CONDITIONING SUPER-BROWNIAN MOTION ON ITS BOUNDARY STATISTICS; FRAGMENTATION AND A CLASS OF WEAKLY EXTREME X-HARMONIC FUNCTIONS

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Abstract. Let X be a super-Brownian motion (SBM) defined on $\mathbb{R}^n$ and (X_D) be its exit measures indexed by sub-domains of $\mathbb{R}^d$. We pick a bounded sub-domain D, and condition the super-brownian motion inside this domain on its "boundary statistics", random variables defined on an auxiliary probability space generated by sampling from the exit measure X_D. Among these, two particular examples are: conditioning on a Poisson random measure with intensity βX_D; and X_D itself. We find the conditional laws as h-transforms of the original SBM law using Dynkin's formulation of X-harmonic functions. For each conditioning we consider, we give an explicit expression for the corresponding X-harmonic function. We also obtain an explicit construction of some of these conditional laws in terms of branching particle systems. For example, we give a fragmentation system description of the law of super-Brownian motion conditioned on $X_D = \nu$, in terms of a particle system, called the backbone. In the backbone, each particle is labeled at its birth by a measure $\tilde{\nu}$. The spatial motion of the particle is an h-transform of Brownian motion, where h is a potential that depends on $\tilde{\nu}$. The label $\tilde{\nu}$ represents the particle and its descendants' total contribution to the exit measure. At the particle's death two new particles are born and $\tilde{\nu}$ is passed to the newborns by fragmentation into two pieces. The X-harmonic function H_x^ν corresponding to conditioning on $X_D = \nu$ is of special interest, as it can be thought as the analogue of the Poisson or Martin kernel. An open problem is to show that H_x^ν is extreme at least for some ν, when D is a smooth domain. An equivalent problem is to show the tail sigma field of SBM in D is trivial with respect to P_μ^ν. We prove a weaker version of this result. We show that for any A in the tail sigma field of X, $P_\mu^{X_D}(A) = 0$ or 1.

1. INTRODUCTION

Studying conditioned Markov processes is a kind of inverse problem – given information about how the process ends up, one tries to infer how it got there, at least in terms of probabilities. In the context of Brownian motion and finite dimensional Markov processes one can make very explicit calculations, starting with the work of J. L. Doob [3]. Attempts to make similar calculations for super Brownian motion are more recent. These studies typically aim to recover the conditional law of a superprocess as the law of a distinct probabilistic object. Several authors have succeeded in coming up with such descriptions for certain conditionings, and produced models with remarkably rich structure. The first of these models was Evans/Evans and Perkins's immortal particle system [9], [8] for super-Brownian motion in $\mathbb{R}^d$ conditioned on survival. In this model, an immortal particle moves according to a Brownian motion, and throws off mass at a uniform rate, and then this mass evolves in the space as an unconditioned super-Brownian motion. Following Evans and Perkins, Salisbury and Verzani [12] considered a super Brownian motion X in a domain D, with conditioning based on the *exit measure* X_D from D. More specifically, they conditioned X on the event that the support of X_D contains certain points $z_1, \ldots, z_k$, and recovered the resulting conditional law in terms of a branching backbone system. The branching backbone is a random tree with k leaves reaching the points $z_1 \ldots z_k$. Similar to Evans and Perkins' model, there is

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mass uniformly created along the branching backbone which follows the law of an unconditioned super-process independent of the points $z_1, \dots, z_k$. Giving such an explicit characterization of a conditioned process is an interesting problem from a probabilistic modeling point of view. For example, in population dynamics, one can view it as an analogue of a host of biological problems in which one has information about the state of the population at certain times or locations, and one wishes to infer the genealogical structure of the populations of the ancestors (e.g. the "out of Africa" problem of human origins).

It turns out that there is more to the conditioning problem than described above. A conditioned process represents a special case of a Girsanov transformation, or a martingale change of measure. For concreteness, let us consider the following example: Let ξ be Brownian motion in a domain E. Let τ_E be the exit time from E. We compute the conditional law, Π_x^z of ξ given $\xi_\tau = z$, by a martingale change of measure from Π_x, the law of ξ. This martingale change of measure is given in terms of a certain harmonic function $h^z(\cdot)$. More precisely, for any domain D strictly contained in E, and any Y measurable with respect to $\mathcal{F}_{\tau_D}$, we have $\Pi_x^z(Y) = \Pi_x(Y h^z(X_{\tau_D})/h^z(x))$. A Girsanov transformation defined in terms of an harmonic function is called an h-transform, and typically conditional laws of Markov processes are formulated as h-transforms of their original laws. This relationship between harmonic functions and conditioning a Markov process has lead to an elegant probabilistic formulation of the Martin boundary theory for elliptic differential operators.

In the context of super-processes, the analogue of harmonic functions are X-harmonic functions. Following Dynkin's definition [4], let us consider a super-Brownian motion $X = (X_D)_D$, a family of random measures (*exit measures*) indexed by sub-domains D of a given domain E in $\mathbb{R}^d$. Let $(P_\mu)_\mu$ be the family of probability laws associated to X indexed by finite initial measures μ with support in E. A non-negative function H is *X-harmonic* if for any subdomain D and any finite measure μ with support in D,

$$P_\mu(H(X_D)) = H(\mu).$$

Note that this property resembles the mean value property of a harmonic function, hence the name X-harmonic. Moreover, the X-harmonic functions are related to conditioning super-Brownian in the same way as harmonic functions are related to conditioning Brownian motion; they give us the explicit Girsanov transformation to switch from the unconditioned probability law to the conditioned probability law. An H-transform P_μ^H is obtained from P_μ by setting

$$P_\mu^H(Y) = P_\mu(H(X_D)Y)$$

for Y non-negative and $\mathcal{F}_D$-measurable, where $\mathcal{F}_D$ is the σ-algebra generated by $X_{D'}, D' \supset D$. In his book [4], Dynkin points us to a new direction in investigating the solutions of the p.d.e. $\frac{1}{2}\Delta = 2u^2$, that is to explore X-harmonic functions, with the idea of thinking them as the analogue of harmonic functions, and ultimately, to build a Martin boundary theory for this non-linear p.d.e. In this case, Martin boundary is defined as the set of extreme elements of the convex set of all X-harmonic functions. A concrete understanding of extreme X-harmonic functions might yield further insights about the solutions of the p.d.e $\frac{1}{2}\Delta u = 2u^2$.

Fascinated by the rich structure of the underlying probabilistic objects as well as this fresh direction towards a Martin boundary theory, our goal in this paper is to explore various ways of conditioning of Super-Brownian motion and the corresponding X-harmonic functions. Here is a summary of our contributions: We have developed a new way of conditioning a super-Brownian motion, so called "conditioning on boundary statistics". What is new here is that the random variables which we condition on are defined on an auxiliary probability space, and generated by sampling from the exit measure X_D. For example, we can condition X on a Poisson random measure with intensity βX_D, where $\beta > 0$. It turns out that this way we recover some of the X-harmonic functions studied earlier in the literature as the X-harmonic functions corresponding to these conditionings. Moreover, we can use these X-harmonic functions to approximate a more significant class of X-harmonic functions. One such class is found by conditioning a super-Brownian motion in D on its exit measure X_D, which the rest of our paper is devoted to. This class consists of X-harmonic functions H^ν_D, indexed by subdomains D of E and finite measures ν on the boundary of D. Note that for fixed D, H^ν_D can be considered as the analogue of the Poisson kernel in D. H^ν_D appeared first in a paper by Dynkin [7], where he showed that if H is an extreme X-harmonic function in E then for every μ, and for every sequence D_k exhausting E

$$H(\mu) = \lim_{k\to\infty} H^{X_{D_k}}_{D_k}(\mu)$$

P^H_μ almost surely. We fix a subdomain D, and let $H^\nu = H^\nu_D$. First, we give an infinite particle fragmentation system description of P^ν_μ, the conditional law of X in D given $X_D = \nu$, in terms of a particle system, called the backbone as in [12], along which a mass is created uniformly. In the backbone, each particle is assigned a measure $\tilde{\nu}$ at its birth. The spatial motion of the particle is an h-transform of Brownian motion, where h is a potential that depends on $\tilde{\nu}$. The measure $\tilde{\nu}$ represents the particle's contribution to the exit measure. At the particle's death two new particles are born and $\tilde{\nu}$ is passed to the newborns by fragmentation into two bits. Here, we used the techniques of [12] applied to a more general setting. This description bridges the theory of conditioned super-processes to the growing literature on infinite fragmentation and coalescent processes, (see e.g. [1] for a comprehensive exposition). In the last part of our paper we prove that for certain ν, the X-harmonic function H^ν is extreme in D. Two key ideas are used in the proof. In [4], Dynkin gives a proof of a result which is originally due to Evans [8], which states that the X-harmonic function $\langle \mu, 1\rangle$ is extreme in $\mathbb{R}^d$. The conditioning corresponding to this X-harmonic function is Evans and Perkins's conditioning, i.e. conditioning on survival. The idea in this proof is to formulate an equivalent problem in terms of the tail σ-field of the immortal particle. In his reformulation of Evans's proof, Dynkin shows us in great clarity and detail how to make a similar idea work for our case. Indeed, we can reduce our problem to showing that the tail sigma field of the branching backbone is trivial. However, we will do something slightly different, instead we will look at the tail σ-field of the branching back bone that corresponds to H^{n,ν_n}, the X-harmonic function corresponding to conditioning on a Poisson random measure with intensity nX_D. (We investigate these X-harmonic functions in section 3, where we also discover that they belong to the class of X-harmonic functions considered by Salisbury and Verzani [12].) This branching backbone is simpler since there are only finitely many particles involved. Then we approximate H^ν by H^{n,ν_n} to prove our result.

2. PRELIMINARIES

2.1. **Super-Brownian motion.** We will follow Dynkin's definition of super-Brownian motion (SBM). Let E be a domain of $\mathbb{R}^d$ and let $\mathcal{M}_E$ be the positive finite measures on E. A super-Brownian motion is a family of measures $X = (X_D)$ indexed by sub-domains of E and a family of probability laws for X, P_μ, where μ are finite measures on E, with the following properties:

(a) Exit property: $P_\mu(X_D(D) = 0) = 1$ every μ, and if $\mu(D) = 0$ then $P_\mu(X_D = \mu) = 1$.

(b) Markov property: If $Y \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\subset D}$ generated by $X_{D'}, D' \subset D$ and $Z \geq 0$ is measurable with respect to the σ-algebra $\mathcal{F}_{\supset D}$ generated by $X_{D''}$, $D'' \supset D$ then

$$P_\mu(YZ) = P_\mu(Y P_{X_D} Z)$$

(c) Branching property: For any non-negative Borel f, $P_\mu(e^{-\langle X_D, f\rangle}) = e^{-\langle \mu, V_D f\rangle}$ where

$$V_D f(y) = -\log P_y(e^{-\langle X_D, f\rangle})$$

and $P_y = P_{\delta_y}$.

(d) Integral equation for the log-Laplace functional: V_Df solves the integral equation

$$u + G_D(2u^2) = K_Df$$

where G_D and K_D are respectively Green and Poisson operators for Brownian motion in D. In other words, if ξ_t is a Brownian motion starting from x, under a probability measure Π_x, then $K_Df(x) = \Pi_x f(\xi_{\tau_D})$, where τ_D is the exit time from D. Likewise, $G_Df(x) = \Pi_x(\int_0^{\tau_D} f(\xi_t)dt)$.

Under certain regularity conditions on D and f (see e.g. [4]), the integral equation in (d) is equivalent to the boundary value problem

$$\begin{aligned} \frac{1}{2}\Delta u &= 2u^2 \\ u(x) &= f(x),\, x \in \partial D. \end{aligned}$$

X_D represents the exit measure from D, and the first property simply means that X_D is concentrated on D^c, so that exiting is instantaneous if we start outside D. The third property means that distinct clumps of initial mass evolve independently. The fourth property restricts attention to finite variance branching, and normalizes the branching rate. The normalizing factor 2 in front of u^2 is chosen to be consistent with [10] and [12].

2.2. **Infinite divisibility and Poisson representation.** It is well known that X_D has an infinitely divisible distribution for each D. This property leads to the construction of a new measure, $\mathbb{N}_x$, called the super-Brownian excursion law starting from x. Under $\mathbb{N}_x$, X evolves as a super-Brownian motion, but $\mathbb{N}_x$ will be σ-finite not a probability. Thus, it is technically more complicated than P_μ. But $\mathbb{N}_x$ is actually a more basic object heuristically, under which the genealogies are simpler, because all the mass starts from a single particle located initially at x.

In fact P_μ can be built up from a Poisson random measure with intensity $\theta(d\chi) = \int \mathbb{N}_x(d\chi)\mu(dx)$. More precisely, let

$$\Pi(d\chi) = \sum_i \delta_{\chi^i}$$

be such a Poisson random measure, where the χ^i are random measure valued paths. Then $X = \sum \chi^i = \int \chi \Pi(d\chi)$ is a super-Brownian motion with initial state μ. In terms of X_D, this yields the following formula (see Theorem 5.3.4 of [5]): Let F be a non-negative measurable function defined on $\mathcal{M}_E$. Then

$$P_\mu(F(X_D)) = e^{-\mathcal{R}_\mu(\mathcal{M}_\mathcal{E})}F(0) + \sum_{n=1}^{\infty} \frac{1}{n!} e^{-\mathcal{R}_\mu(\mathcal{M}_\mathcal{E})} \int \mathcal{R}_\mu(d\nu_1)\cdots\mathcal{R}_\mu(d\nu_n)F(\nu_1 + \cdots + \nu_n) \tag{1}$$

where $\mathcal{R}_\mu$ is the canonical measure of X_D with respect to P_μ and can be derived from $\mathbb{N}_x$ by $\mathcal{R}_\mu(A) = \langle \mu, \mathbb{N}_{(\cdot)}(X_D \in A, X_D \neq 0)\rangle$. In other words, we obtain X_D as a superposition of a Poisson number of more "basic" exit measures, each descended from a single initial individual. These "basic" exit measures arise as the atoms of a Poisson random measure whose characteristic measure is $\mathcal{R}_\mu$

Note that a special case of the above representation gives us $V_Df(x) = \mathbb{N}_x(1 - e^{-\langle X_D, f\rangle})$. Note further that we will in future write this as

$$P_\mu(F(X_D)) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{-\mathcal{R}_\mu(\mathcal{M}_\mathcal{E})} \int \mathcal{R}_\mu(d\nu_1)\cdots\mathcal{R}_\mu(d\nu_n)F(\nu_1 + \cdots + \nu_n)$$

by taking the convention that the $n = 0$ term is the first expression in (1).

The measure $\mathbb{N}_x$ was first considered by J.F. Le Gall [10], in his random snake formulation of super-Brownian motion. As we follow Dynkin's framework to study our problem, we refer the reader to [5] for a systematic account of the theory of the measures $\mathbb{N}_x$ and their applications. Note that the latter has a general branching function ψ, but for us, this is taken to be $\psi(u) = 2u^2$.

2.3. **Moment measures of super Brownian motion.** Among the key tools in our analysis are the recursive moment formulae of SBM. The moment measures of SBM are the following measures:

Let $f_0, f_1 \ldots, f_n$ be bounded Borel functions. and write $f = (f_0, \ldots, f_n)$. For $C \subset \{1, \ldots, n\}$, let

$$n_C(f, x) = \mathbb{N}_x e^{-\langle X_D, f_0\rangle} \Pi_{i\in C}\langle X_D, f_i\rangle \tag{2}$$

$$p_C(f, \mu) = P_\mu e^{-\langle X_D, f_0\rangle} \Pi_{i\in C}\langle X_D, f_i\rangle. \tag{3}$$

Let K_D^l and G_D^l be the Poisson and Green's operator for the operator

$$\mathcal{L}^l = \frac{1}{2}\triangle - l$$

where $l(x) = 4V_D f_0$. In other words, let ξ_t be a diffusion starting from x, with generator $\mathcal{L}^l$ under a probability measure Π_x^l. Then $K_D^l f(x) = \Pi_x^l f(\xi_{\tau_D})$, where τ_D is the exit time from D. Likewise, $G_D^l f(x) = \Pi_x^l(\int_0^{\tau_D} f(\xi_t)dt)$.

For $C = \{i\}$ we have the Palm formula

$$n_C(f, x) = K_D^l f_i(x), \tag{4}$$

and for general C we have the following recursive formula (see e.g. Theorem 5.1.1 of [5], or Lemma 2.6 of [12]):

$$n_C(f, x) = \frac{1}{2}\sum_{A\subset C, A\neq\emptyset, C} G_D^l(4n_A(f,\cdot)n_{C\setminus A}(f,\cdot)) \tag{5}$$

$$p_C(f, \mu) = e^{-\langle\mu, V_D(f_0)\rangle}\sum_{\pi(C)}\langle\mu, n_{C_1}\rangle\cdots\langle\mu, n_{C_r}\rangle \tag{6}$$

Here $\pi(C)$ is the set of partitions of C. These formulas will allow us to construct a variety of X-harmonic functions of polynomial type.

2.4. **Absolute continuity.** The moment formulae together with the Markov property and Poisson representation yield an important theorem due to Dynkin. Let $\mathcal{M}_D^c$ be the space of finite measures compactly supported in D.

Theorem 1. *(Theorem 5.3.2 of [5]) Suppose $A \in \mathcal{F}_{\supset D}$. Then either $P_\mu(A) = 0$ for all $\mu \in \mathcal{M}_D^c$ or $P_\mu(A) > 0$ for all $\mu \in \mathcal{M}_D^c$.*

3. X-harmonic functions and conditioning

Recall the definition of X-harmonic functions, from the introduction: A non-negative function $H : \mathcal{M}_E^c \to [0, \infty)$ is *X-harmonic* if for any subdomain D and any finite measure $\mu \in \mathcal{M}_D^c$,

$$P_\mu(H(X_D)) = H(\mu).$$

We are going to touch upon three different kinds of X-harmonic functions, which are derived from conditioning SBM on its various boundary statistics. These boundary statistics are

(a) a Poisson random measure with characteristic measure βX_D;
(b) a random variable Z drawn from the probability distribution $\frac{X_D}{\langle X_D, 1\rangle}$ if $X_D \neq 0$, and set equal to some given $\Delta \notin \partial D$ if $X_D = 0$;
(c) $L(X_D)$, where L is a linear map from $\mathcal{M}_{\partial D}$ to a topological vector space V (for example, $L(\mu) = \mu$ or $L(\mu) = \langle\mu, 1\rangle$ means we condition on X_D or on its total mass)

Let S be any one of the above statistics. Let Σ be the state space of S endowed with the appropriate σ-algebra $\mathcal{S}$. Given $X_D = \nu$ we let P_S^ν denote the conditional distribution of S. For example, $P_S^\nu(f)$ equals $\langle \nu, f\rangle / \langle \nu, 1\rangle$ in the second case (provided $\nu \neq 0$), and $f(L(\nu))$ in the third.

P_μ denotes a probability measure in which X is an SBM started from $\mu \in \mathcal{M}_D$, and in which S is then drawn (if necessary) by further sampling. In other words, P_μ is a probability defined on the σ-field $\mathcal{G} = \mathcal{F}_{\subset D} \vee \sigma\{S\}$. When $\mu = \delta_x$ we set $P_\mu = P_x$. By construction, $P_\mu(f(S) \mid \mathcal{F}_{\subset D}) = P_S^{X_D}(f)$. In other words, for any $\mathcal{F}_{\subset D}$-measurable Y we have that

$$P_\mu(f(S)Y) = P_\mu(P_S^{X_D}(f)Y).$$

Likewise, we let $P_{\mu,S}$ and $P_{x,S}$ denote the marginal distribution of S under P_μ and P_x, so $P_{\mu,S}(f) = P_\mu(f(S))$.

What we want is the conditional law of X given $S = s$, which should therefore be a transition kernel $P_\mu^s(Y)$ from Σ, the state space of S, to the $\mathcal{F}_{\subset D}$ measurable functions. The following is Theorem 1.1 of Dynkin [7] in the case $S = X_D$. We follow Dynkin's proof, with some modifications.

Let us fix a point $x \in D$.

Theorem 2. *An X-harmonic version of*

$$H_x^s(\mu) = \frac{dP_{\mu,S}}{dP_{x,S}}(s) \tag{7}$$

exists. Moreover,

$$P_\mu^s(Y) = \frac{1}{H_x^s(\mu)} P_\mu(Y H_x^s(X_{D'})) \tag{8}$$

for all $D' \subset D$, and $\mathcal{F}_{D'}$-measurable Y. Any two versions will coincide for every μ, for $P_{x,S}$-a.e. s.

Proof. Let $\bar{H}^s(\mu)$ be any version of $\frac{dP_{\mu,S}}{dP_{x,S}}(s)$ that is jointly measurable with respect to μ and s. We know the existence of such a version from Theorem A.1. of [7]. Let O be a subdomain compactly contained in D. Then

$$P_\mu \bar{H}^s(X_O) = \bar{H}^s(\mu) \tag{9}$$

for $P_{x,S}$-a.e. s, $\forall \mu \in \mathcal{M}_D$. Dynkin [7] proves this when $S = X_D$, and the proofs for the other cases are almost identical to his. Next, we want to construct a version of $\frac{dP_{\mu,S}}{dP_{x,S}}(s)$ so that (9) holds $\forall \mu$, for $P_{x,S}$ a.e. s. To do this, we choose a countable base O_n (w.l.o.g. closed under finite unions), and probability measures $\mu_n \in \mathcal{M}_{O_n}^c$, and we let

$$R(d\eta) = \sum 2^{-n} P_{\mu_n}(X_{O_n} \in d\eta).$$

Note that (9) implies

$$P_\mu \bar{H}^s(X_{O_n}) = \bar{H}^s(\mu)$$

for $R \times P$-a.e. (μ, s). By Fubini's theorem we deduce that there exists a $P_{x,S}$-null set $\mathcal{N}$ s.t.

$$P_\mu \bar{H}^s(X_{O_n}) = \bar{H}^s(\mu) \ \forall n, \text{ for } R\text{-a.e. } \mu \in \mathcal{M}_D^c, \ \forall s \in \mathcal{N}^c. \tag{10}$$

For $s \in \mathcal{N}^c$ and $\mu \in \mathcal{M}_D^c$, we choose O_n containing the support of μ and define

$$H^s(\mu) = P_\mu \bar{H}^s(X_{O_n}).$$

We set $H^s(\mu)$ to some arbitrary value for $s \in \mathcal{N}$. The definition $H^s(\mu)$ is independent of the choice of O_n since, if $O_k \supset O_m$ then

$$\begin{aligned} P_\mu \bar{H}^s(X_{O_k}) &= P_\mu P_{X_{O_n}} \bar{H}^s(X_{O_k}) \\ &= P_\mu \bar{H}^s(X_{O_n}). \end{aligned}$$

The first equality is due to Markov property. The second equality is due to (10) and the fact that $P_\mu(X_{O_n} \in (\cdot))$ is absolutely continuous with respect to R, by Theorem 1.

Clearly, by (9), $H^s(\mu)$ is a version of $\frac{dP_{\mu,S}}{dP_{x,S}}(s)$ for each $\mu \in \mathcal{M}_D^c$. Next, we show that $H^s(\mu)$ is X-harmonic for each $s \in \mathcal{N}^c$. Let μ and O be s.t. $\mu \in \mathcal{M}_O^c$ and pick O_n s.t. O is compactly contained in O_n. Then, by definition,

$$H^s(\mu) = P_\mu(\bar{H}^s(X_{O_n}))$$

and

$$P_\mu H^s(X_O) = P_\mu P_{X_O}(\bar{H}^s(X_{O_n})).$$

By the Markov property these two are equal.

Let us define P_μ^s as in (8). it remains to prove that P_μ^s is the desired transition kernel. Let $Y \in \mathcal{F}_{D'}$. Then

$$\begin{aligned}
P_\mu(f(S)P_\mu^S(Y)) &= \int f(s)P_\mu^s(Y)\, P_{\mu,S}(ds) \\
&= \int f(s)\frac{1}{H_x^s(\mu)} \int Y(\omega)H_x^s(X_{D'}(\omega))P_\mu(d\omega)P_{\mu,S}(ds) \\
&= \int Y(\omega) \int \frac{1}{H_x^s(\mu)} f(s)H_x^s(X_{D'}(\omega))P_{\mu,S}(ds)P_\mu(d\omega) \\
&= \int Y(\omega) \int f(s)H_x^s(X_{D'}(\omega))P_{x,S}(ds)P_\mu(d\omega) \\
&= \int Y(\omega) \int f(s)P_{X_{D'}(\omega),S}(ds)P_\mu(d\omega) \\
&= P_\mu(YP_{X_{D'}}(f(S))) \\
&= P_\mu(f(S)Y).
\end{aligned}$$

Here we are using the definition of $P_{\mu,S}$, the definition of P_μ^s, Fubini's theorem, the definition of H_x^s, the definition of $P_{\nu,S}$, and the Markov property of X.

Uniqueness follows by a similar argument. Suppose H^s and $\tilde{H}^s$ are two X-harmonic versions of the density (7). Then

$$H^s(\mu) = \tilde{H}^s(\mu) \text{ for } P_{x,S}\text{-a.e. } s, \forall \mu$$

because they are densities. With $R(d\mu)$ as before, there is therefore a $P_{x,S}$-null set $\mathcal{N}$ such that

$$H^s(\mu) = \tilde{H}^s(\mu) \text{ for } R\text{-a.e. } \mu \text{ and for } s \notin \mathcal{N}.$$

Let $\mu \in \mathcal{M}_D^c$ and $s \notin \mathcal{N}$. Choose O_n such that $\mu \in \mathcal{M}_{O_n}^c$. Then by absolute continuity and X-harmonicity,

$$H^s(\mu) = P_\mu H^s(X_{O_n}) = P_\mu \tilde{H}^s(X_{O_n}) = \tilde{H}^s(\mu).$$

□

In the remainder of §3 we consider the three special cases described above. Our goal is to obtain relatively explicit formulae for H_x^s in each case.

3.1. **Conditioning on a Poisson random measure with characteristic measure βX_D.** Let N be a Poisson random variable with mean $\langle X_D, \beta\rangle$. Let $Z = \{Z_1, Z_2 \ldots\}$ be an i.i.d. sequence of random variables from $X_D/\langle X_D, 1\rangle$. Let

$$Y_\beta = \sum_{i=1}^N \delta_{Z_i}$$

Note that conditioned on X, Y_β is a Poisson random measure with characteristic measure βX_D, and that the construction makes sense even if $X_D = 0$, because then both N and Y equal 0.

Taking $S = Y_\beta$, Theorem 2 gives an X-harmonic function (which we denote $H_x^{\beta,\nu}$ to make explicit the dependence on β) for conditioning on $Y_\beta = \nu$. Here ν is an atomic measure. We let $P_\mu^{\beta,\nu}$ denote the law of the corresponding conditioned process. In principle this is only uniquely defined for a.e. ν, but we will find an explicit form that is valid more generally.

It will be convenient to also define variants of these objects. This time take $S = (Z_1, \dots, Z_k)$ if $N = k$, and $S = \Delta \notin D$ otherwise. Extend the domain of measurable $f : (\partial D)^k \to \mathbb{R}$ by setting $f(\Delta) = 0$. Set $X_D^k(dz_1, \dots, dz_k)$ to be the product measure $X_D(dz_1) \times \cdots \times X_D(dz_k)$, and write $P_\mu^{\beta,k}$ for the marginal measure $P_{\mu,S}$. In other words,

$$
\begin{aligned}
P_\mu^{\beta,k}(f) &= P_{\mu,S}(f) = P_\mu(f(S)) \\
&= \frac{1}{k!} P_\mu\Big((\int_{(\partial D)^k} \frac{f}{\langle X_D, 1\rangle^k} dX_D^k) \langle X_D, \beta\rangle^k e^{-\langle X_D,\beta\rangle} 1_{\{X_d \neq 0\}} \Big) \\
&= \frac{\beta^k}{k!} P_\mu\Big((\int_{(\partial D)^k} f dX_D^k) e^{-\langle X_D,\beta\rangle} \Big)
\end{aligned}
$$

So for any $\beta > 0$, positive integer k, and any k-tuple $z = \{z_i\}$ of elements of ∂D, Theorem 2 defines for us an X-harmonic function by

$$
H_x^{\beta,k,z}(\mu) = \frac{dP_\mu^{\beta,k}}{dP_x^{\beta,k}}(z_1, z_2, \dots, z_k). \tag{11}
$$

The law of the corresponding conditioned process is denoted $P_\mu^{\beta,k,z}$. If $k = 0$ we simply have scalars $P_\mu^{\beta,0} = P_\mu(e^{-\langle X_D,\beta\rangle}) = e^{-\langle \mu, l_\beta\rangle}$ and $P_x^{\beta,0} = e^{-l_\beta(x)}$, and take $H_x^{\beta,0}(\mu)$ to simply be the ratio

$$
H_x^{\beta,0}(\mu) = \frac{e^{-\langle \mu, l_\beta\rangle}}{e^{-l_\beta(x)}}.
$$

Let $l \geq 0$ be a bounded Borel function on D. For $x \in D$, we let $m_x^l(dz) = \Pi_x^l(\xi_{\tau_D} \in dz)$ denote harmonic measure on ∂D for the operator L^l. Then m_x^l and m_y^l are mutually absolutely continuous, for $x, y \in D$. Let

$$
k_x^l(y, z) = \frac{dm_y^l}{dm_x^l}(z)
$$

denote the density. If D were sufficiently regular, this would be a version of the Martin kernel for the operator L^l, but we make no such regularity assumptions at this point. We take k^l to be a jointly measurable version of this density that is harmonic in y, for each $z \in \partial D$. In the case $l = 0$ we write $m_x(dz) = m_x^0(dz)$ and $k_x(y, z) = k_x^0(y, z)$.

The particular case of interest is $l_\beta = 4V\beta$. Suppose that $k \geq 1$ and that $z_1, \dots, z_k \in \partial D$. For $C \subset K = \{1, \dots, k\}$ recursively define

$$
\rho_C^\beta = \begin{cases} k^{l_\beta}(\cdot, z_i), & \text{for } C = \{i\} \\ \sum_{A \subset C, \emptyset \neq A \neq C} G_D^{l_\beta}(4\rho_A^\beta \rho_{C\setminus A}^\beta), & \text{for } |C| > 1. \end{cases}
$$

Finally, set

$$
\rho_\mu^{\beta,k}(z_1, \dots, z_k) = e^{-\langle \mu, l_\beta\rangle} \sum \langle \mu, \rho_{C_1}\rangle \dots \langle \mu, \rho_{C_r}\rangle,
$$

where the sum ranges over all partitions $\{C_1, \dots, C_r\}$ of K.

Theorem 3. *Let D be a domain, and $\beta \geq 0$. Then*

(a) $H_x^{\beta,\nu}(\mu) = H_x^{\beta,k,z}(\mu)$ *for almost every* ν, *where* k *and* z *are such that*

$$\nu(dx) = \sum_1^k \delta_{z_i}(dx). \tag{12}$$

(b) *For* $(m^{l_\beta})^k$*-a.e.* $(z_1,\ldots,z_k)$, *we have* $\rho_\mu^{\beta,k}(z_1,\ldots,z_k) < \infty$, *and*

$$H_x^{\beta,k,z}(\mu) = \frac{\rho_\mu^{\beta,k}(z_1,\ldots,z_k)}{\rho_x^{\beta,k}(z_1,\ldots,z_k)}. \tag{13}$$

(c) *If* D *is smooth and bounded, then in fact* $\rho_\mu^{\beta,k}(z_1,\ldots,z_k) < \infty$ *whenever* $z_1,\ldots,z_k$ *are distinct.*

Proof. (a) $P_\mu^{\beta,k}(f)$ remains unchanged if we permute the arguments of f. Thus we can choose the densities $H_x^{\beta,k,z}(\mu)$ to be both X-harmonic and invariant under permutations of the z_i. A simple way to confirm this is to replace an X-harmonic choice of $H_x^{\beta,k,z}(\mu)$ by $\frac{1}{k!}\sum_\sigma H_x^{\beta,k,\sigma(z)}(\mu)$, where the sum is over permutations σ. The latter is still X-harmonic, and a version of the density $dP_\mu^{\beta,k}/dP_x^{\beta,k}$, but is also clearly invariant under permutations.

For a finite atomic measure ν, all of whose atoms have mass 1, find k and $z_1,\ldots,z_k$ such that (12) holds. Then define

$$\tilde H_x^{\beta,\nu}(\mu) := H_x^{\beta,k,z}(\mu).$$

Note that $\tilde H_x^{\beta,\nu}(\mu)$ is well defined, since $H_x^{\beta,k,z}$ depends only on $z^k := (z_1,\ldots,z_k)$ and is invariant under permuting z^k. (Note, if two sequences z and $\tilde z$ satisfy (12), then z^k and $\tilde z^k$ must be permutations of each other.)

If $\nu = \sum_{i=1}^k \delta_{z_i}$ write $f_k(z)$ for $f(\nu)$. If $k=0$ then f_0 is just the scalar $f(0)$. Also take the convention that $f(\Delta)=0$ if $X_D=0$. To finish the proof it is enough to observe

$$\begin{aligned}
P_{x,Y_\beta}(\tilde H_x^{\beta,(\cdot)}(\mu) f(\cdot)) &= P_x(\tilde H_x^{\beta,Y_\beta}(\mu) f(Y_\beta)) \\
&= \sum_{k=0}^\infty P_x\Big(e^{-\langle X_D,\beta\rangle}\frac{\langle X_D,\beta\rangle^k}{k!}1_{\{X_D\neq 0\}}\int H_x^{\beta,k,z}(\mu) f_k(z)\frac{X_D^k(dz)}{\langle X_D,1\rangle^k}\Big) \\
&= \sum_{k=0}^\infty \frac{\beta^k}{k!} P_x\Big(e^{-\langle X_D,\beta\rangle}\int H_x^{\beta,k,z}(\mu) f_k(z) X_D^k(dz)\Big) \\
&= \sum_{k=0}^\infty P_x^{\beta,k}(H_x^{\beta,k,(\cdot)}(\mu) f_k(\cdot)) \\
&= \sum_{k=0}^\infty P_\mu^{\beta,k}(f_k) \\
&= \sum_{k=0}^\infty P_\mu\Big(e^{-\langle X_D,\beta\rangle}\frac{\langle X_D,\beta\rangle^k}{k!}1_{\{X_D\neq 0\}}\int f_k(z)\frac{X_D^k(dz)}{\langle X_D,1\rangle^k}\Big) \\
&= P_\mu(f(Y_\beta)) \\
&= P_{\mu,Y_\beta}(f).
\end{aligned}$$

(b) Define $\tilde H_x^{\beta,k,z}(\mu)$ to be the right hand side of (13). Following an argument of [5], chapter 5 one can show that $\rho_\mu^{\beta,k}$ is the density of $P_\mu^{\beta,k}$ with respect to $(m^{l_\beta})^k$. The argument uses the moment formulae (4), (5), (6) and then pulls k factors of harmonic measure out of the resulting expressions, leaving the densities k^{l_β} behind. It follows that $\tilde{H}_x^{\beta,k,z}(\mu)$ is a version of the Radon Nikodym derivative in (11). The finiteness condition for $\rho_\mu^{\beta,k}$ follows immediately.

Furthermore, $\tilde{H}_x^{\beta,k,z}$ falls in the family of functions considered by Salisbury and Verzani in [12], Theorem 3.1 therefore shows that $\tilde{H}_x^{\beta,k,z}$ is X-harmonic. Thus $\tilde{H}_x^{\beta,k,z} = H_x^{\beta,k,z}$ for $(m^{l_\beta})^k$-a.e. z, which is the sense up to which $H_x^{\beta,k,z}$ is well defined.

(c) The argument for (c) is a straightforward modification of the estimates used in Theorem 5.3 of [12].

□

Remarks: (i). The conclusion is that we have obtained an explicit formula for $H_x^{\beta,\nu}(\mu)$. The abstract definition of this X-harmonic function was valid only up to an unspecified null set of ν's. Whereas the canonical expression we have obtained is well defined as long as ν is a finite atomic measure, all of whose atoms have mass 1.
(ii). The arguments of this section would work equally well for conditioning on the value of a Poisson random measure with characteristic measure $\beta(x)X_D(dx)$, where $\beta(x)$ is now a bounded measurable function on ∂D.
(iii). If D is smooth, then instead of taking $k_x^l(y,z)$ to be the density of m_y^l with respect to m_x^l, we could use the Poisson kernel in its place, and get a similar result. In other words, we could take the density of m_y^l with respect to the surface measure γ on ∂D, rather than the density with respect to m_x^l.

3.2. **Conditioning on a r.v. Z sampled from measure $\frac{X_D}{\langle X_D,1\rangle}$.** Recall that the random variable Z is drawn from the probability distribution $\frac{X_D}{\langle X_D,1\rangle}$ if $X_D \neq 0$, and set equal to some given $\Delta \notin \partial D$ if $X_D = 0$. Applying Theorem 2 gives us a family of X-harmonic functions

$$H_x^z = \frac{dP_{\mu,Z}(z)}{dP_{x,Z}(z)}$$

indexed by points z of $\{\Delta\} \cup \partial D$. We denote the law of the corresponding conditional process by P_μ^z.

Recall that ξ_t is a Brownian motion under Π_y. For $z \in \partial D$, we let Π_y^z be a probability under which ξ_t is a $k(\cdot,z)$-transform of Brownian motion. In other words,

$$\Pi_y^z(f(\xi_t), t < \tau_D) = \frac{1}{k(y,z)}\Pi_y(f(\xi_t)k(\xi_t,z), t < \tau_D)$$

for every bounded measurable f.

The following result establishes a concrete formula for H_x^z that is defined for m_c-a.e. $z \in \partial D$ when D is a general domain. When D is smooth, the same argument as in the previous section gives a canonical version, defined for all $z \in \partial D$.

Theorem 4. *Let D be a domain, and fix a base point $c \in D$. Then for $m_c(dz)$-almost all $z \in \partial D$,*

$$H_x^z(\mu) = \frac{\int_0^\infty \langle \mu, k(\cdot,z)\Pi_{(\cdot)}^z(e^{-\phi(u_\beta)})\rangle e^{-\langle u_\beta,\mu\rangle}d\beta}{\int_0^\infty k(x,z)\Pi_x^z(e^{-\phi(u_\beta)})e^{-u_\beta(x)}d\beta} \tag{14}$$

for every μ and x, where $u_\beta = V_D\beta$ and the random variable $\phi(u_\beta)$ is defined as

$$\phi(u_\beta) = 4\int_0^{\tau_D} u_\beta(\xi_t)dt. \tag{15}$$

Proof. We first find the Radon Nikodym derivative of $P_{\mu,Z}$ w.r.t. harmonic measure $m_c(dz)$ on the boundary of D. We observe that

$$P_{\mu,Z}(f) = P_\mu\Big(\frac{\langle X_D, f\rangle}{\langle X_D, 1\rangle}1_{\{X_D\neq 0\}}\Big) = -\int_0^\infty \frac{d}{d\lambda}P_\mu\Big(e^{-\lambda\langle X_D,f\rangle-\beta\langle X_D,1\rangle}\Big)|_{\lambda=0}\,d\beta. \tag{16}$$

Note that the above derivative equals 0 when $X_D = 0$. By the branching property,

$$P_\mu(e^{-\lambda\langle X_D,f\rangle-\beta\langle X_D,1\rangle}) = e^{-\langle \mu,u_{\lambda f+\beta}\rangle} \tag{17}$$

where

$$u_{\lambda f+\beta} = V_D(\lambda f + \beta) = \mathbb{N}_{(\cdot)}(1 - e^{-\langle X_D,\lambda f+\beta\rangle}).$$

Taking the derivative of the right side of (17), and evaluating at $\lambda = 0$ we get

$$P_\mu\Big(\frac{\langle X_D, f\rangle}{\langle X_D, 1\rangle}1_{\{X_D\neq 0\}}\Big) = \int_0^\infty \langle \mu, \mathbb{N}_{(\cdot)}(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle})\rangle e^{-\langle\mu,u_\beta\rangle}d\beta. \tag{18}$$

Differentiation under the integral sign is easily justified. By the Palm formula,

$$\begin{aligned}\mathbb{N}_y(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle}) = \Pi_y(f(\xi_{\tau_D})e^{-\phi(u_\beta)}) &= \int_{\partial D}\Pi^z_y(e^{-\phi(u_\beta)})f(z)\,m_y(dz)\\ &= \int_{\partial D}\Pi^z_y(e^{-\phi(u_\beta)})k_c(y, z)f(z)\,m_c(dz).\end{aligned}$$

So,

$$\langle \mu, \mathbb{N}_{(\cdot)}(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle})\rangle = \int_{\partial D}\langle \mu, k_c(\cdot, z)\Pi^z_{(\cdot)}(e^{-\phi(u_\beta)})\rangle f(z)\,m_c(dz).$$

Hence

$$\begin{aligned}&\int_0^\infty \langle \mu, \mathbb{N}_{(\cdot)}(\langle X_D, f\rangle e^{-\beta\langle X_D,1\rangle})\rangle e^{-\langle\mu,u_\beta\rangle}d\beta\\ &\qquad = \int_{\partial D} f(z)\Big(\int_0^\infty \langle \mu, k_c(\cdot, z)\Pi^z_{(\cdot)}(e^{-\phi(u_\beta)})\rangle e^{-\langle\mu,u_\beta\rangle}d\beta\Big)\,m_c(dz).\end{aligned}$$

Therefore, both $P_{\mu,Z}$ and $P_{x,Z}$ have densities with respect to $m_c(dz)$, given by

$$\int_0^\infty \langle \mu, k_c(\cdot, z)\Pi^z_{(\cdot)}(e^{-\phi(u_\beta)})\rangle e^{-\langle\mu,u_\beta\rangle}d\beta$$

and

$$\int_0^\infty k_c(x, z)\Pi^z_x(e^{-\phi(u_\beta)})e^{-u_\beta(x)}d\beta$$

respectively. The ratio of these two is a version of the desired Radon Nikodym derivative. To show that it equals H^z_x for almost all z, it simply remains to show that it is X-harmonic.

The denominator is simply a normalizing factor, so consider the numerator. For $l_\beta = 4u_\beta$, it is known (see Theorem 1.1 of [12]) that $\mu \mapsto \langle \mu, v\rangle e^{-\langle\mu,u_\beta\rangle}$ is X-harmonic whenever v is $\mathcal{L}^{l_\beta}$-harmonic. And in our case, we have that $\langle \mu, k_c(\cdot, z)\Pi^z_{(\cdot)}(e^{-\phi(u_\beta)})\rangle e^{-\langle\mu,u_\beta\rangle} = \langle \mu, k^{l_\beta}_c(\cdot, z)\rangle e^{-\langle\mu,u_\beta\rangle}$ as required. □

Note that all the arguments of this section could be extended to the case that Z is chosen according to the distribution

$$\frac{h(z)X_D(dz)}{\langle X_D, h\rangle},$$

where h is a positive, bounded, measurable function on ∂D.

3.3. **Conditioning on a linear function of X_D.** Let L be a linear and continuous map from the linear cone of positive finite measures $\mathcal{M}_{\partial D}$ on ∂D to a complete, separable topological vector space V. Let V_+ be the image of $\mathcal{M}_{\partial D}$, and write $V^* = V_+ \setminus \{0\}$. Assume that $L\mu = 0$ implies $\mu = 0$.

Let $\mathcal{B}(V_+)$ be the Borel subsets of V_+. Let T^n be the map $V_+^n \to V_+^n$ defined by:

$$T_n(v_1, \dots, v_n) \mapsto (v_1 + v_2 + \dots + v_n, v_1, \dots, v_{n-1}),$$

and for $A \in \mathcal{B}(V_+)$, let

$$N_{x,L(X_D)}(A) = \mathbb{N}_x(L(X_D) \in A, X_D \neq 0).$$

We fix a base point $c \in D$ and define a reference measure R on V^* by

$$R(A) = P_{c,L(X_D)}(A, X_D \neq 0).$$

Its total mass is $r_0 = P_{c,L(X_D)}(V^*) = 1 - e^{-u(c)}$, where $u(y) = -\log P_y(X_D = 0)$. By Theorem 2 we have X-harmonic functions $H_x^v(\mu) = dP_{\mu,L(X_D)}/dP_{x,L(X_D)}(v)$. We will base our analysis mainly on the choice $x = c$, in which case $H_c^v(d\mu) = dP_{\mu,L(X_D)}/dR(v)$.

Lemma 5. *There exists a strictly positive, measurable function $\gamma_v(x)$ such that*

$$N_{x,L(X_D)}(dv) = \gamma_v(x) R(dv). \tag{19}$$

In addition, there exists a measurable kernel $K_n(v; dv_1, dv_2, \dots, dv_{n-1})$ from V^* to $(V^*)^{n-1}$, such that

$$R^n \circ T^{-1}(dv, dv_1, dv_2, \dots, dv_{n-1}) = K_n(v; dv_1, dv_2, \dots, dv_{n-1}) R(dv). \tag{20}$$

Moreover $K_n(v, \cdot)$ is a strictly positive measure, for R-a.e. v.

Proof. Recall that

$$P_{\mu,L(X_D)}(A) = P_\mu(L(X_D) \in A).$$

Because all $P_\mu(X_D \in \cdot)$ are equivalent so are the $P_{\mu,L(X_D)}$, as are their restriction to V^*. Thus all $P_{\mu,L(X_D)}$ (when restricted to V^*) are equivalent to R. Moreover, since

$$N_{x,L(X_D)}(A) = \mathbb{N}_x(L(X_D) \in A, X_D \neq 0)) = \mathbb{N}_x(P_{X_{D'}}(L(X_D) \in A, X_D \neq 0))$$

for $x \in D' \subset D$, $N_{x,L(X_D)}$ is also equivalent to R. Thus we get the (19) with a measurable $\gamma_v(x)$ strictly positive.

It will be convenient for the proof to write $R(dv) = r_0 \tilde{R}(dv)$, where $\tilde{R}$ is a probability measure. In other words, $\tilde{R}(dv) = P_c(L(X_D) \in dv \mid X_D \neq 0)$. Note that with this choice of $\tilde{R}$, $\tilde{R}^n \circ T^{-1}(dv, dv_1, dv_2 \dots, dv_{n-1})$ is the joint distribution of $(X_1 + \dots + X_n, X_1, \dots, X_{n-1})$ where X_i are independent random variables with distribution $\tilde{R}$. Let R_n be the marginal distribution of $X_1 + \dots + X_n$, where the X_i are as above. The following decomposition is then immediate:

$$\tilde{R}^n \circ T^{-1}(dv, dv_1, dv_2 \dots, dv_{n-1}) = \tilde{K}_n(v; dv_1, dv_2, \dots, dv_{n-1}) R_n(dv)$$

where $\tilde{K}_n(v; dv_1, dv_2, \dots, dv_{n-1})$ is the conditional probability kernel for $(X_1, \dots, X_{n-1})$ given $X_1 + \dots + X_n$.

We now show that R_n is absolutely continuous with respect to R. Let $X_D^1, \dots, X_D^n$ be n independent realizations of the exit measure under the law P_c. Then the distribution of $X_D^1 + \dots + X_D^n$ is given by the $P_{n\delta_c}$ distribution of X_D.

Let F be s.t. $R(F) = 0$, i.e.

$$P_c(F(L(X_D)) 1_{\{X_D \neq 0\}}) = 0 \tag{21}$$

Because P_c and $P_{n\delta_c}$ are absolutely continuous, (21) implies

$$P_{n\delta_c}(F(L(X_D)) 1_{\{X_D \neq 0\}}) = 0 \tag{22}$$

Since

$$\begin{aligned} P_{n\delta_c}(F(L(X_D))1_{\{X_D\neq 0\}}) &= (P_c)^n(F(L(X_D^1+\cdots+X_D^1))1_{\{X_D^1+\cdots+X_D^n\neq 0\}}) \\ &\geq (P_c)^n(F(L(X_D^1)+\cdots+L(X_D^1))1_{\{X_D^1\neq 0\}}\cdots 1_{\{X_D^n\neq 0\}}) \\ &= r_0^n R_n(F), \end{aligned}$$

this implies that $R_n(F)=0$, so indeed, R_n is absolutely continuous with respect to R.

If $h^n(v)$ is the Radon Nikodym derivative of R_n with respect to R, we get (20) with

$$K_n(v;dv_1,dv_2,\dots,dv_{n-1}) = \tilde{K}_n(v;dv_1,dv_2,\dots,dv_{n-1})h^n(v).$$

It remains only to show that $K_n(v,\cdot)$ is strictly positive. Because $\tilde{K}_n(v,\cdot)$ is, this amounts to showing the converse to the absolute continuity result above, namely that R is absolutely continuous with respect to R_n.

Our approach is to use the Poisson representation, as in the absolute continuity argument in [5]. Suppose $R_n(F(X_D))=0$ and $0\leq F\leq 1$. The Poisson representation gives that

$$\begin{aligned} P_\mu(F(X_D),X_D\neq 0) &= \sum_{k=1}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{k!}\int\int F(L(\nu_1)+\cdots+L(\nu_k))\mathbb{N}_{x_1}(X_D\in d\nu_1,X_D\neq 0) \\ &\qquad \cdots\mathbb{N}_{x_n}(X_D\in d\nu_k,X_D\neq 0)\mu(dx_1)\cdots\mu(dx_k) \\ &= \sum_{k=1}^{\infty}\frac{e^{-\langle\mu,u\rangle}}{k!}\int f_k(x_1,\dots,x_k)\mu(dx_1)\cdots\mu(dx_k), \end{aligned}$$

where

$$f_k(x_1,\dots,x_k) = \int F(L(\nu_1)+\cdots+L(\nu_k))\mathbb{N}_{x_1}(X_D\in d\nu_1,X_D\neq 0)\cdots\mathbb{N}_{x_k}(X_D\in d\nu_k,X_D\neq 0).$$

Let $c\in D'\subset D$. Then

$$\begin{aligned} P_c(F(X_D),X_D\neq 0) &= P_c(P_{X_{D'}}(F(X_D),X_D\neq 0)) \\ &= P_c(\sum_{k=1}^{\infty}\frac{e^{-\langle X_{D'},u\rangle}}{k!}\int f_k(x_1,\dots,x_k)X_{D'}(dx_1)\cdots X_{D'}(dx_k)). \end{aligned}$$

There is a similar Poisson representation for $R_n(F(X_D))$, involving a sum of integrals of the f_k for $k\geq n$. Since $R_n(F(X_D))=0$, we conclude that for each $k\geq n$ there are $x_1,\dots,x_k\in D$ such that $f_k(x_1,\dots,x_k)=0$. By absolute continuity, we conclude that $f_k(x_1,\dots,x_k)=0$ for every $x_1,\dots,x_k$.

Since $F\leq 1$ we obtain the bound

$$f(x_1,\dots,x_k)\leq\prod_{j=1}^{k}\mathbb{N}_{x_j}(X_D\neq 0) = u(x_1)\cdots u(x_k).$$

Therefore

$$P_c(F(X_D),X_D\neq 0)\leq\sum_{k=1}^{n-1}P_c(\frac{e^{-\langle X_{D'},u\rangle}}{k!}\langle X_{D'},u\rangle^k).$$

Because $u\to\infty$ on ∂D, all terms $e^{-\langle X_{D'},u\rangle}\langle X_{D'},u\rangle^k$ have stochastic boundary value 0 as we let $D'\uparrow D$ (see [5]). In particular, this sum converges to 0 as we let $D'\uparrow D$, showing that $R(F(X_D))=0$. In other words, R is absolutely continuous with respect to R_n, as required.

□

We can now establish our first canonical version of the X-harmonic functions $H_x^v(\mu)$.

Theorem 6. *Fix a base point $c \in D$. For $P_{c,L(X_D)}$ almost every v,*

$$H_x^v(\mu) = \frac{\tilde{H}_c^v(\mu)}{\tilde{H}_c^v(\delta_x)} \tag{23}$$

for every μ and x, where

$$\tilde{H}_c^v(\mu) = \begin{cases} e^{-\langle \mu,u\rangle + u(c)}, & \text{if } v = 0 \\ e^{-\langle \mu,u\rangle}\langle \mu, \gamma_v\rangle + \sum_{n=2}^{\infty} \int \frac{1}{n!} e^{-\langle \mu,u\rangle} K_n(v; dv_1, \dots, dv_{n-1}))\times \\ \quad \times\langle \mu, \gamma_{v_1}\rangle \cdots \langle \mu, \gamma_{v_{n-1}}\rangle \langle \mu, \gamma_{v-(v_1+\cdots+v_{n-1})}\rangle, & \text{if } v \neq 0 \end{cases}$$

Proof. Let F be a non-negative Borel function on V_+.

$$\begin{aligned}
&\int P_{c,L(X_D)}(dv) F(v) \tilde{H}_c^v(\mu) \\
&= P_{c,L(X_D)}(\{0\}) F(0) \tilde{H}_c^0(\mu) + \int F(v) \tilde{H}_c^v(\mu) R(dv) \\
&= e^{-u(c)} F(0) e^{-\langle \mu,u\rangle + u(c)} + e^{-\langle \mu,u\rangle} \int F(v) \langle \mu, \gamma_v\rangle R(dv) + \\
&\qquad + \sum_{n=2}^{\infty} \frac{e^{-\langle \mu,u\rangle}}{n!} \int\int \langle \mu, \gamma_{v_1}\rangle \cdots \langle \mu, \gamma_{v_{n-1}}\rangle \langle \mu, \gamma_{v-(v_1+\cdots+v_{n-1})}\rangle \times \\
&\qquad\qquad \times F(v) K_n(v; dv_1, \dots, dv_{n-1}) R(dv) \\
&= e^{-\langle \mu,u\rangle} F(0) + \sum_{n=1}^{\infty} \frac{e^{-\langle \mu,u\rangle}}{n!} \int F(v_1 + \cdots + v_n) \langle \mu, \gamma_{v_1}\rangle \cdots \langle \mu, \gamma_{v_n}\rangle R(dv_1) \cdots R(dv_n) \\
&= e^{-\langle \mu,u\rangle} F(0) + \sum_{n=1}^{\infty} \frac{e^{-\langle \mu,u\rangle}}{n!} \int F(v_1 + \cdots + v_n) \langle \mu, N_{(\cdot),L(X_D)}(dv_1)\rangle \cdots \langle \mu, N_{(\cdot),L(X_D)}(dv_n)\rangle \\
&= \sum_{n=0}^{\infty} \frac{e^{-\langle \mu,u\rangle}}{n!} \int F(L(\nu_1) + \cdots + L(\nu_n)) \mathcal{R}_\mu(d\nu_1) \cdots \mathcal{R}_\mu(d\nu_n) \\
&= P_\mu(F(L(X_D)) = P_{\mu,L(X_D)}(F)
\end{aligned}$$

where we are using the Poisson representation. Thus $\tilde{H}_c^v(\mu)$ is a version of the density defining $H_c^v(\mu)$, and the normalization of (23) establishes the same statement with c replaced by x. The result will therefore follow once we show that $\tilde{H}_c^v$ is an X-harmonic function. This is clear for $v = 0$. For $v \neq 0$, the proof will be a special case of Theorem 12. The reader may verify this once the latter proof is given

□

Remarks:

(i) We worked in Theorem 2 to prove abstractly that $H_x^v(\mu)$ was well defined for every μ, for almost all ν. This property is now also clear by construction, from the explicit formulae of Theorem 6. Of course, this result does not remove all ambiguity in the choice of H_x^v, since γ_v and $K(v;\cdot)$ are only well defined for a.e. v. Ideally we would like to prove continuity properties of these objects in v, that would then specify them uniquely. But we have not succeeded in doing that. In subsequent sections we will however be able to clarify the structure of these objects, and show how they determine the behaviour of the $H^v - x$-transformed super-Brownian motion.

(ii) $H^{\beta,k,z}$ falls in the family of X-harmonic functions considered by Salisbury and Verzani in [12]. This family of X-harmonic functions are characterized by a function g, and $\mathcal{L}^{4g}$-harmonic functions $v_1, \dots, v_k$. In our example the function g is $u_\beta = V_D\beta$, and the harmonic functions v_i

are the functions $k_c^{l_\beta}(\cdot, z_i)$. In [12] it is shown that for D Lipschitz of dimension $d \geq 4$, $g = 0$, and $v_i = k_c(\cdot, z_i)$ where $z_1, \ldots, z_k$ are distinct points chosen on the boundary, the resulting X-harmonic function corresponds to conditioning SBM to hit the points z_i. The same argument would work in dimension $d = 3$, at least when D is smooth.

(iii) An interesting question is what happens to H^{n,ν_n} as $n^{-1}\nu_n$ converges to a finite measure ν on the boundary. In section 5 we will show that

$$P_\mu^{n,Y_n} \to P_\mu^{X_D}$$

weakly almost surely, in a sense. A stronger result would be that H_x^{n,ν_n} converges to H_x^ν, whenever $n^{-1}\nu_n \to \nu$, but we cannot show this.

(iv) An important case is when $L(X_D) = X_D$. The corresponding X-harmonic function H^ν can be thought as the analogue of the Martin kernel. Indeed in section 5 we will show something close to the statement that if D is a smooth domain, then for almost all ν, H^ν is extreme.

4. Fragmentation system description of P_μ^ν

The results of this section apply in general to the conditional law P_μ^v given $L(X_D) = v$. For simplicity, however, we will carry out the computations for the case $L(\nu) = \nu$.

Recall that $u > 0$ is the maximal solution of the nonlinear PDE

$$\frac{1}{2}\Delta u = 2u^2$$

on D. We have

$$u(x) = \mathbb{N}_x(X_D > 0) = \lim_{\beta\to\infty} V_D\beta(x) \tag{24}$$

Recall the reference measure $R(d\nu)$ of Lemma 5. In that result, $\gamma_\nu(x)$ was any jointly measurable version of the Radon Nikodym derivative of N_{x,X_D} with respect to R. In the following, we refine this choice.

Lemma 7. *There exists a version of the function $\gamma_\nu(x)$ of Lemma 5 such that for R almost every ν*

$$\gamma_\nu(x) = \mathbb{N}_x(H_c^\nu(X_{D'})) \tag{25}$$

for every x, and every $D' \subset D$, where H_c^ν is as defined in Theorem 6.

Proof. Let $\bar{\gamma}_x^\nu$ be any jointly measurable version of the Radon-Nikodym density of N_{x,X_D} with respect to R, as in Lemma 5. In other words,

$$\mathbb{N}_x(X_D \in A, X_D \neq 0) = \int_A \bar{\gamma}_\nu(x) R(d\nu). \tag{26}$$

Define $\bar{H}_c^\nu$ as in Theorem 6, using the functions $\bar{\gamma}_\nu$. Then set $\gamma_\nu(x) = \mathbb{N}_x(\bar{H}_c^\nu(X_{D'}))$. This choice does not depend on D', because

$$\mathbb{N}_x(\bar{H}_c^\nu(X_{D'})) = \mathbb{N}_x(P_{X_{D''}}(\bar{H}_c^\nu(X_{D'}))) = \mathbb{N}_x(\bar{H}_c^\nu(X_{D''}))$$

whenever $D'' \subset D'$, since $\bar{H}_c^\nu$ is X-harmonic.

We now show that for fixed x, this gives a version of $\bar{\gamma}_\nu(x)$. If $D' \subset D$ then

$$\begin{aligned}\int_{\{\nu\neq 0\}} F(\nu) N_{x,X_D}(d\nu) &= \mathbb{N}_x\big(F(X_D)1_{\{X_D\neq 0\}}\big)\\ &= \mathbb{N}_x\big(P_{X_{D'}}(F(X_D)1_{\{X_D\neq 0\}})\big)\\ &= \mathbb{N}_x\Big(\int_{\{\nu\neq 0\}} \bar{H}_c^\nu(X_{D'})F(\nu)R(d\nu)\Big)\\ &= \int_{\{\nu\neq 0\}} F(\nu)R(d\nu)\mathbb{N}_x(\bar{H}_c^\nu(X_{D'})) = \int_{\{\nu\neq 0\}} F(\nu)\gamma_\nu(x)R(d\nu).\end{aligned}$$

Now define $H_c^\nu(\mu)$ as in Theorem 6 using the functions $\gamma_\nu(x)$. Then $H_c^\nu(\mu)$ and $\bar{H}_c^\nu(\mu)$ are both X-harmonic and versions of the same Radon-Nikodym density, therefore they are equal for every μ, for R-almost all ν. Hence $\gamma_\nu(x) = \mathbb{N}_x(\bar{H}_c^\nu(X_D')) = \mathbb{N}_x(H_c^\nu(X_{D'}))$ for every D' and every x, for R-almost all ν, which implies (25). □

In the remainder of this section, we assume that γ_ν satisfies (25), and that H_c^ν is defined as in Theorem 6. If $D' \subset D$ we may then define a change of measure by

$$\mathbb{N}_x^\nu(Z) = \frac{1}{\gamma_\nu(x)}\mathbb{N}_x(ZH_c^\nu(X_{D'}))$$

for positive $\mathcal{F}_{D'}$-measurable Z. Because, H_c^ν is X-harmonic, $\mathbb{N}_x^\nu$ is defined consistently on $\mathcal{F}_D$, and we have that $\mathbb{N}_x^\nu$ is a probability law because of Lemma 7.

In the remainder of this section we turn to the problem of giving an explicit probabilistic construction of $\mathbb{N}_x^\nu$ in terms of a backbone along which unconditioned mass is created. We do this in two steps. Let $H_1^\nu(\mu) = e^{-\langle\mu,u\rangle}\langle\mu,\gamma_\nu\rangle$, and for $n \geq 2$ let

$$H_n^\nu(\mu) = \int \frac{e^{-\langle\mu,u\rangle}}{n!} K_n(\nu; d\nu_1,\dots,d\nu_{n-1}))\langle\mu,\gamma_{\nu_1}\rangle\cdots\langle\mu,\gamma_{\nu_{n-1}}\rangle\langle\mu,\gamma_{\nu-(\nu_1+\nu_2,\dots,\nu_{n-1})}\rangle. \tag{27}$$

Then $H_c^\nu = \sum_{n\geq 1} H_n^\nu$. We will first use the Palm formula to establish an inductive relationship for H_n^ν, and then compare this to the inductive relationship coming from the first branch of the backbone.

We will make use of a stochastic process ξ_t under various measures Π_x or Π_x^{4u}. In either case we use the shorthand

$$\mathcal{N}_t^{D'}(\phi) = e^{-\int_0^t 4\mathbb{N}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})\,ds}$$

where D' is a subdomain of D and let $\tau_{D'}$ be the exit time of ξ from D'.

Lemma 8. *If $n = 1$ then*

$$\mathbb{N}_x\big(e^{-\langle X_{D'},\phi\rangle}H_1^\nu(X_{D'})\big) = \Pi_x\big(\gamma_\nu(\xi_{\tau_{D'}})\mathcal{N}_{\tau_{D'}}^{D'}(\phi+u)\big).$$

If $n \geq 2$ then

$$\begin{aligned}\mathbb{N}_x(e^{-\langle X_{D'},\phi\rangle}H_n^\nu(X_{D'})) \quad = \quad &\sum_{m=1}^{n-1}\int \Pi_x\bigg(\int_0^{\tau_{D'}} 2\mathcal{N}_t^{D'}(\phi+u)\mathbb{N}_{\xi_t}(e^{-\langle X_{D'},\phi\rangle}H_m^{\nu'})\\ &\mathbb{N}_{\xi_t}(e^{-\langle X_{D'},\phi\rangle}H_{n-m}^{\nu-\nu'}(X_{D'})\,dt\bigg)\ K(\nu,d\nu').\end{aligned}$$

Proof. If $n = 1$ then $H_1^\nu(\mu) = e^{-\langle\mu,u\rangle}\langle\mu,\gamma_\nu\rangle$ and the result is an immediate consequence of the basic Palm formula (4).

If $n \geq 2$, by the recursive moment formulae (5),

$$\mathbb{N}_x(e^{-\langle X_{D'},\phi\rangle} H_n^\nu(X_{D'})) = \frac{1}{n!}\int \mathbb{N}_x(e^{-\langle X_{D'},\phi+u\rangle}\Pi_i\langle X_{D'},\gamma_{\nu_i}\rangle)\, K_n(\nu; d\nu_1,\ldots,d\nu_{n-1})$$

$$= \frac{1}{2\cdot n!}\int \sum_{\substack{M\subset N\\ \emptyset,N\neq M}} \Pi_x\Big(\int_0^{\tau_{D'}} 4\mathcal{N}_t^{D'}(\phi+u)\cdot$$

$$\cdot\, \mathbb{N}_{\xi_t}(e^{-\langle X_{D'},\phi+u\rangle}\Pi_{i\in M}\langle X^{D'},\gamma_{\nu_i}\rangle)\mathbb{N}_{\xi_t}(e^{-\langle X_{D'},\phi+u\rangle}\Pi_{i\notin M}\langle X^{D'},\gamma_{\nu_i}\rangle)\,dt\Big)\, K_n(\nu; d\nu_1,\ldots,d\nu_n).$$

There are $\binom{n}{m}$ possible choices of M in the above expression, with cardinality m. Also note that we can decompose K_n as

$$K_n(\nu; d\nu_1,\ldots,d\nu_{n-1}) = \int_{\nu'} K_m(\nu'; d\nu_1,\ldots,d\nu_m)K_n(\nu-\nu'; d\nu_{m+1},\ldots,d\nu_{n-1})K(\nu; d\nu').$$

Therefore, by rearranging the indices, we get

$$\mathbb{N}_x(e^{-\langle X_{D'},\phi\rangle} H_n^\nu(X_{D'})) = \sum_{m=1}^{n-1}\int K(\nu; d\nu')\Pi_x\Big(\int_0^{\tau_{D'}} 2\mathcal{N}_t^{D'}(\phi+u)\cdot$$

$$\cdot\Big[\frac{1}{m!}\int \mathbb{N}_{\xi_t}(e^{-\langle X_{D'},\phi+u\rangle}\Pi_{i=1}^m\langle X_{D'},\gamma_{\nu_i}\rangle)\, K_m(\nu'; d\nu_1,\ldots,d\nu_{m-1})\Big]$$

$$\cdot\Big[\frac{1}{(n-m)!}\int \mathbb{N}_{\xi_t}(e^{-\langle X_{D'},\phi+u\rangle}\Pi_{i=1}^{n-m}\langle X_{D'},\gamma_{\nu_i}\rangle)\, K_{n-m}(\nu-\nu'; d\nu_1,\ldots,d\nu_{n-m-1})\Big]dt\Big).$$

Now apply induction. □

Now set $\Gamma_\nu = 2\int \gamma_{\nu'}\gamma_{\nu-\nu'}\, K(\nu; d\nu')$.

Theorem 9. *The function $\gamma_\nu(x)$ is $\mathcal{L}^{4u}$-superharmonic, and hence lower-semi-continuous in x. For R-almost all ν it is in fact an $\mathcal{L}^{4u}$-potential, and satisfies*

$$\gamma_\nu = G_D^{4u}\Big[2\int \gamma_\nu\gamma_{\nu-\nu'}\, K(\nu; d\nu')\Big]. \tag{28}$$

Proof. Let D_k be a sequence of domains exhausting D. Since $V_{D_k}u = u$, we have that $\mathcal{N}_t^{D_k}(u) = e^{-\int_0^t 4u(\xi_s)\,ds}$ for $t < \tau_{D_k}$. Thus

$$\begin{aligned}
\gamma_\nu(x) &= \mathbb{N}_x(H_c^\nu(X_{D_k})) = \sum_{n=1}^\infty \mathbb{N}_x(H_n^\nu(X_{D_k}))\\
&= \mathbb{N}_x(H_1^\nu(X_{D_k})) + \\
&\qquad + \sum_{n=2}^\infty\sum_{m=1}^{n-1}\int \Pi_x\Big(\int_0^{\tau_{D_k}} 2\mathcal{N}_t^{D_k}(u)\mathbb{N}_{\xi_t}(H_m^{\nu'})\mathbb{N}_{\xi_t}(H_{n-m}^{\nu-\nu'})\,dt\Big)\, K(\nu; d\nu')\\
&= \mathbb{N}_x(H_1^\nu(X_{D_k})) + \sum_{m=1}^\infty\sum_{j=1}^\infty\int \Pi_x^{4u}\Big(\int_0^{\tau_{D_k}} 2\mathbb{N}_{\xi_t}(H_m^{\nu'})\mathbb{N}_{\xi_t}(H_j^{\nu-\nu'})\,dt\Big)\, K(\nu; d\nu'),\\
&= \mathbb{N}_x(H_1^\nu(X_{D_k})) + 2\int \Pi_x^{4u}(\int_0^{\tau_{D_k}} \gamma_{\nu'}(\xi_t)\gamma_{\nu-\nu'}(\xi_t)\,dt)\, K(\nu; d\nu')\\
&= \mathbb{N}_x(H_1^\nu(X_{D_k})) + \Pi_x^{4u}\Big(\int_0^{\tau_{D_k}} \Gamma_\nu(\xi_t)\,dt\Big).
\end{aligned}$$

The first term is $\mathcal{L}^{4u}$-harmonic on D_k by Lemma 8, and the second term is an $\mathcal{L}^{4u}$-potential, so γ_ν is $\mathcal{L}^{4u}$-superharmonic on each D_k. Thus it is so on D as well.

Moreover

$$\begin{aligned}
\int \mathbb{N}_x(H_1^\nu(X_{D_k}))R(d\nu) &= \mathbb{N}_x(e^{-\langle X_{D_k},u\rangle}\int \langle X_{D_k},\gamma_\nu\rangle R(d\nu)) \\
&= \mathbb{N}_x(e^{-\langle X_{D_k},u\rangle}\int\int X_{D_k}(dy)\gamma_\nu(y)R(d\nu)) \\
&= \mathbb{N}_x(e^{-\langle X_{D_k},u\rangle}\int X_{D_k}(dy)\mathbb{N}_y(X_D\neq 0)) \\
&= \mathbb{N}_x(e^{-\langle X_{D_k},u\rangle}\langle X_{D_k},u\rangle).
\end{aligned}$$

Because $u\to\infty$ on ∂D, the latter has stochastic boundary value 0 (see [5]). Thus

$$\int \mathbb{N}_x(H_1^\nu(X_{D_k}))R(d\nu)\to 0$$

as $k\to\infty$. By Fatou's lemma,

$$\liminf_{k\to\infty}\mathbb{N}_x(H_1^\nu(X_{D_k}))=0$$

for R-a.e. ν. But the second term in our expression for $\gamma_\nu(x)$ is monotone in k, and thus in fact $\gamma_\nu(x)=\Pi_x^{4u}(\int_0^{\tau_{D_k}}\Gamma_\nu(\xi_t)\,dt)$ for R-a.e. ν, for every x.

Choosing a countable dense set $x_1,x_2,\ldots,$ we therefore have a null set B for R, such that the above equality holds for every x and for every $\nu\notin B$. Since both functions are lower-semi-continuous, and agree on a countable dense set, it follows that (28) holds for every $\nu\notin B$. □

Suppose now that ν satisfies (28). Let $\hat{\mathbb{N}}_x$ be the excursion measure of a super-process whose spatial motion is killed at at rate u. In other words, for any $D'\subset D$, we have

$$\hat{\mathbb{N}}_x(F(X_{D'}))=\mathbb{N}_x\left(e^{-\langle X_{D'},u\rangle}F(X_{D'})\right).$$

We define a probability Q_x^ν under which there is a branching diffusion and an associated measure valued process evolving as follows. We start a γ_ν-transform of a $\mathcal{L}^{4u}$ process off at x. Since γ_ν is a potential, this process dies before reaching ∂D. Say it dies at y. Then we choose ν' at random with distribution density

$$\frac{2}{\Gamma_\nu(y)}\gamma_{\nu'}(y)\gamma_{\nu-\nu'}(y)K(\nu;d\nu').$$

Note that to do this we do need $\Gamma_\nu>0$. In other words, this is a point where we require $K(\nu,d\nu')$ to be a strictly positive measure.

Almost surely, (28) holds for both ν' and $\nu-\nu'$, so we may start two new processes at y, following γ_ν and $\gamma_{\nu-\nu'}$ transforms of L^{4u}. We may repeat this process infinitely often. This defines a branching particle system. Let Υ_t denote the measure-valued process putting a unit point mass at each particle alive at time t. We then create mass uniformly along this set of particle paths, which then evolves according to the excursion law $\hat{\mathbb{N}}$. More properly, generate a Poisson random measure with intensity

$$\int_0^\infty 4\Upsilon_t(dz)\hat{\mathbb{N}}_z(\cdot)\,dt$$

and add up the resulting measure-valued processes to form X.

If we wish to write down the corresponding exit measures $X_{D'}$, $D'\subset D$ such that $x\in D'$, we can use the truncated particle system in which particles are killed (and not restarted) upon exiting D'. Denote the associated measure-valued process $\Upsilon_t^{D'}$. Then by definition,

$$Q_x^\nu(e^{-\langle X_{D'},\phi\rangle})=Q_x^\nu\big(e^{-\int_0^\infty 4\langle\Upsilon_t^{D'},\hat{\mathbb{N}}.(1-e^{-\langle X_{D'},\phi\rangle})\rangle\,dt}\big).$$

The main result of this section is

Theorem 10. *Assume that ν satisfies (28). Then $\mathbb{N}_x^\nu(e^{-\langle X_{D'},\phi\rangle}) = Q_x^\nu(e^{-\langle X_{D'},\phi\rangle})$ for every D' and ϕ.*

Proof. Write $\Upsilon^{D'} \sim n$ if exactly n particles of $\Upsilon^{D'}$ exit D'. We will show by induction on n that

$$\mathbb{N}_x(e^{-\langle X_{D'},\phi\rangle} H_n^\nu(X_{D'})) = \gamma_\nu(x) Q_x^\nu(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim n).$$

The theorem then follows by summing on n.

Note first that

$$\begin{aligned}
&e^{-\int_0^t 4u(\xi_s)\,ds} e^{-\int_0^t 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})} \\
&= \exp(-4\int_0^t [u(\xi_s) + \mathbb{N}_{\xi_s}(e^{-\langle X_{D'},u\rangle} - e^{-\langle X_{D'},\phi+u\rangle})]\,ds) \\
&= \exp -4\int_0^t [u(\xi_s) - \mathbb{N}_{\xi_s}(1-e^{-\langle X_{D'},u\rangle}) + \mathbb{N}_{\xi_s}(1-e^{-\langle X_{D'},\phi+u\rangle})]\,ds \\
&= \mathcal{N}_t^{D'}(\phi+u)
\end{aligned}$$

If $n = 1$ then

$$\begin{aligned}
&\gamma_\nu(x) Q_x^\nu(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim 1) \\
&= \gamma_\nu(x) \Pi_x^{4u,\gamma_\nu}\Big(e^{-\int_0^{\tau_{D'}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})\,ds} 1_{\tau_{D'}<\zeta}\Big) \\
&= \Pi_x^{4u}\Big(\gamma_\nu(\xi_{\tau_{D'}}) e^{-\int_0^{\tau_{D'}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})\,ds} 1_{\tau_{D'}<\zeta}\Big) \\
&= \Pi_x\Big(\gamma_\nu(\xi_{\tau_{D'}}) e^{-\int_0^{\tau_{D'}} 4u(\xi_s)\,ds} e^{-\int_0^{\tau_{D'}} 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})\,ds}\Big) \\
&= \Pi_x\big(\gamma_\nu(\xi_{\tau_{D'}}) \mathcal{N}_{\tau_{D'}}^k(\phi+u)\big) \\
&= \mathbb{N}_x\big(e^{-\langle X_{D'},\phi\rangle} H_1^\nu(X_{D'})\big).
\end{aligned}$$

If $n \geq 2$ then

$$\begin{aligned}
&\gamma_\nu(x) Q_x^\nu(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim n) \\
&= \gamma_\nu(x) \Pi_x^{4u,\gamma_\nu}\Big(e^{-\int_0^\zeta 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})\,ds} 1_{\zeta<\tau_{D'}} \sum_{m=1}^n \int \frac{2\gamma_{\nu'}(\xi_\zeta)\gamma_{\nu-\nu'}(\xi_\zeta)}{\Gamma_\nu(\xi_\zeta)} \\
&\qquad \cdot Q_{\xi_\zeta}^{\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim m) Q_{\xi_\zeta}^{\nu-\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim n-m)\, K(\nu; d\nu')\Big) \\
&= \Pi_x^{4u}\Big(\int_0^{\tau_{D'}} \gamma_\nu(\xi_t) 1_{t<\zeta} e^{-\int_0^t 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})\,ds} \sum_{m=1}^n \int \frac{2\gamma_{\nu'}(\xi_t)\gamma_{\nu-\nu'}(\xi_t)}{\Gamma_\nu(\xi_t)} \\
&\qquad \cdot Q_{\xi_t}^{\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim m) Q_{\xi_t}^{\nu-\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim n-m)\, K(\nu; d\nu')\, dt\Big) \\
&= \Pi_x\Big(\int_0^{\tau_{D'}} e^{-\int_0^t 4u(\xi_s)\,ds} e^{-\int_0^t 4\hat{\mathbb{N}}_{\xi_s}(1-e^{-\langle X_{D'},\phi\rangle})\,ds} \sum_{m=1}^n \int 2\gamma_{\nu'}(\xi_t)\gamma_{\nu-\nu'}(\xi_t) \\
&\qquad \cdot Q_{\xi_t}^{\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim m) Q_{\xi_t}^{\nu-\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim n-m)\, K(\nu; d\nu')\, dt\Big) \\
&= \sum_{m=1}^n \int \Pi_x\Big(\int_0^{\tau_{D'}} 2\mathcal{N}_t^{D'}(\phi+u) \cdot \gamma_{\nu'}(\xi_t) Q_{\xi_t}^{\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim m) \\
&\qquad \cdot \gamma_{\nu-\nu'}(\xi_t) Q_{\xi_t}^{\nu-\nu'}(e^{-\langle X_{D'},\phi\rangle}, \Upsilon^{D'} \sim n-m)\, dt\Big)\, K(\nu; d\nu').
\end{aligned}$$

Comparing with the recurrence relation for $\mathbb{N}_x(e^{-\langle X_{D'},\phi\rangle} H_n^\nu(X_{D'}))$, the result now follows by induction. □

A careful examination of the above proof shows that it uses only the hypothesis (28), rather than the way that our particular functions γ_ν arise as densities. In other words, we did not use the fact that H^ν_c is X-harmonic, neither did we assume (25). Instead one could start with any family of solutions γ_ν to (28), define H^ν according to the formula of Theorem 6, and still have the above argument work. In other words,

Corollary 11. *Suppose that the γ_ν satisfy (28). Let H^ν be as defined in Theorem 6 with this $\gamma_\nu(x)$. Then $\gamma_\nu(x) = \mathbb{N}_x(H^\nu(X_{D'}))$ and H^ν is an X-harmonic function.*

Proof. We may construct Q^ν_x as above. By passing to historical processes (as in [2]) one can in fact show that $\mathbb{N}^\nu_x(Z) = Q^\nu_x(Z)$ for every $\mathcal{F}_{D'}$-measurable $Z \geq 0$. It follows that the in this setting, $\mathbb{N}^\nu_x$ are consistently defined probability measures on $\mathcal{F}_{D'}$ as D' varies. Hence by Girsanov's theorem H^ν is X-harmonic. Since the $\mathbb{N}^\nu_x$ are actually probabilities also $\gamma_\nu(x) = \mathbb{N}_x(H^\nu(X_{D'}))$ as required. □

Note that the X-harmonic functions H^ν_x were defined by Dynkin in [7], and we have followed this approach throughout. The results of this section should be viewed as an attempt to clarify the structure of these X-harmonic functions, as well as structure of the conditioned superprocesses that are obtained from them.

4.1. **Description of P^ν_μ.** Above we described the conditional law $\mathbb{N}^\nu_x$. Now we move to an arbitrary initial measure μ, and so need to handle multiple lines of descent starting from time 0. In other words, we are going to describe the distribution of $X_{D'}$, $D' \subset D$ with respect P^ν_μ such that $\mu \in \mathcal{M}_{D'}$.

Let Q_1 and Q_2 be two probability measures on the space of measure valued paths. Let $\chi_1 \sim Q_1$ and $\chi_2 \sim Q_2$ be independent. We will let $Q_1 \star Q_2$ denote the law of $\chi_1 + \chi_2$ which is also a probability measure on the space of measure valued paths.

Let

$$K_n(\nu; d\nu_1, \ldots, d\nu_n) := K_n(\nu; d\nu_1, \ldots, d\nu_n) \times \delta_{\nu-(\nu_1+\ldots+\nu_{n-1})}(d\nu_n).$$

and

$$\Lambda_\nu(\mu) \;\; := \;\; \sum_{n=1}^{\infty} \int_{x_1,\ldots,x_n} \int_{\nu_1,\ldots,\nu_n} K_n(\nu; d\nu_1, \ldots, d\nu_n)\gamma_{\nu_1}(x_1)\ldots\gamma_{\nu_n}(x_n)\mu(dx_1)\ldots\mu(dx_n).$$

Finally we let $\hat{P}_\mu$ be the law of super-Brownian with spatial motion killed at rate u.

Theorem 12. *Let $D' \subset D$. Then*

$$\begin{aligned} P^\nu_\mu(F(X_{D'})) \;\; &= \;\; (\Lambda_\nu(\mu))^{-1} \sum_{n=1}^{\infty} \int_{x_1,\ldots,x_n} \int_{\nu_1,\ldots,\nu_n} \mathbb{N}^{\nu_1}_{x_1} \star \mathbb{N}^{\nu_2}_{x_2} \ldots \star \mathbb{N}^{\nu_n}_{x_n} \star \hat{P}_\mu(F(X_{D'})) \\ &\qquad K_n(\nu; d\nu_1, \ldots, d\nu_n)\gamma_{\nu_1}(x_1)\ldots\gamma_{\nu_n}(x_n)\mu(dx_1)\ldots\mu(dx_n). \end{aligned}$$

This theorem says the following. Initially we pick a random cluster of points (x_i) each labelled with a random fragment ν_i of ν according to the distribution

$$K_n(\nu; d\nu_1, \ldots, d\nu_n)\gamma_{\nu_1}(x_1)\cdots\gamma_{\nu_n}(x_n)\mu(dx_1)\cdots\mu(dx_n).$$

At each point we start a branching particle system. We let these systems evolve independently from each other and according the the law $N^{\nu_i}_{x_i}$. At the same time we independently start a super-Brownian motion whose spatial motion is killed at rate u from initial distribution μ. The P^ν_μ distribution of $X_{D'}$ is the mass distribution absorbed by $\partial D'$ from the mass created from the two systems.

Lemma 13. *For $1 \leq r \leq n$, let $C_1, \ldots, C_r$ be a partition of $\{1, \ldots, n\}$. Let $n_i = |C_i|$. Then*

$$K_n(\nu; d\nu_1, \ldots, d\nu_n) \;\; = \;\; \int K_r(\nu; d\nu_1, \ldots, d\nu_r) \prod_{i=1}^{r} K_{n_i}(\tilde{\nu}_i; d\nu_{C_i}).$$

Proof. Let $Y_1, \dots, Y_n$ be i.i.d random variables from R. Let $Y = Y_1 + \dots + Y_n$. Let $Y_{C_i} = (Y_{i_1}, ..., Y_{i_{r_i}})$ for $i_k \in C_i$. The result is a simple consequence of the fact that the Y_{C_i} are conditionally independent of each other and of Y given the random variables $Y_{i_1} + ... + Y_{i_{r_i}}$. $\square$

Proof. **(of Theorem 12)**

$$
\begin{aligned}
&P_\mu(e^{-\langle X_{D'},\phi\rangle} H_c^\nu(X_{D'})) \\
&= P_\mu\left(e^{-\langle X_{D'},\phi\rangle} \sum_{n=1}^\infty \int \frac{e^{-\langle X_{D'},u\rangle}}{n!} \langle \mu, \gamma_{\nu_1}\rangle \cdots \langle \mu, \gamma_{\nu_n}\rangle \right) K_n(\nu; d\nu_1, \dots, d\nu_n) \\
&= \sum_{n=1}^\infty \frac{1}{n!} \int P_\mu \left(e^{-\langle X_{D'},u+\phi\rangle} \langle \mu, \gamma_{\nu_1}\rangle \cdots \langle \mu, \gamma_{\nu_n}\rangle \right) K_n(\nu; d\nu_1, \dots, d\nu_n) \\
&= \sum_{n=1}^\infty \frac{1}{n!} \int e^{-\langle \mu, V_D(u+\phi)\rangle} \sum_{\pi(n)} \langle \mu, \mathbb{N}_{(.)}\left(e^{-\langle X_{D'},u+\phi\rangle} \langle X_{D'} \gamma_{\nu_{C_1}}\rangle \right)\rangle \cdots \langle \mu, \mathbb{N}_{(.)}\left(e^{-\langle X_{D'},u+\phi\rangle} \langle X_{D'} \gamma_{\nu_{C_r}}\rangle \right)\rangle \\
&\quad K_n(\nu; d\nu_1, \dots, d\nu_n) \\
&= e^{-\langle \mu, V_D(u+\phi)\rangle} \sum_{n=1}^\infty \frac{1}{n!} \sum_{\pi(n)} \int \prod_{i=1}^r \langle \mu, \mathbb{N}_{(.)}\left(e^{-\langle X_{D'},u+\phi\rangle} \left[\int \langle X_{D'}, \gamma_{\nu_{C_i}}\rangle K_{n_i}(\tilde{\nu}_i; d\nu_{C_i}) \right] \right)\rangle \\
&\quad K_r(\nu; d\nu_1, \dots, d\nu_r) \\
&= e^{-\langle \mu, V_D(u+\phi)\rangle} \sum_{n=1}^\infty \frac{1}{n!} \sum_{r=1}^n \int K_r(\nu; d\nu_1, \dots, d\nu_r) \sum_{n_1,\dots,n_r} \frac{n!}{n_1! \cdots n_r!} \\
&\quad \prod_{i=1}^r \langle \mu, \mathbb{N}_{(.)}\left(e^{-\langle X_{D'},u+\phi\rangle} \left[\int \langle X_{D'}, \gamma_{\nu_{C_i}}\rangle K_{n_i}(\tilde{\nu}_i; d\nu_{C_i}) \right] \right)\rangle \\
&= e^{-\langle \mu, V_D(u+\phi)\rangle} \sum_{r=1}^\infty \int K_r(\nu; d\nu_1, \dots, d\nu_r) \prod_{i=1}^r \sum_{n_i=1}^\infty \frac{1}{n_i!} \langle \mu, \mathbb{N}_{(.)}\left(e^{-\langle X_{D'},u+\phi\rangle} \left[\int \langle X_{D'}, \gamma_{\nu_{C_i}}\rangle K_{n_i}(\tilde{\nu}_i; d\nu_{C_i}) \right] \right)\rangle \\
&= e^{-\langle \mu, V_D(u+\phi)\rangle} \sum_{r=1}^\infty \int K_r(\nu; d\nu_1, \dots, d\nu_r) \prod_{i=1}^r \langle \mu, \gamma_{\tilde{\nu}_i} \mathbb{N}_{(.)}^{\tilde{\nu}_i}(e^{-\langle X_{D'},\phi\rangle})\rangle
\end{aligned}
$$

$\square$

5. Is H^ν extreme?

In this section we are going to investigate the question whether the X-harmonic functions H_D^ν are extreme. Our main result is the following:

Theorem 14. *Let D be a smooth domain. Let A be in tail σ-field of X in D. For R-almost all ν,*

$$P_\mu^\nu(A) = 0 \text{ or } 1 \tag{29}$$

for all μ.

This result does not (quite) imply that H^ν is extreme. To show that H^ν is extreme, we would need a stronger version of Theorem 14 where R-almost all ν, (29) holds for every A in the tail σ-field.

To prove Theorem 14 we need the following construction. Let G be the space of finite atomic measures on ∂D. Let $(\mathcal{G}_n)_{n\geq 1}$ be the filtration generated by the finite projection maps W_n on G^∞, (i.e. $W_n(g) = (g_1, \dots g_n)$) and

$$\mathcal{G}_\infty = \bigvee_{n=1}^{\infty} \mathcal{G}_n.$$

For any finite measure ν on ∂D, let P^ν be the law on G^∞, corresponding to the law of independent and identically distributed Poisson random measures with characteristic measure ν. We define the following probability law on $\Omega \times G^\infty$:

$$\bar{P}_\mu(YZ) = P_\mu(YP^{X_D}(Z)) \tag{30}$$

We extend both $X_{D'}$, $D' \subset D$ and W_n, $n \geq 1$, to $\Omega \times G^\infty$ by letting $X_{D'}(\omega, g) = X_{D'}(\omega)$ and $W_n(\omega, g) = W_n(g)$. Clearly, $\bar{P}_\mu$ law of $X = (X_{D'})_{D' \subset D}$ is the same as its law under P_μ. We also extend $\mathcal{F}_{\subset D}$, $\sigma(X_D)$ and $\mathcal{G}_n$ to σ-algebras on $\Omega \times G^\infty$ in the obvious way. If $W_n = V_1, \dots, V_n$, let $U_n = V_1 + \dots + V_n$.

Theorem 15. *Let $A \in \mathcal{F}_{\subset D}$. $\bar{P}_\mu$ almost surely following holds:*
i) U_n is a unit atomic measure for every n.
ii) $\frac{1}{n}U_n \Rightarrow X_D$
iii) $P_\mu^{X_D}(A) = \lim_{n\to\infty} P_\mu^{n,U_n}(A)$

Proof. Note that we can think of W_n as a boundary statistic, even though it is not listed among the statistics of Section 3. The arguments of Section 3 would equally work for conditioning on W_n, which tell us that there exists an X-harmonic function H^{w_n} such that the conditional law of $(X_{D'})_{D' \subset D}$ given $\mathcal{G}_n$ is given by $P_\mu^{H^{W_n}}$. Therefore we have

$$\bar{P}_\mu(A|\mathcal{G}_n) = P_\mu^{H^{W_n}}(A)$$

by Theorem 2. Moreover, by a similar argument as in the proof of Theorem 3 we have that

$$P_\mu^{H^{W_n}}(A) = P_\mu^{n,U_n}(A).$$

Here note that conditional on $X_D = \nu$, U_n is a Poisson random measure with characteristic measure $n\nu$. Also from Section 3.1 we know that U_n is a unit atomic measure $\bar{P}_\mu$ almost surely, clearly this implies (i). Hence, the corresponding X-harmonic function for conditioning on U_n can be chosen as the X-harmonic function of Section 3.1 with $\beta = n$, for every n. By conditional independence of G_∞ and $\mathcal{F}_{\subset D}$ given X_D, for any $A \in \mathcal{F}_{\subset D}$ we have $\bar{P}$ a.s.

$$\bar{P}_\mu(A|G_\infty \vee \mathcal{F}_D] = \bar{P}_\mu[A|\mathcal{F}_D] \tag{31}$$

Note that $\bar{P}$ a.s.

$$\lim_{n\to\infty} \frac{1}{n} U_n = X_D,$$

where the limit is in the weak convergence sense. If for some (ω, g) the above limit does not hold, we exclude those pairs from $\Omega \times G^\infty$. This way we have that X_D is measurable with respect to $\mathcal{G}_\infty$, therefore $P_\mu[A|\mathcal{F}_D]$ is measurable with respect to $\mathcal{G}_\infty$ as well. This and Equation 31 imply

$$P_\mu[A|\mathcal{F}_D] = P_\mu[A|\mathcal{G}_\infty].$$

By the martingale convergence theorem we get that

$$P_\mu[A|\mathcal{F}_D] = \lim_{n\to\infty} P_\mu[A|\mathcal{G}_n].$$

The left side is equal to $P_\mu^{X_D}(A)$, and the left side is equal to $\lim_{n\to\infty} P_\mu^{n,U_n}(A)$. So, except on a $\bar{P}$ null set, for all (ω, g), we have

$$P_\mu^{X_D}(A) = \lim_{n\to\infty} P_\mu^{n,U_n}(A).$$

□

We will also need the following technical lemma in the proof of Theorem 14. Let $(D_k)_{k\geq 0}$ be an exhausting sequence of domains for D. Let m_k be a sequence s.t.

$$\sup_{D_k} u_{m_k}(x) - u(x) < \frac{1}{k}$$

where u_{m_k} is the solution of $\Delta u = \frac{1}{2}u^2$ which blows up on ∂D_{m_k} and u is the solution which blows up on ∂D.

Lemma 16. *For any μ, there exist a sequence n_l such that $\bar{P}_\mu$ almost surely,*

$$\lim_{l\to\infty} P_\mu^{n_l,U_{n_l}}(\liminf_k e^{-\langle X_k, u_{D_{m_k}} - u_n\rangle}) = 1.$$

Proof. Since

$$P_\mu^{n,U_n} \liminf_k e^{-\langle X_{D_k}, u_{D_{m_k}} - u_n\rangle} = P_\mu(\liminf_k e^{-\langle X_{D_k}, u_{D_{m_k}} - u_n\rangle} | U_n)$$

we have

$$\begin{aligned} P_\mu[P_\mu^{n,U_n} \liminf_k e^{-\langle X_k, u_{D_{m_k}} - u_n\rangle}] &= P_\mu(\liminf_k e^{-\langle X_k, u_{D_{m_k}} - u_n\rangle}) \\ &= \liminf_k e^{-\langle \mu, V_{D_k}(u_{D_{m_k}} - u_n)\rangle} \\ &\geq \liminf_k e^{-\langle \mu, (u-u_n)\rangle} e^{-\langle \mu, \frac{1}{k}\rangle} \\ &= e^{-\langle \mu, u-u_n\rangle}. \end{aligned}$$

Note

$$\lim_n e^{-\langle \mu, u-u_n\rangle} = 1$$

which implies that $P_\mu^{n,U_n}[\liminf_k e^{-\langle X_k, u_{D_{m_k}} - u_n\rangle}]$ converges to 1 in probability, therefore there exists a subsequence n_l s.t. $P_\mu^{n_l,U_{n_l}}[\liminf_k e^{-\langle X_k, u_{D_{m_k}} - u_n\rangle}]$ converges to 1 a.s. □

Our strategy to prove Theorem 14 is to get a formula for $P_\mu^{n,\nu_n}(A)$ when A is in the tail σ field and show that we can choose a subsequence n_l and ν_{n_l} so that $\lim_{k\to 0} P_\mu^{n_l,\nu_{n_l}}(A)$ is 0 or 1.

In [6] Dynkin proves that if D equals to $\mathbb{R}^n$ then the X harmonic function $\langle \mu, 1\rangle$ is extreme. His method consists of the following steps. First, he formulates an equivalent problem in terms of a Markov chain $(X_{D_k})_{k\geq 0}$ where (D_k) is a sequence of domains exhausting D. Second, he considers the immortal particle description of the h-transform of Super-Brownian motion with the X-harmonic function $\langle \mu, 1\rangle$. This is also known as super-Brownian motion conditioned on non-extinction. This immortal particle system consists of a Brownian motion ξ, and mass uniformly created along the path of ξ. Reformulating the problem in terms of this particle system, Dynkin shows that the probability of a tail event of the Markov Chain $(X_{D_k})_{k\geq 0}$ is 0 or 1, if and only if the probability of a corresponding event in the tail σ-field of ξ is 0 or 1. It is well known that the tail σ-field of Brownian motion is trivial, hence the result follows.

Our proof is analogous to Dynkin's proof. Consider again the exhausting sequence of domains $(D_k)_{k\geq 1}$ we selected before Lemma 16. Let H^{n,ν_n} be the X-harmonic function of Section 3.1 where ν_n is a finite and unit atomic measure on ∂D. We will consider two Markov Chains. The first one is $X_k = X_{D_k}, k \geq 1$, associated to the family $P_{r,\mu}^{n,\nu_n}$, indexed by (r, μ) such that μ is a finite measure supported on ∂D_r and $r = 1, 2, \ldots$. $P_{r,\mu}^{n,\nu_n}$ is the restriction of the law $P_\mu^{H^{n,\nu_n}}$ to $\sigma(X_{D_{r+1}}, X_{D_{r+2}}, \ldots)$.

We construct the second Markov chain using the branching forest construction of $P_\mu^{H^{n,\nu_n}}$, which we obtain from the results of [12].

Let Θ be the space of finite point measures η on the pairs (x, m) such that $x \in D$, and each m is a finite point measure and if the support of η consists of p points, $(x_1, m_1), \ldots, (x_p, m_p)$, then $m_1 + \cdots + m_p = \nu_n$. Recall, $\mathcal{M}_D$ is the space of finite measures compactly supported in D.

Let $\mu \in \mathcal{M}_D$ and $\eta \in \Theta$. We will consider a probability law Q_η^{n,ν_n} on $\Omega_1 := \mathcal{M}_D \times \Theta$-valued paths, according to which the exit measures $(Y_{D'}, \Upsilon_{D'})_{D' \subset D}$ evolve as follows:

Let $z_1, \ldots, z_k$ be the points in the support of ν_n. η gives us a finite partition $\gamma = (C_1, \ldots, C_p)$ of $\{1, \ldots, k\}$ and a cluster of points $x_1, \ldots, x_p$ in D. We start a branching backbone system labeled with z_{C_i} from each x_i . Here $z_C = (z_i, i \in C)$. The evolution of each branching backbone is as follows: Let $u_n(x) := -\log(e^{-\langle X_D, n\rangle})$. A particle starts from x_i following a ψ_{C_i} transform of a $\mathcal{L}^{u_n}$ diffusion. Here ψ_C is defined by the recursive moment formula with $\psi_{\{i\}}(x) = k^n(x, z_i)$. As ψ_{C_i} is a potential, this particle dies somewhere in D, say at y. Two new particles start at y, each assigned a new label $z_{C'}$ and $z_{C \setminus C'}$ respectively where C' is a proper subset of C_i and chosen randomly proportional to $\psi_{C'}(y)\psi_{C_i \setminus C'}(y)$. We repeat the same procedure for each particle unless the particle is assigned the label z_i for some i, in which case we let it evolve as a $k^n(\cdot, z_i)$ transform of an $\mathcal{L}^{u_n}$ diffusion.

Let $\Upsilon_{D'}$ be the point measure which puts mass on the pair of points (y, m) where the y are the first exit points from D' of the particles in the branching forest system whose ancestors were all born inside D', and the m are the point measures associated to their labels. (That is $m = \sum_{i \in C} \delta_{z_i}(dz)$ if C is the label of the particle exited at y). Let $Y_{D'}$ be the exit distribution on $\partial D'$ created uniformly along the backbone system.

On $\Omega_2 :=$ the space of $\mathcal{M}_D$-valued paths, we consider the family $P_\mu^{(n)}$ the law of a super-Brownian motion whose spatial motion is killed at rate u_n, and the exit measures $(X_{D'})_{D' \subset D}$.

Now, we define a process $(Z_{D'}, \Upsilon_{D'})$ on $\Omega_1 \times \Omega_2$ such that

$$(Z_{D'}(\omega_1, \omega_2), \Upsilon_{D'}(\omega_1, \omega_2)) = (X_{D'}(\omega_2) + Y_{D'}(\omega_1), \Upsilon_{D'}(\omega_1))$$

Consider again the exhausting sequence of domains $(D_k)_{k \geq 1}$. Let $(Z_k, \Upsilon_k)_{k \geq 1} = (Z_{D_k}, \Upsilon_{D_k})_{k \geq 1}$. By construction $(Z_k, \Upsilon_k)_{k \geq 1}$ is a Markov chain with respect to the family of probability laws $P_{r,\mu,\eta}^{n,\nu_n}$, the restriction of $Q_\eta^{n,\nu_n} \times P_\mu^{(n)}$ to $\sigma((Z_{r+1}, \Upsilon_{r+1}), (Z_{r+2}, \Upsilon_{r+2}), \ldots)$.

Proof. **(of Theorem 14)**

The tail σ-field of $(X_k)_{k \geq 1}$ coincides with the intersection of all $\mathcal{F}_{\supset D'}$, $D' \subset D$ [6]. If C is in tail σ-field of $(X_k)_{k \geq 1}$, and $P_{1,\mu}^\nu(C) = 1$ for some μ supported on ∂D_1, this implies $P_\mu^\nu(C) = 1$. By the absolute continuity of the measures P_μ, this implies $P_\mu^\nu(C) = 1$ for all μ. Therefore without loss of generality in what follows we fix μ, a finite measure supported on ∂D_1.

Step 0

Let μ be a measure supported on ∂D_1. We fix C in the tail σ-field of $X_k = X_{D_k}, k \geq 1$. Let Ω^0 be the set of (ω, g) such that $\frac{1}{n} U_n(g) \Rightarrow X_D$, $P_\mu^{X_D(\omega)}(C) = \lim P_\mu^{n, U_n(g)}(C)$, $U_n(g)$ is a unit atomic measure for every n, and for a certain subsequence n_l, $\lim_{l \to \infty} P_\mu^{n_l, U_{n_l}}(\liminf_k e^{-\langle X_k, u_{D_{m_k}} - u_n \rangle}) = 1$. (Existence of the subsequence n_l follows from Lemma 16). From Theorem 15 and Lemma 16 we know that $\bar{P}_\mu(\Omega^0) = 1$. Let $\nu = X_D(\omega, g)$ and $\nu_{n_l} = U_{n_l}(\omega, g)$.

Assume $P_{1,\mu}^\nu(C) > 0$. Our goal is to show that $P_{1,\mu}^\nu(C) = 1$. Note that this is enough to prove Theorem 14, since $R(\{\nu : \nu = X_D(\omega, g), \text{for some } (\omega, g) \in \Omega^0\}^c) = 0$. Let $\epsilon \equiv P_{1,\mu}^\nu(C)$. Then there exists a number K, s.t. for all $l \geq K$

(a) $\tilde{P}_{1,\mu}^{n_l,\nu_{n_l}}(\liminf_k P_{k,X_k}^{n_l}(X_{m_k} = 0)) > 1 - \frac{\epsilon}{2}$, and $\lim_{l \to \infty} \tilde{P}_{1,\mu}^{n_l,\nu_{n_l}}(\liminf_k P_{k,Z_k}(X_{m_k} = 0)) = 1$.

(b) $P_\mu^{n_l,\nu_{n_l}}(C) > \frac{\epsilon}{2}$.

Note that the choice of ν and ν_{n_l} depends on C and μ. Note also that K depends on ν, $(\nu_{n_l})_{l \leq 0}$, C, μ.

Step 1

We show that there exists an event $\tilde{C}$ in the tail σ-field of $(Z_k, \Upsilon_k)_{k\geq 1}$ s.t.

$$P^{n,\nu_n}_{1,\mu}(C) = \int \hat{\mu}(d\eta) P^{n,\nu_n}_{1,\mu,\eta}(\tilde{C})$$

To see this, let $\hat{\mu}$ be a probability measure on Θ constructed as follows. We pick a random partition γ of $\{1,\ldots,k\}$ with probability proportional to $\Pi_{A\in\gamma}\langle\mu,\psi_A\rangle$. For each $A\in\gamma$, we then independently choose a starting point x with law $\frac{1}{\langle\mu,\psi_A\rangle}\psi_A(x)\mu(dx)$. $\hat{\mu}$ is the law of a point measure which puts unit mass on each (x_i, m_i), where if $\gamma = (A_1,\ldots,A_p)$ then $m_i = \sum_{j\in A_i}\delta_{z_j}(dz)$.

Let $\tilde{P}^{n,\nu_n}_{1,\mu}$ be the law defined by

$$\tilde{P}^{n,\nu_n}_{1,\mu}(d\omega) = \int \hat{\mu}(d\eta) P^{n,\nu_n}_{1,\mu,\eta}(d\omega)$$

From [12] we know that $\tilde{P}^{n,\nu_n}_{1,\mu}$-distribution of Z_k, $k\geq 1$ is the same as the distribution of $(X_k)_{k\geq 1}$ w.r.t. $P^{n,\nu_n}_{1,\mu}$, so for any measurable non-negative $f_1,\ldots,f_m$ we have

$$P^{n,\nu_n}_{\mu}[f_1(X_k)\ldots f_m(X_{k+m})] = \tilde{P}^{n,\nu_n}_{1,\mu}(f_1(Z_k),\ldots,f_m(Z_{k+m}))$$

hence by a straightforward application of monotone class theorem we get the result.

Step 2:

Following Dynkin, we show that for any $\tilde{C}$ in the tail σ field of the Markov Chain (Z_k,Υ_k), we have

$$P^{n,\nu_n}_{r,\mu,\eta}(\tilde{C}) = P^{(n)}_{r,\mu}(X_m = 0) P_{r,0,\eta}(\tilde{C}) + J^{\nu,\nu_n}_{r,m,\mu,\eta}(\tilde{C})$$

where

$$0 \leq J^{n,\nu_n}_{r,m,\mu,\eta}(\tilde{C}) \leq (1 - P^n_{r,\mu}(X_m = 0)).$$

(Here please note that μ is a generic μ supported on ∂D_r not the measure μ we fixed in Step 0). To see this, use the Markov property of (Z_k,Υ_k), to get

$$P^{n,\nu_n}_{r,\mu,\eta}(\tilde{C}) = P^{n,\nu_n}_{r,\mu,\eta}(g_m(Z_m \times \Upsilon_m))$$

where

$$g_m(\tilde{\mu},\tilde{\eta}) = P_{n,\tilde{\mu},\tilde{\eta}}(\tilde{C}).$$

Therefore,

$$P^{n,\nu_n}_{r,\mu,\eta}(\tilde{C}) = I_m + J_m$$

where

$$\begin{aligned} I_m &= \int P^{(n)}_{r,\mu}(d\omega_1) Q_\eta(d\omega_2) 1_{\{X_m(\omega_1)=0\}} g_m[Z_m(\omega_1,\omega_2),\Upsilon_m(\omega_1,\omega_2)] \\ &= P^{(n)}_{r,\mu}\{X_m = 0\} P^{n,\nu_n}_{r,0,\eta}(\tilde{C}) \end{aligned}$$

and

$$\begin{aligned} J_m &= \int P^{(n)}_{r,\mu}(d\omega_1) Q_\eta(d\omega_2) 1_{\{X_m(\omega_1)\neq 0\}} g_m[Z_m(\omega_1,\omega_2),\Upsilon_m(\omega_1,\omega_2)] \\ &\leq P^{(n)}_{r,\mu}\{X_m \neq 0\}. \end{aligned}$$

Step 3:

Following Dynkin, we show for any $\tilde{C}$ in the tail σ-field of $(Z_k,\Upsilon_k)_{k\geq r}$, $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}$ a.s.

$$1_{\tilde{C}} = \lim_{k\to\infty} P^{n_l,\nu_{n_l}}_{k,Z_k,\Upsilon_k}(\tilde{C}).$$

This is simply a consequence of the Markov property and the martingale convergence theorem.

Step 4:

Step 3 and Step 1 imply that $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}$ a.s.

$$1_{\tilde{C}} = \lim_{k\to\infty} \left(P^{n_l}_{k,Z_k}(X_{m_k}=0) P^{n_l,\nu_{n_l}}_{k,0,\Upsilon_k}(\tilde{C}) + J^{n_l,\nu_{n_l}}_{k,m_k,Z_k,\Upsilon_k}(\tilde{C}) \right)$$

Step 5:

We argue that $\liminf_k P^{n_l,\nu_{n_l}}_{k,0,\Upsilon_k}(\tilde{C})$ is constant, $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}$ a.s. This will follow once we show that the tail σ-field of (Υ_k) is trivial. Moreover, it is enough if we show that the tail σ-field of the process $(\Upsilon_t)_{t\geq 0}$ is trivial. (Υ_t is defined on Ω_1.) Let τ be following stopping time:

$$\tau := \inf\{t \geq 0 : |\Upsilon_t| = k\}.$$

For η such that $|\eta| = k$, the Q_η-distribution of Υ_t is the same as the distribution of

$$\sum_{i=1}^{k} \delta_{\xi_i(t),\delta_{z_i}}$$

where ξ_i are independent and each ξ_i is a $k^{u_n}(\cdot, z_i)$ transform of an $\mathcal{L}^n$ diffusion with starting point $y_i \in D$ determined by η. If A is in the tail σ field of Υ_t, then for such η, $Q_\eta(A)$ is 0 or 1 since each ξ has a trivial σ-field with respect to their laws. Now, by the Markov property of Υ_t,

$$\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}(A) = Q_{\Upsilon_\tau}(A)$$

which is 0 or 1 as we have just argued.

Step 6: Note

$$P^{(n)}_{k,Z_k}(X_{m_k} = 0) = e^{-\langle Z_k, u_{D_{m_k}} - u_n\rangle},$$

and $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}$ distribution of $(Z_k)_{k\geq 1}$ is the same as the distribution of $(X_k)_{k\geq 1}$ under P^{n,ν_n}_{μ}. Hence

$$\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu} \liminf_k P^{n}_{k,Z_k}(X_{m_k}=0) = P^{n_l,\nu_{n_l}}_{\mu} \liminf_k e^{-\langle X_k, u_{D_{m_k}} - u_n\rangle}.$$

Let K be the constant we set in Step 0. From Step 0, and from the equality above we get that $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu} \liminf_k P^{n_l}_{k,Z_k}(X_{m_k} = 0)$ is always less than $\epsilon/2$ for $l \geq K$ and converges to 0 as $l \to \infty$. Since $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}(\tilde{C}) = P^{n_l,\nu_{n_l}}_{1,\mu}(C)$, from Step 0, we also have that $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}(\tilde{C}) > \epsilon/2$, for $l \geq K$.

Let $l \geq K$. Step 5 implies that either (1) $\liminf_k P^{n_l,\nu_{n_l}}_{k,0,\Upsilon_k}(\tilde{C}) < 1$, $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}$ a.s.; or (2) $\liminf_k P^{n_l,\nu_{n_l}}_{k,0,\Upsilon_k}(\tilde{C}) = 1$, $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}$ a.s. If (1) is correct, then this would imply, if $\omega \in \tilde{C}$ is such that the equality in step (5) holds, then $\liminf_k P_{k,Z_k(\omega)}(X_{m_k} = 0)) = 0$. Hence if (1) were correct we would get

$$\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}(\liminf_k P_{k,Z_k}(X_{m_k}=0)) = 0) \geq \tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}(\tilde{C}).$$

which is a contradiction, since according to what we argued in the previous paragraph, the left side is less than $\epsilon/2$, where the right side is strictly greater than $\epsilon/2$.

Hence we conclude that $\liminf_k P^{n_l,\nu_{n_l}}_{k,0,\Upsilon_k}(\tilde{C}) = 1$, $\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}$ a.s. for $l \geq K$.

Step 7:

Step 6 and Step 5 imply that

$$\tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}(\tilde{C}^c) \leq \tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}\{\liminf_k P_{k,Z_k}(X_{m_k}=0) = 0)\}.$$

The limit of right side is 0 as $l \to \infty$, hence,

$$P^{\nu}_{1,\mu}(C) = \lim_{l\to\infty} P^{n_l,\nu_{n_l}}_{1,\mu}(C) = \tilde{P}^{n_l,\nu_{n_l}}_{1,\mu}(\tilde{C}) = 1.$$

□

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