A combinatorial result with applications to self-interacting random walks.

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Abstract

We give a series of combinatorial results that can be obtained from any two collections (both indexed by $\mathbb{Z}\times\mathbb{N}$) of left and right pointing arrows that satisfy some natural relationship. When applied to certain self-interacting random walk couplings, these allow us to reprove some known transience and recurrence results for some simple models. We also obtain new results for one-dimensional multi-excited random walks and for random walks in random environments in all dimensions.

1 Introduction

Coupling is a powerful tool for proving certain kinds of properties of random variables or processes. A coupling of two random processes X and Y typically refers to defining random variables X' and Y' *on a common probability space* such that $X' \sim X$ (i.e. X and X' are identically distributed) and $Y' \sim Y$. There can be many ways of doing this, but generally one wants to define the probability space such that the *joint distribution* of (X', Y') has some property. For example, suppose that $X = \{X_n\}_{n\geq 0}$ and $Y = \{Y_n\}_{n\geq 0}$ are two nearest-neighbour simple random walks in 1 dimension with drifts $\mu_X \leq \mu_Y$ respectively. One can define $X' \sim X$ and $Y' \sim Y$ on a common probability space so that X' and Y' are independent (this is the so-called product probability space), but one can also define $X'' \sim X$ and $Y'' \sim Y$ on a common probability space so that $X''_n \leq Y''_n$ for all n with probability 1.

Consider now a nearest-neighbour random walk $\{X_n\}_{n\geq 0}$ on $\mathbb{Z}^d$ that has transition probabilities $(2d)^{-1}$ of stepping in each of the $2d$ possible directions, except on the *first departure from each site*. On the first departure, these are also the transition probabilities for stepping to the left and right in any coordinate direction other than the first. But in the first coordinate, the transition probabilities are instead $(2d)^{-1}(1+\beta)$ (right) and $(2d)^{-1}(1-\beta)$ (left), for some fixed parameter $\beta \in [0,1]$. This is known as an excited random walk [1] and the behaviour of these and more general walks of this kind has been studied in some detail since 2003. For this particular model, it

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is known [2] that for $d \geq 2$ and $\beta > 0$, there exists $v_\beta = (v_\beta^{[1]}, 0, \ldots, 0) \in \mathbb{Z}^d$ with $v_\beta^{[1]} > 0$ such that $\lim_{n\to\infty} n^{-1} X_n = v_\beta$ with probability 1. When $d = 1$ the model is recurrent (0 is visited infinitely often) except in the trivial case $\beta = 1$. It is plausible that $v_\beta^{[1]}$ should be a non-decreasing function of β (i.e. increasing the local drift should increase the global drift) but this is not known in general.

A natural first attempt at trying to prove such a monotonicity result would be as follows: given $0 < \beta_1 < \beta_2 \leq 1$, construct a coupling of excited random walks X and Y with parameters β_1 and $\beta_2 > \beta_1$ respectively such that with probability 1, $X_n^{[1]} \leq Y_n^{[1]}$ for all n. Thus far no one has been able to construct such a coupling, and the monotonicity of $v_\beta^{[1]}$ as a function of β remains an open problem in dimensions $2 \leq d \leq 8$. In dimensions $d \geq 9$ this result has been proved [4] using a somewhat technical expansion method, as well as rigorous numerical bounds on simple random walk quantities. More general models in 1 dimension have been studied, and some monotonicity results [7] have been obtained via probabilistic arguments but without coupling. This raises the question of whether or not one can obtain proofs of these kinds of results using a coupling argument that has weaker aims e.g. such that $\max_{m\leq n} X_m^{[1]} \leq \max_{m\leq n} Y_m^{[1]}$ for all n, rather than $X_n^{[1]} \leq Y_n^{[1]}$ for all n.

This paper addresses this issue in 1-dimension. We study relationships between completely deterministic (non-random) 1-dimensional systems of arrows that may prove to be of independent interest in combinatorics. Each system $\mathcal{L}$ of arrows defines a sequence L of integers. We show that under certain natural local conditions on arrow systems $\mathcal{L}$ and $\mathcal{R}$, one obtains relations between the corresponding sequences such as $\max_{m\leq n} L_m^{[1]} \leq \max_{m\leq n} R_m^{[1]}$ for all n (while it's still possible that $L_n^{[1]} > R_n^{[1]}$ for some n).

These may be applied to certain random systems of arrows, to give self-interacting random walk couplings. Doing so, one can obtain results about the (now random) sequence R_n *if* L_n (also random) is well understood, and vice versa. This yields alternative proofs of some existing results, as well as new non-trivial results about so-called multi-excited random walks in 1 dimension and some models of random walks in random environments in all dimensions – see e.g. [5]. To be a bit more precise, in [5] a projection argument applied to some models of random walks in random environments (in all dimensions) gives rise to a one-dimensional random walk Y, which can be coupled with a one-dimensional multi-excited random walk Z (both walks depending on a parameter p) so that for every $j \in \mathbb{Z}$ and every $r \geq 1$:

(i) If Y goes left on its rth visit to j then so does Z (if such a visit occurs), and therefore

(ii) If Z goes right on its rth visit to j then so does Y (if such a visit occurs).

Explicit conditions ($p > \frac{3}{4}$ in this case) governing when $Z_n \to \infty$ as $n \to \infty$ are given in [7]. One would like to conclude that also $Y_n \to \infty$ (whence the original random walk in d-dimensions returns to its starting point only finitely many times) when $p > \frac{3}{4}$. This can be achieved by applying the result of this paper to the coupling mentioned above.

The main contributions of this paper are: combinatorial results concerning sequences defined by arrow systems satisfying certain natural local relationships (see Theorems 1.4 and 1.5); some non-trivial counterintuitive examples; and application of these combinatorial results with non-monotone couplings to obtain new results in the theory of random walks.

1.1 Arrow systems

A collection $\mathcal{E} = (\mathcal{E}(x,r))_{x\in\mathbb{Z},r\in\mathbb{N}}$, where $\mathcal{E}(x,r) \in \{\leftarrow, \rightarrow\}$ is the arrow above the vertex $x \in \mathbb{Z}$ at level $r \in \mathbb{N}$, is called an *arrow system.* This should be thought of as an infinite (ordered) stack of arrows rising above each vertex in $\mathbb{Z}$.

In a given arrow system $\mathcal{E}$, let $\mathcal{E}_{\leftarrow}(j,r)$ denote the number of $\leftarrow$ arrows, out of the first r arrows above j. As r increases, this quantity counts the number of $\leftarrow$'s appearing in the arrow columns above j. Similarly define $\mathcal{E}_{\rightarrow}(j,r) = r - \mathcal{E}_{\leftarrow}(j,r)$. We can define a sequence $E = \{E_n\}_{n\geq 0}$ by setting $E_0 = 0$ and letting E evolve by taking one step to the left or right (at unit times), according to the lowest arrow of the $\mathcal{E}$-stack at its current location, and then deleting that arrow. In other words, if $\#\{0 \leq m \leq n : E_m = E_n\} = k$ then $E_{n+1} = E_n + 1$ if $\mathcal{E}(E_n, k) = \rightarrow$ (resp. $E_{n+1} = E_n - 1$ if $\mathcal{E}(E_n, k) = \leftarrow$). In Section 5 we'll briefly discuss the connection between these sequences and arrow systems and the theory of excursions and trees.

Definition 1.1 ($\mathcal{L} \preccurlyeq \mathcal{R}$)**.** *Given two arrow systems $\mathcal{L}$ and $\mathcal{R}$, we write $\mathcal{L} \preccurlyeq \mathcal{R}$ if for each $j \in \mathbb{Z}$ and each $r \in \mathbb{N}$,*

$$\mathcal{L}_{\leftarrow}(j,r) \geq \mathcal{R}_{\leftarrow}(j,r) \qquad (\textit{and hence also } \mathcal{L}_{\rightarrow}(j,r) \leq \mathcal{R}_{\rightarrow}(j,r)).$$

Definition 1.2 ($\mathcal{L} \trianglelefteq \mathcal{R}$)**.** *We write $\mathcal{L} \trianglelefteq \mathcal{R}$ if for each $j \in \mathbb{Z}$ and each $r \in \mathbb{N}$,*

$$\mathcal{L}(j,r) = \rightarrow \quad \Rightarrow \mathcal{R}(j,r) = \rightarrow .$$

It is easy to see that $\mathcal{L} \trianglelefteq \mathcal{R}$ implies $\mathcal{L} \preccurlyeq \mathcal{R}$.

Now define two paths/sequences $\{L_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ in $\mathbb{Z}$ according to the arrows in $\mathcal{L}$ and $\mathcal{R}$ respectively as above (in particular $L_0 = R_0 = 0$). Since each arrow system determines a unique sequence, but a given sequence may be obtained from multiple different arrow systems, we write $L \preccurlyeq R$ (resp. $L \trianglelefteq R$) if there exist $\mathcal{L} \preccurlyeq \mathcal{R}$ (resp. $\mathcal{L} \trianglelefteq \mathcal{R}$) whose corresponding sequences are L and R respectively. Note that when $\mathcal{L} \trianglelefteq \mathcal{R}$, the paths $Z = L$ and $Y = R$ constructed from $\mathcal{L}$ and $\mathcal{R}$ as above automatically satisfy the conditions (i) and (ii) appearing at the beginning of Section 1.

An arrow system $\mathcal{E}$ is said to be *o-right recurrent* if in the new system $\mathcal{E}_+$ defined by $\mathcal{E}_+(0,i) = \rightarrow$ for all $i \geq 1$, and $\mathcal{E}_+(x,i) = \mathcal{E}(x,i)$ for all $i \geq 1$ and $x > 0$, $E_{+,n} = o$ infinitely often. Similarly $\mathcal{E}$ is *o-left recurrent* if in the system $\mathcal{E}_-$ defined by $\mathcal{E}_-(0,i) = \leftarrow$ for all $i \geq 1$, and $\mathcal{E}_-(x,i) = \mathcal{E}(x,i)$ for all $i \geq 1$ and $x < 0$, $E_{-,n} = o$ infinitely often.

Definition 1.3 ($|\mathcal{I}| \preccurlyeq \mathcal{E}$)**.** *We write $|\mathcal{I}| \preccurlyeq \mathcal{E}$ if:*

- *for each $j > 0$ and $r \geq 1$, $\mathcal{I}_{\leftarrow}(j,r) \geq \mathcal{E}_{\leftarrow}(j,r)$, and*
- *for each $j < 0$ and $r \geq 1$, $\mathcal{I}_{\rightarrow}(j,r) \geq \mathcal{E}_{\rightarrow}(j,r)$.*

That is, if $\mathcal{I}$ has higher counts for arrows pointing toward o.

The main results of this paper are the following two theorems, in which $n_{E,t}(x) = \#\{k \leq t : E_k = x\}$.

Theorem 1.4. *Suppose that* $\mathcal{L} \preccurlyeq \mathcal{R}$. *Then*

(i) $\liminf_{n\to\infty} L_n \leq \liminf_{n\to\infty} R_n$*;*

(ii) $\limsup_{n\to\infty} L_n \leq \limsup_{n\to\infty} R_n$*;*

(iii) Let $a_n \leq n$ *be any increasing sequence, with* $a_n \to \infty$*. If there exists* $x \in \mathbb{Z}$ *such that* $L \geq x$ *infinitely often then* $\limsup_{n\to\infty} \frac{L_n}{a_n} \leq \limsup_{n\to\infty} \frac{R_n}{a_n}$.

(iv) If $n_{R,t}(x) > n_{L,t}(x)$ *then* $n_{R,t}(y) \geq n_{L,t}(y)$ *for every* $y > x$*.*

See also Corollary 3.10 in the case that L is transient to the right.

Theorem 1.5. *Suppose that* $|\mathcal{I}| \preccurlyeq \mathcal{E}$. *Then*

(i) If $\mathcal{E}$ *is o-right recurrent then so is* $\mathcal{I}$*.*

(ii) If $\mathcal{E}$ *is o-left recurrent then so is* $\mathcal{I}$*.*

As $\frac{L_n}{n}$ represents the average speed of the sequence L, up to time n, in many applications the sequence of interest in Theorem 1.4 (iii) will be $a_n = n$. Part (ii) of Theorem 1.4 actually follows from part (i) by a simple mirror symmetry argument. There is a symmetric version of (iii), but one must be careful. Part (iii) obviously implies that if $u = \lim n^{-1} R_n$ and $l = \lim n^{-1} L_n$ both exist then $l \leq v$, however we show in Section 4.1 that $L \trianglelefteq R$ does not imply that $\liminf \frac{L_n}{n} \leq \liminf \frac{R_n}{n}$. The mirror image (about 0) of the counterexample in Section 4.1 also shows that (iii) is not true in general if we drop the condition that $L \geq x$ infinitely often, for some x. One might also conjecture that if $L \trianglelefteq R$ then the amount of time that $R > L$ is at least as large as the amount of time that $R < L$. This is also false as per a counterexample in Section 4.2.

The remainder of the paper is organised as follows. Section 2 contains the basic combinatorial relations which are satisfied by the arrow systems and their corresponding sequences. These will be needed in order to prove our first results. Section 3 gives various consequences of the relationship $\mathcal{L} \preccurlyeq \mathcal{R}$ between two arrow systems, and includes the proofs of the main results of the paper. Section 4 contains the counterexamples described above. Section 5 briefly discusses the relationship between the existing theory of excursions and trees, and our arrow systems. Finally Section 7 contains applications of our results in the study of self-interacting random walks.

2 Basic relations

Given an arrow system $\mathcal{E}$ and $t \geq 0$, let $n_{E,t}(x) = \#\{k \leq t : E_k = x\}$ and $n_{E,t}(x, y) = \#\{k \leq t : E_{k-1} = x, E_k = y\}$. Then the following relationships hold:

$$n_{E,t}(x) = \delta_{x,0} + n_{E,t}(x-1, x) + n_{E,t}(x+1, x) \tag{2.1}$$

$$n_{E,t}(x) = \delta_{E_t,x} + n_{E,t}(x, x+1) + n_{E,t}(x, x-1) \tag{2.2}$$

$$t + 1 = \sum_{i=-\infty}^{\infty} n_{E,t}(i). \tag{2.3}$$

Relation (2.1) says that every visit to x is either from the left or right, except for the first visit if $x = 0$. Relation (2.2) is similar, but in terms of departures from x. The sum in (2.3) is in fact a finite sum since $n_{E,t}(i) = 0$ for $|i| > t$.

Next

$$\begin{aligned} n_{E,t}(x, x+1) =& \mathcal{E}_{\rightarrow}(x, n_{E,t}(x) - I_{E_t=x}) && (2.4)\\ n_{E,t}(x, x-1) =& \mathcal{E}_{\leftarrow}(x, n_{E,t}(x) - I_{E_t=x}), && (2.5) \end{aligned}$$

where e.g. relation (2.4) says that the number of departures from x to the right is the number of "used" right arrows at x.

Finally,

$$n_{E,t}(x, x+1) + I_{x+1\le 0} I_{E_t \le x} = n_{E,t}(x+1, x) + I_{x\ge 0} I_{E_t \ge x+1}, \tag{2.6}$$

which says that the number of moves from x to $x + 1$ is closely related to the number of moves from $x + 1$ to x. They may differ by 1 depending on the position of x relative to 0 and the current value of the sequence. For example, if $0 \le x < E_t$ then the number of moves from x to $x + 1$ up to time t is one more than the number of moves from $x + 1$ to x up to time t.

3 Implications of $\mathcal{L} \preccurlyeq \mathcal{R}$.

In this section we always assume that $\mathcal{L} \preccurlyeq \mathcal{R}$. The results typically have symmetric versions using the fact that $\mathcal{L} \preccurlyeq \mathcal{R} \iff -\mathcal{R} \preccurlyeq -\mathcal{L}$, which is equivalent to considering arrow systems reflected about 0. We divide the section into two subsections based roughly on the nature of the results and their proofs.

For $x \in \mathbb{Z}$ and $k \ge 0$, let $T_L(x, k) = \inf\{t \ge 0 : n_{L,t}(x) = k\}$, and $T_R(x, k) = \inf\{t \ge 0 : n_{R,t}(x) = k\}$.

3.1 Results obtained from the basic relations

The proofs in this section are based on applications of the basic relations of Section 2. The first few results are somewhat technical, but will be used in turn to prove some of the more appealing results. Roughly speaking they describe how the relative numbers of visits of L and R to neighbouring sites $x - 1$ and x relate to each other.

Lemma 3.1. *If L hits x at least $k \ge 1$ times and R is eventually to the left of x after fewer than k visits to x, then there exists a site $y < x$ that R hits at least $n_{L,T_L(x,k)}(y)$ times.*

Proof. Fix x, k and let $T = T_L(x, k)$ and $y_0 := \inf\{z \le x : n_{L,T}(z) > 0\} \le 0$. If $y_0 = x$ then the first $k-1$ arrows at x are all right arrows, i.e. $\mathcal{L}_{\rightarrow}(y_0, k-1) = k-1$. Then also $\mathcal{R}_{\rightarrow}(y_0, k-1) = k-1$ so R cannot be to the left of x after fewer than k visits. Similarly if $y_0 < x$ then the first $n_{L,T}(y_0)$ arrows at y_0 are all right arrows, i.e. $\mathcal{L}_{\rightarrow}(y_0, n_{L,T}(y_0)) = n_{L,T}(y_0)$, and so also $\mathcal{R}_{\rightarrow}(y_0, n_{L,T}(y_0)) = n_{L,T}(y_0)$. Therefore either R visits y_0 at least $n_{L,T}(y_0)$ times or it stays in (y_0, x) infinitely often, whence it must visit some site $y \in (y_0, x)$ at least $n_{L,T}(y)$ times as required. ■

Let $n_L(x) = n_{L,\infty}(x)$ and $n_R(x) = n_{R,\infty}(x)$.

Lemma 3.2. *If R hits $x-1$ at least $n_L(x-1)$ times then either*

(a) $n_R(x) \geq n_L(x)$, or

(b) R is always to the right of x after fewer than $n_L(x)$ visits. ($\Rightarrow \liminf_{n\to\infty} R_n > x$)

Proof. Assume that the first claim fails, so in particular $n_R(x) < \infty$. Let $T = \inf\{t : n_{L,t}(x) = n_R(x) + 1\}$. Then $T < \infty$ so $L_T = x$. Choose r sufficiently large so that $R_t \neq x$ for any $t \geq r$, $R_r \neq x-1$, and $n_{R,r}(x-1) \geq n_{L,T}(x-1)$. Then by (2.1) applied to L at time T, and also to R at time r,

$$\begin{aligned} n_{R,r}(x) + 1 =& n_R(x) + 1 = n_{L,T}(x) = n_{L,T}(x-1,x) + n_{L,T}(x+1,x) + \delta_{0,x} \\ n_{R,r}(x) =& \delta_{x,0} + n_{R,r}(x-1,x) + n_{R,r}(x+1,x). \end{aligned}$$

Subtracting one from the other and rearranging we obtain

$$n_{R,r}(x-1,x) - n_{L,T}(x-1,x) + n_{R,r}(x+1,x) + 1 = n_{L,T}(x+1,x).$$

Now $n_{L,T}(x+1,x) = n_{L,T}(x,x+1) + I_{x+1\leq 0}$ from (2.6), so

$$n_{R,r}(x+1,x) + 1 + [n_{R,r}(x-1,x) - n_{L,T}(x-1,x)] = n_{L,T}(x,x+1) + I_{x+1\leq 0}. \tag{3.1}$$

Using (2.4) and the fact that $R_r \neq x$, then $\mathcal{L} \preccurlyeq \mathcal{R}$, then the fact that $n_{L,T}(x) = 1 + n_{R,r}(x)$, and finally again using (2.4) and the fact that $L_T = x$ we obtain

$$n_{R,r}(x,x+1) = \mathcal{R}_{\to}(x, n_{R,r}(x)) \geq \mathcal{L}_{\to}(x, n_{R,r}(x)) = \mathcal{L}_{\to}(x, n_{L,T}(x) - 1) = n_{L,T}(x,x+1). \tag{3.2}$$

Using this bound in (3.1) yields

$$n_{R,r}(x+1,x) + 1 + [n_{R,r}(x-1,x) - n_{L,T}(x-1,x)] \leq n_{R,r}(x,x+1) + I_{x+1\leq 0}. \tag{3.3}$$

Using the fact that $R_r \neq x-1$ and applying (2.4) to R_r at $x-1$, then using $n_{R,r}(x-1) \geq n_{L,T}(x-1)$, then $\mathcal{L} \preccurlyeq \mathcal{R}$, and finally using the fact that $L_T \neq x-1$ and applying (2.4) to L_T at $x-1$, we have that

$$\begin{aligned} n_{R,r}(x-1,x) &= \mathcal{R}_{\to}(x-1, n_{R,r}(x-1)) \geq \mathcal{R}_{\to}(x-1, n_{L,T}(x-1)) \\ &\geq \mathcal{L}_{\to}(x-1, n_{L,T}(x-1)) = n_{L,T}(x-1,x). \end{aligned}$$

Therefore by (3.3), and then (2.6)

$$n_{R,r}(x+1,x) + 1 \leq n_{R,r}(x,x+1) + I_{x+1\leq 0} \leq n_{R,r}(x+1,x) + I_{R_r \geq x+1}. \tag{3.4}$$

Therefore $R_r \geq x+1$, so in fact $R_t > x$ for every $t \geq r$. Moreover $n_{R,r}(x) = n_R(x) < n_L(x)$, which shows (b). ■

Lemma 3.3. *Let $x \in \mathbb{Z}$, and suppose that for some $k > 0$, $n_L(x) \geq k$ and $n_R(x) \geq k$. Then $n_{R,T_R(x,k)}(x-1) \leq n_{L,T_L(x,k)}(x-1)$.*

Proof. Let $T = T_L(x,k) < \infty$ and $S = T_R(x,k) < \infty$. Then $R_S = x > x-1$, so from (2.6) and (2.5)

$$n_{R,S}(x-1,x) = n_{R,S}(x,x-1) + I_{x\geq 1} = \mathcal{R}_{\leftarrow}(x,k-1) + I_{x\geq 1}.$$

Similarly

$$n_{L,T}(x-1,x) = n_{L,T}(x,x-1) + I_{x\geq 1} = \mathcal{L}_{\leftarrow}(x,k-1) + I_{x\geq 1}.$$

Since $\mathcal{R}_{\leftarrow}(x,k-1) \leq \mathcal{L}_{\leftarrow}(x,k-1)$ it follows that $n_{R,S}(x-1,x) \leq n_{L,T}(x-1,x)$. Finally,

$$\mathcal{R}_{\rightarrow}(x-1,n_{R,S}(x-1)) = n_{R,S}(x-1,x) \text{ and } n_{L,T}(x-1,x) = \mathcal{L}_{\rightarrow}(x-1,n_{L,T}(x-1))$$

whence $\mathcal{R}_{\rightarrow}(x-1,n_{R,S}(x-1)) \leq \mathcal{L}_{\rightarrow}(x-1,n_{L,T}(x-1))$. Since the $n_{R,S}(x-1)$-th arrow at $x-1$ is $\rightarrow$ by definition of S (and similarly for $n_{L,T}(x-1)$ and T) this implies that $n_{R,S}(x-1) \leq n_{L,T}(x-1)$ as required. ■

Lemma 3.4. *If $T = T_L(x,k) < \infty$ and R stays to the right of x after fewer than k visits to x then $n_R(x-1) \leq n_{L,T}(x-1)$.*

Proof. Assume that $n_R(x-1) > 0$, otherwise there is nothing to prove. Let $S' = \sup\{t : R_t = x\}$. Then $R_{S'} = x$, $\mathcal{R}(x-1,n_{R,S'}(x-1)) = \rightarrow$ and $\mathcal{R}(x,n_{R,S'}(x)) = \rightarrow$. By (2.6) applied at $x-1$, and then using (2.5), and finally the fact that $\mathcal{R}(x,n_{R,S'}(x)) = \rightarrow$,

$$n_{R,S'}(x-1,x) = n_{R,S'}(x,x-1) + I_{x\geq 1} = \mathcal{R}_{\leftarrow}(x,n_{R,S'}(x)-1) + I_{x\geq 1} = \mathcal{R}_{\leftarrow}(x,n_{R,S'}(x)) + I_{x\geq 1}.$$

Therefore by (2.4),

$$\mathcal{R}_{\rightarrow}(x-1,n_{R,S'}(x-1)) = n_{R,S'}(x-1,x) = \mathcal{R}_{\leftarrow}(x,n_{R,S'}(x)) + I_{x\geq 1}. \tag{3.5}$$

Since $n_{R,S'}(x) < k = n_{L,T}(x)$ we have $\mathcal{R}_{\leftarrow}(x,n_{R,S'}(x)) \leq \mathcal{L}_{\leftarrow}(x,n_{L,T}(x)-1)$, therefore the right hand side of (3.5) is bounded above by

$$\begin{aligned}\mathcal{L}_{\leftarrow}(x,n_{L,T}(x)-1) + I_{x\geq 1} =& n_{L,T}(x,x-1) + I_{x\geq 1}\\ =& n_{L,T}(x-1,x) = \mathcal{L}_{\rightarrow}(x-1,n_{L,T}(x-1)),\end{aligned}$$

where we have used (2.5), followed by (2.6), and then (2.4). We have shown that

$$\mathcal{R}_{\rightarrow}(x-1,n_{R,S'}(x-1)) \leq \mathcal{L}_{\rightarrow}(x-1,n_{L,T}(x-1)).$$

Since $\mathcal{R}(x-1,n_{R,S'}(x-1)) = \rightarrow$, this implies that $n_{R,S'}(x-1) \leq n_{L,T}(x-1)$ as required. ■

3.2 Results obtained by contradiction

The results in this section include less technical results than those of the previous section. Roughly speaking their proofs will be based on contradiction arguments that proceed as follows. Suppose that we have already proved a statement A whenever $\mathcal{L} \preccurlyeq \mathcal{R}$. We now want to prove a statement B whenever $\mathcal{L} \preccurlyeq \mathcal{R}$. Assume that for some $\mathcal{L}$, $\mathcal{R}$ with $\mathcal{L} \preccurlyeq \mathcal{R}$, B is false. Construct two new systems $\mathcal{L}' \preccurlyeq \mathcal{R}'$ from $\mathcal{L}$ and $\mathcal{R}$ such that statement A is violated for $\mathcal{L}'$ and $\mathcal{R}'$. This gives a contradiction, hence there was no such example where $\mathcal{L} \preccurlyeq \mathcal{R}$ but B is false.

Lemma 3.5. *Let $x \in \mathbb{Z}$, and suppose that $n_R(x) < k \le n_L(x)$. Then $n_R(x-1) \le n_{L,T_L(x,k)}(x-1)$ and $\liminf R_n > x$ (i.e. R is forever to the right of x after fewer than k visits to x and at most $n_{L,T_L(x,k)}(x-1)$ visits to $x-1$).*

Proof. By Lemma 3.4, it is sufficient to prove that under the hypotheses of the lemma, R is to the right of x infinitely often. Suppose instead that R is forever to the left of x (after fewer than k visits to x). Then we may define two new systems $\mathcal{R}'$ and $\mathcal{L}'$ by forcing every arrow at x at level k and above to be $\rightarrow$. To be precise, given an arrow system $\mathcal{E}$ we'll define $\mathcal{E}'$ by $\mathcal{E}'(y,\cdot) = \mathcal{E}(y,\cdot)$ for all $y \neq x$, $\mathcal{E}'(x,j) = \mathcal{E}(y,j)$ for all $j < k$, and $\mathcal{E}'(x,j) = \rightarrow$ for every $j \ge k$. Clearly $\mathcal{L}' \preccurlyeq \mathcal{R}'$ and $T' = T_{L'}(x,k) = T$. The sequences R and R' are identical since we have not changed any arrow used by R anyway. The sequences L and L' agree up to time T, while $L'_n \ge x$ for all $n \ge T$, since L' can never go left from x after time T. It follows that $n_{L'}(z) = n_{L,T}(z) < \infty$ for every $z < x$.

Let $y_1 := \max\{z < x : n_{R'}(z) \ge n_{L',T}(z)\}$. By Lemma 3.1, $-\infty < y_1 < x$. By Lemma 3.2 (applied to L', R') either R' hits y_1+1 at least $n_{L'}(y_1+1) \ge n_{L,T}(y_1+1)$ times, or R' is forever to the right of y_1+1 after fewer than $n_{L'}(y_1+1)$ visits. In either case, $y_1+1 < x$ (as $n_{R'}(x) < k$ and R' lies eventually to the left of x). So there exists some $y_2 \in (y_1, x)$ such that $n_{R'}(y_2) \ge n_{L'}(y_2) = n_{L',T}(y_2)$. This contradicts the definition of y_1. ■

Corollary 3.6. *If $n_{R,t}(x-1) > n_{L,t}(x-1)$ then $n_{R,t}(x) \ge n_{L,t}(x)$.*

Proof. Suppose instead that $n_{R,t}(x) < n_{L,t}(x)$. Let $k = n_{R,t}(x)+1$, so that $T = T_L(x,k) \le t$ and $S = T_R(x,k) > t$. Then

$$n_{R,S}(x-1) \ge n_{R,t}(x-1) > n_{L,t}(x-1) \ge n_{L,T}(x-1).$$

This violates Lemma 3.3 (if $n_R(x) \ge k$) or Lemma 3.5 (if $n_R(x) < k$). ■

Corollary 3.7. *Fix $x > 0$, and let $T = T_L(x,1) = \inf\{t : L_t = x\}$ and $S = T_R(x,1)$. Then $S \le T$.*

Proof. If $T = \infty$ then the result is trivial. So assume $T < \infty$. Lemma 3.5 with $k = 1$ implies that $S < \infty$ as well (R cannot be to the right of $x > 0$ without ever passing through x). For each $i < x$, the number of times that L hits i before T is $n_{L,T}(i)$, so $T = \sum_{i=-\infty}^{x-1} n_{L,T}(i)$. Moreover, $n_{L,T}(i)$ is the number of times that L hits i before hitting $i+1$ for the $n_{L,T}(i+1)$-th time (by definition of T, the last visit to $i < x$ up to time T occurs before the last visit to $i+1$ up to time T). By Lemma 3.3 with $k = 1$ we get that $n_{R,S}(x-1) \le n_{L,T}(x-1)$. Set $k_0 = 1$.

Now apply Lemma 3.3 with $x-1$ instead of x and with $k_1 = n_{R,S}(x-1)$ to get

$$n_{R,T_R(x-1,k_1)}(x-2) \le n_{L,T_L(x-1,k_1)}(x-2).$$

But $n_{R,T_R(x-1,k_1)}(x-2) = n_{R,S}(x-2)$ since R cannot visit $x-2$ at times in $(T_r(x-1,k_1), S]$ (in other words, the last visit to $x-2$ occurs before the last visit to $x-1$). Furthermore, $n_{L,T_L(x-1,k_1)}(x-2) \le n_{L,T}(x-2)$ since $n_{L,T}(x-1) \ge k_1 \Rightarrow T_L(x-1,k_1) \le T$. We have just shown that

$$n_{R,S}(x-2) = n_{R,T_R(x-1,k_1)}(x-2) \le n_{L,T_L(x-1,k_1)}(x-2) \le n_{L,T}(x-2).$$

Iterating this argument while $k_j = n_{R,S}(x-j) > 0$ by applying Lemma 3.3 at $x-j$ with $k = k_j$ (there is nothing to do once $n_{R,S}(x-j) = 0$ for some j), we obtain by induction that $n_{R,S}(i) \le n_{L,T}(i)$ for every $i < x$. Thus $S = \sum_{i=-\infty}^{x-1} n_{R,S}(i) \le \sum_{i=-\infty}^{x-1} n_{L,T}(i) = T$ as required. ■

It follows immediately from Corollary 3.7 that

$$\overline{R}_n := \max_{k \le n} R_k \ge \max_{k \le n} L_k =: \overline{L}_n. \tag{3.6}$$

Of course by mirror symmetry we also have $\underline{R}_n := \min_{k \le n} R_k \ge \min_{k \le n} L_k = \underline{L}_n$. The following result extends this idea to the number of visits of the two paths to $\overline{R}_n$ by time n.

Lemma 3.8. *For each $t \ge 0$, $n_{R,t}(\overline{R}_t) \ge n_{L,t}(\overline{R}_t)$ and $n_{L,t}(\underline{L}_t) \ge n_{R,t}(\underline{L}_t)$.*

Proof. Let $\mathcal{L} \preccurlyeq \mathcal{R}$ and suppose the first claim fails. Let $T = \inf\{t \ge 0 : n_{R,t}(\overline{R}_t) < n_{L,t}(\overline{R}_t)\} < \infty$. Let $\mathcal{N}_t = n_{L,t}(\overline{R}_t) - n_{R,t}(\overline{R}_t)$. Then $\mathcal{N}_{t+1} - \mathcal{N}_t \le 1$ if $\overline{R}_{t+1} = \overline{R}_t$, and by (3.6), $\mathcal{N}_{t+1} = 0$ or -1 if $\overline{R}_{t+1} > \overline{R}_t$. Therefore by definition of T we must have $R_T < \overline{R}_T$, $L_T = \overline{R}_T$, and $n_{L,T}(\overline{R}_T) = 1 + n_{R,T}(\overline{R}_T)$. Moreover this happens regardless of the arrows of $\mathcal{L}$ or $\mathcal{R}$ at $\overline{R}_T$ above level $n_{R,T}(\overline{R}_T)$. Define new arrow systems $\mathcal{L}', \mathcal{R}'$ by setting all arrows at $\overline{R}_T$ at level $1 + n_{R,T}(\overline{R}_T)$ and above to be →. By construction $\mathcal{L}' \preccurlyeq \mathcal{R}'$, and $(L_n, R_n) = (L'_n, R'_n)$ for $n \le T$. However $\overline{L}'_{T+1} = \overline{R}_T + 1 > \overline{R}_T = \overline{R}'_{T+1}$ which violates the fact that $\overline{R}'_n \ge \overline{L}'_n$ for all $n \ge 0$.

The second result follows by mirror symmetry. ■

For each $z \in \mathbb{Z}$, $t \in \mathbb{Z}_+$, let $\overline{z}_t = \max(n_{L,t}(z), n_{R,t}(z))$.

Lemma 3.9. *If there exist t, y such that $R_t \le y < L_t$ and $n_{R,t}(y) > n_{L,t}(y)$ then $n_{R,t}(x) \ge n_{L,t}(x)$ for every $x \in [y, L_t]$.*

Proof. Suppose that t and y satisfy the above hypotheses, but the conclusion fails for some $x \in [y, L_t]$. In other words, $y < x \le L_t$ and $n_{R,t}(x) < n_{L,t}(x)$. Define new arrow systems $\mathcal{L}'$ and $\mathcal{R}'$ by setting:

- all arrows at y at level $n_{R,t}(y) + I_{\{R_t \ne y\}}$ and above to be ←;
- all arrows at x at level $n_{L,t}(x) + I_{\{L_t \ne x\}}$ and above to be →; and
- for each $z > x$ set all arrows above level $\overline{z}_t$ to be →.

The resulting arrow systems satisfy $\mathcal{L}' \preccurlyeq \mathcal{R}'$ with $(L_n, R_n) = (L'_n, R'_n)$ for $n \leq t$. By construction $L'_n \to \infty$ as $n \to \infty$, since L'_n never again goes below x, and can make at most finitely many more $\leftarrow$ moves. But also $R'_n \leq y$ for all $n \geq t$, which contradicts the fact that $\overline{R}'_n \geq \overline{L}'_n$ for all $n \geq 0$. ■

We say that a sequence $\{L_n\}_{n\geq 0}$ on $\mathbb{Z}$ is *transient to the right* if for every $x \in \mathbb{Z}$ there exists $n_x \geq 0$ such that $L_n > x$ for all $n \geq n_x$ (i.e. if $\liminf_{n\to\infty} L_n = +\infty$).

Corollary 3.10. *If* $\liminf_{n\to\infty} L_n = +\infty$ *then* $n_R(x) \leq n_L(x)$ *for every* x *and* $\liminf_{n\to\infty} R_n = +\infty$.

Proof. Suppose that L is transient to the right. Then $n_L(y) < \infty$ for each y. Suppose that for some x, $n_R(x) > n_L(x)$. Let $T = T_R(x, n_L(x) + 1)$. Define new systems $\mathcal{L}' \preccurlyeq \mathcal{R}'$ by setting every arrow at x above level $n_L(x)$ to be $\leftarrow$. Then $L' = L$, so $L' \to \infty$, but $R'_t \leq x$ for every $t \geq T$. This violates (3.6) for L', R'. Therefore $n_R(x) \leq n_L(x)$ for every x, which establishes the first claim.

For the second claim, suppose that R is not transient to the right. Then R is either transient to the left or it visits some site x infinitely often. In either case there is some site x such that $n_R(x) > n_L(x)$ which cannot happen by the first claim. ■

Corollary 3.11. $R \geq L$ *infinitely often.*

Proof. If R is not bounded above, this follows by considering the times at which R extends its maximum. It follows similarly if L is not bounded below, using times at which L extends its minimum. The only remaining possibility is that R is bounded above and L is bounded below, in which case by (3.6) both paths visit only finitely many vertices. In this case consider the sets of vertices that R and L visit infinitely often. Let $x_\infty = \sup\{z \in \mathbb{Z} : n_R(z) = \infty\}$ and $y_\infty = \sup\{z \in \mathbb{Z} : n_L(z) = \infty\}$. If $x_\infty < y_\infty$ then Lemma 3.5 is violated (apply it to $x = y_\infty$ for $k > n_R(y_\infty)$). Therefore $x_\infty \geq y_\infty$, so $R_t \geq L_t$ at all sufficiently large t for which $R_t = x_\infty$. ■

3.2.1 Proof of Theorem 1.4

To prove (i) we show that if $L_n \geq x$ for all n sufficiently large, then $R_n \geq x$ for all n sufficiently large. Suppose instead that $R_n < x$ infinitely often. Then choose N sufficiently large so that $L_n \geq x$ for all $n \geq N$, but $R_N < x$ and $n_{R,N}(R_N) > n_{L,N}(R_N)$. Define two new arrow systems $\mathcal{L}', \mathcal{R}'$ by switching all arrows at R_N from level $n_{R,N}(R_N)$ and above to be $\leftarrow$. Then $\mathcal{L}' \preccurlyeq \mathcal{R}'$ but Lemma 3.9 is violated, as is Corollary 3.11. This establishes (i). Applying (i) to $-\mathcal{R} \preccurlyeq -\mathcal{L}$ establishes (ii).

If $L_n \geq x$ infinitely often then $\limsup L_n/a_n \geq \limsup x/a_n = 0$. If equality holds then the required result is trivial since we've just proved that $R_n \geq x$ infinitely often. Otherwise let $0 < M < \infty$ be such that $\limsup L_n/a_n > M$. Then L_n visits infinitely many sites > 0. Let T_i be the times at which L extends its maximum, i.e. $T_0 = 0$ and for $i \geq 1$, $T_i = \inf\{n > T_{i-1} : L_n = 1 + \max_{k<n} L_k\}$. We first verify the (intuitively obvious) statement that $\frac{L_{T_i}}{a_{T_i}} > M$ infinitely often. If $\frac{L_{T_i}}{a_{T_i}} > M$ only finitely often then for all i sufficiently large, $\frac{L_{T_i}}{a_{T_i}} \leq M$. But for all

$n \in [T_i, T_{i+1})$, $\frac{L_n}{a_n} \le \frac{L_{T_i}}{a_n} \le \frac{L_{T_i}}{a_{T_i}}$. So $\frac{L_n}{a_n} \le M$ for all but finitely many n, contradicting the fact that $\limsup L_n/a_n > M$.

Let S_i be the times at which R extends its max. By definition, $L_{T_i} = i = R_{S_i}$ and from Corollary 3.7, $i \le S_i \le T_i$. It follows immediately that for infinitely many i,

$$\frac{R_{S_i}}{a_{S_i}} \ge \frac{L_{T_i}}{a_{T_i}} > M, \tag{3.7}$$

whence $\limsup_{n\to\infty} \frac{R_n}{a_n} \ge M$. This establishes part (iii)

Finally, suppose that (iv) does not hold, and let τ be the first time at which this fails. In other words

$$\tau = \inf\{t \ge 0 : \text{ there exist } y, x < y \text{ such that } n_{R,t}(x) > n_{L,t}(x) \text{ and } n_{R,t}(y) < n_{L,t}(y)\}.$$

Let x_0 be the largest such x, i.e. $x_0 = \sup\{x \in \mathbb{Z} : n_{R,\tau}(x) > n_{L,\tau}(x), \exists y > x \text{ such that } n_{R,\tau}(y) < n_{L,\tau}(y)\}$ and $y_0 = \inf\{y > x_0 : n_{R,\tau}(y) < n_{L,\tau}(y)\}$. Then $x_0 \le y_0 - 2$ or else Corollary 3.6 is violated. By definition of x_0 and y_0 we have $n_{R,\tau}(y_0 - 1) \ge n_{L,\tau}(y_0 - 1)$. Let $k = n_{L,\tau}(y_0)$. Then $n_{L,\tau}(y_0 - 1) \ge n_{L,T_L(y_0,k)}(y_0 - 1)$ so $n_{R,\tau}(y_0 - 1) \ge n_{L,T_L(y_0,k)}(y_0 - 1)$. On the other hand $n_{R,\tau}(y_0) < k$, so $\tau < T_R(y_0, k)$. If $R_\tau < y_0 - 1$ then $n_{R,T_R(y_0,k)}(y_0 - 1) \ge n_{R,\tau}(y_0 - 1) + 1 > n_{L,T_L(y_0,k)}(y_0 - 1)$. This contradicts one of the Lemmas 3.3 or 3.5 (depending on whether $n_R(y_0) \ge k$), so we must have instead that $R_\tau \ge y_0 - 1 > x_0$. Therefore $n_{R,\tau-1}(x_0) = n_{R,\tau}(x_0) > n_{L,\tau}(x_0) \ge n_{L,\tau-1}(x_0)$. Similarly if $L_\tau > x_0 + 1$ we get a contradiction to the symmetric versions of Lemmas 3.3 or 3.5, so we must have $L_\tau \le x_0 + 1 < y_0$, and therefore $n_{L,\tau-1}(y_0) = n_{L,\tau}(y_0) > n_{R,\tau-1}(y_0)$. This contradicts the definition of τ. ■

3.2.2 Proof of Theorem 1.5

Let $|\mathcal{I}| \preccurlyeq \mathcal{E}$. That is, $\mathcal{I}_+ \preccurlyeq \mathcal{E}_+$ and $\mathcal{E}_- \preccurlyeq \mathcal{I}_-$. Therefore $\underline{I}_{+,t} = 0$ for every t, so by Lemma 3.8, $n_{I_+,t}(o) \ge n_{E_+,t}(o)$. If $\mathcal{E}$ is o-right recurrent, then $n_{E_+,t}(o) \to \infty$, so $n_{I_+,t}(o) \to \infty$ as well, establishing (i). Part (ii) follows by symmetry. ■

4 Counterexamples

4.1 $L \trianglelefteq R$ does not imply that $\liminf \frac{L_n}{n} \le \liminf \frac{R_n}{n}$

In general, $L \trianglelefteq R$ does not imply that $\liminf \frac{L_n}{n} \le \liminf \frac{R_n}{n}$, as we shall see in the following example.

Let us first define the two systems as follows, starting with $\mathcal{L}$. At 0 the first three arrows are $\to$. At every $x > 0$ the first two arrows are $\leftarrow$ and the next three arrows are $\to$. It is easy to check that such a system results in a sequence L that takes steps with the pattern $\to\leftarrow\to\leftarrow\to$ repeated indefinitely (without ever needing to look at arrows other than those specified above). Thus $\lim_{n\to\infty} \frac{L_n}{n} = \frac{3-2}{5} = \frac{1}{5}$.

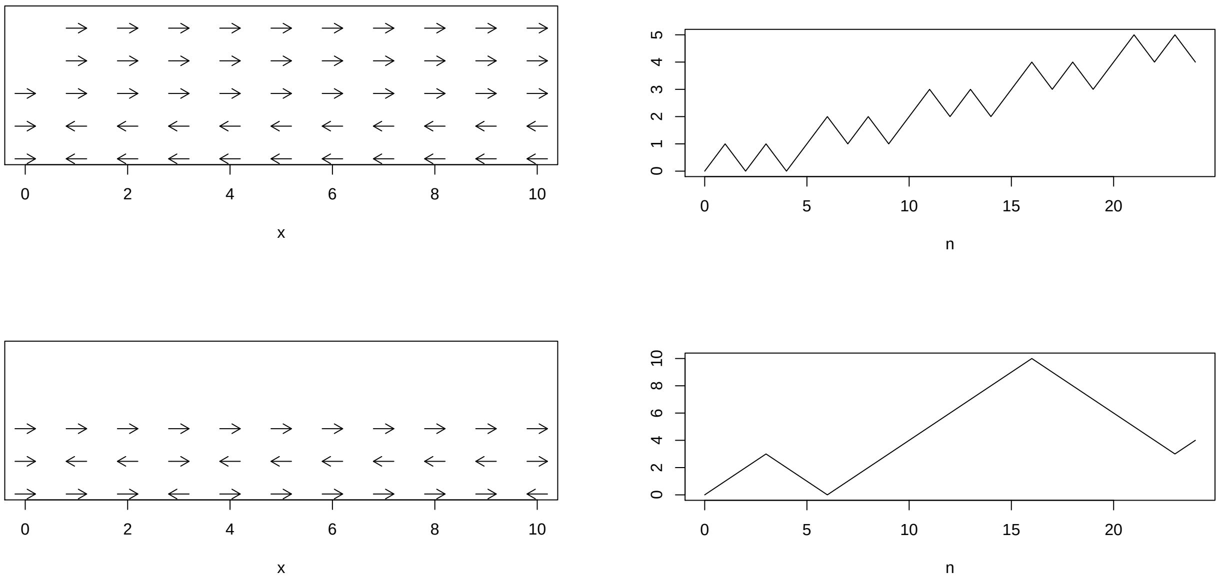

Figure 1: Parts of the systems $\mathcal{L}$ (top) and $\mathcal{R}(3)$ (bottom) and their corresponding sequences L and R, defined in Section 4 such that $\liminf n^{-1}L_n \geq \liminf n^{-1}R_n$. Each site in $\mathbb{N}$ appears five times in the sequence L and three times in the sequence R.

Let us now define a system $\mathcal{R} = \mathcal{R}(N)$, according to a parameter N as follows. At 0 the first three arrows are $\rightarrow$. At each site $x_k = x_k(N)$ of the form

$$x_k = \sum_{m=1}^{k} N^m - \sum_{m=1}^{k-1}\sum_{r=0}^{m}(-1)^{m-r}N^r, \qquad k \geq 1 \tag{4.1}$$

the first arrow is $\leftarrow$ and the next two arrows are $\rightarrow$. At all remaining sites $x > 0$, the first three arrows are $\rightarrow, \leftarrow, \rightarrow$. See Figure 1 for parts of the systems $\mathcal{L}$ and $\mathcal{R}(3)$. By definition of these systems the arrows to the left of 0 and above those shown are irrelevant, so we can set them to be the same (for example, all $\rightarrow$)).

By construction $L \trianglelefteq R$ for each $N \geq 1$, but we will show that $\liminf \frac{R_n}{n} \leq \frac{1}{2N+1} < \frac{1}{5}$ for $N \geq 3$ (also $\limsup \frac{R_n}{n} \geq \frac{N}{N+2}$).

The first site of the form (4.1) is $x_1 = N$. The walk R first encounters a $\leftarrow$ at its first visit to this site and then sees a $\rightarrow$ at site 0 (second visit to 0). The walk R then visits site x_1 for the second time, whence it sees a $\rightarrow$. It continues moving right, visiting every site between x_1 and x_2 exactly once before reaching x_2 at this point it sees a $\leftarrow$, moves to $x_2 - 1$ (for the second visit to that site) and continues seeing $\leftarrow$ at every site in (x_1, x_2) until reaching x_1 for the third time. It then sees $\rightarrow$ at every site in $[x_1, x_2)$ (third visit to each of those sites), but also at every site in $[x_2, x_3)$ (second visit to x_3 and first visit to each site in (x_3, x_4). Continuing in this way, the walk turns left at every x_i on the first visit, and continues left (second visit at interior sites) until reaching x_{i-1} for the third time, and then continues to go right until reaching x_{i+1} for the first time.

The distance between two sites of the form (4.1) is

$$x_{k+1} - x_k = \frac{N^{k+2} + (-1)^{k+1}}{N+1}. \tag{4.2}$$

The lengths of the kth up and down periods respectively are therefore

$$x_k - x_{k-2} = N^k, \quad \text{ and } x_k - x_{k-1} = \sum_{r=0}^{k} (-1)^{k-r} N^r. \tag{4.3}$$

At time $t_k = \sum_{m=1}^{k} N^m + \sum_{m=1}^{k-1} \sum_{r=0}^{m} (-1)^{m-r} N^r$ the walk is at position $x_k = \sum_{m=1}^{k} N^m - \sum_{m=1}^{k-1} \sum_{r=0}^{m} (-1)^{m-r} N^r$ for the first time. At these times we have

$$\frac{R_{t_k}}{t_k} = \frac{\sum_{m=1}^{k} N^m - \sum_{m=1}^{k-1} \sum_{r=0}^{m} (-1)^{m-r} N^r}{\sum_{m=1}^{k} N^m + \sum_{m=1}^{k-1} \sum_{r=0}^{m} (-1)^{m-r} N^r}. \tag{4.4}$$

Relatively simple calculations then give

$$\lim_{k\to\infty} \frac{R_{t_k}}{t_k} = \lim_{k\to\infty} \frac{\frac{N^{k+1}-1}{N-1} - \frac{N^{k+1}}{(N+1)(N-1)}}{\frac{N^{k+1}-1}{N-1} + \frac{N^{k+1}}{(N+1)(N-1)}} = \lim_{k\to\infty} \frac{(N+1)(N^{k+1}-1) - N^{k+1}}{(N+1)(N^{k+1}-1) + N^{k+1}} = \frac{N}{N+2} \tag{4.5}$$

This gives rise to the limit supremum claimed.

Similarly at times $s_k = \sum_{m=1}^{k} N^m + \sum_{m=1}^{k} \sum_{r=0}^{m} (-1)^{m-r} N^r$ the walk is at position $x_{k-1} = \sum_{m=1}^{k} N^m - \sum_{m=1}^{k} \sum_{r=0}^{m} (-1)^{m-r} N^r$ for the last time. After some simple calculations we obtain

$$\lim_{n\to\infty} \frac{R_{s_k}}{s_k} = \frac{1}{2N+1} \tag{4.6}$$

This gives rise to the limit infimum claimed.

4.2 L can be in the lead more than R

Given two sequences L and R with $L \preccurlyeq R$, let $A_{R,t} = \{n \le t : R_n > L_n\}$ and $A_{L,t} = \{n \le t : R_n < L_n\}$. It is not unreasonable to expect that for every $t \in \mathbb{N}$, $|A_{R,t}| \ge |A_{L,t}|$ which essentially says that R is ahead of L more than L is ahead of R. It turns out that this does not hold even when $L \trianglelefteq R$.

To see this, consider the partial arrow systems $\mathcal{R}$ and $\mathcal{L}$ on the left hand side of Figure 2. These two systems differ only at the first arrow at 0, whence $\mathcal{L} \trianglelefteq \mathcal{R}$ (if we set all other arrows to be equal, for example). The first 28 terms of the sequences L and R are plotted on the right of the figure. At any place where the solid line is above the dotted line, $R > L$. In particular $R_n > L_n$ only for $1 \le n \le 7$. Similarly $L > R$ when the dotted line lies above the solid line, which happens at times $9, 10, 14, 15, 19, 20, 24, 25, 26$. Thus we have $|A_{R,25}| = 7 < 8 = |A_{L,25}|$ and similarly $|A_{R,26}| = 7 < 9 = |A_{L,26}|$.

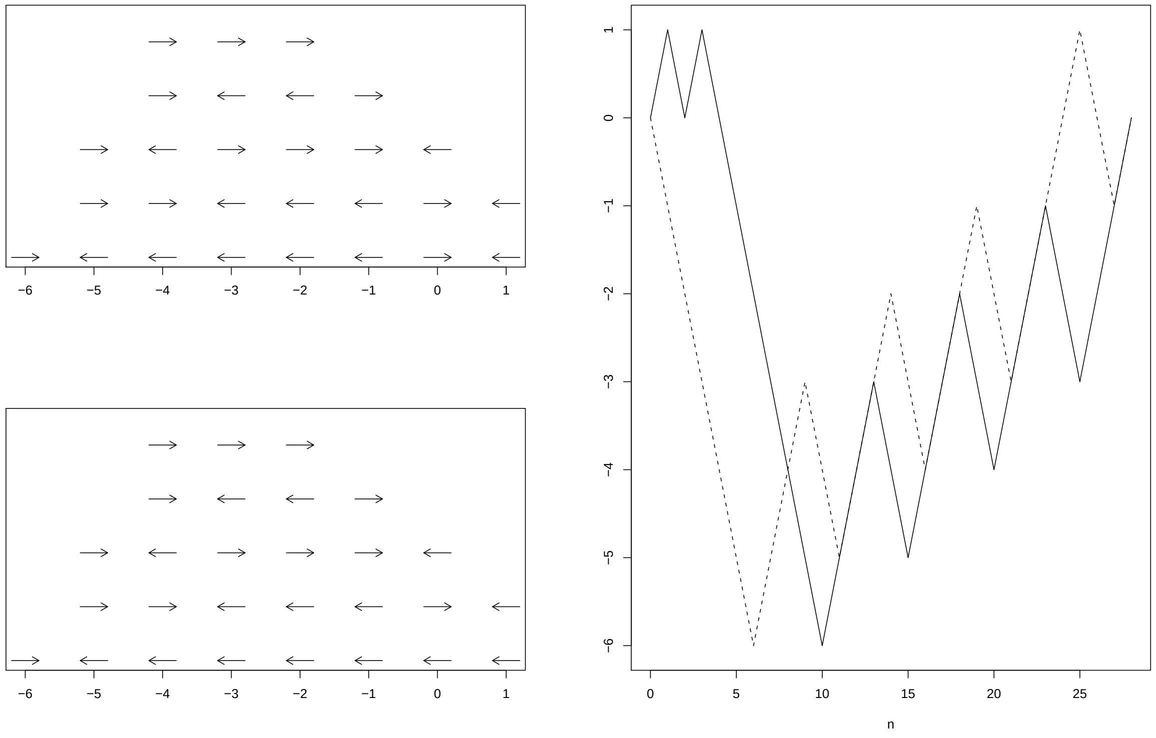

Figure 2: Parts of arrow systems $\mathcal{R}$ (top) and $\mathcal{L}$ (bottom) with $\mathcal{L} \trianglelefteq \mathcal{R}$, along with the corresponding paths R_n (solid) and L_n (dotted). Here, $|A_{R,26}| = 7 < 9 = |A_{L,26}|$.

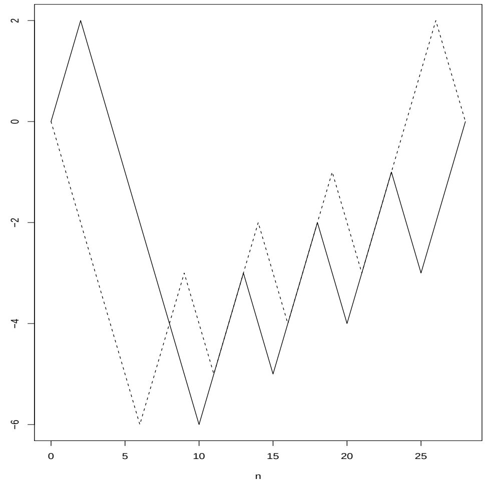

Figure 3: Paths R'_n (solid) and L'_n (dotted) with $L'_n \preccurlyeq R'_n$ and $|A_{R,28}| = 7 < 10 = |A_{L,28}|$. The walks have visited each site the same number of times.

We can modify these systems slightly to get another interesting example. Define $\mathcal{R}'$ from $\mathcal{R}$ by switching the second arrow at 0 to ←, the first arrow at 1 to be → and setting the first arrow at 2 to be ←. Define $\mathcal{L}'$ from $\mathcal{L}$ by switching the first arrow at 1 to be → and setting the first arrow at 2 to be ←. The resulting partial systems satisfy $\mathcal{L}' \preccurlyeq \mathcal{R}'$. At time $t = 28$, $|A_{R,28}| < |A_{L,28}|$, the number of visits to each site is identical, and $L_{28} = R_{28} = 0$ (see Figure 3). This means we can define a system which repeats such a pattern indefinitely. We can add any common steps that we wish in between repetitions of this pattern and hence we can have recurrent, transient, or even ballistic sequences satisfying $L \preccurlyeq R$ but such that $t^{-1}(|A_{L,t}| - |A_{R,t}|) \to v > 0$ as $t \to \infty$.

5 The excursion/genealogy perspective

Given a sequence $E = (E_n)_{n\in\mathbb{Z}_+}$ of integers with $E_0 = 0$ and $E_{n+1} - E_n \in \{-1, 1\}$, an excursion of length $2k$ from 0 is a part of the sequence $E_m = 0, E_{m+1}, \ldots, E_{m+2k} = 0$ (for some $k \in \mathbb{N}$ such that $E_{m+j} \neq 0$ for any $j \in \{1, 2, \ldots, 2k-1\}$. An excursion of finite length $2k$ defines a unique tree, and vice versa). In the case of an infinite excursion $E_m = 0$, $E_{m+2k} \neq 0$ for all $k \in \mathbb{N}$, the excursion defines part of an infinite tree-like structure. The relationship between random walk excursions and branching processes (random trees) has been well studied, possibly beginning with Harris [3]. Indeed, some of the random walk models of Section 7 have been studied via branching processes (see e.g. [6] and the references therein).

In the context of our paper, the entire tree above, whether finite or infinite can be described in terms of our arrow system. The arrows at the origin can be considered as the great ancestors of every vertex in the tree. There are two kinds, the right arrows and the left arrows. Consider for the moment just the right arrows at 0. Let $Z_1^{(1)}$ denote the number of right arrows at 1, before the first left arrow. Similarly for $i \in \mathbb{N}$, let $Z_1^{(i)}$ denote the number of right arrows between the $(i-1)$st and ith left arrows at 1. For each $i \in \mathbb{N}$, these $Z_1^{(i)}$ consecutive right arrows can be considered as the children of the ith right arrow at 0. More generally if the number $Z_x^{(i)}$ of $\rightarrow$ between the $(i-1)$st and ith $\leftarrow$ at $x \in \mathbb{N}$ is considered as the number of children of the ith $\rightarrow$ at $x-1$, this describes a branching or tree structure.

As an alternative to the methods of Sections 3.1 and 3.2, one can try to prove these results by considering what happens to the corresponding branching structures when a left arrow at x in a system $\mathcal{L}$ is exchanged with some right arrow above it (corresponding to $\mathcal{L} \preccurlyeq \mathcal{R}$), or flipped to a right arrow. We have attempted this in some cases, but did not find significant simplifications. The former change can be interpreted in terms of the branching structure by saying that an earlier labelled right arrow at $x-1$ has adopted one or more children, while subsequent right arrows at $x-1$ have had their children changed as well, some having adopted and some having given up children for adoption. In terms of excursions, as long as the first few (sufficiently many) excursions from x are finite, this has the effect of simply switching the order of excursions from x. Excursions to the right appearing earlier than previously and excursions to the left appearing later. Otherwise an infinite excursion to the left from x can vanish if an infinite excursion to the right supplants it. Right transience is equivalent to an infinite right excursion from every site ≥ 0, which is in turn equivalent to having infinitely many generations of descent for the corresponding branching structure. Then for example Corollary 3.10 can roughly be interpreted as coming from the fact that no children of any generation in the branching structure are lost by changes of the underlying arrow system via the relation $\preccurlyeq$.

6 Open problems:

Open problem 6.1 (Monotonicity with respect to starting point)**.** *Reprove the results in Section 3 when* $L_0 = x_0 < 0 = R_0$ *(assuming still that* $\mathcal{L} \preccurlyeq \mathcal{R}$*).*

Note that it's sufficient to prove this in the same system.

Open problem 6.2. *Find* $\rho_{\preccurlyeq}$ *and* $\rho_{\trianglelefteq}$*, where*

$$\rho_{\preccurlyeq} = \sup\left\{\rho : \text{ there exist } L \preccurlyeq R \text{ such that } \limsup_{t\to\infty} \frac{|A_{L,t}|}{t} \geq \rho\right\},$$

and similarly for $\rho_{\trianglelefteq}$*.*

7 Applications

In this section we briefly describe some of the applications of our main results in the theory of self-interacting random walks.

7.1 Multi-excited random walks on $\mathbb{Z}$

A cookie environment is an element $\omega = (\omega(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$ of $[0,1]^{\mathbb{Z}\times\mathbb{N}}$. Given two cookie environments ω and ω' we write $\omega \trianglelefteq \omega'$ if $\omega(x,n) \le \omega'(x,n)$ for every $x \in \mathbb{Z}$ and $n \in \mathbb{N}$.

A (multi-)excited random walk in (possibly random) cookie environment ω, starting from the origin, is a sequence of random variables $X = \{X_n\}_{n\ge 0}$ defined on a probability space (and adapted to a filtration $\mathcal{F}_n$) such that

$$P_\omega\big(X_{n+1} = X_n + 1\big|\mathcal{F}_n\big) = \omega(x, \ell(n)) = 1 - P_\omega\big(X_{n+1} = X_n - 1\big|\mathcal{F}_n\big),$$

where $\ell(n) = \ell_X(n) = \sum_{m=0}^{n} 1_{\{X_m = X_n\}}$. In other words, if you are currently at x and this is the kth time that you have been at x then your next step is to the right with probability $\omega(x,k)$, independent of all other information.

Let $\mathbf{U} = (U(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$ be a collection of independent standard uniform random variables. For each $x \in \mathbb{Z}$ and $n \in \mathbb{N}$ set

$$\mathcal{L}_{\omega,\mathbf{U}}(x,n) = \begin{cases} \to & \text{, if } U(x,n) < \omega(x,n) \\ \leftarrow & \text{, otherwise.} \end{cases} \tag{7.1}$$

Then $\mathcal{L}_{\omega,\mathbf{U}}$ is an arrow system determined entirely by the pairs $(\omega(x,n), U(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$.

Theorem 7.1. *If $\omega \trianglelefteq \omega'$, then there exists a probability space on which $L = \{L_n\}_{n\ge 0}$ and $R = \{R_n\}_{n\ge 0}$ are multi-excited random walks in cookie environments ω and ω' respectively, such that $L \trianglelefteq R$ with probability 1.*

Proof. Define $\mathcal{L}$ as in (7.1) and

$$\mathcal{R}_{\omega',\mathbf{U}}(x,n) = \begin{cases} \to & \text{, if } U(x,n) < \omega'(x,n) \\ \leftarrow & \text{, otherwise.} \end{cases} \tag{7.2}$$

Then $\mathcal{L} \trianglelefteq \mathcal{R}$ and the result follows trivially. ■

Theorem 7.1 then implies monotonicity in the sense of Theorem 1.4 on that probability space. Moreover, on the event that $\liminf_{n\to\infty} L_n = \infty$, for every $x \in \mathbb{Z}$ the corresponding walk R visits x no more times than L visits x.

An immediate consequence is that for any two random environments ω and ω' on a probability space, we can embed that space in a larger space (i.e. including a family $\mathbf{U}$ of uniform random variables) on which multi-excited random walks L and R are defined such that $L \trianglelefteq R$ almost surely on the event that $\omega \trianglelefteq \omega'$.

Definition 7.2. *A random cookie environment ω is said to be i.i.d. if the random vectors $\omega(x,\cdot)$ are i.i.d. as x varies over $\mathbb{Z}$.*

The following result follows from [7] and [6]. But using our coupling, it can now also be obtained directly from [6], which proves it in the case $\omega(o,n) = \frac{1}{2}$ for $n > M$.

Lemma 7.3. *Let ω be an i.i.d. cookie environment. Suppose that there exist $M \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ such that* *(i)* $\omega(o,n) \geq \frac{1}{2}$ *for every* $n > M$ *almost surely, and*

(ii) $\mathbb{E}[\sum_{i=1}^{M}(2\omega(o,i)-1)] > \alpha$.

If $\alpha > 1$ *then the excited random walk* X *in this (i.i.d.) cookie environment is transient to the right. If* $\alpha > 2$ *then* $\limsup n^{-1}X_n > 0$.

Some of the known results for excited random walks in i.i.d. or ergodic environments can be extended to more general self-interacting random walks with a bounded number of positive drifts per site.

Definition 7.4. *A nearest neighbour self-interacting random walk is any sequence of random variables* $X_n \in \mathbb{Z}$ *such that* $|X_{n+1} - X_n| = 1$ *a.s.* $\forall n$.

Lemma 7.5. *Let* X_n *be a nearest-neighbour self-interacting random walk and* $\mathcal{F}_n = \sigma(X_k, k \leq n)$. *Suppose that there exist* $M \in \mathbb{N}$ *and* $(\eta_k)_{k \leq M} \in [0,1)^M$ *such that*

- $\mathbb{P}(X_{n+1} = X_n + 1 | \mathcal{F}_n) I_{\ell(n)=k} \leq \eta_k$ *for all* $k \leq M$ *and all* $n \in \mathbb{Z}_+$ *almost surely, and*
- $\mathbb{P}(X_{n+1} = X_n + 1 | \mathcal{F}_n) I_{\ell(n)=k} \leq \frac{1}{2}$ *for all* $k > M$ *and all* $n \in \mathbb{Z}_+$*, almost surely.*

If $\alpha = \sum_{k=1}^{M}(2\eta_k - 1) \leq 1$ *then* X *is not transient to the right, almost surely. If* $\alpha \leq 2$ *then* $\limsup n^{-1}X_n \leq 0$*, almost surely.*

Proof. Define $\eta_k = \frac{1}{2}$ for $k > M$. For each $x \in \mathbb{Z}$, let $\omega(x,k) = \eta_k$ for $k \in \mathbb{N}$. Let $\mathbf{U} = (U(x,m))_{x \in \mathbb{Z}, m \in \mathbb{N}}$ be i.i.d. standard uniform random variables. and define $\mathcal{R}$ by

$$\mathcal{R}(x,k) = \begin{cases} \rightarrow, & \text{if } U(x,k) \leq \eta_k \\ \leftarrow, & \text{otherwise.} \end{cases}$$

The corresponding walk R_n has the law of an excited random walk in the (non-random) environment ω. By [6], if $\alpha = \sum_{k=1}^{M}(2\eta_k - 1) \leq 1$ then R is not transient to the right, almost surely. If $\alpha \leq 2$ then $\limsup n^{-1}R_n \leq 0$, almost surely.

For a nearest neighbour sequence $x_0, \dots, x_n$ define

$$P_{n,k}(x_0, \dots, x_n) = \mathbb{P}(X_{n+1} = X_n + 1 | X_0 = x_0, \dots, X_n = x_n) I_{\ell_x(n)=k}.$$

Define a nearest neighbour self-interacting random walk X' by setting $X'_0 = 0$ and

$$X'_{n+1} = \begin{cases} X'_n + 1, & \text{if } U(X'_n, k) \leq P_{n,k}(X'_0, \dots, X'_n) \\ X'_n - 1, & \text{otherwise.} \end{cases}$$

Then X' has the law of X. Since $P_{n,k} \leq \eta_k$ almost-surely, we have that $X' \trianglelefteq R$ almost surely. The result now follows by Cor. 3.10. ■

For excited random walks defined by i.i.d. cookie environments in 1 dimension, it is known up to a high level of generality that right transience and the existence of a positive speed $v > 0$ do not depend on the order of the cookies. One might expect that the value of v should depend on this order. It is therefore natural to wonder whether or not there is an $\preccurlyeq$ analogue of Theorem 7.1. The answer is yes. The following theorem (and easy extensions of it) implies that one cannot decrease the $(\limsup)$-speed of a cookie random walk by moving stronger cookies to earlier in the pile at each site.

Theorem 7.6. *Let ω be a cookie environment such that $\omega(x,k) \geq \omega(x,j)$ for some $x \in \mathbb{Z}$, $j < k \in \mathbb{N}$. Define $\omega' = \omega'(x,k,j)$ such that $\omega'(x,k) = \omega(x,j)$ and $\omega'(x,j) = \omega(x,k)$ and $\omega'(y,m) = \omega(y,m)$ for all $(y,m) \notin \{(x,k),(x,j)\}$. Then there exists a probability space on which $L = \{L_n\}_{n\geq 0}$ and $R = \{R_n\}_{n\geq 0}$ are multi-excited random walks in cookie environments ω and ω' respectively, such that $L \preccurlyeq R$ with probability 1.*

Proof. Given ω, x, k, j as in the conditions of the Theorem, let $\mathbf{U} = \big(U_{x,k,j}, (U(y,n))_{y\in\mathbb{Z}, n\in\mathbb{N}}\big)$ be a family of independent standard uniform random variables. We want to define environments $\mathcal{L} = \mathcal{L}_{\omega,\mathbf{U}}$ and $\mathcal{R}$ with $\mathcal{L} \preccurlyeq \mathcal{R}$. Define $\mathcal{L}(y,n)$ for all $(y,n) \notin \{(x,k),(x,j)\}$ as in (7.1). Further define

$$(\mathcal{L}(x,j),\mathcal{L}(x,k)) = \begin{cases} (\rightarrow,\rightarrow) & \text{, if } U_{x,k,j} < \omega(x,j)\omega(x,k) \\ (\rightarrow,\leftarrow) & \text{, if } \omega(x,j)\omega(x,k) \leq U_{x,k,j} < \omega(x,j) \\ (\leftarrow,\rightarrow) & \text{, if } \omega(x,j) \leq U_{x,k,j} < \omega(x,j) + \omega(x,k)(1-\omega(x,j)) \\ (\leftarrow,\leftarrow) & \text{, otherwise.} \end{cases} \tag{7.3}$$

Finally define $\mathcal{R} = \mathcal{L}_{\omega',\mathbf{U}}$ (i.e. as above, except with ω' instead of ω). Then $\mathcal{L}$ and $\mathcal{R}$ have the same arrows everywhere, except possibly at (x,k) and (x,j). If $(\mathcal{L}(x,j),\mathcal{L}(x,k)) = (\rightarrow,\rightarrow)$ then $U_{x,k,j} < \omega(x,j)\omega(x,k) = \omega'(x,k)\omega'(x,j)$ so also $(\mathcal{R}(x,j),\mathcal{R}(x,k)) = (\rightarrow,\rightarrow)$. Otherwise if $(\mathcal{L}(x,j),\mathcal{L}(x,k)) = (\rightarrow,\leftarrow)$ then $U_{x,k,j} < \omega(x,j) \leq \omega(x,k) = \omega'(x,j)$ so $\mathcal{R}(x,j) =\rightarrow$. This proves that $\mathcal{L} \preccurlyeq \mathcal{R}$ (almost surely) as claimed. Finally, one can check that the sequences L and R are random walks in cookie environments ω and ω' respectively. ∎

Corollary 7.7. *Let ω, ω', L, R be as in Thm 7.6. If L is transient to the right then* $\limsup L_n/n \leq \limsup R_n/n$.

Proof. Theorems 1.4 and 7.5 ∎

7.2 RWDRE in dimension d with no negative drift

Suppose that at each site in $\mathbb{Z}^2$, independently of other sites, we place either $\rightarrow$ (with probability p), $\longleftrightarrow\!\!\!\!\!\!\!\updownarrow\;\;$ arrows (with probability $p_1 \leq 1-p$) or $\leftrightarrow$ arrows (with probability $p_2 = 1 - p_1 - p$). Let $\{X_n\}_{n\geq 0}$ be a (nearest-neighbour) random walk that follows an arrow chosen uniformly from all possible arrows at its current location, independently of all previous choices, given such a random environment. We call this a *random walk in a degenerate random environment* because the usual ellipticity assumption for RWRE fails to hold. The projection Y_n of X_n in the $\searrow$ direction can be realized as a random walk in a (non-i.i.d.) cookie environment ω'. It turns out that ω' can be coupled to an i.i.d. cookie environment ω so that $\omega \trianglelefteq \omega'$. Theorem 7.3 then implies that a multi-excited walk in the environment ω' is transient to the right, when $\frac{p}{1-p} > 1$. Theorem 1.4 then implies transience of Y under the same conditions. More precisely:

Theorem 7.8 (see [5]). *When $\frac{p}{1-p} > 1$, for $\mathbb{P}$-almost every such environment the random walk is transient in direction $\searrow$. When $\frac{p}{1-p} > 2$, for $\mathbb{P}$-almost every such environment the random walk is ballistic in direction $\searrow$.*

This type of problem was in fact our original motivation for developing the arguments of the current paper.

A result of this kind holds more generally (in general dimensions and with non-uniform steps). For example, if we replace $\rightarrow$ with $\updownarrow\!\!\!\rightarrow$ arrows, the relevant quantity becomes $\frac{p}{3(1-p)}$. What is really required in this argument is that with high probability the walker experiences a significant drift in some direction u, but with probability zero there is a drift in direction $-u$.

7.3 Walks with drift toward the origin

Given a parameter $\beta > -1$, define a nearest-neighbour *once-reinforced random walk* (ORRW) $X = (X_n)_{n\geq 0}$ on $\mathbb{Z}$ by setting $X_0 = o$, $\vec{X}_n = (X_0, \dots, X_n)$ and for $n \geq 1$,

$$\mathbb{P}(X_{n+1} - X_n = 1|\mathcal{F}_n) = \frac{1 + \beta I_{\{X_n+1 \in \vec{X}_{n-1}\}}}{2 + \beta[I_{\{X_n+1 \in \vec{X}_{n-1}\}} + I_{\{X_n-1 \in \vec{X}_{n-1}\}}]}, \tag{7.4}$$

where $x \in \vec{X}_n$ is notation for $x = X_i$ for some $i \leq n$. When $\beta > 0$ this walk has a preference for stepping to locations that it has visited before. We can also define a one-sided version of this walk, i.e. a ORRW $X^+ = (X^+_n)_{n\geq 0}$ on $\mathbb{Z}_+$ by setting $X^+_0 = o$, and for $n \geq 1$, $\mathbb{P}(X^+_{n+1} - X^+_n = 1|\mathcal{F}_n) = 1$ if $X^+_n = 0$ and otherwise exactly as in (7.4).

An immediate Corollary of Theorem 1.5, and coupling to simple random walk, is the result that any random walk on $\mathbb{Z}$ that never experiences a drift away from the origin is recurrent, i.e. if $\mathbb{P}\big((X_{n+1}-X_n)\cdot\text{sign}(X_n) \leq 0\big|\mathcal{F}_n\big) \geq \frac{1}{2}$ for all $n \geq 0$ almost surely then $\mathbb{P}(X_n = 0 \text{ infinitely often}) = 1$. In particular the ORRW with $\beta \geq 0$ is recurrent

Our method can be used to prove some less obvious recurrence results (e.g. versions of Lemmas 7.3 and 7.5 but for recurrence), where the random walk can sometimes experience a drift away from the origin, by coupling the appropriate random walk with a 1-dimensional recurrent multi-excited random walk.

Stronger results can be obtained in the one-sided context. We can couple various recurrent excited random walk models on $\mathbb{Z}_+$ together so that those with obviously smaller right drift are "more recurrent" in terms of the number of visits to the origin by time t for all t. Another example is contained in the following theorem.

Theorem 7.9. *There exists a probability space on which*

- *for each $\beta > -1$ there is a once reinforced random walk $X^+(\beta)$ on $\mathbb{Z}_+$*
- *$X^+(\beta) \trianglelefteq X^+(\zeta)$ whenever $\beta \geq \zeta$.*

Proof. Let $\mathbf{U} = (U(x,n))_{x\in\mathbb{Z},n\in\mathbb{N}}$ be a family of i.i.d. standard uniform random variables. Define an arrow system $\mathcal{I}_\beta$ as follows. Let $\mathcal{I}_\beta(o,k) = \rightarrow$ for all $k \in \mathbb{N}$. Define $A_{x,i}(\beta) = \cup_{j=1}^{i}\{U(x,j) < 1/(2+\beta)\}$. For $x > 0$ define

$$\mathcal{I}_\beta(x,1) = \begin{cases} \rightarrow, & \text{if } U(x,1) < \frac{1}{2+\beta} \\ \leftarrow, & \text{otherwise,} \end{cases} \tag{7.5}$$

and for $k > 1$

$$\mathcal{I}_\beta(x,k) = \begin{cases} \rightarrow, & \text{if } U(x,k) < \frac{1}{2+\beta} \\ \rightarrow, & \text{if } U(x,k) < \frac{1}{2} \text{ and } A_{x,k-1}(\beta) \text{ occurs} \\ \leftarrow, & \text{otherwise.} \end{cases} \tag{7.6}$$

Suppose $\beta \geq \zeta > -1$. We claim that $\mathcal{I}_\beta \trianglelefteq \mathcal{I}_\zeta$. To see this note that

$$\mathcal{I}_\beta(x,1) = \rightarrow \quad \iff \quad U(x,1) < \frac{1}{2+\beta} \quad \Rightarrow \quad U(x,1) < \frac{1}{2+\zeta} \quad \iff \quad \mathcal{I}_\zeta(x,1) = \rightarrow .$$

Similarly $A_{x,i}(\beta) = \cup_{j=1}^{i}\{U(x,j) < 1/(2+\beta)\} \subset \cup_{j=1}^{i}\{U(x,j) < 1/(2+\zeta)\} = A_{x,i}(\zeta)$ so that

$$\left\{U(x,k) < \frac{1}{2}\right\} \cap A_{x,k-1}(\beta) \subset \left\{U(x,k) < \frac{1}{2}\right\} \cap A_{x,k-1}(\zeta), \tag{7.7}$$

and hence $\big[\mathcal{I}_\beta(x,k) = \rightarrow\big] \Rightarrow \big[\mathcal{I}_\zeta(x,k) = \rightarrow\big]$ as required. It remains to show that the corresponding sequence $I(\beta)$ is a once-reinforced random walk on $\mathbb{Z}_+$. To see this, suppose that at time n, I is at $x > 0$ for the k-th time. If $I_n + 1 \notin \vec{I}_{n-1}$ then the first $k-1$ arrows at x are $\leftarrow$, so that $A_{x,k-1}(\beta)$ does not occur. Then $\mathbb{P}(I_{n+1} - I_n = 1|\mathcal{F}_n) = \mathbb{P}(U(x,k) < \frac{1}{2+\beta}) = \frac{1}{2+\beta}$. Otherwise if $I_n + 1 \in \vec{I}_{n-1}$ then at least one of the arrows at x up to level $n-1$ is $\rightarrow$, so $A_{x,k-1}(\beta)$ occurs and $\mathbb{P}(I_{n+1} - I_n = 1|\mathcal{F}_n) = \mathbb{P}(U(x,k) < \frac{1}{2}) = \frac{1}{2}$. ■

It follows immediately from Theorem 7.9 that all of the conclusions of Section 3 hold. For example, we have coupled once-reinforced random walks on $\mathbb{Z}_+$ with all parameter values $\beta > -1$, such that the number of visits to o by time t is monotone increasing in β [by Lemma 3.8], the maximum up to time t is decreasing in β [by (3.6)], and the joint distribution of the local times for all $x \geq 0$ and $\beta > -1$ satisfy (iv) of Theorem 1.4. The first two results hold for the standard coupling of ORRW on $\mathbb{Z}_+$ (see below), under which $X_k^+(\beta) \leq X_k^+(\zeta)$ for all k and $\beta \geq \zeta$. But (iv) of Theorem 1.4 need not hold for that coupling, as can easily be seen by example.

Remark 7.10 (Standard coupling for ORRW)**.** *Let $\mathbf{U} = (U_n)_{n\geq 0}$ be a family of independent standard uniform random variables. Given $\beta > -1$ define $X_0^+ = 0$ and (conditionally on $X_0^+, \dots, X_n^+$), if $X_n^+ = 0$ then $X_{n+1}^+ = 1$, while if $X_n^+ > 0$ then*

$$X_{n+1}^+ = \begin{cases} X_n^+ - 1 & , \text{ if } U_n < \frac{1}{2} \text{ and } X_n^+ + 1 \in \vec{X}_{n-1}^+ \\ X_n^+ - 1 & , \text{ if } U_n < \frac{1+\beta}{2+\beta} \text{ and } X_n^+ + 1 \notin \vec{X}_{n-1}^+ \\ X_n^+ + 1 & , \text{ otherwise.} \end{cases} \tag{7.8}$$

One can show that $X_n^+(\beta) \leq X_n(\zeta)^+$ when $\beta \geq \zeta > -1$.

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References

[1] I. Benjamini and D. B. Wilson. Excited random walk. *Electron. Comm. Probab.*, 8:86–92 (electronic), 2003.

[2] J. Bérard and A. Ramírez. Central limit theorem for excited random walk in dimension $d \geq 2$. *Electr. Comm. Probab.*, 12:300–314, 2007.

[3] T. E. Harris. First passage and recurrence distributions. *Transactions of the American Mathematical Society*, 73(3):471–486, 1952.

[4] R. van der Hofstad and M. Holmes. Monotonicity for excited random walk in high dimensions. *To appear, Probab. Theory Relat. Fields.*, 2009.

[5] M. Holmes and T. Salisbury. Random walks in degenerate random environments. In preparation, 2009.

[6] E. Kosygina and M. Zerner. Positively and negatively excited random walks on integers, with branching processes. *Electron. J. Probab.*, 13:1952–1979, 2008.

[7] M. Zerner. Multi-excited random walks on integers. *Probab. Theory Relat. Fields.*, 133:98–122, 2005.