\epsilon^1(B_t)v_\epsilon^2(B_t) dt)}{v_\epsilon^1(x)v_\epsilon^2(x)} \leq c_2(K_x(y, z_1) + K_x(y, z_2)).$$

So we have proved the lemma. \square

We are now ready to state and prove Theorem 4.1 assuming (A2) and not (A1), in the case $n = 2$.

Theorem 4.3. *Let $n = 2$, and let D be a bounded C^2 domain in dimension $d \geq 4$. Assume (A2) and that $a_1 = 0$. Then for $x, y \in D$,*

$$(4.25) \quad \lim_{\epsilon \rightarrow 0} \frac{v_\epsilon^i(y)}{v_\epsilon^i(x)} = K_x^D(y, z_i) \text{ for } i = 1, 2$$

$$(4.26) \quad \lim_{\epsilon \rightarrow 0} \frac{v_\epsilon^A(y)}{\prod_{i \in A} v_\epsilon^i(x)} = \psi^{(2)}(0)U(K_x^D(\cdot, z_1)K_x^D(\cdot, z_2))(y),$$

Proof : The proof of (4.25) is as in Step 1 of the proof of Theorem 4.1. It essentially follows verbatim. Lemma 4.4 was used but that does not require (A1) or (A2). The proof of (4.26) follows as in the induction step of the proof of Theorem 4.1. The ingredients for identifying the limit and application of dominated convergence are available immediately from (4.18), (4.19), (4.20), (4.23), (4.26) and (4.21) respectively. \square

Remark 4.1. *From the above it seems likely that the same idea could work for $n \geq 3$. To do so would require a suitable appeal to Taylor's Theorem (as in the proof of (4.18)). We have not succeeded in carrying this out, and so have presented the proof given earlier, under the stronger assumption (A1).*

4.3. Branching backbone for the limiting process, ψ analytic. The analysis of Section 3 can also be carried out for the limiting conditioned process obtained above. We will simply state the conclusions one obtains, without repeating the derivation.

Recall that

$$(4.27) \quad M_k^N = \sum_{\sigma \in \mathcal{P}(N)} \prod_{C \in \sigma} \langle X^k, K_x^C \rangle,$$

where the K_x^C are given in Theorem 4.1. For $\Phi_k \in \mathcal{F}_k$, we let

$$\mathbb{M}_y^N(\Phi_k) = \frac{1}{K_x^N(y)} \mathbb{N}_y(\Phi_k M_k^N)$$

be the limiting measure arising in Theorem 4.2. Then the following statements hold, under the conditions of that result.

- M_k^N is a martingale with respect to \mathcal{F}_k , so the associated Girsanov transform \mathbb{M}_k^N defines a consistent probability measure on $\vee_k \mathcal{F}_k$. In other words, in Dynkin's terminology, M_k^N is an X -harmonic function of X , and \mathbb{M}_y^N is its X -transform.

- This probability can equivalently be described in terms of a branching backbone throwing off mass, that is, as a superprocess with immigration along a random set obtained from a branching tree of particles.
- The backbone starts with a single particle, located initially at y . It performs a K_x^N -transform of Brownian motion, which dies somewhere in the interior of D . Say it dies at \hat{y} .
- A random partition Σ of N is chosen, so given \hat{y} , the probability that $\Sigma = \sigma$ is

$$\frac{1}{V^N(\hat{y})} b(|\sigma|, 0, \hat{y}) \prod_{A \in \sigma} K_x^A(\hat{y}),$$

where $\sigma \in \mathcal{P}(N)$, $|\sigma| \geq 2$. Here

$$V^N(\hat{y}) = \sum_{\sigma \in \mathcal{P}(N), |\sigma| \geq 2} b(|\sigma|, 0, \hat{y}) \prod_{A \in \sigma} K_x^A(\hat{y}).$$

- For each $A \in \Sigma$, a particle is born at \hat{y} which proceeds to carry out a K_x^A -transform of Brownian motion. If $A = \{z_i\}$, this particle survives to exit D at z_i . If $|A| \geq 2$ then the particle dies in the interior of D and is replaced by a random number of children, labeled by a random partition of A in the manner described above. This process repeats until all partitions consist of singletons. This process produces a branching tree of particles, with precisely n leaves, corresponding to particles exiting D at the n points of N .
- Mass is created/immigrated at points of D where the backbone branches. Given that j particles are born because of a branch at \hat{y} , the mass created is a random variable $R \geq 0$ whose conditional law $\mu^j(dr)$ is

$$\mu^j = \begin{cases} \frac{r^j \pi(dr)}{\int_0^\infty r^j \pi(dr)}, & j \geq 3 \\ \frac{r^2 \pi(dr)}{2a_2 + \int_0^\infty r^2 \pi(dr)}, & j = 2. \end{cases}$$

- Mass is also created continuously along the backbone according to a Lévy process, with Lévy exponent

$$\eta(\lambda) = \psi'(\lambda) - \psi'(0) = 2a_2\lambda + \int_0^\infty r(1 - e^{-\lambda r}) \pi(dr).$$

- Once created, the mass evolves as the (unconditioned) ψ -super Brownian motion. In other words, with excursion law \mathbb{N} .

As remarked earlier, note that unlike (3.1), there is no factorial factor in (4.27). This is simply because of the indexing scheme for the backbones. In Section 3 we ordered the n points and then labeled them at random. While now we label using the natural order given by the partition. In that sense, the current indexing scheme parallels that of [29].

The argument is an induction, based on the Palm formula (Lemma 2.3), as in Theorem 3.1. But note that each of the local characteristics of the backbone and mass evolution given above are consistent with sending $\epsilon \rightarrow 0$ in the characteristics of Section 3

4.4. Explosion of Mass as $\epsilon \rightarrow 0$. The explosion effect we wish to understand is already present when $n = 2$, so we will focus mainly on that case. To reiterate the representation for analytic ψ in this special case, write $K_x^i = K_x^D(\cdot, z_i)$. Then the process of interest is the Girsanov transform by the martingale

$$M_k^2 = \langle X^k, K_x^1 \rangle \langle X^k, K_x^2 \rangle + m_2 \langle X^k, U(K_x^1 K_x^2) \rangle,$$

where

$$m_2 = -\frac{\psi''(0)}{2} = a_2 + \frac{1}{2} \int_0^\infty r^2 \pi(dr).$$

The backbone has a single branch, and follows a $U(K_x^1 K_x^2)$ -transform till it dies, whereupon two particles are born, one doing a K_x^1 -transform, and the other doing a K_x^2 -transform. Mass is created at the branch point, according to the law μ^2 , and continuously along the backbone, according to the Lévy exponent η . The mass then evolves as an unconditioned super-Brownian motion.

Consider the stable branching function

$$\psi_\beta(\lambda) = \int_0^\infty r^{-(\beta+2)} (e^{-\lambda r} - 1 + \lambda r) dr = c_\beta \lambda^{1+\beta}$$

(for c_β chosen appropriately), which satisfies (2.4) for $0 < \beta < 1$. (A1) fails for ψ_β , but it does apply to

$$\psi_\beta^\gamma(\lambda) = \int_0^\infty r^{-(\beta+2)} e^{-\gamma r} (e^{-\lambda r} - 1 + \lambda r) dr,$$

and clearly $\psi_\beta^\gamma \rightarrow \psi_\beta$ as $\gamma \downarrow 0$. Thus our construction applies to ψ_β^γ , giving an X -transform $\mathbb{M}_y^{2,\gamma}$ with density

$$\tilde{M}_k^\gamma = \frac{1}{K_x^{\{1,2\}}(y)} M_k^{2,\gamma} = \frac{\langle X^k, K_x^1 \rangle \langle X^k, K_x^2 \rangle + m_2^\gamma \langle X^k, U(K_x^1 K_x^2) \rangle}{m_2^\gamma U(K_x^1 K_x^2)(y)},$$

where $m_2^\gamma = \frac{1}{2} \int_0^\infty r^{-\beta} e^{-\gamma r} dr \rightarrow \infty$ as $\gamma \downarrow 0$. Thus

$$\tilde{M}_k^\gamma \rightarrow \tilde{M}_k = \frac{1}{U(K_x^1 K_x^2)(y)} \langle X^k, U(K_x^1 K_x^2) \rangle,$$

which does not represent a valid Girsanov transform, because \tilde{M}_k is not a martingale in k under \mathbb{N}_y . It is not a surprise that something goes wrong here, because expectations of terms like $\langle X^k, K_x^1 \rangle \langle X^k, K_x^2 \rangle$ should blow up as $\gamma \downarrow 0$ – after all, stable random variables don't have finite second moments. Our original motivation for carrying out the analysis of this paper was understanding precisely what goes wrong in the stable case. In other words, of understanding how the singularity arises.

The problem is not the backbone, since the description of the backbone does not even depend on γ . Nor is the problem the continuous mass creation or the subsequent evolution of mass, since those approach the corresponding mechanisms for the ψ_β -super Brownian motion. The problem is precisely the creation mechanism $\mu^{2,\gamma}(dr)$ of mass at the branch point, since its density is proportional to $r^{-\beta} e^{-\gamma r}$, which fails to be tight when $\gamma \downarrow 0$. In other words, as $\gamma \downarrow 0$, the mass born at the branch point blows up.

While this heuristic analysis is sufficient to explain the singularity, one can use it to rigorously explain the change of measure by the \tilde{M}_k . Let ζ be the time the original particle in the backbone dies. Let τ_k be the lesser of the time it dies and the time it exits D_k .

Theorem 4.4. *\tilde{M}_k is a supermartingale. Let $\phi \geq 0$. Then for $y \in D_k$,*

$$\mathbb{N}_y(\tilde{M}_k \exp -\langle X^k, \phi \rangle) = \lim_{\gamma \downarrow 0} \mathbb{M}_y^{2,\gamma}(\exp -\langle X^k, \phi \rangle, \tau_k < \zeta)$$

Proof. Just follow the argument of Theorem 3.1. In the notation of that result, the event $\{\tau_k < \zeta\}$ is exactly the same as $\{\Upsilon^k \approx (\{1, 2\})\}$, and the probabilities of such events entered into the proof of Theorem 3.1. \square

The interpretation of the supermartingale property is that we lose absolute continuity between the two measures in the limit, with $\mathbb{M}_y^{2,\gamma}$ increasingly concentrating mass near a set of \mathbb{N}_y -measure 0. Namely the set where mass explodes.

A similar analysis can be carried out in the case $n > 2$. One shows by induction that as $\gamma \rightarrow 0$, one has $b(j, 0, y) = (-1)^j \psi_\beta^\gamma(0) \sim c_{\beta,j}^1 \gamma^{1+\beta-j}$, and $K_x^A \sim c_\beta^{|A|} \gamma^{1+\beta-|A|} U(\prod_{k \in A} K_x^k)$ for $|A| > 2$. It follows that asymptotically there is a single branch point with $|A|$ branches, at which the backbone changes from a single K_x^A transform, to $|A|$ transforms by K_x^k , $k \in A$. Moreover, the mass created at this branch point blows up as $\gamma \rightarrow 0$. For example, in the case $n = 3$, $\sigma = \{\{1\}, \{2\}, \{3\}\}$ has probability proportional to $\gamma^{\beta-2}$, while $\sigma = \{\{1, 2\}, 3\}$ has probability proportional to $\gamma^{\beta-1} \gamma^{\beta-1} = \gamma^{2\beta-2}$ which is of lower order of magnitude.

4.5. Other orders of limits. Still in the analytic case, with $n = 2$, the same arguments would have handled a slightly more general conditioning. Namely, condition on the exit measure charging both $\Delta_{\epsilon_1}^{z_1}$ and $\Delta_{\epsilon_2}^{z_2}$, where $\epsilon_1 > 0$ and $\epsilon_2 > 0$. First send $\epsilon_1 \rightarrow 0$ and then $\epsilon_2 \rightarrow 0$. It is not hard to check that this gives precisely the same limiting object as before. So if we apply this procedure to the ψ_β^γ -super Brownian motion, and then let $\gamma \downarrow 0$, the mass should still blow up, but the backbone should take the form of a $U(K_x^1 K_x^2)$ -transform, splitting into a K_x^1 -transform and a K_x^2 -transform.

With this modification, one should be able to treat a different approach to the stable case than given above. Namely, start out with the actual ψ_β -super Brownian motion. Condition its exit measure to charge $\Delta_{\epsilon_1}^{z_1}$ and $\Delta_{\epsilon_2}^{z_2}$. Then send $\epsilon_1 \rightarrow 0$ followed by $\epsilon_2 \rightarrow 0$. The following informal calculation suggests how the backbone should behave in the limit.

For $i = 1, 2$ set

$$\begin{aligned} u^i &= v^i = u_{\epsilon_i}^i = v_{\epsilon_i}^i = \mathbb{N}.(X^D(\Delta_{\epsilon_i}^{z_i}) > 0) \\ u^{12} &= u_{\epsilon_1, \epsilon_2}^{12} = \mathbb{N}.(X^D(\Delta_{\epsilon_1}^{z_1} \cup \Delta_{\epsilon_2}^{z_2}) > 0) \\ v^{12} &= v_{\epsilon_1, \epsilon_2}^{12} = u^1 + u^2 - u^{12} = \mathbb{N}.(X^D(\Delta_{\epsilon_1}^{z_1}) > 0, X^D(\Delta_{\epsilon_2}^{z_2}) > 0). \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2}\Delta v^{12} &= \frac{1}{2}\Delta[u^1 + u^2 - u^{12}] = \psi_\beta(u^1) + \psi_\beta(u^2) - \psi_\beta(u^{12}) \\ &= \psi_\beta(v^1) + \psi_\beta(v^2) - \psi_\beta(v^1 + v^2 - v^{12}) \\ &= \psi_\beta(v^1) + [\psi_\beta(v^2) - \psi_\beta(v^1 + v^2)] + [\psi_\beta(v^1 + v^2) - \psi_\beta(v^1 + v^2 - v^{12})] \\ &\sim c_\beta \left([v^1]^{1+\beta} - [1 + \beta][v^2]^\beta v^1 + [1 + \beta][v^1 + v^2]^\beta v^{12} \right). \end{aligned}$$

Fix x and rescale, letting $\bar{v}^1 = v^1(\cdot)/v^1(x)$, $\bar{v}^2 = v^2(\cdot)/v^2(x)$, $\bar{v}^{12} = v^{12}(\cdot)/v^1(x)v^2(x)^\beta$. If all these quantities remain bounded, then sending $\epsilon_1 \rightarrow 0$ should give

$$\frac{1}{2}\Delta \bar{v}^{12} \sim (1 + \beta)c_\beta \left([v^2]^\beta \bar{v}^{12} - [\bar{v}^2]^\beta \bar{v}^1 \right).$$

So sending $\epsilon_1 \rightarrow 0$ and then $\epsilon_2 \rightarrow 0$ should give

$$\frac{1}{2}\Delta \bar{v}^{12} \sim -(1 + \beta)c_\beta [\bar{v}^2]^\beta \bar{v}^1.$$

Though we again expect this conditioning to have a mass which blows up in the limit, the above suggests that the backbones should still converge weakly. But this time the limit should be a $U([K_x^2]^\beta K_x^1)$ -transform, splitting into a K_x^1 -transform and a K_x^2 -transform.

Call the first part of the backbone the *trunk*. Taking limits in the other order should produce a trunk that is a $U([K_x^1]^\beta K_x^2)$ -transform. So taking limits in which both $\epsilon_1 \rightarrow 0$ and $\epsilon_2 \rightarrow 0$ in a coordinated way should give trunks that are transforms by

$$U\left([K_x^1 K_x^2]^\beta [\theta K_x^1 + (1 - \theta)K_x^2]^{1-\beta}\right),$$

for arbitrary $0 \leq \theta \leq 1$.

We will not try to make the above informal argument rigorous. But we originally found it puzzling. Not so much because uniqueness of limits breaks down in the case of the ψ_β -super Brownian motion. But rather because the $U(K_x^1 K_x^2)$ trunk we obtained earlier does not appear among the possible limits when taken this other way. This suggests that there should be a way of capturing a broader class of limits, encompassing both types obtained above. Or at least, of obtaining both types of limits by a common procedure. Carrying this out rigorously, in the current context, seems more work than it is perhaps worth, and we do not claim to have done so in full detail. But in the next section, we will find a simpler setting, in which this can be done more more easily.

4.6. A class of martingales, with $n = 2$. In this section, we focus on a technically simpler collection of martingales, exhibiting some of the same behaviour as found above. We only present the heuristic idea and do not present detailed proofs in this section. The heuristic development can be made rigorous but we have chosen not to include this, as our aim in this section is to illustrate how a spectrum of limits can be obtained via various limiting procedures.

Let ψ satisfy (A1). Let $u, f, g, v > 0$ be bounded solutions on D to the following equations:

$$(4.28) \quad \begin{cases} \frac{1}{2}\Delta u = \psi(u) \\ \frac{1}{2}\Delta f - \psi'(u)f = 0 \\ \frac{1}{2}\Delta g - \psi'(u)g = 0 \\ \frac{1}{2}\Delta v - \psi'(u)v = -\psi''(u)fg \end{cases}$$

as well as $v = 0$ on ∂D (so v is an $L_{\psi' \circ u}$ -potential). Call \mathcal{M}_2 the set of (u, f, g, v) so obtained.

Under \mathbb{M}_x , for every $(u, f, g, v) \in \mathcal{M}_2$ define:

$$M_k(u, f, g, v) = e^{-\langle X^k, u \rangle} \left[\langle X^k, v \rangle + \langle X^k, f \rangle \langle X^k, g \rangle \right].$$

It can be verified that $M_k(u, f, g, v)$ is a \mathcal{F}_k martingale. Moreover, we can realize the Girsanov transform

$$\mathbb{M}_y^{(u, f, g, v)}(\Phi_k) = \frac{1}{v(y)} \mathbb{N}_y(\Phi_k M_k^{(u, f, g, v)})$$

by such martingales, as follows: Let $(u, f, g, v) \in \mathcal{M}_2$. Then $\mathbb{M}_y^{(u, f, g, v)}$ can be described as follows: Start a v -transform of the $L_{\psi' \circ u}$ process till it dies in D . At this point, start two paths that run to ∂D , respectively an f -transform and a g -transform of the $L_{\psi' \circ u}$ process. Mass is created at the branch point, according to the law μ^2 , and continuously along the backbone, according to the Lévy exponent η . The mass then evolves as an unconditioned super-Brownian motion.

In this simplified context, the analogue to the question considered in previous sections is the following: Let $(u_n, f_n, g_n, v_n) \in \mathcal{M}_2$, with each term $\rightarrow 0$. Find the limit points, either of $\mathbb{M}_y^{(u_n, f_n, g_n, v_n)}$, or of the backbone.

When ψ satisfies (A1), there is a systematic answer to this question. For a fixed x , we renormalize by constants a_n and b_n , so that

$$\bar{f}_n = \frac{f_n}{a_n}, \quad \text{and} \quad \bar{g}_n = \frac{g_n}{b_n},$$

converge to non-zero values. Then \bar{f}_n , \bar{g}_n , and $\bar{v}_n = \frac{v_n}{a_n b_n}$ all have non-zero limits f, g, v which satisfy

$$\frac{1}{2}\Delta f = \frac{1}{2}\Delta g = 0, \quad v = \psi''(0)U(fg).$$

Moreover $\mathbb{M}_y^{(u_n, f_n, g_n, v_n)}$ converges in the weak topology to $\mathbb{M}_y^{(0, f, g, v)}$.

In the case of ψ_β , the actual Girsanov transforms $\mathbb{M}_y^{(u_n, f_n, g_n, v_n)}$ degenerate as before, because mass blows up. We look at the limits of the backbones instead.

Let a_n, b_n, c_n all be sequences of positive reals, that all $\rightarrow 0$. Assume that $a_n b_n = o(c_n^{1-\beta})$. Let ϕ_f, ϕ_g, ϕ_u be smooth functions on ∂D , that are bounded away from 0. Define (u_n, f_n, g_n, v_n) to be the solutions to (4.28), using boundary conditions $a_n \phi_f$ for f_n , $b_n \phi_g$ for g_n , and $c_n \phi_u$ for u_n . Set $d_n = \beta(1 + \beta)a_n b_n c_n^{-(1-\beta)}$ and

$$\bar{u}_n = \frac{u_n}{c_n}, \quad \bar{f}_n = \frac{f_n}{a_n}, \quad \bar{g}_n = \frac{g_n}{b_n}, \quad \bar{v}_n = \frac{v_n}{d_n}.$$

(4.28) gives that

$$\frac{1}{2}\Delta v_n - (1 + \beta)u_n^\beta v_n = -\beta(1 + \beta)u_n^{-(1-\beta)}a_n b_n \bar{f}_n \bar{g}_n,$$

so

$$\frac{1}{2}\Delta \bar{v}_n - (1 + \beta)u_n^\beta \bar{v}_n = -\bar{u}_n^{-(1-\beta)}\bar{f}_n \bar{g}_n.$$

Now taking limits we have $\bar{u}_n \rightarrow u$, $\bar{f}_n \rightarrow f$, $\bar{g}_n \rightarrow g$, and $\bar{v}_n \rightarrow v$ where u, f , and g are the harmonic functions with boundary values ϕ_u, ϕ_f, ϕ_g , and $v = U(u^{-(1-\beta)}fg)$. The backbones converge weakly to a v -transform branching into an f -transform and a g -transform.

We conclude this section with two of examples analogous to the limits obtained in the previous sections.

Example 4.1. Take $\phi_u = 1$. The trunk is a $U(fg)$ -transform, as in Section 4.4

Example 4.2. Take $\phi_u = \phi_f$. The trunk is a $U(f^\beta g)$ -transform, as in Section 4.5

In other words, this gives a general class of limiting objects, which encompasses examples analogous to both types of limits obtained earlier. In that sense, the model of this section interpolates between these examples, and helps explain the variety of limits obtained.

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