Mortality Derivatives: Valuation and Hedging of the Ruin-Contingent Life Annuity (RCLA)

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Abstract

This paper analyzes a type of mortality contingent-claim called a ruin-contingent life annuity (RCLA). This product fuses together a type of equity put option with a personal longevity call option. The annuitant's (i.e. long position) payoff from a generic RCLA is \$1 of income per year for life, akin to a defined benefit pension, but deferred until a pre-specified financial diffusion process hits zero. We derive the PDE and relevant boundary conditions satisfied by the RCLA value (i.e. the hedging cost) assuming a complete market where No Arbitrage is possible. We then describe some efficient numerical techniques and provide estimates of a typical RCLA under a variety of realistic parameters. The motivation for studying the RCLA is that it is now embedded in approximately \$800 billion worth of U.S. variable annuity (VA) policies which have recently attracted scrutiny from analysts and regulators.

1 Introduction

Amongst the expanding universe of derivative securities priced off non financial state variables, a recent innovation has been the mortality-contingent claim. As its name suggests, a mortality-contingent claim is a derivative product whose payoff is dependent or linked to the mortality status of an underlying reference life or pool of lives. The simplest and perhaps the most trivial mortality-contingent claim is a personal life insurance policy with a face value of one million dollars for example. In this case, the underlying state variable is the (binary) life status of the insured. If-and-when it jumps from the value of one (alive) to the value of zero (dead) the beneficiary of the life insurance policy receives a payout of one million dollars. Another equally trivial example is a life or pension annuity policy which provides monthly income until the annuitant dies. Payment for these options can be made up-front, as in the case of a pension income annuity, or by installments as in the case of a life insurance policy. Indeed, the analogy to credit default swaps is obvious and it is said that much of the technology — such as Gaussian copulas and reduced form hazard rate models – which are (rightfully or wrongfully) used for pricing credit derivatives can be traced to the actuarial science behind the pricing of insurance claims.

Yet, in the past these pure mortality-contingent claims have been (perhaps rightfully) ignored[1] by the mainstream quant community primarily because of the law of large numbers. It dictates that a large-enough portfolio of policies held by a large insurance company should diversify away all risk. Under this theory pricing collapsed to rather trivial time-value-of-money calculations based on cash-flows that are highly predictable in aggregate.

However this conventional viewpoint came into question when, in the early part of this decade, a number of large insurance companies began offering equity put options with rather complex optionality that was directly tied to the mortality status of the insured. These variable annuity (VA) policies, as they are commonly known, have been the source of much public and regulatory consternation in late 2008 and early 2009, as the required insurance reserves mushroomed. We refer the interested reader to popular media reports such as the 3/7/2009 article in the *Wall Street Journal* describing the insurance industry's woes. An additional source of interest, not directly addressed in this paper, is the emergence of actuarial evidence that mortality itself contains a stochastic component.

Motivated by all of this, in this paper we value and provide hedging guidance on a type of product called a ruin-contingent life annuity (RCLA). The RCLA provides the buyer with a type of insurance against the joint occurrence of two separate (and likely independent) events; the two events are *under average* investment returns and *above average* longevity. The RCLA behaves like a pension annuity that provides lifetime income, but only in *bad*

[1] There are some exceptions, for example the 2006 article in the Journal of Derivatives by Stone and Zissu on the topic of securitizing life insurance settlements.

economic scenarios. In the good scenarios, properly defined, it pays nothing. The RCLA is obviously (much) cheaper than a generic life annuity which provides income under all economic scenarios. We will argue that the RCLA is a fundamental mortality-contingent building block of all VA "income guarantees" in the sense that it is not muddled by tax-frictions and other institutional issues. At the same time it retains many of the real-world features embedded within these policies. At the very least this article should provide an introduction to what we label *finsurance* – products that combine financial and insurance options in one package.

Huang, Milevsky and Salisbury (2009) motivated the need for a stand-alone ruin-contingent life annuity (RCLA), albeit without deriving any valuation relationships. In this article we provide the valuation and hedging machinery for the RCLA, in a complete market setting (i.e. assuming No Arbitrage). In terms of its position within the actuarial and finance literature, the RCLA is effectively a type of annuity option, and so this work is related to Ballotta and Haberman (2003) as well as Boyle and Hardy (2003) in which similar complete market techniques are relied upon. In a subsequent paper we plan to describe the impact of incomplete markets and other frictions.

1.1 How Does the RCLA Work?

The RCLA is based on a reference portfolio index (RPI), a.k.a. the state variable, upon which the income/pension annuity start-date is based. The RPI is initiated at an artificial level of \$100, for example, and consists of a broad portfolio of stocks (for example the SP500 or Russell 3000 Index). However at the end of each day, week or month the RPI is adjusted for total returns (plus or minus) and by a fixed cash outflow (minus) that reduces the RPI. The cash outflow can be constant in nominal terms or constant in real terms or something in between. The income annuity embedded within the RCLA begins payments if-and-when the RPI hits zero. Figure #1 provides an example of a possible sample-path for the RPI in discrete time.

Figure #1 Placed Here

Here is a detailed example that should help explain the mechanics of the RPI and the stochastic annuity start date. Assume that the Russell 3000 index is at a level of \$100 on January 1st, 2009. If under a pre-specified withdrawal rate of \$7 we assume that during January 2009 the Russell 3000 total return was a nominal 2%, then the level of a vintage 2009 RPI on the first day of February 2009 would be $\$100(1.02) - (7/12) = \$101.42$. The annual withdrawal rate of \$7 is divided by 12 to create the monthly withdrawal, which can also be adjusted for inflation. The same calculation algorithm continues each month. Think of the RPI as mimicking the behavior of a retirement drawdown portfolio.

Now, if and when this (vintage) 2009 RPI ever hits zero, the insurance company would then commence making \$1 for life payments (either nominal or inflation-adjusted) to the annuitant who bought the product in January 2009, as long as they were still alive. Figure #2 graphically illustrates how the performance of the RPI would trigger the lifetime income payment. Under path #1 in which the RPI hits zero twenty years after purchase, the income would start at the age of 80. Under path #2 where the RPI never hits zero, the annuitant would receive nothing from the insurance company.

Figure #2 Placed Here

A generic RCLA is defined in units of \$1, so if the annuitant purchased 7 units, they would continue to receive the same \$7 of income without any disruptions to their standard of living. At inception the retiree buying the RCLA could select from a range of withdrawal rates, for example 5%, 6% or 7%, assuming the insurance company was willing to offer a menu of spending rates (at different prices, of course.) Likewise, the annuitant could specify nominal payments of \$1 for life or real payments of \$1 for life, which would obviously impact pricing as well.

To be precise, and when necessary, we will use the notation $RPI_t(z, \gamma)$ to denote the level/value of the reference portfolio index in year $(z + t)$, where the initial withdrawal rate in year z, was set at γ percent of the initial value I_0. More generically we will refer to $W_t := RPI_t(z, \gamma)$ as the state variable underlying the derivative's payout function.

It is worth pointing out that, from the point of view of the insurance company offering an RCLA, this is a complete-markets product, that can be perfectly hedged (at least under our assumptions). Thus the price or value we will compute below is really measuring the company's hedging cost. This may differ from the economic value an individual client places on the product, since from the client's point of view, the market is incomplete and mortality risk is unhedgeable. What makes a hedge possible for the company is the law of large numbers – after selling many individual contracts, the total cash flows due to mortality become essentially deterministic, leaving only cash flows due to market fluctuations to be hedged. We will comment further on this issue below.

1.2 Agenda for the Paper

In section #2 we briefly review the pricing of generic life annuities, which also helps introduce notation and provides some basic intuition for the RCLA. Section #3 formally introduces the concept of "ruin" under the relevant diffusion process, which becomes the trigger for the RCLA. Section #4, which is the core of the paper, introduces, values, and then describes the hedge for a basic RCLA. Section #5 describes some advanced products in which the payoff and ruin-trigger are non-constant. It also discusses the connection between RCLA values and the popular Guaranteed Living Withdrawal Benefits (GLWB) that are sold with variable annuity (VA) products in the U.S. We provide numerical examples in all sections and then conclude the paper in Section #6 with some direction for future research.

2 Valuation of the Income Annuity

In this section we very briefly review the valuation of single premium immediate (income) annuities, mainly in order to introduce notation and terminology for the remainder of the paper and provide background for those unfamiliar with mortality-contingent claims. We refer the interested reader to any basic actuarial textbook, such as Promislow (2006) or Milevsky (2006), for the assumptions we gloss over.

The value of a life annuity that pays \$1 per annum in continuous-time, is denoted by $ALDA(x,\rho|\tau)$, where x denotes the purchase age, ρ denotes the (insurance company) valuation discount rate and τ is the start date. The ALDA is an acronym for *Advanced Life Delayed Annuity*. When the ALDA start date is immediate ($\tau = 0$) we use the shorthand notation $SPIA(x,\rho) := ALDA(x,\rho|0)$ (for *Single Premium Immediate Annuity*). Either way, the annuity valuation factor is equal to:

$$ALDA(x,\rho|\tau) := E\left[\int_{\tau}^{T_x} e^{-\rho t}dt\right] = E\left[\int_{\tau}^{\infty} 1_{\{T_x>t\}}e^{-\rho t}dt\right] = \int_{\tau}^{\infty} {}_tp_x \; e^{-\rho t}dt, \tag{1}$$

where T_x denotes the future lifetime random variable conditional on the current (purchase) age x of the annuitant and $({}_tp_x)$ denotes the conditional probability of survival to age $(x+t)$. In the above expression τ is deterministic and denotes the delay time period before the insurance company begins making lifetime payments to the annuitant. It is an actuarial identity that:

$$ALDA(x,\rho|\tau) := SPIA(x+\tau,\rho) \times {}_{\tau}p_x \times e^{-\rho\tau}, \tag{2}$$

which is the product of the age–$(x+\tau)$ SPIA factor multiplied by the conditional probability of surviving to age $(x+\tau)$ multiplied by the relevant discount factor $e^{-\rho\tau}$. In other words, the cost of a delayed annuity can be written in terms of an (older) immediate annuity, the survival probability and the discount rate. This actuarial identity will be used later when τ itself is randomized.

Note that the expectation embedded within equation (1) is taken with respect to the physical (real world) measure underlying the distribution of T_x, which, due to the law of large numbers and the ability to eliminate all idiosyncratic mortality risk is also equal to the risk-neutral measure. While outside the scope of this paper which deals exclusively with complete markets, in the event the realized force of mortality itself is stochastic, it may in fact generate a mortality risk premium in which case the physical (real world) and risk-neutral measure might not be the same. We leave this for other research.

Under any continuous law of mortality specified by a deterministic function $\lambda_t > 0$, the expectation in equation (1), the annuity factor, can be re-written as:

$$ALDA(x, \rho|\tau) = \int_{\tau}^{\infty} e^{-\int_0^t \lambda_q \, dq} e^{-\rho t} \, dt = \int_{\tau}^{\infty} e^{-\int_0^t (\lambda_q + \rho) dq} dt. \tag{3}$$

For most of the numerical examples within the paper we will assume that λ_t obeys the Gompertz-Makeham (GM) law of mortality. The canonical GM force of mortality (see the paper by Carrière (1994) or Frees, Carrière and Valdez (1996) for example), can be represented by:

$$\lambda_t = \lambda + \frac{1}{b} e^{\left(\frac{x+t-m}{b}\right)}, \tag{4}$$

where $\lambda \geq 0$ is a constant non-age dependent hazard rate, $b > 0$ denotes a dispersion coefficient and $m > 0$ denotes a modal value. Our notation for λ_t assumes four embedded parameters: x the current age, λ, m and b. Note that when $m \to \infty$, and $b > 0$, the GM collapses to a constant force of mortality λ, and the future lifetime random variable is exponentially distributed. We will obtain some limiting expressions in this case. For the more general and practical GM law, our RCLA valuation expressions will be stated as solutions to a PDE.

As far as the basic *ALDA* factor is concerned, in the case of GM mortality, one can actually obtain a closed-form expression for equation (3), which was first suggested by Mereu (1962) and fully derived in Milevsky (2006) on pages 116-117. Namely:

$$ALDA(x, \rho|\lambda, m, b|\tau) = \frac{b\Gamma(-(\lambda + \rho)b, \exp\{\frac{x-m+\tau}{b}\})}{\exp\left\{(m - x)(\lambda + \rho) - \exp\left\{\frac{x-m}{b}\right\}\right\}}, \tag{5}$$

where $\Gamma(x, y)$ denotes the incomplete Gamma function. The annuity factor itself is a decreasing function of age x, delay period τ, and the valuation rate ρ. To see this, Figure #3 plots the annuity factor in equation (5), for a continuum of ages from $x = 40$ to $x = 80$ assuming the valuation rates, $\rho = 3\%, 5\%$ and 7% and $\tau = 0$ deferral period.

Figure #3 Placed Here

Table #1 displays some numerical values for a basic SPIA (immediate) and ALDA (delayed) annuity factor, under the Gompertz Makeham ($m = 86.3, b = 9.5$) continuous law of mortality. For example under an insurance valuation rate of $\rho = 5\%$, at the age of $x = 40$, a buyer pays \$16.9287 for an income stream of \$1 per year for life, starting immediately. If the annuity is purchased at the same age but the start of income is delayed for $\tau = 10$ years, the buyer pays \$9.1010 for \$1 per year for life, starting at age 50. In contrast, under the same $r = 5\%$ rate, at age 65 the annuity value is \$11.3828 per dollar of lifetime income, starting immediately and only \$4.0636 if the start of the income is delayed for $\tau = 10$ years. In general, for higher valuation rates, advanced ages and longer delay periods, the annuity factor is lower. Note that the above Gompertz-Makeham assumptions imply the conditional expectation of life at age 65 is $EPX(65 \mid 0, 86.3, 9.5) = 18.714$ years, which can be easily obtained by substituting an insurance valuation rate of $\rho = 0\%$ into the annuity factor. Note that no death benefits or guarantee periods are assumed in these valuation expressions. Thus, the occurrence of death prior to the end of the delay period will result in a complete loss of premium.

Table #1 Placed Here

The ruin-contingent life annuity and its variants, which we will formally define in the next section, can be viewed as generalizations of the ALDA factor, but where the delay period τ is stochastic and tied to the performance of a reference portfolio index.

3 Retirement Spending and Lifetime Ruin

The RCLA is an income annuity that begins payment when a reference portfolio index (RPI) hits zero, or is ruined. In this section we describe the mechanics of the state variable which triggers the payment. To begin with we assume investment returns are generated by a lognormal distribution so that the RPI obeys the classic "workhorse" of financial economics:

$$dW_t = (\mu W_t - \gamma I_0)\, dt + \sigma W_t dB_t, \qquad W_0 = I_0. \tag{6}$$

The parameter μ denotes the drift rate and σ denotes the diffusion coefficient. The constant γI_0 denotes the annual spending rate underlying the RPI. Note that when $\gamma = 0$ the process W_t collapses to a geometric Brownian motion (GBM) which can never access zero in finite time. The presence of γ reduces the drift and makes zero accessible in finite time. The greater the value of γ the greater is the probability, all else being equal, that W_t hits zero[2].

We define the ruin time R of the diffusion process as a hitting-time or level-crossing time, which should be familiar from the classical insurance or queueing theory literature. Formally it is defined as:

$$R := \inf \{t; W_t \leq 0 \mid W_0 = I_0\}. \tag{7}$$

There is obviously the possibility that $R = \infty$ and the RPI never hits zero. See the paper by Huang, Milevsky and Wang (2004) or the paper by Dhaene, Denuit, Goovaerts, Kaas and Vyncke (2002), as well as Norberg (1999), for a detailed and extensive description of the various analytic and moment-matching techniques that can be used to compute the probability distribution of R. Likewise, see the paper by Young (2004) for a derivation of asset allocation control policies on (μ, σ) that can be used to minimize ruin probabilities within the context of retirement spending. Our focus is not on controlling R or explicitly

[2]The evolution of retirement wealth implied by equation (6) is often studied as an alternative to annuitization in the pension and retirement planning literature. See, for example, the paper by Albrecht and Maurer (2002) or Kingston and Thorp (2005), in which γI_0 is set equal to the relevant SPIA factor times the initial wealth at retirement.

estimating $\Pr[T_x \geq R]$ which is the lifetime ruin probability. We are simply interested in using R as a deferral time for an income annuity.

The probability distribution of the ruin time $\Pr[R \leq T]$ is denoted by $RCDF(I_0, T)$ and the corresponding density function is denoted by $RPDF(I_0, t)$. And, although neither $RCDF(I_0, s)$ or $RPDF(I_0, T)$ is available analytically, they both can be expressed as a solution to a fundamental PDE, which we describe later. Note however that $RCDF(I_0, T = \infty)$, which is the probability of ruin over the time $[0, \infty)$ is indeed available in closed analytic form and discussed in the references given after equation (7).

4 The Ruin-Contingent Life Annuity (RCLA)

Like the generic annuity, the ruin-contingent life annuity (RCLA) is acquired with a lump-sum premium now, and eventually pays \$1 of income per year for life. However, the income payments do not commence until time $\tau = R$, when the reference portfolio index (RPI) hits zero. And, if the RPI never hits zero – or the annuitant dies prior to the RPI hitting zero – the RCLA expires worthless. Thus, the defining structure of the RCLA is similar to the annuity factor in equation (1), albeit with a stochastic upper *and* lower bound:

$$\text{RCLA}(x, \rho|\lambda, m, b|I_0, \gamma, \mu, \sigma|\tau, \beta) = E\left[\int_R^{R \vee T_x} e^{-\rho t} dt\right] \tag{8}$$

The \$1 of annual lifetime income starts at time R and continues until time $\max\{R, T_x\}$. Thus, if the state-of-nature is such that $T_x < R$, and the annuitant is dead prior to the ruin time, the integral from R to R is zero and the payout is zero. Each RCLA unit entitles the annuitant to \$1 of income. Thus, if one thinks of an RCLA as "insuring" a γI_0 drawdown plan, then buying γI_0 RCLA units, would continue to pay γI_0 dollars upon ruin.

Now, in order to derive a valuation relationship for the RCLA defined by equation (8) we do the following. First, we simplify notation by writing the annuity factor $ALDA(x, \rho|\xi)$ as $F(\xi)$. In other words,

$$F(\xi) = \int_\xi^\infty {}_tp_x \, e^{-\rho t} \, dt = E\left[\int_\xi^{\xi \vee T_x} e^{-\rho t} dt \mid \xi\right] \tag{9}$$

Our problem then becomes to calculate:

$$E[F(R)] = E\left[E\Big[\int_R^{R \vee T_x} e^{-\rho t} dt \mid R\Big]\right] = E\left[\int_R^{R \vee T_x} e^{-\rho t} dt\right] = RCLA. \tag{10}$$

Note that once again we rely on the law of large numbers – from the perspective of the insurance company – to diversify away any idiosyncratic longevity risk and value the RCLA based on (subjective, physical) mortality expectations.

Now, if $\mathcal{F}_t$ is the filtration generated by W_t, the reference portfolio index, then $M_t = E[F(R)|\mathcal{F}_t]$ is a martingale. By the Markov property, it can be represented in the form $f(t \wedge R, W_t)$, so applying Ito's lemma leads to the familiar (Kolmogorov backward) PDE:

$$f_t + (\mu w - \gamma I_0) f_w + \frac{1}{2}\sigma^2 w^2 f_{ww} = 0 \tag{11}$$

for $w > 0$ and $t > 0$. We now have an expression for (8) as

$$\text{RCLA} = f(0, I_0). \tag{12}$$

Equation (11) differs from the famous Black-Scholes-Merton PDE by the presence of the γI_0 constant multiplying the space derivative f_w. Also, our boundary conditions are different from the linear ones for call and put options. Two of our boundary conditions are that $f(t, w) \to 0$ as either $t \to \infty$ or $w \to \infty$. Intuitively, the RCLA is worthless in states of nature where the underlying RPI never gets ruined, and/or only gets ruined after the annuitants have all died. The boundary condition we require is that $f(t, 0) = F(t)$, defined by equation (9). The intuition here is that if-and-when the RPI hits zero at some future time ξ, a live annuitant will be entitled to lifetime income whose actuarially discounted value is the annuity factor $F(\xi)$.

Moreover, when $\lambda_t = \lambda$ is constant we recover the simple expression $F(\xi) = e^{-(\lambda+\rho)\xi}/(\lambda + \rho)$ and one can simplify the entire problem to obtain a solution of the form $f(t, w) = e^{-(\lambda+\rho)t} h(w)$, where the new one-dimensional function $h(w)$ satisfies the ODE:

$$(\mu w - \gamma I_0) h_w(w) + \frac{1}{2}\sigma^2 w^2 h_{ww}(w) - (\lambda + \rho) h(w) = 0, \tag{13}$$

where h_w and h_{ww} denote the first and second derivatives respectively. The two boundary conditions are $h(\infty) = 0$ and $h(0) = 1/(\lambda + \rho)$. But, when λ_t is non-constant and obeys the full GM law, we must use the full expression $F(\xi) = {}_{\xi}\overline{a}_x(\rho)$ for the boundary condition, which was displayed in equation (5). Note that we then have a parabolic PDE, which can be solved numerically.

Note that in both equations (11) and (13) we maintain a distinction between the drift rate μ and the insurance valuation rate ρ. One reason for doing so is to leave open the possibility of using our valuation equation to calculate the expected $RCLA$ returns under the physical measure, in which μ could be the growth rate under the physical measure even if $\rho = r$ is the risk-free interest rate. Another reason is that even if we are interested in calculating prices (or the costs of manufacturing or hedging the products), and so take $\mu = r$ to be the risk-free interest rate, an $RCLA$ contract could still in principal specify a different value for the insurance valuation rate ρ. We will discuss this further in Section 4.3. However, in our numerical examples below we will take $\mu = \rho = r$ (the risk-free rate) as in the Black-Scholes-Merton economy, etc.

There are also extensions of this analysis that should be possible. It would be natural, given this product's role in retirement savings, to incorporate real inflation adjustment factors into the RCLA payouts. Since the product is envisioned as having a long horizon, it would also be worthwhile to incorporate stochastic volatility into the model for the underlying asset price, as well as variable interest rates. Finally, we have assumed complete diversification of mortality risk, due to the law of large numbers and the sale of a very large number of contracts. This is only a first approximation to actuarial practice, in which adjustments are made to account for the non-zero mortality risk still present when only a finite number of contracts are sold. We hope to treat several of these effects in subsequent work, but note that in some cases this means moving to techniques suitable for incomplete markets.

4.1 Solution Technique

To solve the PDE for $f(t,w)$ which is displayed in equation (11), we first use the following transformation:

$$f(t,w) = F'(t)u(t,w), \tag{14}$$

where without any loss of generality $u(t,w)$ is defined as a new (possibly) two-dimensional function. By taking partial derivatives and the chain rule, it is easy to verify that:

$$f_t = F''u + F'u_t,\ f_w = F'u_w,\ f_{ww} = F'u_{ww}, \tag{15}$$

where once again we use shorthand notation f_t, f_w and f_{ww} for the three derivatives of interest. By substituting equation (15) into equation (11), the valuation PDE for $f(t,w)$ can be written in terms of the known function $F(t)$ and the yet-to-be-determined function $u(t,w)$ as:

$$\frac{F''}{F'}u + (\mu w - \gamma I_0)u_w + \frac{1}{2}\sigma^2 w^2 u_{ww} + u_t = 0. \tag{16}$$

Now, since by construction,

$$F(\xi) = \int_\xi^\infty e^{-\int_0^s (\lambda_q+\rho)dq} ds, \tag{17}$$

we have that

$$F'(\xi) = -e^{-\int_0^\xi (\lambda_q+\rho)dq} ds,\ F''(\xi) = -(\lambda_\xi + r)F'(\xi). \tag{18}$$

Thus, expressed in units of time t, the PDE for $u(t,w)$ becomes

$$-(\lambda_t + \rho)u + (\mu w - \gamma I_0)u_w + \frac{1}{2}\sigma^2 w^2 u_{ww} + u_t = 0, \tag{19}$$

where u is shorthand for $u(t,w)$, and u_t, u_w, u_{ww} are shorthand notations for the time, space and second space derivatives, respectively. Now, going back to the decomposition of $f(t,w)$ in equation (14), and using the boundary condition for $f(t,w)$ at $w = 0$, we have

$$F(t) = f(t,0) = F'(t)u(t,0), \tag{20}$$

and

$$F'(t) = F'(t)u_t(t,0) + F''(t)u(t,0), \tag{21}$$

from which we obtain

$$u_t(t,0) = (\lambda_t + \rho)u(t,0) + 1. \tag{22}$$

For the numerical procedure, we first generate values of $u(t,w)$ by solving equation (19) with boundary conditions from equation (22) and condition $u(w,t) \to 0$ as $w \to \infty$ and $t \to \infty$. Then we multiply $u(t,w)$ by $F'(t)$ to generate the RCLA values of $f(t,w)$.

Figure #4 Placed Here

If necessary, values can also be calculated simultaneously for multiple values of γ by rescaling. This is the case, for example, in the numerical examples and tables found below. We let $\tilde{w} = w/\gamma I_0$ and define $\tilde{u}(t,\tilde{w}) = u(t,w)$. Then the PDE for $\tilde{u}$ is seen to be

$$-(\lambda_t + \rho)\tilde{u} + (\mu\tilde{w} - 1)\tilde{u}_{\tilde{w}} + \frac{1}{2}\sigma^2\tilde{w}^2\tilde{u}_{\tilde{w}\tilde{w}} + \tilde{u}_t = 0, \tag{23}$$

with the same boundary conditions as before. The parameter γ no longer appears, so only one PDE needs to be solved, after which we can calculate

$$f(t,w) = F'(t)u(t,w) = F'(t)\tilde{u}\Big(t, \frac{w}{\gamma I_0}\Big) \tag{24}$$

for any desired value of γ. In fact, we will drop the "tilde" notation, since $\tilde{u}$ is just u in the special case $\gamma I_0 = 1$. Thus, if we have computed that particular function u we then get RCLA values for other γ's as

$$\text{RCLA} = F'(0)u(0, I_0/\gamma I_0) = F'(0)u(0, 1/\gamma). \tag{25}$$

In Figure #4 we plot $f(t,w)$, which is the RCLA value, assuming $\mu = \rho = r = 0.06$ (i.e. for risk neutral pricing) $m = 86.3$, $b = 9.5$ and $x = 40$ (all three embedded mortality parameters) and $\lambda_0 = 0.003$, which is the age-independent component of the Gompetz-Makeham law. The computation is done by solving the equation for $u(t,w)$ for $0 < t < 80$ (corresponding to a maximum age of death of 120) and $0 < w < 25$, and using a normalized value of $\gamma I_0 = 1$. As mentioned above, the function $f(t,w)$ is recovered by multiplying $u(t,w)$ by $F'(t)$, evaluated by numerical quadrature based on Simpson's rule. We can then use f to value RCLA's with different withdrawal rates. Thus, for example, the point $f(0,10)$ corresponds to the price of a \$1 per year for life RCLA, purchased at the age of 40, assuming a spending rate of $\gamma = 1/10 = 10\%$ of the RPI level $I_0 = 100$.

Note that we experimented with different domain sizes up to $w = 100$ and no visible differences in results were observed, relative to the case when $w = 25$. (A single-run took a few seconds for a grid resolution of $\delta w = 0.1$ on a MacBook Pro.) Note that when $\lambda_t = \lambda$ is a constant, $u_t = 0$ and we recover the above-referenced special case mentioned prior to equation (13).

4.2 Numerical Examples

Table #2a displays the (risk neutral) value of the RCLA – which pays \$1 per year of lifetime income – assuming the Reference Portfolio Index (RPI) is allocated to LOW volatility investments with $\sigma = 10\%$. The spending γ denotes the fixed percentage of the initial RPI level I_0 that is withdrawn annually (and in continuous time) until ruin. When $\gamma = \infty$ the RPI hits zero immediately and the RCLA collapses to a basic annuity priced in Table #1. The mortality is assumed Gompertz with parameters $m = 86.3$ and $b = 9.5$. Thus, for example, at the age of 65 the value of a 5% withdrawal RCLA on a "low volatility" index is \$0.6872 under a valuation rate of $\rho = 3\%$ and a mere \$0.1384 under a valuation rate of $\rho = 5\%$. In fact, even at the young age of $x = 50$ and under a relatively high spending percentage of $\gamma = 7\%$, the value of the RCLA is only \$2.4921 per dollar of lifetime income upon ruin, under the 5% valuation rate. Predictably, at advanced ages the same 7% withdrawal RCLA is valued at only a fraction of this cost. For example, at age $x = 75$, and under a valuation rate $\rho = 5\%$, the value of the RCLA is only \$0.1965. This is the impact of low ($\sigma = 10\%$) investment volatility; naturally when σ and γ are low, the probability of lifetime ruin is very small. In contrast, Table #2b which is identical in structure to #2a displays the (risk neutral) value of the RCLA assuming the Reference Portfolio Index (RPI) is allocated to HIGH volatility investments with $\sigma = 25\%$. Once again the RPI spending rate γ denotes the fixed percentage withdrawn.

Table #2a,#2b Placed Here

Note the impact of the higher volatility rate on the RCLA value. The 5% withdrawal RCLA that cost \$0.6872 at the age of 65, under a valuation rate of $\rho = 3\%$ and low investment volatility in Table #2a is now valued at \$2.3015 in Table #2b under an investment volatility of $\sigma = 25\%$. Similarly, the value for a 7% withdrawal RCLA at age $x = 7$ and under $\rho = 5\%$ quadruples to \$0.8470.

As one might expect intuitively, the value of an RCLA is also extremely sensitive to the withdrawal percentage γ underlying the RPI. For example, at the age of 65 and under a valuation rate of $\rho = 3\%$, a withdrawal percentage of $\gamma = 7\%$ on a high volatility RPI leads to an RCLA value of \$3.6732, but is worth less than half at \$1.6103 under a $\gamma = 4\%$ withdrawal percentage. One can interpret these results as indicating that insuring lifetime income against ruin at a 7% withdrawal rate is roughly 125% more expensive than insuring against ruin at a 4% withdrawal rate. This provides an economic benchmark by which different spending strategies can be compared.

4.3 Hedging

Our price, determined by risk-neutral valuation in previous sections, represents a hedging cost. It is worth making the hedging argument explicit (and evaluating Delta), even though this has certainly been implicit in what we described above.

The partial differential equations given in the preceding sections evaluate expectations. In the complete markets setting, the expectations are risk-neutral, and represent hedging costs. In that setting, we normally choose the equity growth rate μ and the insurance valuation rate ρ to both coincide with the risk free interest rate r: $\mu = \rho = r$. This is the setting used for the numerical examples given above. But we could also use the PDE's to work out discounted expected cash flows under the real-world or physical measure, a problem that can arise in aspects of risk management other than pricing. In that case we would apply the above formulas with μ equalling the real-world equity growth rate, and $\rho = r$ to be the risk-free rate.

By generalizing the RCLA slightly, we can also imagine using the PDE when $\mu = r$ (so our measure is risk-neutral and we're looking at pricing and hedging), but $\rho < r$. As we shall see below, this would be the case if payments from the RCLA were not fixed at \$1 per year for life, but rather at $e^{\delta t}$, where $\delta = \mu - \rho$. This would correspond to an inflation-enhanced RCLA in which a fixed inflation rate δ is incorporated into the contract, so payments increase over time at rate δ. The standard RCLA described earlier is just the case $\delta = 0$. In this subsection (and this subsection only) we will work out the hedging portfolio assuming a complete market, with risk-free rate $\mu = r$ and a valuation rate $\rho = r - \delta$. We do not change the definition of the reference portfolio.

Note that we do not hedge the RCLA "derivative" using the reference portfolio index (RPI) W_t, satisfying $dW_t = (rW_t - \gamma I_0)\,dt + \sigma W_t\,dB_t$ and $W_0 = I_0$, since that quantity incorporates withdrawals and is not readily tradeable. Instead we use a stock index S_t without withdrawals (which is assumed tradeable), on which the RPI is based. In other words,

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t. \tag{26}$$

We assume that a large number N of RCLA's is sold at time 0, to age-x individuals. The company hedges these with a portfolio worth V_t at time t. Then

$$V_t = \Delta_t S_t + R_t \tag{27}$$

where Δ_t is the number of stock index units held, and R_t is a position in a money market account with interest rate r. Since the number of contracts is large, a predictable fraction ${}_tp_x$ of contract holders are still alive at time t, leading to outflows from the hedging portfolio of $e^{\delta t}{}_tp_xN$, if ruin has occurred by time t. Thus

$$\begin{aligned} dV_t &= \Delta_t\,dS_t + rR_t\,dt - e^{\delta t}{}_tp_xN1_{\{R<t\}}\,dt \\ &= rV_t\,dt + \Delta_t\sigma S_t\,dB_t - e^{\delta t}{}_tp_xN1_{\{R<t\}}\,dt. \end{aligned} \tag{28}$$

We obtain a positive solution by taking

$$V_t = Ne^{rt}f(t, W_t) \tag{29}$$

and

$$\Delta_t = \begin{cases} Ne^{rt} W_t f_w(t, W_t)/S_t, & W_t > 0 \\ 0, & W_t = 0. \end{cases} \tag{30}$$

The verification is a simple consequence of Ito's lemma, the fact that $f(t, w)$ solves (11) when $w > 0$, and the observation that $Ne^{rt} f_t(t, 0) = Ne^{rt} F'(t) = -Ne^{rt} e^{-\rho t}{}_t p_x = -e^{\delta t}{}_t p_x N$. Put another way, the value of the stock position in the hedge, per initial contract sold, is just

$$\Delta_t S_t / N = e^{rt} W_t f_w(t, W_t). \tag{31}$$

This expression reflects the fact that our solution is written using W_t rather than S_t, and the observation that f is already a discounted quantity (being a martingale). Note that the relation between W_t and S_t could be made explicit, but is path dependent.

Finally, the initial hedging cost, per contract, is just $V_0/N = f(0, I_0)$ as in (12). Of course, in reality a company would simultaneously hedge a book of RCLA's with different purchase dates, and sold to clients with a range of ages. But the above analysis serves to illustrate the connection between hedging and pricing.

5 More Exotic Time-Dependent Payouts

We now describe two additional types of RCLA, both of which are motivated by real-world products. In the first modification the spending rate γI_0 increases to $\gamma \max_{0 \le s \le t}\{W_s\}$, which accounts for good performance, each time the underling RPI reaches a new maximum. In other words, this product could be used to insure a drawdown plan, in which withdrawals ratchet or step up. At ruin time R, this product pays \$1 per year for life akin to the generic RCLA. In the second modification the spending rate increases in a similar manner, but the lifetime income – which starts upon the RPI's ruin time – will be increased as well. Both of these RCLA variants are embedded within the latest generation of variable annuity (VA) policies sold around the world with guaranteed lifetime withdrawal benefits (GLWB). We now proceed to describe and value them in detail.

5.1 The Fast-RCLA

Once again, we let T_x denote the remaining lifetime random variable under a deterministic hazard rate λ_t, and we assume the RPI process W_t is independent of T_x and satisfies the following diffusion equation:

$$dW_t = (\mu W_t - g(t, M_t))dt + \sigma W_t dB_t, \qquad W_0 = I_0 \tag{32}$$

where the new function M_t is defined as:

$$M_t = \max_{0 \le s \le t} W_s. \tag{33}$$

Both W_t and M_t are now defined up until the time R that W_t hits zero. Note that the drift term in equation (32) now includes a more general specification and is not necessarily a constant deterministic term γI_0, as in the basic RCLA case. The modified product that we call a *Fast RCLA* differs from the basic RCLA in that the spending function is defined in the following manner.

$$g(t,m)=\begin{cases} 0 & t\leq\tau \\ \gamma\max\{m,W_0e^{\beta\tau}\} & t>\tau \end{cases}, \tag{34}$$

where the new constant β denotes a "bonus rate" for delaying τ years prior to spending/withdrawals. Note that τ is now a delay period before the RPI begins withdrawals. The constant γ multiplying the max function in equation (34) serves the same role as γ in the basic RCLA. It is a pre-specified percentage rate of some initial RPI value.

Thus, for example, assume that $W_0 = I_0 = 100$ and that during the first ten years $(t \leq \tau = 10)$ the reference portfolio index W_t grows at some (lognormally distributed) rate and without any withdrawals. Then, after ten years $(t > \tau = 10)$ the RPI starts to pay-out the greater of (i) $\gamma = 5\%$ of the the maximum RPI value M_{10} observed to date, and (ii) $\gamma = 5\%$ of $100e^{(0.05)10} = 164.87$, which is \$8.2 per year. Then, each time the process W_t reaches a new high, so that $M_t = W_t$, the spending rate $g(t, M_t)$ is reset to $(0.05)W_t = (0.5)M_t$. Then, if-and-when the RPI hits zero the insurance company makes payments of \$1 per year for life, to the annuitant.

The value of the Fast RCLA is (still) defined as:

$$\text{F-RCLA}(x,\rho|\lambda,m,b|I_0,\gamma,\mu,\sigma|\tau,\beta) := f(0,I_0,I_0) \tag{35}$$

where for $0 < w \leq m$,

$$f(t,w,m)=E\left[\int_R^{R\vee T_x} e^{-\rho s}ds \mid W_t=w, M_t=m\right]. \tag{36}$$

The only difference between the F-RCLA and the RCLA is in the structure of the ruin time R. When $\tau = 0$ and the RPI begins immediate withdrawals, the (generic) F-RCLA is more expensive compared to a basic RCLA because the ruin-time R under the diffusion specified by equation (32) will occur prior to (or at the same time) as the ruin-time generated by the constant withdrawal implicit within equation (6).

To solve this valuation equation we go back to the PDE for the basic RCLA which we derived in the previous section. Note that the original PDE, displayed in equation (11), did not involve the hazard rate function λ_t. Rather, the mortality was embedded into the boundary conditions. We take advantage of the same idea for the Fast-RCLA.

First, we tinker with the definition of the $g(t, m)$ spending function. We re-scale by starting M_t at $W_0e^{\beta\tau}$ rather than at W_0. So let $\overline{M}_t = W_0e^{\beta\tau} \vee \max_{0\leq s\leq t} W_s$. We then define a "moneyness" variable $Y_t = W_t/\overline{M}_t$, satisfying $0 \leq Y_t \leq 1$. Let $\bar{g}(t,\overline{m})$ be $g(t,m)$ in terms of the new variables, so that:

$$\overline{g}(t,\overline{m}) = \begin{cases} 0, & t \leq \tau \\ \gamma\overline{m}, & t > \tau. \end{cases} \tag{37}$$

Our problem is now to calculate the value of a new function defined as

$$h(t,y,\overline{m}) = E[F(R) \mid Y_t = y, \overline{M}_t = m] \tag{38}$$

where $F(\xi)$ is defined as above, and R is the ruin time of W_t. Then the F-RCLA value $f(t,w,m) = h(t,y,\overline{m})$ where $\overline{m} = m \vee W_0 e^{\beta\tau}$ and $w = y\overline{m}$.

The next step is to calculate h using that

$$E[F(R) \mid \mathcal{F}_t] = h(t \wedge R, Y_t, \overline{M}_t) \tag{39}$$

is a martingale. To apply Ito's lemma, we need to write down the stochastic equations for Y_t (the new moneyness variable) and $\overline{M}_t$ (the new maximum diffusion value). Note that $\overline{M}_t$ is increasing, and defining $dL_t = d\overline{M}_t/\overline{M}_t$, we have that L_t is a process that increases only when $Y_t = 1$, and

$$d\overline{M}_t = \overline{M}_t \, dL_t. \tag{40}$$

Likewise

$$\begin{aligned} dY_t &= \frac{1}{\overline{M}_t} dW_t - \frac{W_t}{\overline{M}_t^2} d\bar{M}_t \\ &= \frac{\mu W_t - \overline{g}}{\overline{M}_t}\, dt + \frac{\sigma W_t}{\overline{M}_t}\, dB_t - \frac{W_t}{\overline{M}_t}\, dL_t \\ &= (\mu Y_t - \widehat{g}(t))\, dt + \sigma Y_t \, dB_t - Y_t \, dL_t \end{aligned} \tag{41}$$

where we use yet another function,

$$\widehat{g}(t) = \frac{\overline{g}(t,\overline{m})}{\overline{m}} = \begin{cases} 0, & t \leq \tau \\ \gamma, & t > \tau. \end{cases} \tag{42}$$

We interpret (41) as a "Skorokhod equation" and L_t as a "local time" of Y_t at 1, the effect of which is to pull Y_t down when it reaches 1, to ensure that it does not ever exceed 1. In particular, L_t is determined by the process Y_t. Note that $\overline{M}_t$ has now entirely disappeared from the stochastic equation for Y_t, so in fact Y_t is a one-dimensional Markov process all by itself. Because R is determined by Y, in fact

$$h(t,y,\overline{m}) = h(t,y) \tag{43}$$

does not depend on $\overline{m}$ at all. We are able to make all of these simplifications because of the simple structure of the original spending rate $g(t,m)$ in equation (33). If we had a more general withdrawal rate, say of the form $g(t, w, m)$ where g is a more complicated function than the one used above, then we would have to keep track of the maximum state variable $\overline{m}$ in addition to the moneyness state variable y.

Now, applying Ito's lemma, we get that for $t < R$,

$$dh(t, Y_t) = [h_t + (\mu Y_t - \widehat{g}(t))h_y + \frac{1}{2}\sigma^2 Y_t^2 h_{yy}]\, dt + \sigma Y_t h_y\, dB_t - Y_t h_y\, dL_t. \tag{44}$$

For this to be a martingale, both the dt and dL_t terms must vanish. So in particular,

$$h_t + (\mu y - \widehat{g}(t))h_y + \frac{1}{2}\sigma^2 y^2 h_{yy} = 0 \tag{45}$$

and $h_y = 0$ when $y = 1$ (recall that $dL_t = 0$ unless $Y_t = 1$). The latter is one boundary condition, and $h(t, 0) = F(t)$, $h(t, y) \to 0$ as $t \to \infty$ are the others. Note the similarity between the PDE we must solve for the F-RCLA in equation (45) and the original valuation PDE for the RCLA displayed in equation (11). Besides the boundary conditions, the only difference is that γI_0 is replaced by $\widehat{g}(t)$. So, in the Gompertz case there is one time variable and one spatial variable.

5.2 The Super-RCLA

In the previously analyzed F-RCLA, the spending/withdrawal stepped-up over time, but when ruin occurs the F-RCLA payout is the same as for the RCLA, namely \$1 per year for life. This type of product is relevant in some contexts but not in others. Sometimes the lifetime income that is promised upon ruin can be greater than the originally guaranteed rate, and is linked to the function $g(t, m)$ itself. Therefore, in this sub-section we examine the case in which the lifetime income paid by the annuity is linked to the increasing level of spending/withdrawals. As before, the RPI value satisfies the process:

$$dW_t = (\mu W_t - g(t, M_t))\, dt + \sigma W_t\, dB_t, \tag{46}$$

under the same (μ, σ) parameters and where the withdrawal function $g(t, m)$ satisfies:

$$g(t, w, m) = \begin{cases} 0, & t < \tau \\ \gamma m, & t \geq \tau \end{cases} \tag{47}$$

and $M_t = W_0 e^{\beta\tau} \vee \max_{0 \leq s \leq t} W_s$. Recall that β is a bonus rate (during the delay period) and τ denotes the length of delay period, measured in years. In this sense, the underlying diffusion and ruin-time dynamics are identical to the previously discussed F-RCLA case.

However, in contrast to the \$1 of lifetime income payoff from the F-RCLA, we define the Super-RCLA value as:

$$\text{S-RCLA}(x, \rho | \lambda, m, b | I_0, \gamma, \mu, \sigma | \tau, \beta) := \frac{f(0, I_0, I_0)}{g(0, I_0)} \tag{48}$$

$$f(t, w, m) := E\left[g(R, m) \int_R^{R \wedge T_x} e^{-\rho s}\, ds \right].$$

The S-RCLA starts paying income for life when the process in equation (46) is ruined, but the income will not be \$1. Instead, it will be equal to the withdrawal amount itself, $g(R,m)$, just prior to the time of ruin R, divided by the initial withdrawal rate $g(0,I_0)$. If there was no step-up in the withdrawal spending prior to ruin, then the payout will simply be \$1 for life, just like the F-RCLA and the original RCLA. We have decided to define the function $f(t,w,m)$ so that we do not have to carry around the denominator $g(0,I_0)$ of equation (48) during the entire derivation.

Either way, our boundary condition must change even though large parts of the solution are similar to the F-RCLA and RCLA. We define the moneyness variable $Y_t = W_t/M_t$ so that $0 \le Y_t \le 1$. Also, let L_t be the local time of Y at 1, so

$$dY_t = (\mu Y_t - \widehat{g}(t))\,dt + \sigma Y_t\,dB_t - dL_t \tag{49}$$

where the (new) scaled variable $\widehat{g}(t)$ is now defined as:

$$\widehat{g}_t = \begin{cases} 0, & t < \tau \\ \gamma I_0, & t \ge \tau. \end{cases} \tag{50}$$

By construction, we also have that $dM_t = M_t\,dL_t$. Moreover, the S-RCLA value defined by:

$$E\left[g(R,m)\int_R^{R\wedge T_x} e^{-\rho s}\,ds \mid \mathcal{F}_t\right] \tag{51}$$

will be a martingale. By the Markov property the S-RCLA value will be of the form $f(t\wedge R, W_{t\wedge R}, M_{t\wedge R})$ for some function f. There is a scaling relationship $f(t,cw,cm) = cf(t,w,m)$, from which we conclude that $f(t,w,m) = mh(t,y)$ for some function h (where $y = w/m$). Applying Ito's lemma,

$$d\left(M_t h(t,Y_t)\right) = (h_t + h_y(\mu Y_t - \widehat{g}(t)) + \frac{1}{2}\sigma^2 Y_t^2 h_{yy})\,dt + \sigma Y_t h_y\,dB_t + M_t(h - h_y)\,dL_t. \tag{52}$$

We conclude that

$$h_t + h_y(\mu y - \widehat{g}(t)) + \frac{1}{2}\sigma^2 y^2 h_{yy} = 0 \tag{53}$$

for $0 < y < 1$, with boundary condition $h(t,1) = h_y(t,1)$ at $y = 1$. There will again be a boundary condition $h(t,w) \to 0$ as $t \to \infty$. At $y = 0$ the boundary condition is that:

$$h(t,0) = \begin{cases} 0, & t < \tau \\ \gamma I_0 F(t), & t \ge \tau \end{cases} \tag{54}$$

where $F(t)$ is defined as before. Note that we are multiplying γI_0 by the annuity factor $F(t)$ since the payoff is now specified in terms of the spending rate and not single dollars. Also, since the equation is parabolic we only need a boundary condition in time at $t = \infty$.

After solving this PDE for h, we recover the S-RCLA value as:

$$f(0, w_0, m_0) = f(0, w_0, w_0 e^{\beta\tau}) = w_0 e^{\beta\tau} h(0, e^{-\beta\tau}). \tag{55}$$

It is worth commenting on the boundary condition $h = 0$ when $w = 0$ and $t < \tau$. This is because the formulation of the S-RCLA implies that the payout rate $g(t, w, m) = 0 \; \forall t$, if it happens that $R < \tau$. However, the RPI cannot get ruined (in a GBM world) before time τ: $P(R < \tau) = 0$. So it is presumably irrelevant what boundary condition we use when $w = 0$ and $t < \tau$.

Table #3 Placed Here

Table #3 displays the (risk neutral) value of the Super RCLA assuming the Reference Portfolio Index (RPI) is allocated to MEDIUM volatility ($\sigma = 17\%$) investments. The initial RPI spending γ denotes the percent of the initial index value that is withdrawn annually (in continuous time.) The factors in Table #3 are not directly comparable to the factors in Table #2 since the lifetime income upon ruin could exceed \$1, if the RPI "does well" prior to ruin. As an example, consider a 67 year old with an initial spending rate of $\gamma = 5.5\%$. Under a valuation rate of $\rho = 5\%$ and investment volatility of $\sigma = 17\%$, an S-RCLA guaranteeing a lifetime payout of at least \$1 upon ruin is valued at \$1.2643. Again, the actual guaranteed payout will be determined by the extent of withdrawal step-ups during the spending period. In these examples, Gompertz mortality is assumed with parameters $m = 86.3$ and $b = 9.5$.

Table #4 Placed Here

Table #4 illustrates the impact of investment (RPI) volatility (σ) on both the RCLA and S-RCLA value, assuming the same Gompertz mortality with parameters $m = 86.3$ and $b = 9.5$. Note that both the RCLA and S-RCLA are represented per guaranteed dollar of lifetime income and the valuation rate (and hence μ) is equal to 5%. We assume no delay ($\tau = 0$) and hence no bonus ($\beta = 0$). The table also displays the percent by which the Super RCLA exceeds the RCLA value, under various volatility assumptions and ages. Thus, for example, at the age of 67, under both a valuation rate $\rho = 5\%$ and a spending percentage $\gamma = 5\%$, the value of an S-RCLA is between 100% and 140% greater than the value of a basic RCLA, depending on the level of volatility assumed in the RPI. It seems that under greater volatility σ, not only are the values of RCLA and S-RCLA higher, but the ratio between S-RCLA and RCLA is greater as well.

5.3 Connection to Guaranteed Living Withdrawal Benefit (GLWB)

As we alluded to in the introduction, variants of RCLA derivatives are embedded within variable annuity (VA) contracts with guaranteed living income benefits (GLiBs) sold in the

U.S., with variants sold in the UK, Japan, and now in Canada. This is now a market with close to \$1 trillion in assets, and with annual sales of over \$100 billion, in 2008. Hence the motivation for studying these products. A GLiB is a broad term that captures a wide variety of annuity riders, including the Guaranteed Minimum Withdrawal Benefit (GMWB), the Guaranteed Lifetime Withdrawal Benefit (GLWB) and the Guaranteed Minimum Income Benefit (GMIB). Thus, for example, a typical GLWB assures the policyholder that if they withdraw no more than \$5 per \$100 of initial investment deposit, they will be entitled to receive these \$5 payments for the rest of their life regardless of the performance of the investments. They can withdraw or surrender the policy and receive the entire account value – net of withdrawals to date – at any time. On the other hand, if the account value ever hits zero the guarantee begins and the annuitant receives lifetime payments.

Although in general the valuation of exotic options within retirement benefits has been analyzed by Sherris (1995) for example, these more specialized GLWB products have been studied by Dai, Kwok and Zong (2008) as well as Chen, Vetzal and Forsyth (2008) and Milevsky and Salisbury (2006). Our paper provides yet another perspective on these types of embedded options and Table #5 can now be interpreted as more than just model values for a theoretical product, but an actual estimate of the discounted value of the embedded insurance offered by a variable annuity with a guaranteed lifetime withdrawal benefit.

Table #5a, #5b Placed Here

Table #5a displays the value of a continuous step-up (a.k.a. super) Guaranteed Lifetime Withdrawal Benefit (GLWB) under a variety of bonus, delay and withdrawal assumptions. We assume precisely the maximum permitted withdrawals after the specified delay, and no lapsation. Thus it is the value of the S-RCLA multiplied by the number of lifetime dollars guaranteed based on an initial deposit of \$100 The mortality is assumed Gompertz with parameters $m = 86.3$ and $b = 9.5$. In contrast, Table #5b displays the same "super" GLWB, but under a low volatility of $\sigma = 10\%$. As in Table #4b, the GLWB value is obtained by multiplying the value of a S-RCLA by the initial number of dollars guaranteed.

So, for example, assume that a 65 year old deposits \$100 into a VA+GLWB that offers a 5% bonus for each year that withdrawals are not made, and it offers a "5% of base" payment for life once the income begins. The underlying base – on which the lifetime income guarantee is based – steps up in continuous time. So, if the individual intends on holding the VA+GLWB for 7 years, and then begins withdrawals, the value of this guaranteed income stream (in addition to the market value of the account itself) is \$10.8703 per \$100 initial deposit, under a 3% valuation rate and \$4.7075 under a 5% valuation rate. This assumes the underlying VA assets are invested in a portfolio of stocks and bonds with expected volatility of $\sigma = 17\%$. Again, note the contrast in GLWB values under a lower investment volatility of $\sigma = 10\%$ in Table #5b. The same two benefits at age $x = 65$ are valued substantially lower at \$4.2610 under $\rho = 3\%$ and \$0.9040 under $\rho = 5\%$.

This number comes from multiplying the S-RCLA value times five, since the initial guaranteed amount is \$5. Of course, for there to be no arbitrage, the ongoing management fees charged on the initial deposit of \$100 would have to cover the discounted (time zero) value of the GLWB option. Once again, the continuously stepped-up GLWB guarantee on a variable annuity policy is just a bundle of S-RCLA units plus a portfolio of managed money in a systematic withdrawal plan. As one would expect, the greater the volatility, the lower the valuation rate and the younger the individual, the higher is the value of the embedded option, at time zero.

6 Conclusion and Discussion

This paper values a type of exotic option that we christened a ruin-contingent life annuity (RCLA). The generic RCLA pays \$1 per year for life, like a classical deferred annuity, but it begins making these payment only once a reference portfolio index is ruined. If this underlying reference index never hits zero, the income never starts. The rationale for buying an RCLA, and especially for a retiree without a Defined Benefit (DB) pension plan, is that it jointly hedges against financial market risk and personal longevity risk, which is cheaper than buying security against both individually. The motivation for studying the RCLA is that this exotic option is now embedded in approximately \$800 billion worth of U.S. variable annuity policies. The impetus for creating stand-alone RCLA products is that they might appeal to the many soon-to-be-retired baby boomers who (i.) are not interested in paying for the entire variable annuity package, and (ii.) would be willing to consider annuitization, but only as a worst case "Plan B" scenario for financing retirement. Indeed, there is a substantial amount of economic and behavioral evidence – see for example the introduction to the book by Brown, Mitchell, Poterba and Warshawsky (2001) – that voluntary annuitization is unpopular as a "Plan A" for retirees. Thus, perhaps a cheaper annuity, and one that has a built-in delay period might appeal to the growing masses of retirees without Defined Benefit (DB) pension plans. This was suggested recently by Webb, Gong and Sun (2007) as well.

Our analysis is done in the classical Black-Scholes-Merton framework of complete markets and fully diversifiable market (via hedging) and longevity (via the law of large numbers) risk. We derived the PDE and relevant boundary conditions satisfied by the RCLA and some variants of the basic RCLA. We then described and used efficient numerical techniques to provide extensive estimates and display sensitivities to parameter values.

Our simple valuation framework only provides a very rough intuitive sense of what these ruin-contingent life annuities might cost in real life. Of course, until a liquid and two-way market develops for these products, it is hard to gauge precisely what they will cost in a competitive market. We are currently working on extending the PDE formulation approach –

by increasing the number of state variables in the problem – to deal with stochastic mortality, which might also be dependent on market returns, as well as the implications of time varying volatility, non-trivial mortality risk, and mean reverting interest rates. Likewise, we are investigating the game-theoretic implications of paying RCLA premiums continuously, as opposed to up-front. In other words, what happens when the RCLA option is purchased via installments, which then endows the option holder (annuitant) to lapse and cease payment? What is the ongoing No Arbitrage premium in this case? The option to lapse leads to a variety of interesting financial economic questions regarding the existence of equilibrium, all of which we leave for future research.

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6.1 Appendix: Summary Notation

- ${}_tp_x(\lambda, m, b)$. Conditional probability of survival under a Gompertz Makeham law of mortality for remaining lifetime. Numerical example: ${}_{65}p_{10}(0, 86.3, 9.5) = 0.8202$

- $EPX(x \mid \lambda, m, b)$. Conditional expectation of remaining lifetime, under a Gompertz Makeham law of mortality. Numerical example: $EPX(65 \mid 0, 86.3, 9.5) = 18.6784$

- $SPIA(x, \rho \mid \lambda, m, b)$. Valuation factor under a discount rate of ρ, issued at age x, for an income annuity that pays \$1 per year for life starting immediately. The Gompertz Makeham law of mortality is assumed with parameters (λ, m, b). Numerical example: $SPIA(65, 0.03 \mid 0, 86.3, 9.5) = \$13.6601$

- $ALDA(x, \rho \mid \lambda, m, b \mid \tau)$. Valuation factor under a discount rate of ρ, issued at age x, for an income annuity that pays \$1 per year for life starting after a delay of τ year. Numerical example: $SPIA(65, 0.03 \mid 0, 86.3, 9.5 \mid 10) = \$5.6499$

- $RCLA(x, \rho \mid \lambda, m, b \mid I_0, \gamma, \mu, \sigma \mid \tau, \beta)$. Valuation factor under a discount rate of ρ, issued at age x, for a basic ruin-contingent life annuity that pays \$1 per year for life starting upon ruin of the Reference Portfolio Index (RPI) denoted by W_t. The RPI's value at time zero, which is age x, is $I_0 = W_0$. The parameter β is a "bonus credit" rate. The parameter τ is a delay period. During the period time $(0, \tau)$, which is the age range $(x, x+\tau)$, there are no withdrawals made from W_t, which in turn follows a basic Geometric Brownian motion (GBM) with drift rate μ and diffusion rate σ. The withdrawals start from the RPI at time τ, at the annual rate of $\gamma I_0 e^{\beta\tau}$ until ruin time. Numerical example #1: $RCLA(65, 0.03 \mid 0, 86.3, 9.5 \mid 100, 0.05, 0.03, 0.25 \mid 0, 0) = \$2.3015$. Numerical example #2: $RCLA(65, 0.03 \mid 0, 86.3, 9.5 \mid 100, 0.05, 0.03, 0.25 \mid 7, 0.05) = \$1.8166$

- $FRCLA(x, \rho \mid \lambda, m, b \mid I_0, \gamma, \mu, \sigma \mid \tau, \beta)$. Valuation factor under a discount rate of ρ, issued at age x, for a FAST ruin-contingent life annuity that pays \$1 per year for life starting upon ruin of the Reference Portfolio Index (RPI) denoted by W_t. The RPI's value at time zero, which is age x, is $I_0 = W_0$. The parameter β is a "bonus credit" rate. The parameter τ is a delay period. During the period time $[0, \tau]$, which is the age range $[x, x+\tau]$, there are no withdrawals made from W_t, which in turn follows a basic Geometric Brownian motion (GBM) with drift rate μ and diffusion rate σ. The withdrawals start from the RPI at time τ, at the annual rate of $g_\tau = \gamma \max[I_0 e^{\beta\tau}, M_\tau]$, where $M_\tau = \sup_{0 \le s \le \tau} W_\tau$ until ruin time. The withdrawals rate g_t increases each time the RPI reaches a new high. Numerical example #1: $FRCLA(65, 0.03 \mid 0, 86.3, 9.5 \mid 100, 0.05, 0.03, 0.25 \mid 0, 0) = \$3.3659$. Numerical example #2: $FRCLA(65, 0.03 \mid 0, 86.3, 9.5 \mid 100, 0.05, 0.03, 0.25 \mid 7, 0.05) = \$2.2883$.

- $SRCLA(x,\rho \mid \lambda,m,b \mid I_0,\gamma,\mu,\sigma \mid \tau,\beta)$.Valuation factor under a discount rate of ρ, issued at age x, for a SUPER ruin-contingent life annuity that pays AT LEAST \$1 per year for life starting upon ruin of the Reference Portfolio Index (RPI) denoted by W_t. The RPI's value at time zero, which is age x, is $I_0 = W_0$. The parameter β is a "bonus credit" rate. The parameter τ is a delay period. During the period time $[0,\tau]$, which is the age range $[x, x+\tau]$, there are no withdrawals made from W_t, which in turn follows a basic Geometric Brownian motion (GBM) with drift rate μ and diffusion rate σ. The withdrawals start from the RPI at time τ, at the annual rate of $g_\tau = \gamma \max[I_0 e^{\beta\tau}, M_\tau]$, where $M_\tau = \sup_{0\le s\le\tau} W_\tau$ until ruin time. The withdrawals rate g_t increases each time the RPI reaches a new high. The lifetime income from the S-RCLA will be $g_R \ge 1$, where R denotes the random ruin time. The lifetime income will be strictly greater than \$1 if-and-only-if there is at least one step-up during the time period $[0,R]$. Numerical example #1: $SRCLA(62, 0.05 \mid 0, 86.3, 9.5 \mid 100, 0.05, 0.05, 0.20 \mid 0, 0) =$ \$2.3325. Numerical example #2: $SRCLA(62, 0.05 \mid 0, 86.3, 9.5 \mid 100, 0.05, 0.05, 0.25 \mid 0, 0) =$ \$3.6467.

- $GLWB(x,\rho \mid \lambda,m,b \mid I_0,\gamma,\mu,\sigma \mid \tau,\beta)$. The value of a "basic" (no step up) Guaranteed Lifetime Withdrawal Benefit issued at age x, under a valuation rate of ρ. It is defined as γI_0 times the value of the corresponding $RCLA$ value.

- $SGLWB(x,\rho \mid \lambda,m,b \mid I_0,\gamma,\mu,\sigma \mid \tau,\beta)$. The value of a "super" (continuous step up) Guaranteed Lifetime Withdrawal Benefit issued at at age x, under a valuation rate of ρ. It is defined as γI_0 times the value of the corresponding $S - RCLA$ value.

- $RPDF(x,s \mid I_0,\gamma,\mu,\sigma \mid \tau,\beta)$. The probability density function (PDF) evaluated at time s, for the ruin time R, of an RPI that is subjected to constant withdrawals of γI_0. Note that this PDF does not integrate to one since there is a probability $R = \infty$.

- $RCDF(x,T \mid I_0,\gamma,\mu,\sigma \mid \tau,\beta)$. The cumulative distribution function (CDF) evaluated at time T, for the ruin time R, of an RPI that is subjected to constant withdrawals of γI_0. It is the probability of ruin prior to time T.

Table #1

Value of Single Premium Immediate/Delayed Annuity				
Purchase Age	Delay	$\rho = 3\%$	$\rho = 5\%$	$\rho = 7\%$
Age = 40	$\tau = 0$ yrs.	\$23.0144	\$16.9287	\$13.1126
	$\tau = 5$ yrs.	\$18.3822	\$12.5148	\$8.9034
	$\tau = 10$ yrs.	\$14.4228	\$9.1010	\$5.9575
	$\tau = 20$ yrs.	\$8.2124	\$4.4665	\$2.4877
Age = 50	$\tau = 0$ yrs.	\$19.7483	\$15.2205	\$12.1693
	$\tau = 5$ yrs.	\$15.1364	\$10.8256	\$7.9778
	$\tau = 10$ yrs.	\$11.2448	\$7.4697	\$5.0815
	$\tau = 20$ yrs.	\$5.3714	\$3.0815	\$1.7921
Age = 65	$\tau = 0$ yrs	\$13.6601	\$11.3828	\$9.6609
	$\tau = 5$ yrs.	\$9.1653	\$7.0974	\$5.5719
	$\tau = 10$ yrs.	\$5.6499	\$4.0636	\$2.9515
	$\tau = 20$ yrs.	\$1.3886	\$0.8577	\$0.5320
Age = 75	$\tau = 0$ yrs.	\$9.2979	\$8.1680	\$7.2460
	$\tau = 5$ yrs.	\$5.0620	\$4.1250	\$3.3839
	$\tau = 10$ yrs.	\$2.2852	\$1.7240	\$1.3062
	$\tau = 20$ yrs.	\$0.1645	\$0.1055	\$0.0677

Note: Table displays the basic annuity factor – with no market contingencies – which is the actuarial present value per \$1 of annual income (in continuous time) for life. The mortality is assumed Gompertz with parameters $\lambda = 0$, $m = 86.3$ and $b = 9.5$. Prices are risk neutral (ie. $\mu = \rho = r =$ risk-free rate). No death benefits or guarantee periods are assumed. Thus, a death prior to the end of the delay period will result in a complete loss of premium.

Table #2a

Ruin-Contingent Life Annuity (RCLA): LOW Volatility ($\sigma = 10\%$)				
Reference Portfolio Index (RPI) Initial Value is $W_0 = I_0 = \$100$				
Lifetime Payout Upon Ruin is \$1 per year				
Initial Purchase	RPI Spending	$\rho = 3.0\%$	$\rho = 5.0\%$	$\rho = 7.0\%$
Age = 50	$\gamma = \infty$	\$19.7483	\$15.2205	\$12.1693
	$\gamma = 10\%$	\$10.0297	\$5.5770	\$2.7307
	$\gamma = 7\%$	\$6.3444	\$2.4921	\$0.6928
	$\gamma = 6\%$	\$4.6797	\$1.4549	\$0.2887
	$\gamma = 5\%$	\$2.9226	\$0.6470	\$0.0820
	$\gamma = 4\%$	\$1.3642	\$0.1853	\$0.0129
	$\gamma = 3\%$	\$0.3716	\$0.0249	\$0.0008
Age = 65	$\gamma = \infty$	\$13.6601	\$11.3828	\$9.6609
	$\gamma = 10\%$	\$4.7321	\$2.6623	\$1.2869
	$\gamma = 7\%$	\$2.2498	\$0.8381	\$0.2217
	$\gamma = 6\%$	\$1.3972	\$0.4024	\$0.0758
	$\gamma = 5\%$	\$0.6872	\$0.1384	\$0.0168
	$\gamma = 4\%$	\$0.2294	\$0.0282	\$0.0019
	$\gamma = 3\%$	\$0.0385	\$0.0024	\$0.0001
Age = 75	$\gamma = \infty$	\$9.2979	\$8.1680	\$7.2460
	$\gamma = 10\%$	\$1.7928	\$0.9691	\$0.4433
	$\gamma = 7\%$	\$0.5818	\$0.1965	\$0.0476
	$\gamma = 6\%$	\$0.2930	\$0.0752	\$0.0130
	$\gamma = 5\%$	\$0.1094	\$0.0194	\$0.0022
	$\gamma = 4\%$	\$0.0253	\$0.0027	\$0.0002
	$\gamma = 3\%$	\$0.0026	\$0.0001	\$0.0000

Notes: Table displays the value of the RCLA – which pays \$1 per year of lifetime income – assuming the Reference Portfolio Index (RPI) is allocated to LOW volatility investments with $\sigma = 10\%$. The spending γ denotes the fixed percentage of the initial RPI level I_0 that is withdrawn annually (and in continuous time) until ruin. When $\gamma = \infty$ the RPI hits zero immediately and the RCLA collapses to a basic annuity displayed in Table #1. The mortality is assumed Gompertz with parameters $\lambda = 0$, $m = 86.3$ and $b = 9.5$. Prices are risk neutral (ie. $\mu = \rho = r =$ risk-free rate).

Table #2b

Ruin-Contingent Life Annuity (RCLA): HIGH Volatility ($\sigma = 25\%$)				
Reference Portfolio Index (RPI) Initial Value is $W_0 = I_0 = \$100$				
Lifetime Payout Upon Ruin is \$1 per year				
Initial Purchase	RPI Spending	$\rho = 3.0\%$	$\rho = 5.0\%$	$\rho = 7.0\%$
Age = 50	$\gamma = \infty$	\$19.7483	\$15.2205	\$12.1693
	$\gamma = 10\%$	\$10.6454	\$6.4788	\$3.8827
	$\gamma = 7\%$	\$8.0694	\$4.4234	\$2.3422
	$\gamma = 6\%$	\$6.9858	\$3.6383	\$1.8159
	$\gamma = 5\%$	\$5.7793	\$2.8227	\$1.3093
	$\gamma = 4\%$	\$4.4570	\$2.0038	\$0.8466
	$\gamma = 3\%$	\$3.0457	\$1.2249	\$0.4571
Age = 65	$\gamma = \infty$	\$13.6601	\$11.3828	\$9.6609
	$\gamma = 10\%$	\$5.4652	\$3.5491	\$2.2451
	$\gamma = 7\%$	\$3.6732	\$2.1443	\$1.2009
	$\gamma = 6\%$	\$2.9976	\$1.6622	\$0.8790
	$\gamma = 5\%$	\$2.3015	\$1.1972	\$0.5899
	$\gamma = 4\%$	\$1.6103	\$0.7719	\$0.3477
	$\gamma = 3\%$	\$0.9645	\$0.4144	\$0.1657
Age = 75	$\gamma = \infty$	\$9.2979	\$8.1680	\$7.2460
	$\gamma = 10\%$	\$2.4354	\$1.6324	\$1.0625
	$\gamma = 7\%$	\$1.4095	\$0.8470	\$0.4882
	$\gamma = 6\%$	\$1.0713	\$0.6113	\$0.3330
	$\gamma = 5\%$	\$0.7531	\$0.4031	\$0.2049
	$\gamma = 4\%$	\$0.4705	\$0.2322	\$0.1082
	$\gamma = 3\%$	\$0.2420	\$0.1072	\$0.0445

Notes: Table – which is identical in structure to #2a – displays the (risk neutral) value of the RCLA assuming the Reference Portfolio Index (RPI) is allocated to HIGH volatility investments with $\sigma = 25\%$. The RPI spending rate γ denotes the fixed percentage withdrawn.

Table #3

Super RCLA Value: MEDIUM Volatility ($\sigma = 17\%$)				
Lifetime Payout Upon Ruin is AT LEAST \$1 per year				
Initial Purchase	Initial Spending Rate	$\rho = 3.0\%$	$\rho = 5.0\%$	$\rho = 7.0\%$
Age = 50	$\gamma = 10.0\%$	\$13.1593	\$8.4032	\$5.2951
	$\gamma = 7.0\%$	\$10.6177	\$6.0704	\$3.2801
	$\gamma = 5.5\%$	\$8.5497	\$4.3654	\$2.0237
	$\gamma = 4.0\%$	\$5.6736	\$2.3663	\$0.8479
Age = 57	$\gamma = 10.0\%$	\$10.2025	\$6.6799	\$4.2748
	$\gamma = 7.0\%$	\$7.7181	\$4.4651	\$2.4178
	$\gamma = 5.5\%$	\$5.8394	\$2.9846	\$1.3756
	$\gamma = 4.0\%$	\$3.4939	\$1.4433	\$0.5122
Age = 65	$\gamma = 10.0\%$	\$6.6981	\$4.4761	\$2.8938
	$\gamma = 7.0\%$	\$4.5249	\$2.6205	\$1.4077
	$\gamma = 5.5\%$	\$3.0899	\$1.5607	\$0.7076
	$\gamma = 4.0\%$	\$1.5749	\$0.6362	\$0.2216
Age = 67	$\gamma = 10.0\%$	\$5.8505	\$3.9217	\$2.5370
	$\gamma = 7.0\%$	\$3.8074	\$2.1997	\$1.1767
	$\gamma = 5.5\%$	\$2.5171	\$1.2643	\$0.5698
	$\gamma = 4.0\%$	\$1.2218	\$0.4897	\$0.1695
Age = 75	$\gamma = 10.0\%$	\$2.8580	\$1.9170	\$1.2303
	$\gamma = 7.0\%$	\$1.5232	\$0.8606	\$0.4481
	$\gamma = 5.5\%$	\$0.8542	\$0.4148	\$0.1809
	$\gamma = 4.0\%$	\$0.3261	\$0.1254	\$0.0420

Notes: Table displays the value of the Super RCLA assuming the Reference Portfolio Index (RPI) is allocated to MEDIUM volatility investments with $\sigma = 17\%$ volatility. The initial RPI spending γ denotes the percent of the initial index value that is withdrawn annually (in continuous time). The factors in Table #3 are not directly comparable to the factors in Table #2 since the lifetime income upon ruin could exceed \$1, if the RPI "does well" prior to ruin. The mortality is assumed Gompertz with parameters $\lambda = 0$, $m = 86.3$ and $b = 9.5$. Prices are risk neutral (ie. $\mu = \rho = r =$ risk-free rate).

Table #4

Valuation of RCLA v.s. Super-RCLA under Differing RPI Volatility ($\rho = 5\%$)				
What is a Step-Up Really Worth?				
Purchase	Volatility (σ)	RCLA ($\gamma = 5\%$)	SRCLA ($\gamma = 5\%$)	"Super Premium"
Age = 57	$\sigma = 8\%$	\$0.2102	\$0.4341	*+106%*
	$\sigma = 15\%$	\$0.8590	\$1.9123	*+123%*
	$\sigma = 20\%$	\$1.4378	\$3.3569	*+133%*
	$\sigma = 25\%$	\$2.0521	\$5.0267	*+145%*
Age = 62	$\sigma = 8\%$	\$0.1096	\$0.2233	*+104%*
	$\sigma = 15\%$	\$0.5617	\$1.2390	*+121%*
	$\sigma = 20\%$	\$1.0088	\$2.3325	*+131%*
	$\sigma = 25\%$	\$1.5052	\$3.6467	*+142%*
Age = 67	$\sigma = 8\%$	\$0.0470	\$0.0939	*+100%*
	$\sigma = 15\%$	\$0.3230	\$0.7043	*+118%*
	$\sigma = 20\%$	\$0.6362	\$1.4534	*+128%*
	$\sigma = 25\%$	\$1.0060	\$2.4051	*+139%*

The above table illustrates the impact of investment (RPI) volatility (σ) on both the RCLA and S-RCLA value, assuming the same Gompertz mortality with parameters $\lambda = 0$, $m = 86.3$ and $b = 9.5$. Note that both the RCLA and S-RCLA are represented per guaranteed dollar of lifetime income (i.e. scaled) and that the valuation rate (and hence μ) is equal to 5%. We assume no delay ($\tau = 0$) and hence no bonus ($\beta = 0$). The table also displays the percent by which the Super RCLA exceeds the RCLA value, under various volatility assumptions and ages.

Table #5a

Super GLWB Value with Delays & Bonus: MEDIUM Volatility ($\sigma = 17\%$)				
\$100 investment into a Guaranteed Lifetime Withdrawal Benefit (GLWB)				
Initial Purchase	Bonus, Delay, Spending	$\rho = 3.0\%$	$\rho = 5.0\%$	$\rho = 7.0\%$
Age = 50	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$39.5199	\$19.0804	\$8.2223
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$40.3168	\$18.4768	\$7.5435
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$38.8829	\$17.0176	\$6.6352
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$34.6250	\$13.7056	\$4.8320
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$28.5642	\$9.9539	\$3.0714
Age = 65	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$13.0509	\$6.1738	\$2.5882
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$10.8703	\$4.7075	\$1.8006
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$9.0896	\$3.6539	\$1.2920
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$5.9436	\$2.0457	\$0.6090
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$3.1648	\$0.9087	\$0.2177
Age = 70	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$7.1354	\$3.3118	\$1.3634
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$5.2707	\$2.1989	\$0.8097
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$4.0458	\$1.5467	\$0.5182
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$2.1754	\$0.6977	\$0.1911
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$0.8564	\$0.2258	\$0.0487
Age = 75	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$3.2413	\$1.4654	\$0.5891
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$2.0320	\$0.8077	\$0.2834
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$1.3834	\$0.4970	\$0.1559
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$0.5568	\$0.1647	\$0.0411
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$0.1368	\$0.0329	\$0.0064

Notes: The table displays the value of a CONTINUOUS step-up Guaranteed Lifetime Withdrawal Benefit (GLWB) under a variety of bonus, delay and withdrawal assumptions. It is the value of the Super-RCLA multiplied by the number of lifetime dollars guaranteed. The mortality is assumed Gompertz with parameters $\lambda = 0$, $m = 86.3$ and $b = 9.5$. Prices are risk neutral (ie. $\mu = \rho = r =$ risk-free rate).

Table #5b

Super GLWB Value with Delays & Bonus: LOW Volatility ($\sigma = 10\%$)				
\$100 investment into a Guaranteed Lifetime Withdrawal Benefit (GLWB)				
Initial Purchase	Bonus, Delay, Spending	$\rho = 3.0\%$	$\rho = 5.0\%$	$\rho = 7.0\%$
Age = 50	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$22.8628	\$6.9956	\$1.3465
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$22.6176	\$6.1226	\$1.0295
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$21.8057	\$5.3677	\$0.8103
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$19.5396	\$3.9733	\$0.4759
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$16.1957	\$2.6430	\$0.2351
Age = 65	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$5.4466	\$1.4094	\$0.2317
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$4.2610	\$0.9040	\$0.1189
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$3.5343	\$0.6506	\$0.0719
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$2.2689	\$0.3214	\$0.0249
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$1.1458	\$0.1229	\$0.0064
Age = 70	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$2.4404	\$0.5805	\$0.0887
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$1.6694	\$0.3148	\$0.0369
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$1.2655	\$0.2038	\$0.0196
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$0.6525	\$0.0793	\$0.0052
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$0.2322	\$0.0209	\$0.0009
Age = 75	$\beta = 5\%, \tau = 1, \gamma = 5\%$	\$0.8428	\$0.1813	\$0.0254
	$\beta = 5\%, \tau = 7, \gamma = 5\%$	\$0.4822	\$0.0793	\$0.0082
	$\beta = 5\%, \tau = 10, \gamma = 5\%$	\$0.3215	\$0.0446	\$0.0037
	$\beta = 5\%, \tau = 15, \gamma = 5\%$	\$0.1190	\$0.0122	\$0.0007
	$\beta = 5\%, \tau = 20, \gamma = 5\%$	\$0.0247	\$0.0018	\$0.0000

Notes: The table displays the value of a CONTINUOUS step-up Guaranteed Lifetime Withdrawal Benefit (GLWB) under a variety of bonus, delay and withdrawal assumptions. It is the value of the Super-RCLA multiplied by the number of lifetime dollars guaranteed. The mortality is assumed Gompertz with parameters $\lambda = 0$, $m = 86.3$ and $b = 9.5$. Prices are risk neutral (ie. $\mu = \rho = r =$ risk-free rate). This Table #5b is based on the RPI allocated to LOW volatility investments.