PROOF OF THE GHAHRAMANI–LAU CONJECTURE

VIKTOR LOSERT, MATTHIAS NEUFANG, JAN PACHL AND JURIS STEPRĀNS

Abstract. The Ghahramani–Lau conjecture is established; in other words, the measure algebra of every locally compact group is strongly Arens irregular.

1. Introduction

The study of the Arens products on the second dual of a Banach algebra has been an active area of functional analysis and abstract harmonic analysis for many years. As is well-known, operator algebras (and their quotient algebras) are Arens regular, i.e., both Arens products on the second dual coincide. However, the situation is radically different for group algebras such as the convolution algebra $L_1(G)$ over a locally compact group G: building on the pioneering work (in the abelian case) of Civin–Yood [1] from 1961, N.J. Young [19] showed in 1973 that $L_1(G)$ is never Arens regular unless G is finite. It was thus natural to ask how irregular the multiplication in the bidual of $L_1(G)$ is – this was only settled in 1988 by Lau–Losert [14] who showed that $L_1(G)$ is strongly Arens irregular (in the terminology established in [3]); in other words, left multiplication by $m \in L_1(G)^{**}$ on $L_1(G)^{**}$ is the same with respect to both Arens products only if $m \in L_1(G)$, and this holds as well for right multiplication by m.

Since the measure algebra $\mathbf{M}(G)$ contains $L_1(G)$ as a closed subalgebra (in fact, as an ideal), it is clear by Young's theorem that $\mathbf{M}(G)$ is only Arens regular for finite groups G. It was conjectured by Lau [12] and Ghahramani–Lau [5] that, as in the case of $L_1(G)$, the measure algebra $\mathbf{M}(G)$ is also strongly Arens irregular for any locally compact group G. In [15], this was established for two classes of locally compact non-compact groups: those whose cardinality is a non-measurable cardinal, and those for which the relation $\kappa(G) \geq 2^{\chi(G)}$ holds, where $\kappa(G)$ denotes the compact covering number, and $\chi(G)$ the local weight, also known as the character.

In this paper, we prove the Ghahramani-Lau conjecture for all locally compact groups.

2. Overview

Set theoretic notation will play a sufficiently significant role in certain of the arguments to be presented that it will be worthwhile setting down some conventions. Ordinals will be defined in the way developed by von Neumann — in particular, every ordinal is equal to the set of its predecessors. For example, this means that for ordinals α and β the assertions $\alpha < \beta$, $\alpha \subsetneq \beta$ and $\alpha \in \beta$ have the same meaning.

Date: August 19, 2010.

It will, at times, prove to be convenient to exploit the fact that if n is a natural number then $n = \{0, 1, \ldots, n-1\}$. A cardinal is identified with the least ordinal of the same cardinality. The cardinality of a set E will be denoted by $|E|$. To avoid confusion with cardinal exponentiation, the set of functions from the set X to the set Y will be denoted by ${}^{X}Y$. If κ and λ are cardinals then κ^{λ} denotes the cardinality of ${}^{\lambda}\kappa$. The cofinality of a cardinal κ is the least cardinal τ such that there is $X \subseteq \kappa$ of cardinality τ such that $\kappa = \cup X$. A cardinal is said to be singular if its cofinality is smaller than its cardinality and it is regular otherwise. For any cardinal κ the least cardinal greater than κ is denoted by κ^{+} and called the cardinal successor of κ. If α is an ordinal number then, following Cantor, the α^{th} infinite cardinal is denoted by $\aleph_\alpha$ — to be precise, $\aleph_\alpha$ is the unique infinite cardinal such that the set of smaller cardinals has order type α. Further details of ordinals, cardinals and transfinite induction can be found in standard sources such as [10] or [9].

Vector spaces are over the reals $\mathbb{R}$ or over the complex field $\mathbb{C}$. The linear span of a set D in a vector space is $\operatorname{span}(D)$. When M is a normed space, $\mathbf{B_1}(M)$ is the unit ball in M, and M^* is the dual of M. For $f \in M^*$, $\mu \in M$, the value of the functional will be written as $\langle f, \mu\rangle$. As usual, M is identified with a subspace of the second dual M^{**} by the canonical embedding.

Now assume that M is a Banach algebra with multiplication $\star$. The product operation $\star$ of M can be extended to M^{**} in two canonical ways, the *left Arens product* and *right Arens product* denoted by $\Box$ and $\Diamond$ respectively. They are defined for $\mathfrak{m}$ and $\mathfrak{n}$ in M^{**} and h in M^* and μ and ν in M as follows:

$$\begin{aligned}
\langle\, \mathfrak{m}\Box\mathfrak{n} \,,\, h \,\rangle &= \langle\, \mathfrak{m} \,,\, \mathfrak{n}\Box h \,\rangle \\
\langle\, \mathfrak{n}\Box h \,,\, \mu \,\rangle &= \langle\, \mathfrak{n} \,,\, h\Box\mu \,\rangle \\
\langle\, h\Box\mu \,,\, \nu \,\rangle &= \langle\, h \,,\, \mu\star\nu \,\rangle \\
\langle\, \mathfrak{m}\Diamond\mathfrak{n} \,,\, h \,\rangle &= \langle\, \mathfrak{n} \,,\, h\Diamond\mathfrak{m} \,\rangle \\
\langle\, h\Diamond\mathfrak{m} \,,\, \mu \,\rangle &= \langle\, \mathfrak{m} \,,\, \mu\Diamond h \,\rangle \\
\langle\, \mu\Diamond h \,,\, \nu \,\rangle &= \langle\, h \,,\, \nu\star\mu \,\rangle.
\end{aligned}$$

Definition 1. *The left and right topological centres of M^{**} are defined respectively by*

$$Z_t^l(M^{**}) = \{\, \mathfrak{m} \in M^{**} \mid \textit{the mapping}\ \mathfrak{n} \mapsto \mathfrak{m}\Box\mathfrak{n}\ \textit{is weak}^*\ \textit{continuous on}\ M^{**} \,\}$$

$$Z_t^r(M^{**}) = \{\, \mathfrak{m} \in M^{**} \mid \textit{the mapping}\ \mathfrak{n} \mapsto \mathfrak{n}\Diamond\mathfrak{m}\ \textit{is weak}^*\ \textit{continuous on}\ M^{**} \,\}.$$

Note that if μ and ν are in M and $h \in M^*$ then

$$\langle\, \mu\Box h \,,\, \nu \,\rangle = \langle\, h\Box\nu \,,\, \mu \,\rangle = \langle\, h \,,\, \nu\star\mu \,\rangle$$

and a similar observation holds for $\Diamond$. Moreover, if $\mu \in M$ then $\langle\, \mu\Box\mathfrak{m} \,,\, h \,\rangle = \langle\, \mathfrak{m} \,,\, h\Box\mu \,\rangle$. It follows that $M \subseteq Z_t^l(M^{**}) \cap Z_t^r(M^{**})$ (see [2, p. 248ff] for a further discussion of Arens products and topological centres).

Definition 2. *M is called* left strongly Arens irregular *if $Z_t^l(M^{**}) = M$ holds and* right strongly Arens irregular *if $Z_t^r(M^{**}) = M$. The algebra M is called* strongly Arens irregular *if $Z_t^l(M^{**}) = Z_t^r(M^{**}) = M$.*

The two products are connected. Using the fact that M is weak* dense in M^{**} it is an easy observation that

$$Z_t^l(M^{**}) = \{\ \mathfrak{m} \in M^{**} \mid (\forall \mathfrak{n} \in M^{**})\mathfrak{m}\Box\mathfrak{n} = \mathfrak{m}\Diamond\mathfrak{n}\ \}$$

$$Z_t^r(M^{**}) = \{\ \mathfrak{m} \in M^{**} \mid (\forall \mathfrak{n} \in M^{**})\mathfrak{n}\Box\mathfrak{m} = \mathfrak{n}\Diamond\mathfrak{m}\ \}$$

When X is a locally compact topological space, $\mathbf{M}(X)$ is the Banach space of real or complex, bounded Radon measures on X with the total variation norm. $C_0(X)$ denotes the space of real or complex valued continuous functions on X vanishing at infinity, equipped with supremum norm. The Riesz Representation Theorem identifies $\mathbf{M}(X)$ with $C_0(X)^*$ by setting $\langle\, \mu\, ,\, f\, \rangle = \int f d\mu$ for $f \in C_0(X)$ and $\mu \in \mathbf{M}(X)$. Using the natural embedding of $C_0(X)$ into $C_0(X)^{**}$ yields that $\langle\, \mu\, ,\, f\, \rangle = \langle\, f\, ,\, \mu\, \rangle$. When $X = G$ is a locally compact topological group, $\mathbf{M}(G)$ is a Banach algebra with convolution $\star$ (see [2, p. 374ff]). The operation $\star$ gives rise to left and right actions of $\mathbf{M}(G)$ on $C_0(G)$ defined by

$$\langle\, \mu \star f\, ,\, \nu\, \rangle = \langle\, f\, ,\, \nu \star \mu\, \rangle$$

$$\langle\, f \star \mu\, ,\, \nu\, \rangle = \langle\, f\, ,\, \mu \star \nu\, \rangle$$

for f in $C_0(G)$ and μ and ν in $\mathbf{M}(G)$.

The following is the main result of this article.

Theorem 3. *For every locally compact group G the measure algebra $\mathbf{M}(G)$ is strongly Arens irregular.*

This was the conjecture of Ghahramani-Lau which was proved by Lau [13] for discrete groups, and by Neufang [15] when $\kappa(G) \geq 2^{\chi(G)}$ (and in some other cases). In order to establish the result in full generality, it suffices to prove that $Z_t^l(M^{**}) = M^{**}$ because the result for the right topological centre will follow by noting that $Z_t^r(A^{**}) = Z_t^l((A^{\mathrm{op}})^{**})$ where, for any algebra $(A, \cdot, +)$, the algebra A^{op} denotes $(A, \cdot^{\mathrm{op}}, +)$ with $\cdot^{\mathrm{op}}$ defined by $a \cdot^{\mathrm{op}} b = b \cdot a$. Details on this can be found in [3, p. 22]. Since $\mathbf{M}(G)^{\mathrm{op}} = \mathbf{M}(G^{\mathrm{op}})$ this suffices. Henceforth, attention will be focused on $Z_t^l(M^{**})$.

If K is a closed subgroup of a topological group G, then G/K denotes the space of left cosets — in other words, cosets of the form gK — with the quotient topology ([6, 5.15]). If X is a topological space, the *compact covering number* of X, denoted $\kappa(X)$, is the least cardinal τ such that X is a union of τ compact subsets. The *density* of a space X, denoted by $d(X)$, is the least cardinal of a dense subset of X . For $x \in X$ define $\chi(x,X)$ (*local weight* or character) to be the least cardinal τ such that x has a base of neighbourhoods of cardinality τ. If the group of homeomorphisms acts transitively on X, in particular for $X = G$ or $X = G/K$, this does not depend on the choice of x and $\chi(x,X)$ can and will be denoted by $\chi(X)$. When K is compact, the coset space G/K is metrizable if and only if $\chi(G/K)\leq\aleph_0$ (the group case is well known: [6, 8.3]; the general case was done by Kristensen: [6, 8.14d]). For any locally compact group G and any compact subgroup K of infinite index, the equality $|G/K|=\kappa(G)\cdot 2^{\chi(G/K)}$ holds (this can be shown as in the group case, see the article by Hodel in [11, pp. 1–61] or the original argument of Pospíšil in [16]). Furthermore, we have $d\big(\mathbf{M}(G/K)\big)=|G/K|$ (this is trivial when K is open, otherwise it follows from [8, 5.5]).

The point mass at $x\in G$ will be denoted by δ_x and, if $X\subseteq G$, then δ_X will denote the set of point masses δ_x with $x\in X$. The weak* closure of δ_X in $\mathbf{M}(G)^{**}$ will be denoted $\overline{\delta_X}$; this should not cause confusion since other closures of this set will not be considered. For $\mu\in\mathbf{M}(G)$ and $x\in G$ write $\mu\star x=\mu\star\delta_x$. Thus $\mu\star x(Ex)=\mu(E)$ when $E\subseteq G$ is a Borel set. When $H\subseteq G$, write $\mu\star H=\{\mu\star h\mid h\in H\}$. When $D\subseteq\mathbf{M}(G)$ and $H\subseteq G$, write $D\star H=\{\mu\star h\mid\mu\in D,h\in H\}$.

When $\mu,\nu\in\mathbf{M}(G)$, $\mu\ll\nu$ means absolute continuity, $\mu\perp\nu$ means that μ and ν are mutually singular. We have $\mu\perp\nu$ if and only if $|\mu|\perp|\nu|$. When $D,D'\subseteq\mathbf{M}(G)$, $D\perp D'$ means that $\mu\perp\nu$ whenever $\mu\in D$ and $\nu\in D'$. If $D,D'\subseteq\mathbf{M}(G)$ and $D\perp D'$ then clearly $\operatorname{span}(D)\cap\operatorname{span}(D')=\{0\}$.

For G a locally compact group a left Haar measure on G will, as usual, be fixed and denoted by λ_G. If G is compact, the λ_G will be chosen normalized. Two facts about Haar measure will be used without further comment:

- If G is compact the left and right Haar measures are the same.
- The null sets for left and right Haar measure are the same family.

If $K\subseteq G$ is a closed, but not necessarily normal, subgroup then λ_K can be thought of as a measure on all of G by setting $\lambda_K(G\setminus K)=0$. The notation $\mathbf{M_a}(G)$ will be used to denote the subspace of those measures in $\mathbf{M}(G)$ that are absolutely continuous with respect to Haar measure λ_G. This will be identified in the usual way with the space $L^1(G)$ defined by λ_G . $\mathbf{M_s}(G)$ is the subspace of the measures $\mu\in\mathbf{M}(G)$ such that $\mu\perp\lambda_G$; equivalently, $\mu\perp\nu$ for every $\nu\in\mathbf{M_a}(G)$. (This notation differs from that in [6, 19.13], where $\mathbf{M_s}(G)$ stands for the space of singular *continuous* measures.)

The next result is easy, but has been isolated as a lemma because it plays a critical, and potentially confusing, role in the arguments leading to the proof of Theorem 3.

Lemma 4. *For G a locally compact group and a compact subgroup $K\subseteq G$ there is a canonical, isometric isomorphism from $\mathbf{M}(G/K)$ onto $\mathbf{M}(G)\star\lambda_K$.*

Proof. First recall that $\mathbf{M}(G/K)=C_0(G/K)^*$. Next, $C_0(G/K)$ can be identified with the continuous, right K-periodic functions on G vanishing at infinity — that is to say that $C_0(G/K)$ consists of those $f\in C_0(G)$ such that $f(x)=f(xk)$ for all $k\in K$. These, in turn, can be identified with $\lambda_K\star C_0(G)$. To see this, observe that if $f\in C_0(G)$ then $\mu\star f(a)=\int f(ab)d\mu(b)$. It follows from the fact that K is a subgroup that $\lambda_K \star f$ is right K-periodic. Conversely, if f is right K-periodic then $\lambda_K \star f(a) = \int f(ab)d\lambda_K(b) = \int f(a)d\lambda_K(b) = f(a)$ and so $f \in \lambda_K \star C_0(G)$. Hence, it suffices to find an isometric isomorphism from $\mathbf{M}(G) \star \lambda_K$ to $(\lambda_K \star C_0(G))^*$. Such an isomorphism Φ is defined by $\langle\, \Phi(\mu \star \lambda_K)\, ,\, \lambda_K \star f \,\rangle = \langle\, \mu\, ,\, \lambda_K \star f \,\rangle$. The facts that Φ is well defined, a homomorphism, an isometry and one-to-one are all routine. To see that Φ is onto, given $h \in (\lambda_K \star C_0(G))^*$ use the Hahn-Banach Theorem and the fact that $\lambda_K \star C_0(G)$ is a closed subspace of $C_0(G)$ to extend h to $\tilde{h} \in C_0(G)^*$. Then there is a measure $\mu \in \mathbf{M}(G)$ such that $\langle\, \tilde{h}\, ,\, f \,\rangle = \langle\, \mu\, ,\, f \,\rangle$ for all $f \in C_0(G)$ and, therefore,

$$\langle\, \Phi(\mu \star \lambda_K)\, ,\, \lambda_K \star f \,\rangle = \langle\, \mu\, ,\, \lambda_K \star f \,\rangle = \langle\, h\, ,\, \lambda_K \star f \,\rangle$$

for all $f \in C_0(G)$. In other words, $\Phi(\mu \star \lambda_K) = h$. □

Lemma 4 renders harmless the abuse of notation that identifies $\mathbf{M}(G) \star \lambda_K$ with $\mathbf{M}(G/K)$ and this will be done routinely from now on. An alternate approach to the lemma is to extend the quotient map from G to G/K to a map from $\mathbf{M}(G)$ to $\mathbf{M}(G/K)$ by taking inverse images of measures under the quotient map. The natural inclusion of $\mathbf{M}(G) \star \lambda_G$ in $\mathbf{M}(G)$ yields

$$\mathbf{M}(G) \star \lambda_G \to \mathbf{M}(G) \to \mathbf{M}(G/K)$$

and the composition of these two maps can be used for the identification.

Corollary 5. *If G is locally compact and $K \subseteq G$ is a compact subgroup then $\mathbf{M}(G/K)$ is a left ideal in $\mathbf{M}(G)$.*

Definition 6. *If τ is a cardinal and G is a locally compact group then $\mu \in \mathbf{M}(G)$ will be called* τ-thin *if there is a set $P \subseteq G$ such that $|P| = \tau$ and $\mu \star p \perp \mu \star p'$ for all distinct p and p' in P.*

The basic observation used in [15] was that $Z_t^l(M^{**})$ must be small whenever there exists $h \in M^*$ such that $\mathbf{B_1}(M^{**})\Box h$ is big. (Lemmas 7 and 8 provide precise conditions). This motivates the search for factorization theorems. In the case that $M = \mathbf{M}(G)$ it is difficult to get information about the behaviour of $\mathfrak{n}\Box h$ for general $\mathfrak{n} \in M^{**}$. However, it will be shown that an even stronger assertion holds: $\delta_G\Box h$ is big. Then factorizations can be constructed using thinness of measures (see Theorem 17). The argument of [15] shows that if $\kappa(G) \geq 2^{\chi(G)}$ then an appropriate factorization condition holds.

But this approach breaks down when $\kappa(G)$ is small (in particular for compact G). As an alternative, we start with the case of singular measures. Theorem 11 will establish that if G is non-discrete, any $\mu \in \mathbf{M_s}(G)$ is $2^{\aleph_0}\kappa(G)$-thin and this will prove Theorem 3 for metrizable groups (and also for all G with $|G| \leq 2^{\aleph_0}\kappa(G)$).

In the non-metrizable case a further refinement of the decomposition of $\mathbf{M}(G)$ is necessary. For the proof of Theorem 3 in the general case, it will be necessary to consider subspaces $\mathbf{M}(G/K)$ where K is a compact subgroup and proceed by induction on $\chi(G/K)$. Instead of singular measures it will prove to be useful to deal with *strongly singular measures*, measures enjoying the property of being $|G/K|$-thin. Corollary 41 will make this precise.

3. Subspaces and Direct sums

This section will look at subspaces, rather than subalgebras, of Banach algebras. The reason for this interest is that the singular measures form a subspace, but not a subalgebra — recall that the canonical measure on the Cantor set is singular, but its convolution with itself is Lebesgue measure on the unit interval. The Lebesgue decomposition of measures yields that $\mathbf{M}(G)$ is a direct sum of subspaces — the singular measures and the measures absolutely continuous with respect to Haar measure — one of which is not a subalgebra. To be more precise, the terminolgy *subspace* will always refer to a closed subspace — and *direct sum*, denoted by $\oplus$, will always refer to the topological direct sum. The canonical identification of $(M \oplus N)^*$ with $M^* \oplus N^*$ will also be used without further mention.

Lemma 7. *Let M be a Banach algebra and assume that $M = M_0 \oplus M_1$ and that there exists $h \in M^*$ such that $\mathbf{B_1}(M^{**})\Box h = \mathbf{B_1}(M_0^*)$. Then $Z_t^l(M^{**}) \subseteq M_0 \oplus M_1^{**}$.*

Proof. Let $\mathfrak{m} \in Z_t^l(\mathbf{M}(G)^{**})$. The mapping $\psi_h : \mathfrak{n} \mapsto \langle\, \mathfrak{m}\Box\mathfrak{n}\,,\, h\,\rangle$ is weak* continuous from $\mathbf{B_1}(M^{**})$ to $\mathbb{C}$. But the mapping $\varphi_h : \mathfrak{n} \mapsto \mathfrak{n}\Box h$ is weak* continuous and $\psi_h = \mathfrak{m} \circ \varphi_h$ by definition. Since the set $\mathbf{B_1}(M^{**})$ is weak* compact, it follows that $\mathfrak{m}$ is weak* continuous on the set $\mathbf{B_1}(M^{**})\Box h = \mathbf{B_1}(M_0^*)$. Using the theorem of Krein-Šmulian (or Banach-Dieudonné) this suffices to conclude that see that $\mathfrak{m}$ is weak* continuous on M_0^*. Therefore $\mathfrak{m} \restriction M_0^* \in M_0$ and, hence, $\mathfrak{m} \in M_0 \oplus M_1^{**}$. □

For the proof of Theorem 3 in the non-metrizable case, a generalization of Lemma 7 is needed.

Lemma 8. *Let M be a Banach algebra and $M_2 \subseteq M$ a subspace such that $M_0 \oplus M_1 = M_2$. Then there exists $h \in M^*$ such that*

$$(\mathbf{B_1}(M^{**})\Box h) \restriction M_2 = \mathbf{B_1}(M_0^*)$$

then $Z_t^l(M^{**}) \cap M_2^{**} \subseteq M_0 \oplus M_1^{**}$.

Proof. This is similar to the proof of Lemma 7. □

4. Separation of singular measures

Let G be a locally compact group. Then $\mathbf{M_s}(G) \star G \subseteq \mathbf{M_s}(G)$, and $\mathbf{M_a}(G) \star G \subseteq \mathbf{M_a}(G)$. In view of the Lebesgue decomposition [6, 19.20] we have $\mathbf{M}(G) = \mathbf{M_s}(G) \oplus \mathbf{M_a}(G)$. This section establishes results to be used in §5. In particular, it will be shown that every measure in $\mathbf{M_s}(G)$ is $|G|$-thin provided that $|G| = 2^{\aleph_0}\kappa(G)$. This is sufficient to establish the Ghahramani-Lau Conjecture for metrizable groups.

Versions of the following lemma are well-known. Saks gives a proof in [18, III.11] for $G = \mathbb{R}^n$ and provides references to original sources. A stronger version for the circle group is proved by Prokaj [17, Th. 1]. If G is a discrete group then $\mathbf{M_s}(G)$ is the null space $\{0\}$. Thus the lemma is of interest only for non-discrete groups, although it holds for discrete groups as well for trivial reasons.

Lemma 9. *Let G be a locally compact group. If $\mu \in \mathbf{M_s}(G)$ and U is any compact neighbourhood of e_G then $\mu \perp (\mu \star x)$ for λ_G-almost all x in U.*

Proof. Since $\mu \perp \lambda_G$ if and only if $|\mu| \perp \lambda_G$, we may assume that $\mu \geq 0$ and $\mu \neq 0$.

There is a G_δ-set $E \subseteq G$ such that $\mu(G \setminus E) = 0$ and $\lambda_G(E) = 0$. Define $f : G \times G \to [0,1]$ by

$$f(x,y) = \begin{cases} 0 & \text{if } yx \notin E \\ 1 & \text{if } yx \in E \end{cases}$$

As E is a G_δ-set, the function f is Borel measurable on $G \times G$.

Then $0 \leq \int_U f(x,y) d\lambda_G(x) = \lambda_G(U \cap y^{-1}E) \leq \lambda_G(y^{-1}E) = 0$ for every $y \in G$, because $\lambda(E) = 0$. On the other hand $\int_G f(x,y) d\mu(y) = \mu(Ex^{-1}) = \mu \star x(E)$ for every $x \in G$. Now apply Fubini's theorem to get

$$\int_U \mu \star x(E)\, d\lambda_G(x) = \int_U \int_G f(x,y)\, d\mu(y)\, d\lambda_G(x) = \int_G \int_U f(x,y)\, d\lambda_G(x)\, d\mu(y) = 0\ ,$$

which means that $\mu \star x(E) = 0$ for λ_G-almost all x in U. Clearly $\mu \perp (\mu \star x)$ for every x for which $\mu \star x(E) = 0$. □

Lemma 10. *Let G be any locally compact group. Let $\mu \in \mathbf{M}(G)$, $\varepsilon > 0$, and let $H \subseteq G$ be a countable set such that the measures $\mu \star h$ are pairwise mutually singular for $h \in H$. Then there is a compact set $C \subseteq G$ such that $|\mu|(G \setminus C) < \varepsilon$ and the sets Ch are pairwise disjoint for $h \in H$.*

Proof. Using singularity, get pairwise disjoint sets $E_h \subseteq G$ $(h \in H)$ such that $\mu \star h$ is concentrated on E_h for all $h \in H$ and $|\mu \star h'|(E_h) = 0$ for all $h, h' \in H$, $h \neq h'$.

Let $E = \bigcap_{h \in H} E_h h^{-1}$. Since $|\mu|(G \setminus E_h h^{-1}) = |\mu \star h|(G \setminus E_h) = 0$, it follows that $|\mu|(G \setminus E) = 0$. Thus there is a compact set $C \subseteq E$ such that $|\mu|(G \setminus C) < \varepsilon$. The sets Ch are pairwise disjoint for $h \in H$ because $Ch \subseteq E_h$. □

Remark. All that is actually needed in Lemma 10 is that H has cardinality less than the additivity of the measure μ — the least cardinal of a family of null sets whose union is not null.

The next result was proved by Prokaj [17, Theorem 10] for $G = \mathbb{R}$.

Theorem 11. *If G is a non-discrete, locally compact group then every $\mu \in \mathbf{M_s}(G)$ is $2^{\aleph_0}\kappa(G)$-thin.*

Proof. (The reader may wish to consult the set theoretic notation introduced in §2 before continuing.) It will, in fact, be shown that for every $\mu \in \mathbf{M_s}(G)$ there exists a K_σ-set $E \subseteq G$ and a set $P \subseteq G$ such that μ is concentrated on E and $|P| = 2^{\aleph_0}\kappa(G)$ and $(Ep) \cap (Ep') = \emptyset$ when p and p' are distinct elements of P.

If $\kappa(G) > 2^{\aleph_0}$ then let H be σ-compact subgroup containing the support of μ. In this case it suffices to let P be any set selecting precisely one element from each coset of H. Hence, it suffices to show that each singular measure is $2^{\aleph_0}$-thin.

To this end, let $\mu \in \mathbf{M_s}(G)$ and assume, without loss of generality, that $\mu \geq 0$ and $\mu \neq 0$. For $j \in \mathbb{N}$ construct by induction

- compact neighbourhoods U_j of e_G
- compact sets $C_j \subseteq G$
- elements $x_j \in G$,

so that the following conditions (1°) – (5°) hold where, for any $k \in \mathbb{N}$ and $d \in {}^k2$ the notation $d_* = x_0^{d(0)} \cdots x_{k-1}^{d(k-1)}$ will be used with the standard convention that $x^0 = e_G$ and $x^1 = x$:

(1°) $(\mu \star d_*) \perp (\mu \star d'_*)$ whenever d and d' are distinct elements of 2^j ,

(2°) $\mu(G \setminus C_j) \leq 2^{-j}$,

(3°) $(C_j\ d_*\ U_{j+1})\ \cap\ (C_j\ d'_*\ U_{j+1}) = \emptyset$ whenever d and d' are distinct elements of 2^j ,

(4°) $U_{j+1} \cap (x_j U_{j+1}) = \emptyset$,

(5°) $U_{j+1} \cup (x_j U_{j+1}) \subseteq U_j$.

Let U_0 be any compact neighbourhood of e_G. Let $n \in \mathbb{N}$ be such that U_n has been constructed as well as C_j and x_j for all $j \in n$ so that (1°) – (5°) hold.

If $n = 0$, put $\nu = \mu$, otherwise $\nu = \sum_{d \in 2^n} \mu \star d_*$. Note that $\nu \in \mathbf{M_s}(G)$ and, by Lemma 9, there is x_n in the interior of U_n such that $\nu \perp (\nu \star x_n)$. From that and from (1°) for $j < n$ it follows that (1°) holds for $j = n + 1$ as well.

By Lemma 10 there is a compact set $C_n \subseteq G$ such that (2°) holds for $j = n$, and

$$(C_n\ d_*) \cap (C_n\ d'_*) = \emptyset \ \text{ for distinct } \ d, d' \in {}^{n+1}2.$$

Thus $\{C_n\ d_* | d \in {}^{n+1}2\}$ is a finite set of pairwise disjoint compact sets, and there is a neighbourhood W of e_G such that

$$(C_n\ d_*\ W)\ \cap\ (C_n\ d'_*\ W) = \emptyset \ \text{ for distinct } \ d, d' \in {}^{n+1}2.$$

Since x_n is in the interior of U_n and $x_n \neq e_G$, there is a compact neighbourhood U_{n+1} of e_G such that $U_{n+1} \subseteq W$ and (4°) and (5°) hold with $j = n$. Since $U_{n+1} \subseteq W$, we get also (3°) for $j = n$.

The construction of U_j, C_j and x_j satisfying (1°) – (5°) is complete.

Next, for each $n \in \mathbb{N}$ define $E_n = \bigcap_{j=n}^{\infty} C_j$ and let $E = \bigcup_n E_n$. Then E is a K_σ-set and $\mu(G \setminus E) = 0$. For every $d \in {}^{\mathbb{N}}2$ and $k \geq 1$ let $d \restriction k$ denote the restriction of d to k and note that

$$K(d) = \bigcap_{n=1}^{\infty} (d \restriction n)_* U_n$$

is nonempty, being the monotone intersection of compact sets. From (4°) we get $K(d) \cap K(d') = \emptyset$ for $d \neq d'$. Form P by taking one element in each $K(d)$. Thus $|P| = 2^{\aleph_0}$. It remains to be proven that $(Ep) \cap (Ep') = \emptyset$ when p and p' are distinct elements of P.

Take $e, e' \in E$ and $p, p' \in P$, $p \neq p'$. From the definition of E and P it follows that $e, e' \in E_n \subseteq C_n$ for some n, and $p \in K(d)$, $p' \in K(d')$ for some distinct d and d' in ${}^{\mathbb{N}}2$. Let $j \geq n$ be such that $d \restriction j \neq d' \restriction j$. Since $p \in (d \restriction j)_* U_{j+1}$, $p' \in (d' \restriction j)_* U_{j+1}$, and $e, e' \in E_n \subseteq C_j$, we get from (3°), $ep \neq e'p'$. □

It may be of interest to note that if G is metrizable then the set P can be chosen to be perfect. All that needs to be added to the proof is an inductive requirement that the diametre of the set U_j is less than $1/j$.

5. Factorization for thin measures

The purpose of this section is to prove Theorem 17 and Corollary 18, which are key steps in the proof of the Theorem 3. An extended version will be given in Theorem 19.

Lemma 12. *Let G be any locally compact group. Let $\mathcal{D}$ be a family of closed subspaces of $\mathbf{M}(G)$ such that $D \perp D'$ when D and D' are distinct elements of $\mathcal{D}$. If $h_D \in \mathbf{B_1}(D^*)$ for each $D \in \mathcal{D}$ then there exists $h \in \mathbf{B_1}(\mathbf{M}(G)^*)$ that agrees with h_D on D for every $D \in \mathcal{D}$.*

Proof. If μ is in the space spanned by $\bigcup \mathcal{D}$ then $\mu = \sum_{k=1}^{m} \mu_k$ for some $m \in \mathbb{N}$ where $\mu_k \in D_k \in \mathcal{D}$, and $D_k \neq D_j$ for $j \neq k$. There is then a unique linear functional h' on the space spanned by $\bigcup \mathcal{D}$ extending each h_D. Note that if $\mu, \nu \in \mathbf{M}(G)$ and $\mu \perp \nu$ then $\|\mu + \nu\| = \|\mu\| + \|\nu\|$ and hence

$$|h'(\mu)| \leq \sum_k |h'(\mu_k)| \leq \sum_k \|h_{D_k}\| \, \|\mu_k\| \leq \sum_k \|\mu_k\| = \|\mu\|$$

which shows that h' has norm ≤ 1 and therefore extends to a linear functional of norm ≤ 1 on $\mathbf{M}(G)$. □

Lemma 13. *Let G be any locally compact group, τ uncountable. Assume that M_0 is a subspace of $\mathbf{M}(G)$ such that if $\mu \in M_0$ then $|\mu| \in M_0$ and μ is τ-thin. If $F \subseteq M_0$ is finite, $D \subseteq \mathbf{M}(G)$ and $|D| < \tau$ then there exists $x \in G$ such that $D \perp (F \star x)$.*

Proof. The measure $\nu = \sum_{\xi \in F} |\xi|$ is τ-thin, hence there is a set $P \subseteq G$ such that $|P| = \tau$ and $\nu \star p \perp \nu \star p'$ for all $p, p' \in P$, $p \neq p'$.

For every $\mu \in D$ the set $P_\mu = \{p \in P \mid \mu \text{ is not singular to } \nu \star p\}$ must be countable because μ is finite. Thus $|\bigcup_{\mu \in D} P_\mu| \leq |D| \cdot \aleph_0 < \tau = |P|$, and there exists $x \in P \setminus \bigcup_{\mu \in D} P_\mu$. Then $D \perp \nu \star x$ and therefore $D \perp (F \star x)$. □

Lemma 14. *Let G be any locally compact group, τ uncountable. Assume that M_0 is a subspace of $\mathbf{M}(G)$ such that if $\mu \in M_0$ then $|\mu| \in M_0$ and μ is τ-thin. If $\{F_\xi\}_{\xi \in \tau}$ is an indexed family of finite subsets of M_0 then there exist $x_\xi \in G$ for $\xi \in \tau$ such that $(F_\beta \star x_\beta) \perp (F_\gamma \star x_\gamma)$ unless $\beta = \gamma$.*

Proof. Construct the x_ξ by transfinite induction, using Lemma 13 at each step. □

Definition 15. *A direct sum* $M_2 = M_0 \oplus M_1$ *of subspaces of* $\mathbf{M}(G)$ *will be called* G*-invariant decomposition of* M_2 *if* $M_0 \star G \subseteq M_0$ *and* $M_1 \star G \subseteq M_1$.

Lemma 16. *Let* G *be a locally compact group,* $M_2 = M_0 \oplus M_1 \subseteq \mathbf{M}(G)$ *be a* G*-invariant decomposition of* M_2*. Let* $\mathcal{O}$ *be an uncountable collection of non-empty weak* open subsets of* $\mathbf{B_1}(M_0^*)$*. Assume that* $\mu \in M_0$ *implies that* $|\mu| \in M_0$ *and that* μ *is* $|\mathcal{O}|$*-thin. Then there exists* $h \in M_0^*$ *such that* $(\delta_G \Box h) \restriction M_0$ *intersects every set from* $\mathcal{O}$.

Proof. Without loss of generality assume that each $U \in \mathcal{O}$ is a basic neighbourhood of the form

$$U = \{f \in \mathbf{B_1}(M_0^*) \ \mid \ |f(\mu) - g_U(\mu)| < \varepsilon_U \text{ for all } \mu \in F_U\}$$

where $F_U \subseteq M_0$ is a finite set, $g_U \in \mathbf{B_1}(M_1^\circ)$ and $\varepsilon_U > 0$. Apply Lemma 14 to $\mathcal{F} = \{F_U\}_{U \in \mathcal{O}}$ to obtain elements $x_U \in G$ for $U \in \mathcal{O}$, such that if $U, V \in \mathcal{O}$, $U \neq V$, then $(F_U \star x_U) \perp (F_V \star x_V)$.

Then let D_U be the space spanned by $F_U \star g_U$ and apply Lemma 12 to the functionals $h_U \in \mathbf{B_1}(D_U^*)$ defined by $h_U(\nu) = g_U(\nu \star x_U^{-1})$ for $\nu \in F_U$. Thus there is $h \in \mathbf{B_1}(M_2^*)$ that agrees with h_U on D_U for every $U \in \mathcal{O}$ and vanishes on M_1. Extending h to an element of $\mathbf{B_1}(\mathbf{M}(G)^*)$ yields that for each $\mu \in F_U$ the following equalities hold

$$\delta_{x_U} \Box h(\mu) = h(\mu \star x_U) = h_U(\mu \star x_U) = g_U(\mu \star x_U \star x_U^{-1}) = g_U(\mu) \,,$$

hence $\delta_{x_U} \Box h \restriction M_0 \in U$. □

The role of the cardinal invariant $d(M_0)$ in Lemma 17 may benefit from some comment. For most groups found in nature $d(\mathbf{M}(G)) = |\mathbf{M}(G)|$. The proof of Corollary 22 will reveal that if $\kappa(G)^{\aleph_0} = \kappa(G) 2^{\aleph_0}$ and $\chi(G) = \aleph_0$ then the equality $d(\mathbf{M}(G)) = |\mathbf{M}(G)|$ holds. Since this equality would simplify somewhat the next series of results, it is worth noting that it does not hold in general. For example it fails for $G \times \mathbb{T}$ if G is a discrete group of cardinality $\aleph_\omega$ and $2^{\aleph_0} < \aleph_\omega$.

Theorem 17 (Factorization for thin measures)**.** *Let* G *be a locally compact group,* $\mathbf{M}(G) = M_0 \oplus M_1$ *a* G*-invariant decomposition of* $\mathbf{M}(G)$*. Assume that* $|\mu| \in M_0$ *for every* $\mu \in M_0$ *and that each measure in* M_0 *is* $d(M_0)$*-thin. Then there exists* $h \in \mathbf{B_1}(\mathbf{M}(G)^*)$ *such that* $\overline{\delta_G} \Box h = \mathbf{B_1}(M_0^*)$.

Proof. If $M_0 = \{0\}$ the result is trivial, so it will be assumed that M_0 contains a non-zero element and hence $d(M_0)$ is infinite. Next, assume that $d(M_0)$ is uncountable. Taking a norm-dense subset of M_0 with cardinality $d(M_0)$, one can find a family $\mathcal{O}$ forming a weak* open base for $\mathbf{B_1}(M_0^*)$ such that $|\mathcal{O}| = d(M_0)$. Now let $h \in \mathbf{B_1}(M_0^*)$ be given by Lemma 16. Since the set $\overline{\delta_G} \Box h$ is weak* closed and the mapping $\mathfrak{n} \mapsto \mathfrak{n} \Box h$ is continuous it follows easily from the properties of h and $\mathcal{O}$ that $\overline{\delta_G} \Box h = \mathbf{B_1}(M_0^*)$.

To see that $d(M_0)$ is, in fact, uncountable assume otherwise. It follows that no non-zero $\mu \in M_0$ is $\aleph_1$-thin — such a non-zero $\mu \in M_0$ would yield an uncountable family of disjoint open sets contradicting the fact that M_0 has a countable dense subset. By Theorem 11 it follows that $M_0 \subseteq \mathbf{M_a}(G)$ and that $G = \bigcup_{k=0}^{\infty} C_k$ where each C_k is compact. Let S be a set such that $|\mu|(S) = \|\mu\|$ and such that for every $X \subseteq S$, if $|\mu|(X) = 0$ then $\lambda_G(X) = 0$ — such a set is easily constructed by removing counterexamples inductively and observing that this process terminates after countably many steps. Since $|\mu|$ is absolutely continuous with respect to λ_G it follows that for all measurable $Z \subseteq S$

$$\lambda_G(Z) = 0 \text{ if and only if } |\mu|(Z) = 0. \tag{1}$$

Let $X \subseteq G$ be a countable set such that $\lambda_G(SX \cap C_k)$ is maximal for each k — noting that the fact $\lambda_G(C_k) < \infty$ makes this possible. Then let $\{g_n\}_{n=0}^{\infty}$ enumerate X and define $\nu = \sum_{n=0}^{\infty} |\mu| \star g_n/2^{n+1}$. From the G-invariance of M_0 it follows that $\nu \in M_0$.

Now note that $|\mu|(G \setminus SX) = 0$. To see this, suppose not and find m such that $|\mu|(C_m \setminus SX) > 0$. Then use Proposition 443D in [4] to find $g \in G$ such that $\lambda_G(Sg \cap C_m \setminus SX) > 0$. This contradicts the maximality of $\lambda_G(SX \cap C_m)$.

Next, it will be established that if $Z \subseteq SX$ is measurable and $g \in G$ and $\lambda_G(Zg^{-1} \setminus SX) = 0$ then $\nu(Z) = 0$ if an only if $\nu(Zg^{-1}) = 0$. To see this observe that if $\nu(Z) > 0$ then there is some m such that $|\mu| \star g_m(Z) = |\mu|(Zg_m^{-1}) > 0$ and hence $\lambda_G(Zg_m^{-1}) > 0$. By equivalence (1) it then follows that $\lambda_G(Zg^{-1}) > 0$. Since $\lambda_G(Zg^{-1} \setminus SX) = 0$ it follows that there is some k such that $\lambda_G(Zg^{-1}g_k^{-1}) > 0$. Hence $|\mu| \star g_k(Zg^{-1}) > 0$ and so $\nu(Zg^{-1}) > 0$. The other direction has a similar argument.

A contradiction will be obtained by showing that ν is not even 2-thin, let alone $\aleph_0$-thin. To this end, suppose that $\nu \perp \nu \star g$. Let $Z \subseteq G$ be such that $0 < \nu \star g(Z) = \nu(Zg^{-1})$. It may as well be assumed that $Z \subseteq SX$. Then, since $|\mu|(G \setminus SX) = 0$ it follows that $\lambda_G(Zg^{-1} \setminus SX) = 0$. Hence $\nu(Z) > 0$ yielding the required contradiction.

□

Corollary 18. *Under the same hypotheses as Theorem 17* $Z_t^l\big(\mathbf{M}(G)^{**}\big) \subseteq M_0 \oplus M_1^{**}$.

Proof. This follows immediately from Lemma 7 and Theorem 17. □

Remark. In this section, we use several times the assumption that $\mu \in M_0$ implies $|\mu| \in M_0$. For a closed subspace M_0 of $\mathbf{M}(G)$ this is in fact equivalent to M_0 being a sub vector lattice (i.e., in the complex case, $\mu \in M_0$ implies that its real and imaginary part belong to M_0 and the set of real measures in M_0 is closed under the lattice operations).

Again there is an extended version, needed for the proof of Theorem 3 in the non-metrizable case.

Theorem 19 (Factorization on subspaces)**.** *Let G be a locally compact group and* $\mathbf{M}(G) \supseteq M_0 \oplus M_1 = M_2$. *Assume that $\mu \in M_0$ implies that $|\mu| \in M_0$ and that μ*

is $d(M_0)$-thin. Furthermore, assume that there exists a G-invariant decomposition $\widetilde{M_2} = \widetilde{M_0} \oplus \widetilde{M_1} \subseteq \mathbf{M}(G)$ such that $M_0 \subseteq \widetilde{M_0}$ and $M_1 \subseteq \widetilde{M_1}$. Then there exists $h \in \mathbf{B_1}(\mathbf{M}(G)^*)$ such that $\overline{\delta_G \Box h} \restriction M_0 = \mathbf{B_1}(M_0^*)$.

Proof. This is similar to the proof of Theorem 17. Let $\mathcal{O}$ be a weak* open base for $\mathbf{B_1}(M_0^*)$ of cardinality $d(M_0)$. It may as well be assumed that each open set in $\mathcal{O}$ is an intersection of sub-basic sets of the form

$$S(\mu, r, \epsilon) = \{f \in \mathbf{B_1}(M_0^*) \mid |\langle\, f\,,\, \mu\,\rangle - r| < \epsilon\}$$

where μ comes from a dense subset of M_0, $r \in \mathbb{C}$ and $\epsilon > 0$. These all lift to open sets in $\widetilde{M_0}^*$ of the form

$$\tilde{S}(\mu, r, \epsilon) = \{f \in \mathbf{B_1}(\widetilde{M_0}^*) \mid |\langle\, f\,,\, \mu\,\rangle - r| < \epsilon\}$$

and the lifted family of open sets will be denoted as $\widetilde{\mathcal{O}}$. Then take $h \in (\widetilde{M_0})^*$ obtained from Lemma 16, applied to $\widetilde{\mathcal{O}}$ and $\widetilde{M_2} = \widetilde{M_0} \oplus \widetilde{M_1}$. It follows immediately that $\delta_G \Box h$ intersects every set from $\widetilde{\mathcal{O}}$ and, hence, $(\delta_G \Box h) \restriction M_0$ intersects every set from $\mathcal{O}$. This implies $\overline{\delta_G \Box h} \restriction M_0 = \mathbf{B_1}(M_0^*)$. ☐

Corollary 20. *Under the same hypotheses as Theorem 19 it follows that $Z_t^l\big(\mathbf{M}(G)^{**}\big) \cap M_2^{**} \subseteq M_0 \oplus M_1^{**}$.*

Proof. This follows from Lemma 8. ☐

6. PROOF OF THEOREM 3— METRIZABLE CASE

Lemma 21. *If G is a locally compact group then $Z_t^l\big(\mathbf{M}(G)^{**}\big) \cap \mathbf{M_a}(G)^{**} \subseteq \mathbf{M}(G)$.*

Proof. Since $\mathbf{M_a}(G)$ is a subalgebra of $\mathbf{M}(G)$, it follows by elementary arguments that $Z_t^l\big(\mathbf{M}(G)^{**}\big) \cap \mathbf{M_a}(G)^{**} \subseteq Z_t^l\big(\mathbf{M_a}(G)^{**}\big)$. Now one can apply the theorem, due to Lau and Losert [14, Theorem 1] that $Z_t^l\big(\mathbf{M_a}(G)^{**}\big) = \mathbf{M_a}(G)$. ☐

Corollary 22. *If G is a locally compact group of cardinality not greater than $\kappa(G)2^{\aleph_0}$ then $Z_t^l(\mathbf{M}(G)^{**}) = \mathbf{M}(G)$.*

Proof. If G is discrete then $\mathbf{M}(G) = \mathbf{M_a}(G)$ and the resut follows from Lemma 21. By [8, 5.5]) it follows that $d(\mathbf{M}(G)) = |G| \leq \kappa(G)2^{\aleph_0}$. By Theorem 11 and the fact that G is not discrete, it follows that each measure in $\mathbf{M_s}(G)$ is $\kappa(G)2^{\aleph_0}$-thin. Since $\mathbf{M}(G) = \mathbf{M_s}(G) \oplus \mathbf{M_a}(G)$ and $d(\mathbf{M_s}(G)) \leq d(\mathbf{M}(G)) \leq \kappa(G)2^{\aleph_0}$ it is possible to apply Corollary 18 to obtain that

$$Z_t\big(\mathbf{M}(G)^{**}\big) \subseteq \mathbf{M_s}(G) \oplus \mathbf{M_a}(G)^{**}.$$

Lemma 21 then yields that $Z_t\big(\mathbf{M}(G)^{**}\big) \subseteq \mathbf{M_s}(G) + \mathbf{M}(G) = \mathbf{M}(G)$. ☐

The Ghahramani–Lau Conjecture for metrizable groups — indeed the conjecture for the class of all groups G of cardinality no greater than $2^{\aleph_0}\kappa(G)$, a class that includes the metrizable groups — is settled by Corollary 22. It should be mentioned that the family of groups of cardinality $2^{\aleph_0}$ is really larger than the class of metrizable groups assuming that $2^{\aleph_1} = 2^{\aleph_0}$ as can be seen by considering the group ${}^{\omega_1}2$. Hence it is consistent with set theory that Corollary 22 applies to some non-metrizable groups. However the proof of the full conjecture requires the following theorem, which also yields Corollary 22 when K is the trivial group. The more detailed result will be needed as the beginning of an induction hypothesis for the proof of Theorem 3 to be described in later sections.

Theorem 23. *Let G be a locally compact group, K a compact subgroup such that $|G/K| \le 2^{\aleph_0}\kappa(G)$ — in particular, this is the case if G/K is metrizable. Then $Z_t\big(\mathbf{M}(G)^{**}\big) \cap \mathbf{M}(G/K)^{**} \subseteq \mathbf{M}(G/K)$.*

Proof. Put $M_2 = \mathbf{M}(G/K)$ and $M_0 = \mathbf{M_s}(G) \cap M_2$ and $M_1 = \mathbf{M_a}(G) \cap M_2$, and $\widetilde{M}_0 = \mathbf{M_s}(G)$, $\widetilde{M}_1 = \mathbf{M_a}(G)$, $\widetilde{M}_2 = \mathbf{M}(G)$ and apply now Theorem 11, Theorem 19 and Lemma 21 and the strategy of Corollary 22. □

7. Compact subgroups and some classes of measures

First, we will give now some formulas for the character of quotient spaces. Then we define some classes of compact subgroups in a non-metrizable group, corresponding classes of measures and decompositions of $\mathbf{M}(G)$. This contains the strongly singular measures mentioned at the end of §2.

Lemma 24. *Let G be a locally compact group.*

(1) *If H_0 and H_1 are compact subgroups of G, $H_0 \supseteq H_1$, then*

$$\chi(G/H_1) = \max\big(\chi(G/H_0), \chi(H_0/H_1)\big)\ .$$

(2) *If H_i are compact subgroups of G for $i \in I$, then*

$$\chi\big(\, G/\bigcap_{i\in I} H_i\,\big) \;\le\; \sup_{i\in I}\big(\,\chi(G/H_i)\,\big) + |I|\ .$$

(3) *If G is σ-compact and H is a closed subgroup, then there is $N \subseteq H$, a closed, normal subgroup of G, such that*

$$\chi(G/N) \le \chi(G/H) + \aleph_0\,.$$

In (3), without σ-compactness, one has $\ \chi(G/N) \le \chi(G/H) + \kappa(G)\,.$

Proof. To prove (1) observe that there is a family of open sets $\mathcal{U}_0$ of cardinality $\chi(G/H_0)$ such that $\cap\mathcal{U}_0 = H_0$. As well, there is a family of open sets $\mathcal{U}_1$ of cardinality $\chi(H_0/H_1)$ such that $\cap\mathcal{U}_1 \cap H_0 = H_1$. Because the character of a point x in locally compact space is the same as the least cardinal of an open family whose intersection is $\{x\}$, the image under the quotient map of $\mathcal{U}_0 \cup \mathcal{U}_1$ is a subbase for H_1 in G/H_1. Hence $\chi(G/H_1) \le \max\big(\chi(G/H_0), \chi(H_0/H_1)\big)$. To show the other direction of the inequality, suppose that $\mathcal{U}$ is a family of open sets in G, closed under finite intersections, whose intersection is H_1. The cardinality of $\mathcal{U}$ must obviously be at least $\chi(H_0/H_1)$. Suppose it is less than $\chi(G/H_0)$ and note that $H_0 \subsetneq H_0U$ for each $U \in \mathcal{U}$. Hence there is $z \in \bigcap_{U\in\mathcal{U}} H_0U \setminus H_0$. For each $U \in \mathcal{U}$ there is, therefore, some $x_U \in H_0$ such that $z \in Ux_U$. Since H_0 is compact, there is a $\subseteq$-cofinal, $\subseteq$-directed subfamily $\mathcal{U}' \subseteq \mathcal{U}$ such that $\lim_{U\in\mathcal{U}'} x_U = x$ for some $x \in H_0$. In other words, $\bigcap\mathcal{U}' = H_1$ but $zx^{-1} \in \cap\mathcal{U}'$. Since $z \notin H_0$ and $x \in H_0$ this yields that $zx^{-1} \notin H_1$ and a contradiction to $\bigcap\mathcal{U}' - H_1$.

To prove (2) simply use the fact that a family of compact neighbourhoods of a point is a neighbourhood base precisely if it is directed downwards and its intersection is the point.

To show (3) let $N = \bigcap_{x\in G} xHx^{-1}$ and, replacing G by G/N, it may be assumed that N is trivial. Let $(V_i)_{i\in I}$ be a family of compact e_G-neighbourhoods such that $\bigcap_{i\in I} V_i \subseteq H$ and $|I| = \chi(G/H)$. Choose compact symmetric e_G-neighbourhoods W_i such that $W_i^3 \subseteq V_i$ and countable subsets D_i in G such that $G = D_iW_i$. Then the family of all finite intersections of the sets xW_ix^{-1} where $x \in D_i$, $i \in I$ gives an e_G-neighbourhood base of cardinality at most $\chi(G/H) + \aleph_0$.
Alternatively, when H is compact, one can prove (3) using Lemma 2 from [7] and formulas (†) and (‡) (which extends to quotient spaces). □

Corollary 25. *Let $\gamma > 0$ be an ordinal number and G a locally compact group. If K_α are compact subgroups of G for $\alpha \le \gamma$, such that $\chi(G/K_0) < \chi(G)$, $K_\alpha \supseteq K_{\alpha+1}$ and $\chi(K_\alpha/K_{\alpha+1}) = \aleph_0$ for all $\alpha < \gamma$, and $K_\beta = \bigcap_{\alpha<\beta} K_\alpha$ for all limit ordinals $\beta \le \gamma$, then $\chi(G/K_\gamma) = |\gamma| + \chi(G/K_0) + \aleph_0$.*

Proof. By induction, Lemma 24 gives $\chi(G/K_\gamma) \le |\gamma| + \chi(G/K_0) + \aleph_0$. Now assume that equality does not hold for some γ and choose γ minimal. Note first that γ must be a successor cardinal. To see this suppose not and let $\tau = \chi(G/K_\gamma) < |\gamma| + \chi(G/K_0)$. There is then $\alpha < \gamma$ such that $\tau < |\alpha|$ and, by the minimality of γ, it follows that $\chi(G/K_\alpha) = |\alpha| + \chi(G/K_0)$. But then if $\mathcal{V}$ is a neighbourhood base of K_γ of cardinality τ it follows that $\{UK_\alpha\}_{U\in\mathcal{V}}$ is neighbourhood base for K_α of cardinality $\tau < |\alpha| + \chi(G/K_0)$.

Observe that for each neighbourhood U of K_γ there is some $\beta(U) \in \gamma$ such that $U \supseteq K_{\beta(U)}$.To see this note that otherwise $\{K_\beta \setminus U\}_{\beta\in\gamma}$ is a filter base of compact sets with empty intersection. If $\mathcal{V}$ is a base for K_γ of cardinality less than γ then, since γ is regular, it follows that there is some $\xi \in \gamma$ such that $\beta(U) < \xi$ for each $U \in \mathcal{V}$. This yields the contradiction that $\bigcap\mathcal{V} \supseteq K_\xi \neq K_\gamma$. □

The following definition will be of most interest in the case that G is a non-metrizable, locally compact group — in other words, $\chi(G) > \aleph_0$.

Definition 26. *For a locally compact group G and an infinite cardinal number τ define*

$$\mathfrak{K}_\tau(G) = \{K \mid K \text{ compact, subgroup of } G \text{ and } \chi(G/K) < \tau\}.$$

These classes of compact subgroups will be used to define classes of measures that will be useful in the induction and can be seen as the key structures in a generalization of the Lebesgue decomposition of measures for non-metrizable groups.

Definition 27. *Let G be a locally compact group and τ an infinite cardinal. The measures of character τ are defined to be*

$$\mathbf{M}_\tau(G) = \cup\{\mathbf{M}(G/K) \mid K \in \mathfrak{K}_{\tau^+}(G)\}.$$

The strongly singular measures of character $\aleph_0$ are defined by

$$\mathbf{M}_{\mathbf{ss},\aleph_0}(G) = \mathbf{M}_{\mathbf{s}}(G) \cap \mathbf{M}_{\aleph_0}(G)$$

and if τ is uncountable then

$$\mathbf{M}_{\mathbf{ss},\tau}(G) = \{\mu \in \mathbf{M}_{\mathbf{s}}(G) \cap \mathbf{M}_\tau(G) \mid \ \mu \perp \mathbf{M}_{\tau_1}(G) \textit{ for all } \tau_1 < \tau\}.$$

The approximately invariant measures of character $\aleph_0$ are defined by

$$\mathbf{M}_{\mathbf{ai},\aleph_0}(G) = \mathbf{M}_{\mathbf{a}}(G) \cap \mathbf{M}_{\aleph_0}(G)$$

and if τ is uncountable they are defined by

$$\mathbf{M}_{\mathbf{ai},\tau}(G) = \{\mu \in \mathbf{M}_\tau(G) \mid \ \mu = \lim_{K \in \mathfrak{K}_\tau(G)} \mu * \lambda_K \ \textit{(norm limit)}\}.$$

When K is a compact subgroup of G and not open, define $\mathbf{M}_{\mathbf{ss}}(G,K) = \mathbf{M}(G/K) \cap \mathbf{M}_{\mathbf{ss},\chi(G/K)}(G)$ and $\mathbf{M}_{\mathbf{ai}}(G,K) = \mathbf{M}(G/K) \cap \mathbf{M}_{\mathbf{ai},\chi(G/K)}(G)$.

The decompositions to be obtained in Theorem 29 can also be described as follows: Every $\mu \in \mathbf{M}_{\mathbf{s}}(G)$ has a unique (up to reorderings) representation $\mu = \sum \mu_i$, where $\mu_i \in \mathbf{M}_{\mathbf{ss},\tau_i}(G)$ for some pairwise different τ_i with $\aleph_0 \leq \tau_i \leq \chi(G)$. Thus $\mathbf{M}_{\mathbf{s}}(G)$ is the l^1-sum of all the spaces $\mathbf{M}_{\mathbf{ss},\tau}(G)$ with $\aleph_0 \leq \tau \leq \chi(G)$. The reader familiar with Riesz decompositions will see the decompositions of Theorem 29 as particular examples of these.

Recall the Kakutani-Kodaira theorem. (See [7] Theorem 3 for a more general version; a more special case, sufficient for our purposes, can be found in [6, 8.7]).

Lemma 28. *Let H be a locally compact, σ-compact group. If U_n, $n \in \mathbb{N}$, are neighbourhoods of e_H then there exists a compact, normal subgroup N such that H/N is metrizable and $N \subseteq U_n$ for all n.*

Theorem 29. *Let G be a non-metrizable locally compact group, τ a cardinal number with $\aleph_0 \leq \tau \leq \chi(G)$, K a non-open compact subgroup of G.*

(i) *$\mathbf{M}_\tau(G)$ and $\mathbf{M}_{\mathbf{ai},\tau}(G)$ are norm closed ideals in $\mathbf{M}(G)$, $\mathbf{M}_{\mathbf{ss},\tau}(G)$ is a norm closed subspace.*

(ii) *$\mathbf{M}_\tau(G), \mathbf{M}_{\mathbf{ai},\tau}(G)$ and $\mathbf{M}_{\mathbf{ss},\tau}(G)$ are L-subspaces (i.e., $\mu \in M$ and $|\nu| \ll |\mu|$ implies $\nu \in M$). We have $\mathbf{M}_\tau(G) = \mathbf{M}_{\mathbf{ss},\tau}(G) \oplus \mathbf{M}_{\mathbf{ai},\tau}(G)$ and $\mathbf{M}_{\mathbf{ss},\tau}(G) \perp \mathbf{M}_{\mathbf{ai},\tau}(G)$.*

(iii) *$\mathbf{M}(G/K)$, $\mathbf{M}_{\mathbf{ai}}(G,K)$ and $\mathbf{M}_{\mathbf{ss}}(G,K)$ are closed subspaces and vector sublattices in $\mathbf{M}(G)$ (i.e., $\mu \in M$ implies $|\mu| \in M$). We have*

$$\mathbf{M}(G/K) = \mathbf{M}_{\mathbf{ss}}(G,K) \oplus \mathbf{M}_{\mathbf{ai}}(G,K) \quad \textit{and} \quad \mathbf{M}_{\mathbf{ss}}(G,K) \perp \mathbf{M}_{\mathbf{ai}}(G,K)\,.$$

Proof. $\mathbf{M}(G/K)$ is always a closed subspace of $\mathbf{M}(G)$ and a left ideal by Corollary 5. Using (2) of Lemma 24, it follows that $\mathbf{M}_\tau(G)$ is a closed subspace and hence, so are $\mathbf{M}_{\mathbf{ai},\tau}(G)$ and $\mathbf{M}_{\mathbf{ss},\tau}(G)$. It also follows that $\mathbf{M}_\tau(G)$ and $\mathbf{M}_{\mathbf{ai},\tau}(G)$ are left ideals. Now take $\nu \in \mathbf{M}(G)$ and $\mu \in \mathbf{M}_\tau(G)$ and let H be an open, σ-compact subgroup containing the support of ν. Then by (3) of Lemma 24 and Lemma 28, there exists a normal subgroup N of G such that $N \in \mathfrak{K}_{\tau^+}(G)$ and $\mu \in \mathbf{M}(G/N)$. By Lemma 4 a measure $\rho \in \mathbf{M}(G)$ belongs to $\mathbf{M}(G/N)$ if and only if $\rho = \rho \star \lambda_N$. Therefore, in order to show that $\mu \star \nu \in \mathbf{M}(G/N)$ it suffices to show that $\mu \star \nu = \mu \star \nu \star \lambda_N$. Since $\mu \star \lambda_N = \mu$, it suffices to show that $\nu \star \lambda_N = \lambda_N \star \nu$. From the discussion preceding the statement of Theorem 3 it suffices to show that $\langle\, \nu \star \lambda_N \,,\, f \,\rangle = \langle\, \lambda_N \star \nu \,,\, f \,\rangle$ for all $f \in C_0(G)$. However, since the support of ν is in H it suffices to show that $\langle\, \nu \star \lambda_N \,,\, f \,\rangle = \langle\, \lambda_N \star \nu \,,\, f \,\rangle$ for all $f \in C_0(H)$. Since N is normal in H it follows that if $f \in C_0(H)$ and $a \in H$ then

$$\lambda_N \star f(a) = \int f(ab)d\lambda_N(b) = \int f(ba)d\lambda_N(b) = f \star \lambda_N(a)$$

and from this it follows that

$$\langle\, \nu \star \lambda_N \,,\, f \,\rangle = \langle\, \nu \,,\, \lambda_N \star f \,\rangle = \langle\, \nu \,,\, f \star \lambda_N \,\rangle = \langle\, \lambda_N \star \nu \,,\, f \,\rangle$$

as required. Thus $\mathbf{M}_\tau(G)$ is a right ideal and a similar argument works for $\mathbf{M}_{\mathbf{ai},\tau}(G)$. This proves (i).

Since $\mathbf{M}_\tau(G)$ is norm closed, in order to show that $\mathbf{M}_\tau(G)$ is an L-subspace, it suffices to show that $h\mu \in \mathbf{M}_\tau(G)$ whenever $\mu \in \mathbf{M}_\tau(G)$ and h is a continuous function of compact support. As above let H be a σ-compact subgroup such that h vanishes outside H. By Lemma 28, the Kakutani-Kodaira Theorem, there exists a compact normal subgroup N of H such that H/N is metrizable and h is N-periodic. If $\mu \in \mathbf{M}(G/K)$ with $K \in \mathfrak{K}_{\tau^+}$ then $h\mu \in \mathbf{M}\big(G/(K \cap N)\big)$ and by (2) of Lemma 24, $K \cap N \in \mathfrak{K}_{\tau^+}$. It is easy to see that this also implies that $\mathbf{M}_{\mathbf{ss},\tau}(G)$ is an L-subspace and a similar argument works for $\mathbf{M}_{\mathbf{ai},\tau}(G)$.

The facts that $\mathbf{M}_{\mathbf{ss},\tau}(G) \perp \mathbf{M}_{\mathbf{ai},\tau}(G)$ and $\mathbf{M}_{\mathbf{ss},\tau}(G) \oplus \mathbf{M}_{\mathbf{ai},\tau}(G) = \mathbf{M}_\tau(G)$ when $\tau = \aleph_0$ follow directly from the definitions and the Lebesgue decomposition of measures. To see that this is also the case when $\tau > \aleph_0$ it will first be shown that $\mathbf{M}_{\mathbf{ss},\tau}(G) \perp \mathbf{M}_{\mathbf{ai},\tau}(G)$. To establish this let $\mu \in \mathbf{M}_{\mathbf{ss},\tau}(G)$ and $\nu \in \mathbf{M}_{\mathbf{ai},\tau}(G)$ be positive measures such that $\|\mu\| = \|\nu\| = 1$. Then $\lim_j \|\nu - \nu \star \lambda_{K_j}\| = 0$ where $K_j \in \mathfrak{K}_\tau(G)$ for each j. It follows from the definition of $\mathbf{M}_{\mathbf{ss},\tau}(G)$ that $\mu \perp \nu \star \lambda_{K_j}$ for each j or, in other words, $\|\nu \star \lambda_{K_j} - \mu\| = 2$ for each j. Hence $\|\mu - \nu\| = 2$ and so $\mu \perp \nu$.

Having established that $\mathbf{M}_{\mathbf{ss},\tau}(G) \perp \mathbf{M}_{\mathbf{ai},\tau}(G)$ it follows that $\mathbf{M}_{\mathbf{ss},\tau}(G) \cap \mathbf{M}_{\mathbf{ai},\tau}(G) = \{0\}$ and so it suffices to show that $\mathbf{M}_\tau(G) = \mathbf{M}_{\mathbf{ss},\tau}(G) + \mathbf{M}_{\mathbf{ai},\tau}(G)$. To this end, take any $\mu \in \mathbf{M}_\tau(G)$ with $\mu \geq 0$. Using the fact that $\mathbf{M}_{\mathbf{ai},\tau}(G)$ is closed it may be assumed that there is some $\nu \in \mathbf{M}_{\mathbf{ai},\tau}(G)$ such that $\|\nu\|$ is maximal among those measures satisfying $0 \leq \nu \leq \mu$. In other words, since $\mathbf{M}_{\mathbf{ai},\tau}(G)$ is an L-subspace, every non-zero measure in $\mathbf{M}_{\mathbf{ai},\tau}(G)$ is singular to $\mu - \nu$. Since $\mathbf{M}_\rho \subseteq \mathbf{M}_{\mathbf{ai},\tau}(G)$ for

each $\rho < \tau$, it follows from the definition of $\mathbf{M}_{\mathbf{ss},\tau}(G)$ that $\mu - \nu \in \mathbf{M}_{\mathbf{ss},\tau}(G)$. This shows that $\mathbf{M}_\tau(G) = \mathbf{M}_{\mathbf{ss},\tau}(G) \oplus \mathbf{M}_{\mathbf{ai},\tau}(G)$.
The statements in (iii) about $\mathbf{M}(G/K)$ and its subspaces are easy consequences of (i) and (ii). ☐

8. Separation of strongly singular measures

In this section Theorem 11 is extended to the case of strongly singular measures (Theorem 36). This will be used in the proof of the general case of Theorem 3.

Lemma 30. *If G is any locally compact group then G has a compact, not necessarily normal, subgroup K_G such that G/K_G is metrizable. If G is non-metrizable, then for any such group it follows that $\chi(K_G) = \chi(G)$.*

Proof. Take any symmetric compact neighbourhood U of e_G. Then $G_0 = \bigcup_{n=1}^{\infty} U^n$ is an open subgroup of G (cf. [6, 5.7]), hence $\chi(G_0) = \chi(G)$. Since G_0 is compactly generated, by Lemma 28 there is a compact normal subgroup K_G of G_0 such that G_0/K_G is metrizable and hence G/K_G is metrizable. If $\chi(G) > \aleph_0$, then $\chi(K_G) = \chi(G)$ since $\chi(G) = \max\big(\chi(G/K_G), \chi(K_G)\big)$ by Lemma 24. ☐

A similar construction yields the next lemma.

Lemma 31. *Let G be a locally compact group and K its compact non-open subgroup. For every $\mu \in \mathbf{M}(G)$ there are an open σ-compact subgroup G_0 of G and a compact normal subgroup N of G_0 with $N \subseteq K \subseteq G_0$, $\chi(G/K) = \chi(G/N)$ and $\mu(G \backslash G_0) = 0$.*

Proof. As in the proof of the previous lemma, there is an open σ-compact subgroup G_0 of G such that $\mu(G \setminus G_0) = 0$ and $K \subseteq G_0$. By (3) of Lemma 24 there is a compact normal subgroup N of G_0 such that $N \subseteq K$ and

$$\chi(G/K) \le \chi(G/N) = \chi(G_0/N) \le \chi(G_0/K) + \aleph_0 = \chi(G/K).$$

☐

Lemma 32. *Let G be a locally compact group, H a closed σ-compact subgroup and V a neighbourhood of e_G. Let $\nu, \nu' \in \mathbf{M}(G)$, $\nu, \nu' \ge 0$. If $\nu \perp \nu'$ then there exist a K_σ-set $E \subseteq G$ and a compact normal subgroup N of H such that $N \subseteq V$ and $\nu(E) = \nu(G)$, $\nu'(EN) = 0$ and $\chi(H/N) \le \aleph_0$. In particular $\nu \star \lambda_N \perp \nu' \star \lambda_N$.*

Proof. Since $\nu \perp \nu'$, there is a K_σ-set $E \subseteq G$ such that $\nu(E) = \nu(G)$ and $\nu'(E) = 0$. Thus there are compact sets C_n such that $E = \bigcup_{n=1}^{\infty} C_n$. For every n and m there exists an open set $W_{n,m} \supseteq C_n$ satisfying $\nu'(W_{n,m}) < 1/m$. Since C_n are compact, there are neighbourhoods $U_{n,m}$ of e_G with $C_n U_{n,m} \subseteq W_{n,m}$. By Lemma 28 there is a compact normal subgroup N of H such that $N \subseteq \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} U_{n,m} \cap V$ and $\chi(H/N) \le \aleph_0$. Then $\nu'(C_n N) \le \nu'(C_n U_{n,m}) \le \nu'(W_{n,m}) < 1/m$ for every m, giving $\nu'(C_n N) = 0$ for every n. Since $EN = \bigcup_{n=1}^{\infty} C_n N$, it follows that $\nu'(EN) = 0$. Thus ν and $\nu * \lambda_N$ are concentrated on EN, whereas $\nu' * \lambda_N(EN) = \nu'(EN) = 0$. ☐

Lemma 33. *Let G be a locally compact group such that $\chi(G) > \aleph_0$, H a closed σ-compact subgroup, V an e_G-neighbourhood and let $\mu \in \mathbf{M}_{\mathbf{ss},\chi(G)}(G)$ with $\mu \geq 0$, $\mu(G) = 1$. For every normal subgroup K_1 of H with $K_1 \in \mathfrak{K}_{\chi(G)}(G)$ there exists a normal subgroup K_2 of H with $K_2 \in \mathfrak{K}_{\chi(G)}(G)$ and a K_σ-set $E \subseteq G$ such that $K_2 \subseteq K_1 \cap V$, $\mu(E) = 1$, $\mu \star \lambda_{K_1}(EK_2) = 0$ and $\chi(K_1/K_2) \leq \aleph_0$.*

Proof. Since $K_1 \in \mathfrak{K}_{\chi(G)}(G)$, we have $\mu \perp \mu \star \lambda_{K_1}$. Let N be as in Lemma 32 with $\nu = \mu$ and $\nu' = \mu \star \lambda_{K_1}$. Put $K_2 = K_1 \cap N$. Recall ([6, 5.33]) that the group K_1/K_2 is topologically isomorphic to the group K_1N/N. Since $\chi(H/N) \leq \aleph_0$, we get that K_1/K_2 is metrizable and by (1) of Lemma 24 that $K_2 \in \mathfrak{K}_{\chi(G)}(G)$. ☐

Lemma 34. *Let G be a locally compact group such that $\chi(G) > \aleph_0$, H a closed σ-compact subgroup, V an e_G-neighbourhood and let $\mu \in \mathbf{M}_{\mathbf{ss},\chi(G)}(G)$ with $\mu \geq 0$, $\mu(G) = 1$. For every compact normal subgroup K_1 of H with $\chi(G/K_1) < \chi(G)$ there is a compact normal subgroup K_2 of H such that $K_2 \subseteq K_1 \cap V$, $\chi(K_1/K_2) = \aleph_0$ and $\mu \star K_2 \perp \mu \star xK_2$ for λ_{K_1}-almost all $x \in K_1$.*

Proof. Take $K_1 \in \mathfrak{K}_{\chi(G)}(G)$. Choose K_2, E by Lemma 33. Now

$$\int_{K_1} \mu \star x(EK_2)\, d\lambda_{K_1}(x) = \int_{K_1} \mu(EK_2x^{-1})\, d\lambda_{K_1}(x) = \mu \star \lambda_{K_1}(EK_2) = 0$$

and therefore $\mu \star x(EK_2) = 0$ for λ_{K_1}-almost all x in K_1.

Take any $x \in K_1$ such that $\mu \star x(EK_2) = 0$ and any $y, y' \in K_2$. Then

$$\mu \star xy(Ey') = \mu \star x \star y(Ey') = \mu \star x(Ey'y^{-1}) \leq \mu \star x(EK_2) = 0\,,$$

while $\mu \star y'(Ey') = \mu(E) = 1$ which shows that $\mu \star y' \perp \mu \star xy$. Thus $\mu \star K_2 \perp \mu \star xK_2$.

If K_2 were open in K_1 then $\lambda_{K_1}(K_2) > 0$ and we would get $\mu \star K_2 \perp \mu \star K_2$. Thus K_1/K_2 is not discrete, giving $\chi(K_1/K_2) = \aleph_0$. ☐

Lemma 35. *Let G be a locally compact group, K_0 a compact subgroup such that $\chi(G) > \chi(G/K_0) + \aleph_0$ and $\mu \in \mathbf{M}_{\mathbf{ss},\chi(G)}(G)$ with $\mu \geq 0$, $\mu(G) = 1$. Then for $0 < \alpha \leq \chi(G)$ there exist compact, normal, non-open subgroups K_α of K_0 and $x_\alpha \in K_\alpha$ such that*

- (i) $K_\alpha \supseteq K_{\alpha+1}$ *and* $\chi(K_\alpha/K_{\alpha+1}) = \aleph_0$ *for all* $\alpha < \chi(G)$;
- (ii) $K_\beta = \bigcap_{\alpha<\beta} K_\alpha$ *for all limit ordinals* $\beta \leq \chi(G)$;
- (iii) $K_{\chi(G)} = \{e_G\}$;
- (iv) $\mu \star K_{\alpha+1} \perp \mu \star x_\alpha K_{\alpha+1}$ *for all* $\alpha < \chi(G)$.

Proof. Let $(V_\alpha)_{\alpha<\chi(G)}$ be a family of e_G-neighbourhoods satisfying $K_0 \subseteq V_0$ and $\bigcap_{\alpha<\chi(G)} V_\alpha = \{e_G\}$. Now use transfinite induction, applying Lemma 34 for each $\alpha < \chi(G)$ to obtain $K_{\alpha+1} \subseteq K_\alpha \cap V_\alpha$ and x_α satisfying (i) and (iv). Use (ii) to define K_β for limit ordinals $\beta \leq \chi(G)$. ☐

Note that by Corollary 25 it follows that $\chi(G/K_\alpha) = |\alpha| + \chi(G/K_0) + \aleph_0$ for $0 < \alpha < \chi(G)$, in particular $K_\alpha \in \mathfrak{K}_{\chi(G)}(G)$.

Theorem 36. *Let G be a non-discrete, locally compact group and $\mu\in\mathbf{M}_{\mathbf{ss},\chi(G)}(G)$. There exists a set $P\subseteq G$ such that $|P| = 2^{\chi(G)}$ and $\mu\star p\perp\mu\star p'$ for all $p,p'\in P$, $p\neq p'$. In other words, μ is $2^{\chi(G)}$-thin.*

Proof. By Theorem 11 the result holds for $\chi(G)=\aleph_0$. We now prove it for $\chi(G)>\aleph_0$. Without loss of generality, assume that $\mu\geq 0$, $\mu(G)=1$. Recall the notational conventions in the proof of Theorem 11.

Let K_α and $x_\alpha\in K_\alpha$ for $\alpha\leq\chi(G)$ be as in Lemma 35, where K_0 is the group $K_G\subseteq G$ from Lemma 30. Induction on $\alpha\in\chi(G)$ will be used to construct elements $d_*\in K_G$ for all $d\in{}^{\alpha}2$ such that

(1•) if $\alpha<\beta<\chi(G)$ and $d\in{}^{\beta}2$ then $(d\restriction\alpha)_*K_\alpha\supseteq d_*K_\beta$;

(2•) if $\beta<\chi(G)$ is a limit ordinal and $d\in{}^{\beta}2$ then

$$\bigcap_{\alpha<\beta}(d\restriction\alpha)_*K_\alpha = d_*\bigcap_{\alpha<\beta}K_\alpha = d_*K_\beta\ ;$$

(3•) if $\beta<\chi(G)$ and d and d' are distinct elements of ${}^{\beta}2$ then $\mu\star d'_*K_\beta\perp\mu\star d_*K_\beta$.

That will conclude the proof, since it is then possible to choose a point

$$p_d\in\bigcap_{\alpha<\chi(G)}(d\restriction\alpha)_*K_\alpha\qquad\text{for every}\ \ d\in{}^{\chi(G)}2$$

and taking $P=\{p_d\mid d\in{}^{\chi(G)}2\}$, we have $|P|={}^{\chi(G)}2$. Moreover, (3•) implies that if $p\neq p'$ then $\mu\star p\perp\mu\star p'$.

To carry out the induction, start with $\alpha=0$ and define $\emptyset_*=e_G$. Now assume that $0\neq\beta<\chi(G)$ and that d_* have been defined for all $d\in{}^{\alpha}2$ and $\alpha<\beta$. If $d\in{}^{\beta}2$ and β is a limit ordinal define d_* to be any point in $\bigcap_{\alpha<\beta}(d\restriction\alpha)_*K_\alpha$. If $\beta=\alpha+1$ then define

$$d_*=\begin{cases}(d\restriction\alpha)_* & \text{when } d(\alpha)=0\\ x_\alpha(d\restriction\alpha)_* & \text{when } d(\alpha)=1\ .\end{cases}$$

Properties (1•) and (2•) are easy to prove from the definition, using the normality of K_α in K_0. To prove (3•), in view of (1•) and (2•) it is enough to consider the case where $\beta=\alpha+1$ and $d,d'\in{}^{\beta}2$ are such that $d'\restriction\alpha=d\restriction\alpha$, $d_*=(d\restriction\alpha)_*$ and $d'_*=x_\alpha(d\restriction\alpha)_*=x_\alpha d_*$.

Since $\mu\star K_{\alpha+1}\perp\mu\star x_\alpha K_{\alpha+1}$, we have also $\mu\star K_{\alpha+1}d_*\perp\mu\star x_\alpha K_{\alpha+1}d_*$. Hence, using that $K_{\alpha+1}$ is a normal subgroup of K_0,

$$\begin{aligned}K_{\alpha+1}d_* &= d_*K_{\alpha+1} &&= d_*K_\beta\\ x_\alpha K_{\alpha+1}d_* &= x_\alpha d_*K_{\alpha+1} &&= d'_*K_\beta\ .\end{aligned}$$

Thus $\mu\star d_*K_\beta\perp\mu\star d'_*K_\beta$. □

9. Proof of Theorem 3 — general case

First we give some further properties of the spaces $\mathbf{M}_{\mathbf{ss}}(G,K)$ and $\mathbf{M}_{\mathbf{ai}}(G,K)$. Then we prove Theorem 45 by transfinite induction, and obtain our main result (Theorem 3) as a corollary.

Lemma 37. *Let G be a locally compact group, K a compact subgroup of G and $\mu \in \mathbf{M}(G)$. We have $\mu \perp \mathbf{M}(G/K)$ iff $\mu \perp |\mu| \star \lambda_K$.*

Proof. One direction is clear. For the other one, assume that $\mu \perp |\mu| \star \lambda_K$ and take any $\nu \in \mathbf{M}(G/K)$. It is enough to consider $\mu \geq 0$ and $\nu \geq 0$. The Lebesgue decomposition of ν with respect to $\mu \star \lambda_K \in \mathbf{M}(G/K)$ yields $\nu = \nu_0 + \nu_1$ where $\nu_0, \nu_1 \in \mathbf{M}(G/K)$, $\nu_0, \nu_1 \geq 0$, $\nu_0 \perp \mu \star \lambda_K$ and $\nu_1 \ll \mu \star \lambda_K$. Since $\nu_0 \perp \mu \star \lambda_K$, there is a K-invariant Borel set $E \subseteq G$ such that $\nu_0(E) = \nu_0(G)$ and $\mu \star \lambda_K(E) = 0$, and then $\mu(E) = 0$, so that $\nu_0 \perp \mu$. From $\nu_1 \ll \mu \star \lambda_K$ and $\mu \perp \mu \star \lambda_K$ we get $\nu_1 \perp \mu$. Hence $\nu \perp \mu$. □

For a locally compact group G and its compact subgroup K, put

$$\mathfrak{K}_K(G) = \{L \in \mathfrak{K}_{\chi(G/K)}(G) \mid L \supseteq K\}.$$

By (2) in Lemma 24 the family $\mathfrak{K}_K(G)$ is downwards directed by inclusion. In the sequel we are mainly interested in the case where G/K is not metrizable, i.e. $\chi(G/K) > \aleph_0$.

Lemma 38. *Let G be a locally compact group, K a compact subgroup of G such that $\chi(G/K) > \aleph_0$, and $\nu \in \mathbf{M}(G)$. We have $\nu \in \mathbf{M_{ai}}(G,K)$ iff $\nu = \lim_{L \in \mathfrak{K}_K(G)} \nu \star \lambda_L$ (norm limit). This is also equivalent to $\inf_{L \in \mathfrak{K}_K(G)} \|\nu - \nu \star \lambda_L\| = 0$. For $\nu \in \mathbf{M}(G/K)$ we have $\nu \in \mathbf{M_{ss}}(G,K)$ iff $\nu \perp |\nu| \star \lambda_L$ for all $L \in \mathfrak{K}_K(G)$.*

Proof. Put $\tau = \chi(G/K)$. If L, L' are compact subgroups of G with $L \subseteq L'$, then $\lambda_{L'} \star \lambda_L = \lambda_{L'}$. Thus $\|\nu \star \lambda_{L'} - \nu \star \lambda_L\| \leq \|\nu \star \lambda_{L'} - \nu\|$ and it follows that $\|\nu \star \lambda_L - \nu\| \leq 2\, \|\nu \star \lambda_{L'} - \nu\|$. This shows that $\nu = \lim_{L \in \mathfrak{K}_K(G)} \nu \star \lambda_L$ is equivalent to $\inf_{L \in \mathfrak{K}_K(G)} \|\nu - \nu \star \lambda_L\| = 0$ and this equivalence persists when $\mathfrak{K}_K(G)$ is replaced by any family of compact subgroups which is directed downwards.
Since $\mathfrak{K}_K(G) \subseteq \mathfrak{K}_\tau(G)$, it follows that $\nu = \lim_{L \in \mathfrak{K}_K(G)} \nu \star \lambda_L$ implies $\nu = \lim_{L \in \mathfrak{K}_\tau(G)} \nu \star \lambda_L$, i.e. $\nu \in \mathbf{M}_{\mathbf{ai},\tau}(G)$. Furthermore $\nu \star \lambda_L \in \mathbf{M}(G/K)$ for $L \in \mathfrak{K}_K(G)$ implies $\nu \in \mathbf{M}(G/K)$.

For the converse, if $L \in \mathfrak{K}_\tau(G)$, let H be an open σ-compact subgroup of G containing K and L . By (3) of Lemma 24 there exists $L' \in \mathfrak{K}_\tau(G)$ such that L' is normal in H and $L' \subseteq L$. Then KL' is a group, by (1) of Lemma 24 we have $KL' \in \mathfrak{K}_K(G)$ and for $\nu \in \mathbf{M}(G/K)$ we get (using $\lambda_{KL'} = \lambda_K \star \lambda_{L'}$) that $\nu \star \lambda_{KL'} = \nu \star \lambda_{L'}$. Thus $\nu = \lim_{L \in \mathfrak{K}_\tau(G)} \nu \star \lambda_L$ implies $\nu = \lim_{L \in \mathfrak{K}_K(G)} \nu \star \lambda_L$.

For the last part, we may assume that $\nu \geq 0$. By Theorem 29, we have $\nu = \nu_1 + \nu_2$, where $\nu_1 \in \mathbf{M_{ss}}(G,K)$, $\nu_2 \in \mathbf{M_{ai}}(G,K)$. If $\nu_2 \neq 0$ we have by the first part $\nu_2 \not\perp \nu_2 \star \lambda_L$ for some $L \in \mathfrak{K}_K(G)$. Since $0 \leq \nu_2 \leq \nu$, this implies $\nu \not\perp \nu \star \lambda_L$. The other direction follows immediately from the definition of $\mathbf{M_{ss}}(G,K)$. □

Corollary 39. *Let G be a locally compact group, K a compact subgroup of G such that $\chi(G/K) > \aleph_0$, and $\nu \in \mathbf{M}(G)$. Let $\mathcal{D}$ be a family of compact subgroups of G that is downwards directed by inclusion and such that $\mathfrak{K}_K(G) \subseteq \mathcal{D} \subseteq \mathfrak{K}_{\chi(G/K)}(G)$.*

We have $\nu \in \mathbf{M}_{\mathbf{ai}}(G,K)$ *iff* $\nu = \lim_{L\in\mathcal{D}} \nu \star \lambda_L$ *(norm limit). This is also equivalent to* $\inf_{L\in\mathcal{D}}\|\nu - \nu\star\lambda_L\| = 0$ *. For* $\nu \in \mathbf{M}(G/K)$ *we have* $\nu \in \mathbf{M}_{\mathbf{ss}}(G,K)$ *iff* $\nu \perp |\nu|\star\lambda_L$ *for all* $L \in \mathcal{D}$ *.* □

Corollary 40. *Let* G *be a locally compact group,* K *a normal compact subgroup of* G *for which* G/K *is not metrizable. When* $\mathbf{M}(G)\star\lambda_K$ *is identified with* $\mathbf{M}(G/K)$ *as in Lemma 4, the space* $\mathbf{M}_{\mathbf{ss}}(G,K)$ *is identified with* $\mathbf{M}_{\mathbf{ss}}(G/K,\{e_{G/K}\})$*, and* $\mathbf{M}_{\mathbf{ai}}(G,K)$ *is identified with* $\mathbf{M}_{\mathbf{ai}}(G/K,\{e_{G/K}\})$*.*

Proof. Let $\pi\colon G \to G/K$ be the quotient mapping. Then $L \mapsto \pi(L)$ defines a one-to-one correspondence between $\mathfrak{K}_K(G)$ and $\mathfrak{K}_{\{e_{G/K}\}}(G/K) = \mathfrak{K}_{\chi(G/K)}(G/K)$. □

Corollary 41. *Let* G *be a locally compact group and* K *its non-open compact subgroup. Every measure in* $\mathbf{M}_{\mathbf{ss}}(G,K)$ *is* $|G/K|$ *-thin.*

Proof. Recall that $|G/K| = \kappa(G)\cdot 2^{\chi(G/K)}$. Since every $\mu \in \mathbf{M}_{\mathbf{ss}}(G,K)$ is $\kappa(G)2^{\aleph_0}$-thin by Theorem 11, it remains to be proved that μ is also $2^{\chi(G/K)}$-thin when $\chi(G/K) > \aleph_0$.

Take any $\mu \in \mathbf{M}_{\mathbf{ss}}(G,K)$ and assume $\chi(G/K) > \aleph_0$. By Lemma 31 there are an open σ-compact subgroup G_0 of G and a compact normal subgroup N of G_0 such that $N \subseteq K \subseteq G_0$, $\chi(G/K) = \chi(G/N)$ and $\mu(G\setminus G_0) = 0$. Let μ' be the restriction of μ to G_0. Then

$$\mu' \in \mathbf{M}_{\mathbf{ss}}(G_0,N) = \mathbf{M}_{\mathbf{ss}}(G_0/N,\{e_{G/N}\}) \subseteq \mathbf{M}_{\mathbf{ss},\chi(G_0/N)}(G_0/N)$$

by Corollary 40 and the definition of $\mathbf{M}_{\mathbf{ss}}(G_0,N)$. It follows that μ' is $2^{\chi(G_0/N)}$-thin by Theorem 36, and thus μ is $2^{\chi(G/K)}$-thin. □

Lemma 42. *Let* G *be a locally compact group and* τ *a successor cardinal such that* $\aleph_0 < \tau \le \chi(G)$*. Let* $\{K_\alpha\}_{\alpha\le\tau}$ *be a family of compact subgroups of* G *such that* $\chi(G/K_0) < \tau$*,* $K_\alpha \supseteq K_{\alpha+1}$ *and* $\chi(K_\alpha/K_{\alpha+1}) = \aleph_0$ *for* $\alpha < \tau$ *, and* $K_\beta = \bigcap_{\alpha<\beta} K_\alpha$ *for limit ordinals* $\beta \le \tau$ *. Put* $K = K_\tau$ *.*

1. *The family* $\{K_\alpha\}_{\alpha\in\tau}$ *is cofinal in* $\mathfrak{K}_K(G)$ *(downwards directed by inclusion).*
2. *For* $\nu \in \mathbf{M}(G)$ *we have* $\nu \in \mathbf{M}_{\mathbf{ai}}(G,K)$ *iff* $\nu = \lim_{\alpha<\tau} \nu\star\lambda_{K_\alpha}$ *(norm limit).*
3. *For* $\nu \in \mathbf{M}(G/K)$ *we have* $\nu \in \mathbf{M}_{\mathbf{ss}}(G,K)$ *iff* $\nu \perp |\nu|\star\lambda_{K_\alpha}$ *for all* $\alpha < \tau$ *.*

The same proof shows that the theorem holds not only for successor cardinals but also for any regular limit cardinal τ.

Proof. The proof of (1) is similar to that of Corollary 25. Since $K = \bigcap_{\alpha<\tau} K_\alpha$, the sets VK_α where V is a neighbourhood of e_G and $\alpha < \tau$ form a neighbourhood base of e_G in G/K. Given $L \in \mathfrak{K}_K(G)$, there exist e_G-neighbourhoods V_i and ordinals $\alpha_i < \tau$ for $i \in I$ such that $\bigcap_{i\in I} V_i K_{\alpha_i} \subseteq L$, where $|I| = \chi(G/L) < \tau$. Since τ is a successor cardinal, it follows that $\beta = \sup\{\alpha_i \mid i \in I\} < \tau$ and then $K_\beta \subseteq L$.
(2) As mentioned in the first part of the proof of Lemma 38, $\nu = \lim_{\alpha<\tau} \nu\star\lambda_{K_\alpha}$ is equivalent to $\inf_{\alpha<\tau}\|\nu - \nu\star\lambda_{K_\alpha}\| = 0$. By Corollary 25, we have $\chi(G/K_\tau) = \tau >$

$\chi(G/K_\alpha)$ for $\alpha < \tau$. Apply (1) and the first part of Lemma 38.
(3) If $L \supseteq K_\alpha$ and $\nu \perp |\nu| \star \lambda_{K_\alpha}$ then $\nu \perp |\nu| \star \lambda_L$ by Lemma 37. Apply (1). $\square$

Lemma 43. *Let G be a locally compact group, K its compact subgroup such that $\chi(G/K) > \aleph_0$, and $\mu \in \mathbf{M_{ss}}(G,K)$. There exist a family $\mathcal{D}$ of compact subgroups of G that is downwards directed by inclusion and such that $\mathfrak{K}_K(G) \subseteq \mathcal{D} \subseteq \mathfrak{K}_{\chi(G/K)}(G)$, and two cofinal subsets $\mathcal{C}$ and $\mathcal{C}'$ of $\mathcal{D}$ such that $\mu \star \lambda_L \perp \mu \star \lambda_{L'}$ for all $L \in \mathcal{C}$, $L' \in \mathcal{C}'$.*

Proof. If $|\mu| \star \lambda_L \perp |\mu| \star \lambda_{L'}$ then $\mu \star \lambda_L \perp \mu \star \lambda_{L'}$. Thus it suffices to prove the lemma for $\mu \geq 0$, $\mu(G) = 1$.

First we prove the lemma with $\mathcal{D} = \mathfrak{K}_K(G)$ under the assumption that the group K is normal in G. In that case G/K is a group, and every compact subgroup of G/K is identified with the quotient group H/K where $H \supseteq K$ is a compact subgroup of G.

With the assumption that K is normal in G, the proof proceeds separately in two cases. In both cases we apply Lemma 35 to obtain a family of subgroups K_α/K, $\alpha < \chi(G/K)$, of G/K; or, equivalently, the corresponding family of subgroups K_α of G. Property (iv) in the lemma then implies that $\mu \star \lambda_{K_\alpha} \perp \mu \star \lambda_{K_\beta}$ for $\alpha < \beta < \chi(G/K)$.

Case I: $\chi(G/K)$ is a limit cardinal. Define

$$\mathcal{C} = \{L \in \mathfrak{K}_K(G) \mid \chi(G/L) \text{ is even and } \mu \star \lambda_L \in \mathbf{M_{ss}}(G,L)\}$$
$$\mathcal{C}' = \{L \in \mathfrak{K}_K(G) \mid \chi(G/L) \text{ is odd and } \mu \star \lambda_L \in \mathbf{M_{ss}}(G,L)\}$$

From the definition of $\mathbf{M_{ss}}(G,L)$ we get $\mu \star \lambda_L \perp \mu \star \lambda_{L'}$ for $L \in \mathcal{C}$, $L' \in \mathcal{C}'$. To show that $\mathcal{C}$ is cofinal in $\mathfrak{K}_K(G)$, take any $K_0 \in \mathfrak{K}_K(G)$. Starting with K_0/K, let K_α/K for $\alpha < \chi(G/K)$ be compact subgroups of G/K with properties (i)–(iv) in Lemma 35. Then $\mu \star \lambda_{K_\alpha} \perp \mu \star \lambda_{K_\beta}$ for $\alpha < \beta < \chi(G/K)$.

As $\chi(G/K)$ is a limit cardinal, there is an even cardinal τ such that

$$\chi(G/K_0) < \tau < \chi(G/K).$$

Then $\chi(G/K_\tau) = \tau$ by Corollary 25 and $\mu \star \lambda_{K_\tau} \in \mathbf{M_{ss}}(G, K_\tau)$ by (3) in Lemma 42, which shows that $\mathcal{C}$ is cofinal.

The same argument with an odd cardinal τ proves that $\mathcal{C}'$ is cofinal.

Case II: $\chi(G/K)$ is a successor cardinal. Choose a compact group $K_0 \subseteq G$ such that $K_0 \supseteq K$ and G/K_0 is metrizable (Lemma 30). Starting with K_0/K, let K_α/K for $\alpha < \chi(G/K)$ be compact subgroups of G/K with properties (i)–(iv) in Lemma 35. Define

$$\mathcal{C} = \{K_\alpha \mid \alpha < \chi(G/K),\ \alpha \text{ is even}\}$$
$$\mathcal{C}' = \{K_\alpha \mid \alpha < \chi(G/K),\ \alpha \text{ is odd}\}$$

(Note that in Case I the sets $\mathcal{C}$ and $\mathcal{C}'$ are defined using even and odd cardinal numbers, while in Case II they are defined using even and odd ordinal numbers.)

Since $\mu \star \lambda_{K_\alpha} \perp \mu \star \lambda_{K_\beta}$ for $\alpha < \beta < \chi(G/K)$, we have $\mu \star \lambda_L \perp \mu \star \lambda_{L'}$ for $L \in \mathcal{C}$,

$L'\in\mathcal{C}'$. The sets $\mathcal{C}$ and $\mathcal{C}'$ are cofinal in $\mathfrak{K}_K(G)$ by (1) in Lemma 42.
That concludes the proof when the group K is normal in G. When K is a compact but not necessarily normal subgroup of G, by Lemma 31 there are an open σ-compact subgroup G_0 of G and a compact normal subgroup N of G_0 such that $N \subseteq K \subseteq G_0$, $\chi(G/K) = \chi(G/N)$ and $\mu(G\setminus G_0) = 0$. If $L\in\mathfrak{K}_K(G)$ then $L\cap G_0\in\mathfrak{K}_K(G)$; it follows that if the lemma holds with G_0 in place of G then it also holds as stated. Thus we may assume that $G = G_0$. Then $\mathfrak{K}_K(G) \subseteq \mathfrak{K}_N(G) \subseteq \mathfrak{K}_{\chi(G/K)}(G)$ and we have proved that there are cofinal subsets $\mathcal{C}$ and $\mathcal{C}'$ of $\mathfrak{K}_N(G)$ such that $\mu\star\lambda_L \perp \mu\star\lambda_{L'}$ for all $L\in\mathcal{C}$, $L'\in\mathcal{C}'$. □

Corollary 44. *Let G be a locally compact group, K its compact subgroup such that $\chi(G/K) > \aleph_0$, and $\mu\in\mathbf{M}_{\mathbf{ss}}(G,K)$. There is a family $\mathcal{D}$ of compact subgroups of G that is downwards directed by inclusion, $\mathfrak{K}_K(G) \subseteq \mathcal{D} \subseteq \mathfrak{K}_{\chi(G/K)}(G)$, and such that one of the following two alternatives holds:*

- *either $\lim_{L\in\mathcal{D}}\langle h, \mu\star\lambda_L\rangle = 0$ for every $h\in\mathbf{M}(G)^*$*
- *or $\lim_{L\in\mathcal{D}}\langle h, \mu\star\lambda_L\rangle$ does not exist for some $h\in\mathbf{M}(G)^\star$.*

In other words, if the weak limit $\mathfrak{m} = \lim_{L\in\mathcal{D}}\mu\star\lambda_L$ exists in $\mathbf{M}(G/K)^{**}$ then $\mathfrak{m} = 0$.*

Proof. Let $\mathcal{D}$ and $\mathcal{C},\mathcal{C}'$ be as in Lemma 43. Assume there is $h_0\in\mathbf{M}(G)^*$ such that $\lim_{L\in\mathcal{D}}\langle h_0, \mu\star\lambda_L\rangle \neq 0$. By Lemma 12 there is $h\in\mathbf{M}(G)^*$ that agrees with h_0 on $\mathcal{C}$ and is 0 on $\mathcal{C}'$. The net $\{\langle h, \mu\star\lambda_L\rangle\}_{L\in\mathcal{D}}$ does not converge. □

Theorem 45. *Let G be a locally compact group, K a compact subgroup. Then $Z_t^l\big(\mathbf{M}(G)^{**}\big)\cap\mathbf{M}(G/K)^{**} \subseteq \mathbf{M}(G/K)$.*

Proof. We will use induction on $\tau = \chi(G/K)$. The case where G/K is metrizable (i.e. $\chi(G/K) \leq \aleph_0$) has been settled in Theorem 23. Thus we can assume that G/K is non-metrizable and that the theorem holds with L in place of K for every $L\in\mathfrak{K}_\tau(G)$, where $\tau = \chi(G/K)$. Put $M_0 = \mathbf{M}_{\mathbf{ss}}(G,K)$, $M_1 = \mathbf{M}_{\mathbf{ai}}(G,K)$, $M_2 = \mathbf{M}(G/K)$ and $\widetilde{M_0} = \mathbf{M}_{\mathbf{ss},\tau}$, $\widetilde{M_1} = \mathbf{M}_{\mathbf{ai},\tau}$, $\widetilde{M_2} = \mathbf{M}_\tau$. Since $d\big(\mathbf{M}_{\mathbf{ss}}(G,K)\big) \leq |G/K|$, Corollaries 20 and 41 show that

$$Z_t^l\big(\mathbf{M}(G)^{**}\big)\cap\mathbf{M}(G/K)^{**} \subseteq \mathbf{M}_{\mathbf{ss}}(G,K)\oplus\mathbf{M}_{\mathbf{ai}}(G,K)^{**}.$$

Since $Z_t^l\big(\mathbf{M}(G)^{**}\big)\cap\mathbf{M}(G/K)^{**}$ is a linear space and $\mathbf{M}_{\mathbf{ss}}(G,K) \subseteq Z_t^l\big(\mathbf{M}(G)^{**}\big)$, it is enough to prove that $Z_t^l\big(\mathbf{M}(G)^{**}\big)\cap\mathbf{M}_{\mathbf{ai}}(G,K)^{**} \subseteq \mathbf{M}(G/K)$.
Take any $\mathfrak{m}\in Z_t^l\big(\mathbf{M}(G)^{**}\big)\cap\mathbf{M}_{\mathbf{ai}}(G,K)^{**}$. It is easy to see (using weak* approximation) that under the standard embedding of biduals (see §2) $\mathbf{M}(G/K)^{**} = \mathbf{M}(G)^{**}\Box\lambda_K$ holds. $Z_t^l\big(\mathbf{M}(G)^{**}\big)$ being a subalgebra, the inductive assumption implies that $\mathfrak{m}\Box\lambda_L = \mu_L \in \mathbf{M}(G/L)$ for all $L\in\mathfrak{K}_\tau(G)$. Let $\mu\in\mathbf{M}(G/K)$ be the measure obtained by restricting $\mathfrak{m}$ to $C_0(G/K)$ ($\subseteq\mathbf{M}(G/K)^*$). We get $\langle\mu_L, f\rangle = \langle\mathfrak{m}\Box\lambda_L, f\rangle = \langle\mu, \lambda_L\Box f\rangle = \langle\mu\star\lambda_L, f\rangle$ for $f\in C_0(G/K)$, hence $\mu_L = \mu\star\lambda_L$.
By Theorem 29 we have $\mu = \mu_0+\mu_1$, where $\mu_0\in\mathbf{M}_{\mathbf{ss}}(G,K)$, $\mu_1\in\mathbf{M}_{\mathbf{ai}}(G,K)$. By Corollary 44 there is a downwards directed family $\mathcal{D}$, $\mathfrak{K}_K(G) \subseteq \mathcal{D} \subseteq \mathfrak{K}_{\chi(G/K)}(G)$, such that either $\lim_{L\in\mathcal{D}}\langle h, \mu_0\star\lambda_L\rangle = 0$ for every $h\in\mathbf{M}(G)^*$ or $\lim_{L\in\mathcal{D}}\langle h, \mu_0\star\lambda_L\rangle$

does not exist for some $h \in \mathbf{M}(G)^\star$.

By Corollary 39, $(\lambda_L)_{L\in\mathcal{D}}$ is a right approximate unit for $\mathbf{M}_{\mathbf{ai}}(G,K)$. We have $\mathfrak{m} \in Z_t^l\big(\mathbf{M}_{\mathbf{ai}}(G,K)^{**}\big)$ because $\mathfrak{m} \in Z_t^l\big(\mathbf{M}(G)^{**}\big)$, and if $\mathfrak{n} \in \mathbf{M}_{\mathbf{ai}}(G,K)^{**}$ is any weak* accumulation point of the net $(\lambda_L)_{L\in\mathcal{D}}$ then $\mathfrak{m}\Box\mathfrak{n} = \mathfrak{m}$ (see [2] Prop. 2.9.16 and its proof). Then from $\mu \in \mathbf{M}(G) \subseteq Z_t^l\big(\mathbf{M}(G)^{**}\big)$ we get by approximation $\mathfrak{m} = \mu\Box\mathfrak{n}$. Since this holds for every accumulation point $\mathfrak{n}$, it follows that $\mathfrak{m} = \lim_{L\in\mathcal{D}} \mu \star \lambda_L$ (weak* limit).

By Corollary 39 we have $\mu_1 = \lim_{L\in\mathcal{D}} \mu_1 \star \lambda_L$ in the norm in $\mathbf{M}(G/K)$ and therefore also in the weak* topology in $\mathbf{M}(G/K)^{**}$. Hence the weak* limit $\lim_{L\in\mathcal{D}} \mu_0 \star \lambda_L$ exists, and thus $\lim_{L\in\mathcal{D}} \mu_0 \star \lambda_L = 0$ in the weak* topology in $\mathbf{M}(G/K)^{**}$. We conclude that $\mathfrak{m} = \lim_{L\in\mathcal{D}} \mu_0 \star \lambda_L + \lim_{L\in\mathcal{D}} \mu_1 \star \lambda_L = \mu_1$, which proves that $\mathfrak{m} \in \mathbf{M}(G/K)$. ☐

Theorem 3 now follows easily from Theorem 45.

Proof of Theorem 3. By Theorem 45 with $K = \{e_G\}$ we get $Z_t^l\big(\mathbf{M}(G)^{**}\big) = \mathbf{M}(G)$. As explained in the discussion following Theorem 3, Theorem 45 with G^{op} in place of G yields $Z_t^r\big(\mathbf{M}(G)^{**}\big) = \mathbf{M}(G)$. ☐

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