MAXIMAL FUNCTIONS AND THE ADDITIVITY OF VARIOUS FAMILIES OF NULL SETS

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ABSTRACT. It is shown to be consistent with set theory that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ planes in Euclidean 3-space whose union is not null. Similar results are obtained for circles in the plane as well as other geometric objects. The proof relies on results from harmonic analysis about the boundedness of certain maximal operators and a measure theoretic pigeonhole principle.

1. INTRODUCTION

Davies has shown [2] that any measurable subset of the plane can be covered by a family of lines whose union has the same measure as the set itself. A question in this same spirit, due to Peter Komjáth, asked [6] whether the assertion that every set of reals of size $\aleph_1$ is Lebesgue null implies that the union of any $\aleph_1$ lines in the plane is also Lebesgue null? This was a prime motivation behind the paper [11] which showed that, for any $\gamma < 1$ it is consistent that there is a set of reals of size $\aleph_1$ which is not null with respect to Hausdorff γ-measure but all sets of size $\aleph_1$ are null with respect to Lebesgue measure. It has been remarked that a Besicovitch duality argument can be used to obtain $\aleph_1$ lines in the plane whose union is not Lebesgue null from a set of reals of size $\aleph_1$ which is not null with respect to Hausdorff γ-measure for $\gamma > 1/2$, thus solving Komjath's question.

The argument uses the map L from $\mathbb{R}^2$ to lines in the plane defined by setting $L(a,b)$ to be the line with slope a and y-intercept b. For any set $S \subseteq \mathbb{R}^2$ and any angle θ, elementary arguments — such as those in §12.1 of [5] or §7.3 of [4] — show that the projection of S onto a line forming angle θ with the x-axis is a linear image of the intersection of $L(S) = \bigcup_{(a,b)\in S} L(a,b)$ with the vertical line intersecting the x-axis at $\tan(\theta)$. If $L(S)$ is a Lebesgue null set then there is a null G_δ set $A \supseteq L(S)$ and, furthermore, the set $A^* = \{(a,b) \mid L(a,b) \subseteq A\}$ is Borel, contains S and $L(A^*) \subseteq A$. Hence for almost all angles θ the projection of A^* onto the line forming angle θ with the x-axis is null. Since A^* is Borel, the Projection Theorem — Theorem 6.1 of [5] or Theorem 6.8 of [4] — implies that the Hausdorff dimension of A^*, and hence A, is less than 1. Therefore the model of [11] with $1/2 < \gamma < 1$ will solve Komjath's problem since Theorem 5.8 of [4] implies that the square of linear set of positive γ-Hausdorff measure will be a subset of the plane of positive 2γ-Hausdorff measure. Observe also that complex inversion — in other words, the map sending z to $1/z$ in the complex plane — sends lines to circles and preserves null sets. Therefore there are also $\aleph_1$ circles whose union is not null in this same model.

However, the geometric nature of duality arguments limits the generality one can expect to obtain from them. After all, Komjáth's question can be set in a much broader context by using the framework of small cardinals. For any family $\mathcal{B}$ of compact subsets of a Polish measure space (X,μ) define the cardinal $\mathrm{add}(\mathcal{B})$ to be the least cardinal of a subfamily $\mathcal{F} \subseteq \mathcal{B}$ such that $\cup\mathcal{F}$ is not μ-null. So, letting $\mathcal{S}$ be the points in $\mathbb{R}$ and $\mathcal{P}^n$ be the hyperplanes in $\mathbb{R}^n$, Komjáth's question becomes whether or not $\mathrm{add}(\mathcal{S}) = \mathrm{add}(\mathcal{P}^2)$. (Note that $\mathrm{add}(\mathcal{S})$ is the well known cardinal usually denoted by $\mathrm{non}(\mathcal{N}ull)$.) Letting $\mathcal{C}^n$ be the surfaces of spheres in $\mathbb{R}^n$ one can ask about the relationships between $\mathrm{add}(\mathcal{C}^n)$ and $\mathrm{add}(\mathcal{P}^k)$ as well as other similarly defined invariants. One can not hope to apply duality arguments to classes $\mathcal{B}$ which consist of much more than planes or spheres or simple geometric transformations of these objects. For example, let $f \subseteq \mathbb{R}^n$ be a smooth curve and let $\mathcal{B}$ consist of all isometric images of f. However, there are results in harmonic analysis which allow a solution to the generalized Komjath problem considerably broader than that provided by [11].

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These results are connected to the problem of reconstructing a measurable function from averages over small sets. The prototypical example here is the Lebesgue Density Theorem which establishes that the value of a measurable function at almost any point $x \in \mathbb{R}^k$ can be approximated by taking averages over small balls centred at x. If the goal is to approximate the value of the function by averaging over even smaller sets then, in many cases, this is also possible. For example Stein [12] showed how to do this for the surfaces of spheres in $\mathbb{R}^k$ for $k \geq 3$ and, later, Bourgain [1] did the same for $k = 2$. Similar results due to Falconer [3] and Marstrand [7] exist for hyperplanes in $\mathbb{R}^k$ for $k \geq 3$ as well. These results, as well as many similar ones, all follow from the boundedness of certain maximal operators associated with the families in question and, in all cases, one gets as a corollary that any set which contains many of the small sets — namely those over which the average is to be calculated — has positive Lebesgue measure.

The main result which will be proved, Theorem 2.1, has as a consequence the consistency of $\text{add}(\mathcal{B}) = \aleph_1 \neq \text{add}(\mathcal{S})$ so long as the boundedness of the maximal operator appropriate for the family $\mathcal{B}$ can be established. The major step in proving Theorem 2.1 will be to use the *norms on possibilities* technology developed in [10] to construct an ω^ω-bounding, proper partial order which forces the ground model reals to be a null set. In §2 the forcing partial order is defined and its main properties are established. The partial order $\mathbb{P}$ will consist of trees of approximations to a cover of the ground model by a null G_δ. The strategy for constructing $\mathbb{P}$ will require choosing a very quickly growing sequence of integers $\{M_n\}_{n=0}^\infty$ and letting $\mathcal{X}_n$ consist of all sets $X \subseteq M_n$ having size less than $M_n/2^n$. These will be used to code open sets of measure less than 2^{-n}. For each n an $\mathbb{N}$-valued norm ν will then be defined on $\mathcal{P}(\mathcal{X}_n)$ which will measure the size of subsets of $\mathcal{X}_n$. The trees belonging to $\mathbb{P}$ are then defined to be subtrees of $\prod_n^{<\infty} \mathcal{X}_n$ such that for each $t \in T$ and each integer k there is some $s \supseteq t$ such that $\nu\left(\Sigma_T(s)\right) \geq k$ where $\Sigma_T(s)$ is the set of successors of s. It will be easy to see that $\mathbb{P}$ forces the ground model reals to become a null set, but to see why, for example, the union of the ground model circles is not Lebesgue null after forcing with such a partial order requires looking at the construction of the norm ν using the boundedness of the appropriate maximal operator.

This construction is completed in §4 by proving a measure theoretic pigeonhole principle which may be of interest in its own right as well. In the simplest 1-dimensional case, the principle says that a measurable mapping of a product of probability spaces either maps most of a vertical section to small set, or, it maps most of the graph of a function to small set. The proof provided in §4 works only for finite probability spaces unless some additional axioms are assumed, but this is enough for the intended applications. The connection between this pigeonhole principle and the norm of §2 is established in §3 by introducing an intermediary norm which bridges the gap between maximal functions and pigeonhole principles.

The applications of the theorem, as well as the results from harmonic analysis upon which they are based, are stated in §5. Some final remarks and questions are also recorded in §5.

2. Defining the Partial Order

The study of the additivity properties of the families mentioned in the introduction will rely a great deal on the fact that these families are not only easily definable, but that they also have well behaved parameterizations by measure spaces. These parameterizations by measure spaces are crucial in that they enable arguments to use concepts such as *many circles* or *most lines*. Definition 2.1 describes the sort of parametrized families which will be studied.

Definition 2.1. Let (P, σ) be an atomless Polish probability space and $\text{Compact}(\mathbb{R}^n)$ be the space of compact subsets of $\mathbb{R}^n$ under the Hausdorff metric. Let λ be Lebesgue measure on $\mathbb{R}^k$ and μ be a regular measure on $\mathbb{R}^k$ such that the set

$$\left\{ C \subseteq \text{Compact}(\mathbb{R}^k) \mid \mu(C) > r \right\}$$

is Borel in $\text{Compact}(\mathbb{R}^n)$. A mapping Λ defined on P will be said to be *μ-appropriate* if:

(1) $\Lambda : P \to \text{Compact}(\mathbb{R}^n)$ is continuous.
(2) $1 \leq \inf_{p \in P} \mu(\Lambda(p)) \leq \sup_{p \in P} \mu(\Lambda(p)) < \infty$
(3) For every $\theta > 0$ and $\delta > 0$ if $J(\theta, \delta)$ is defined by

$$J(\theta, \delta) = \inf \left\{ \lambda\left(A\right) \mid A \subseteq \mathbb{R}^n \text{ is open and } \sigma(\{p \in P \mid \mu(\Lambda(p) \cap A) > \theta\}) > \delta \right\} \tag{2.1}$$

then $J(\theta, \delta) > 0$ and $\lim_{\delta\to 0} J(\theta,\delta) = 0$ for all θ.

In all the cases to be considered here the measure μ will be Hausdorff k-measure for some integer k. Observe that all Hausdorff k-measures μ satisfy the property that the set

$$A = \left\{(C, r) \;\middle|\; C \subseteq \mathbb{R}^k \text{ is compact and } \mu(C) > r\right\}$$

is Borel in Compact$(\mathbb{R}^n) \times \mathbb{R}$. To see this, observe that $(C, r) \notin A$ if and only if for each integer m there are $\{q_i\}_{i=1}^k \subseteq \mathbb{R}^n$ with rational co-ordinates and rational $\{t_i\}_{i=1}^k$ such that $0, t_i < 1/m$ for all i and $\sum_{i=1}^k t_i^s \leq r$ and

$$C \subseteq \bigcup_{i=1}^{k} B_{t_i}(q_i)$$

where $B_{t_i}(q_i)$ represents the open ball of diameter t_i and centre q_i. Compactness yields $\epsilon > 0$ such that the open ball around C in the Hausdorff metric with radius ϵ is contained in $\bigcup_{i=1}^k B_{t_i}(q_i)$. In other words, A is F_σ.

An example might be useful at this point.

Example 2.1. Let $(P, \sigma) = ([1, 2], \lambda)$ and let $\Lambda : P \to \text{Compact}(\mathbb{R}^k)$ be defined by letting $\Lambda(t)$ be the surface of the k-sphere of radius t with centre at the origin. In this case Λ is λ^2-appropriate where λ^2 is Hausdorff 2-measure, noting that this restricts to surface measure on each sphere. The fact that Conditions 1 and 2 are satisfied is clear. That Condition 3 holds can be seen by applying Fubini's Theorem in spherical co-ordinates.

The following two examples illustrating Definition 2.1, in spite of their similarity, will provide boundaries for the application of the methods to be developed; those methods will be applicable to Example 2.2, but not to Example 2.3.

Example 2.2. Let $(P, \sigma) = ((0, 1], \lambda)$ Let $\Lambda : P \to \text{Compact}(\mathbb{R}^3)$ be defined by letting $\Lambda(t)$ be the disc of radius 1 with centre at $(t, 0, 0)$ and perpendicular to the t-axis. The fact that the mapping Λ is λ^2-appropriate follows from an easy application of Fubini's Theorem.

Example 2.3. Let $(P, \sigma) = ((0, 1], \lambda)$ and let $\Lambda : P \to \text{Compact}(\mathbb{R}^2)$ be defined by letting $\Lambda(t)$ be the unit line segment with centre at $(t, 0)$ and perpendicular to the t-axis. Once again, Fubini's Theorem shows that the mapping Λ is λ^1-appropriate.

The following definition is needed to be able to apply results in the harmonic analysis literature to the context of Definition 2.1.

Definition 2.2. Let $(G, \cdot)$ be a locally compact group and let λ_G be its Haar measure. Let $*$ be a continuous left action of G on $\mathbb{R}^k$ that is measure preserving; in other words, $\lambda(A) = \lambda(g * A)$ for each measurable $A \subseteq \mathbb{R}^k$ and $g \in G$. For any μ-appropriate function $\Lambda : P \to \text{Compact}(\mathbb{R}^k)$ the maximal operator $M_{\Lambda,G}$ can be defined on measurable functions on $\mathbb{R}^k$ by

$$M_\Lambda(f)(g) = \sup_{p\in P} \int_{z\in\Lambda(p)} f(g * z)\, d\mu \tag{2.2}$$

The operator M_Λ will be said to be bounded if there are real numbers $q \geq 1$, $p \geq 1$ and a constant K such that

$$\|M_\Lambda(f)\|_p \leq K\|f\|_q \tag{2.3}$$

for all f which are the characteristic functions of bounded open sets in $\mathbb{R}^k$. The norm $\|\|_p$ is calculated with respect to λ_g of course.

Lemma 2.1. *If Λ is a μ-appropriate function and $(G, \cdot)$ is a locally compact group and $*$ a continuous left action of G on $\mathbb{R}^k$ then for every open set $U \subseteq \mathbb{R}^n$ the mapping $\Lambda^U : P \times G \to \mathbb{R}$ defined by $\Lambda^U(p, g) = \mu((g*\Lambda(p))\cap U)$ satisfies the following measurability condition: For every $r \in \mathbb{R}$*

$$\left\{(p, g) \in P \times G \;\middle|\; \Lambda^U(p, g) > r\right\} \tag{2.4}$$

is analytic.

Proof. Note that the inner regularity of μ implies that the set 2.4 is equal to the set of all (p, g) such that there exists $C \in \text{Compact}(\mathbb{R}^n)$ such that

1. $C \subseteq g * \Lambda(p)$
2. $C \subseteq U$
3. $\mu(C) > r$

and each of these conditions is described by a Borel set.

In particular, $C \not\subseteq g * \Lambda(p)$ if and only if there is $c \in C$ such that the Euclidean distance from c to $g * \Lambda(p)$ is greater than 0. Using the continuity of Λ and the continuity of the action $*$ it follows that there is an open neighbourhood V of (C, p, g) in the space $\text{Compact}(\mathbb{R}^n) \times P \times G$ such that $C' \not\subseteq g' * \Lambda(p')$ for each $(C', p', g') \in V$. In other words, if

$$A_1 = \{(C, p, g) \in \text{Compact}(\mathbb{R}^n) \times P \times G \mid C \subseteq g * \Lambda(p)\}$$

then A_1 is closed.

If $A_2 = \{(C, p, g) \in \text{Compact}(\mathbb{R}^n) \times P \times G \mid C \subseteq U\}$ then it is clear that A_2 is open and if

$$A_3 = \{(C, p, g) \in \text{Compact}(\mathbb{R}^n) \times P \times G \mid \mu(C) > r\}$$

then A_3 is Borel by the hypothesis of Definition 2.1.

Therefore the set 2.4 is equal to

$$\{(p, g) \in P \times G \mid (\exists C \in \text{Compact}(G))(C, p, g) \in A_1 \cap A_2 \cap A_3\}$$

establishing that the set 2.4 is analytic. □

Note also that Lemma 2.1 can not be strengthened to obtain that the mapping Λ^U is continuous. For example if $s = 0$, λ is the counting measure, $P = [0, 1]$ and $\Lambda(x) = \{x, x/2\}$ then $\lim_{x\to 0} \lambda(\Lambda(x)) = 2$ while $\lambda(\Lambda(0)) = 1$.

Defining the operator M_Λ only for those functions which are the characteristic functions of bounded open sets may seem somewhat contrived, but this unnatural formulation will be useful when applying the results of §5. It is therefore useful to observe that for any μ-appropriate function $\Lambda : P \to \text{Compact}(\mathbb{R}^k)$, $(G, \cdot)$ a locally compact group and $*$ a continuous, measure preserving left action of G on $\mathbb{R}^k$ the maximal operator M_Λ can defined for the characteristic functions of bounded open sets $U \subseteq \mathbb{R}^k$ by

$$M_\Lambda(U)(g) = \sup_{p \in P} \Lambda^U(p, g) \tag{2.5}$$

In this formulation it is easy to see that $M_\Lambda(U)$ is a measurable function. To see this observe that $M_\Lambda(U)(g) > r$ if and only if

$$(\exists p \in P)\Lambda^U(p, g) > r$$

and the set of all $g \in G$ satisfying this condition is analytic, and hence measurable, by Lemma 2.1. It follows that M_Λ is measurable and Inequality 2.3 is then reformulated as

$$\|M_\Lambda(U)\|_p \leq K\lambda(U)^{p/q} \tag{2.6}$$

Since broadening the scope of the definitions of the maximal operators M_Λ to functions other than characteristic functions of open sets would find no applications in the current work, the implications of the more natural definition will not be pursued here. It should be noted that often the definition of maximal operators involves normalizing the measure μ by dividing by $\mu(\Lambda(p))$ in Equation 2.2. This plays no role here because only the boundedness of these operators is of interest and Condition 2 of Definition 2.1 is always in force. Also, the supremum is usually taken over all reals but once again, since only boundedness is of interest here, this is not significant.

Proposition 2.1. *If $\Lambda : P \to$ Compact$(\mathbb{R}^k)$ and $(G, \cdot)$ is a locally compact group and $*$ a continuous left action of G on $\mathbb{R}^k$ then for every $\epsilon > 0$ and open $A \subseteq [0, 1]^k$ the following weak type (p, q) inequality holds:*

$$\lambda_G\left(\left\{g \in G \mid (\exists x \in P)\Lambda^A(x, g) \geq \epsilon\right\}\right) \leq \frac{K\lambda(A)^{p/q}}{\epsilon^p}. \tag{2.7}$$

provided that Λ is s-appropriate and M_Λ satisfies Inequality 2.6.

Proof. It follows from Inequality 2.6 that

$$\int_{g\in G} \sup_{x\in P}(\Lambda^A(x,g))^p d\lambda \leq K\lambda(A)^{p/q} \tag{2.8}$$

and using that

$$\lambda\left(\left\{g\in G \mid (\exists x\in P)\Lambda^A(x,g)\geq\epsilon\right\}\right)\epsilon^p \leq \int_{g\in G}\sup_{x\in P}(\Lambda^A(x,g))^p d\lambda \tag{2.9}$$

the result follows. ☐

In the arguments to follow, Inequality 2.7 will play a central role and the boundedness of maximal operators will merely be used to conclude that Inequality 2.7 holds. Indeed, Marstrand obtains equivalent inequalities in [7] and [8] without resorting to maximal operators. For the remainder of this section, as well §3, let (P,σ) be an atomless, Polish probability space and $\Lambda : P \to \mathbb{R}^k$ be either s-appropriate or a parametrized family of s-appropriate functions and let $J : (0,1)^2 \to (0,1)$ be as in Definition 2.1. If Λ is a parametrized family of s-appropriate functions then let (P_0,σ_0) and (P_1,σ_1) be Polish probability spaces as required in Definition **??** such that $P = P_0 \times P_1$ and $\sigma = \sigma_0 \times \sigma_1$. Assume also that Inequality 2.7 holds for K, q' and q. For example, this will hold if the operator M_Λ is bounded.

Notation 2.1. If $A \subseteq X \times Y$ and $x \in X$ then $A\langle x\rangle = \{y \in Y \mid (x,y) \in A\}$.

Definition 2.3. Let $\{\epsilon(j)\}_{j=1}^{\infty}$ be a descending sequence of positive reals such that $\epsilon(1) < 1$ and for each $j \in \mathbb{N}$ and any two probability spaces (Q_0,μ_0) and (Q_1,μ_1)

$$\mu_0\left(\{x\in Q_0 \mid \mu_1(A\langle x\rangle) > 2\epsilon(j)\}\right) > 2j^{q'}K^{q'}J(1/j,\epsilon(j))^{q'/q} \tag{2.10}$$

for all $A \subseteq Q_0 \times Q_1$ such that $\mu_0 \times \mu_1(A) \geq 1/(j+1)$. Let $J^*(j) = J(1/j,\epsilon(j))$.

Definition 2.4. For $M \in \mathbb{N}$ and $\xi > 0$ define $\mathcal{P}_\xi(M)$ to be the set of all $X \subseteq M$ such that $|X| < \xi M$. Define a function $\Theta : \mathcal{P}(\mathcal{P}_\xi(M)) \to \mathcal{P}(\mathbb{N})$ by specifying inductively, simultaneously for all $\mathcal{Y} \subseteq \mathcal{P}_\xi(M)$, which integers belong to $\Theta(\mathcal{Y})$.

- For all $\mathcal{Y} \subseteq \mathcal{P}_\xi(M)$, $0 \in \Theta(\mathcal{Y})$ if and only if $\mathcal{Y} \neq \emptyset$.
- For all $\mathcal{Y} \subseteq \mathcal{P}_\xi(M)$, $1 \in \Theta(\mathcal{Y})$ if and only if $\cup\mathcal{Y} = M$.
- For all $\mathcal{Y} \subseteq \mathcal{P}_\xi(M)$, if $j \geq 1$ then $j+1 \in \Theta(\mathcal{Y})$ if and only if for every function F from $\mathcal{Y}$ to the open subsets of $\mathbb{R}^k$ of measure less than $J^*(j)$

  $$\lambda\times\sigma\left(\left\{(x,p)\in[0,1]^k\times P \mid j\notin\Theta(\langle x,p,F\rangle_j)\right\}\right) \leq 1/(j+1) \tag{2.11}$$

  where $\langle x,t,H\rangle_n$ is defined to be $\left\{Y \mid \Lambda^{H(Y)}(t,x) \leq 1/n\right\}$ for $t \in P$, $x \in [0,1]^k$, any integer n and any partial function H from $\mathcal{P}(\mathcal{P}_\xi(M))$ to the open subsets of $\mathbb{R}^k$. Observe that the set of Inequality 2.11 is measurable by Lemma 2.1.

Define $\nu(\mathcal{Y}) = j$ if $j+1$ is the least integer such that $j+1 \notin \Theta(\mathcal{Y})$.

Observe that if $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$ then $\nu(\mathcal{Y}_1) \leq \nu(\mathcal{Y}_2)$ and if $j+1 \in \Theta(\mathcal{Y})$ then $j \in \Theta(\mathcal{Y})$. Now note that since $1/(j+1) < 1$ for each $j \geq 1$ it follows that if $\nu(\mathcal{Y}) \geq j+1$ then, letting F be the function defined by $F(Y) = \emptyset$, it follows that there is some pair (x,p) such that $\nu(\langle x,p,F\rangle_j) \geq j$. Applying this successively yields a subset $\mathcal{Z} \subseteq \mathcal{Y}$ such that $\nu(\mathcal{Z}) \geq 1$ or, in other words, $\cup\mathcal{Z} = M$.

At this point it is not even clear whether there is any integer M such that $\nu(\mathcal{P}_{1/2}(M)) > 2$. This, and much more, will be resolved in §4. For the moment, however, the following is stated as a hypothesis to be justified later in Corollary 4.1.

$$(\forall n\in\mathbb{N})(\exists M(n)\in\mathbb{N})\ \nu\left(\mathcal{X}_n\right) > n \tag{2.12}$$

where $\mathcal{X}_n = \mathcal{P}_{2^{-n}}(M(n))$. Let $T = \prod_{n=0}^{\infty}\mathcal{X}_n$ and let $T^{<\infty} = \bigcup_{i=0}^{\infty}\prod_{n=0}^{i}\mathcal{X}_n$. For $t \in T^{<\infty}$ let $[t]$ denote the open subset of T defined by $[t] = \{c \in T \mid t \subseteq c\}$. Now for any closed $C \subseteq T$ let $C^{<\infty} = \{t \in T^{<\infty} \mid [t] \cap C \neq \emptyset\}$ and, for $t \in C^{<\infty}$ let $\Sigma_C(t)$ be the immediate successors of t in $C^{<\infty}$; in other words,

$$\Sigma_C(t) = \left\{X \in \mathcal{X}_{|t|} \mid (\exists c \in C)t \subseteq c \text{ and } c(|t|) = X\right\}.$$

Let $\mathbb{P}$ consist of all closed subsets of $C \subseteq T$ such that for all $t \in C^{<\infty}$ and for each $j \in \mathbb{N}$ there is some $s \in C^{<\infty}$ such that $t \subseteq s$ and $\nu(\Sigma_C(s)) \geq j$. For an ordinal ξ let $\mathbb{P}_\xi$ be the countable support product of ξ copies of $\mathbb{P}$ and let $\leq$ be the coordinate-wise ordering on $\mathbb{P}_\xi$. Standard fusion arguments show that $\mathbb{P}_\xi$ partially ordered by $\leq$ is proper. Before continuing it will be observed that if κ is a regular cardinal then

$$1 \Vdash_{\mathbb{P}_\kappa} \text{``}\operatorname{add}(\mathcal{S}) = \operatorname{non}(\mathcal{N}ull) \geq \kappa\text{''}$$

and, indeed, this follows from the next lemma.

Lemma 2.2. *If $V \subseteq W$ are both models of set theory then*

$$1 \Vdash_{\mathbb{P}\cap V} \text{``}W \cap \mathbb{R} \textit{ is Lebesgue null''}$$

in the model W.

Proof. For $A \in \mathcal{X}_n$ define

$$A^* = \bigcup_{i \in A} \left[\frac{i}{M(n)}, \frac{i+1}{M(n)}\right]$$

and let

$$S_m(G) = \bigcup_{n=m+1}^{\infty} G(n)^*$$

where G is a name for the $\mathbb{P}$ generic real. It follows that the Lebesgue measure of $S_m(G)$ is no greater than $1/2^m$ and it suffices to show that given $C \in \mathbb{P} \cap V$, $m \in \mathbb{N}$ and $x \in [0,1]$ in the model W there is some $C' \subseteq C$ such that $C' \in \mathbb{P} \cap V$ and $C' \Vdash_{\mathbb{P}\cap V}$ "$x \in S_m(G)$". Choose $t \in C^{<\infty}$ such that $\nu(\Sigma_C(t)) \geq 1$ and $|t| \geq m$. Then, since $\cup\Sigma_C(t) = M(|t|)$ and this is absolute, it is possible to select $X \in \Sigma_C(t)$ such that $x \in X^*$. Then let $C' = C \cap [t^\frown X]$. □

Corollary 2.1. *If κ is a regular cardinal then $1 \Vdash_{\mathbb{P}_\kappa}$ "$\operatorname{add}(\mathcal{S}) = \operatorname{non}(\mathcal{N}ull) \geq \kappa$".*

Proof. Given $G \subseteq \mathbb{P}_\kappa$ generic over V and $X \subseteq \mathbb{R}$ of cardinality less than κ in $V[G]$, the properness of $\mathbb{P}_\kappa$ guarantees that there is $\xi \in \kappa$ such that $X \in W = V[G \cap \mathbb{P}_\xi]$. Lemma 2.2 now applies. □

In the following, note that the continuity of Λ ensures that Λ is defined in any model of set theory so that $\Lambda(p)$ is always interpreted as a compact set.

Theorem 2.1. *For any cardinal κ*

$$1 \Vdash_{\mathbb{P}_\kappa} \text{``} \bigcup_{x \in V \cap \mathbb{R}^k} \bigcup_{v \in V \cap P} (\Lambda(v) + x) \textit{ is not Lebesgue null''}$$

Proof. The proof will be presented in a series of steps, each adding another level of complexity.

To begin, it will be assumed that $\kappa = 1$ and Λ is s-appropriate. If the theorem is false then, without any loss of generality, there is $p \in \mathbb{P}$ such that

$$p \Vdash_{\mathbb{P}} \text{``} \bigcup_{x \in V \cap \mathbb{R}^k} \bigcup_{v \in V \cap P} (\Lambda(v) + x) \subseteq \bigcup_{i=0}^{\infty} \prod_{j=1}^{k} (q_{j,i}, q_{j,i} + r_i) \text{ and } \sum_{i=0}^{\infty} r_i^k < J(1/2, 1/3)\text{''}$$

where it may be assumed that r_i and $q_{j,i}$ are names for rationals. The cube $\prod_{j=1}^k (q_{j,i}, q_{j,i}+r_i)$ will be represented by B_i.

Now construct by induction on n conditions $p_n \in \mathbb{P}$, integers N_n and $K_n \geq 1$ and $A_n \subseteq T^{<\infty}$ for $i \leq n$ such that:

1. $p_0 \leq p$ and $p_{n+1} \leq p_n$ for each n
2. $\{[t]\}_{t \in A_n}$ is a finite open cover of p_n by disjoint non-empty sets
3. if $t \in A_n$ then $|t| > n$
4. $\Sigma_{p_n}(t) = \Sigma_{p_m}(t)$ for each $m \geq n$ and for each $t \in A_n$
5. $\nu(\Sigma_{p_n}(t)) \geq K_n + 1 \geq n$ for each $t \in A_n$
6. there is an open set W such that $p_0 \Vdash$ "$\bigcup_{i=0}^{N_0} B_i = \check{W}$"

(7) for each $t \in A_n$ and $Y \in \Sigma_{p_n}(t)$ there is an open set $W_{t^\frown Y}$ such that

$$p_{n+1} \cap [t^\frown Y] \Vdash \text{“} \bigcup_{i=N_n+1}^{N_{n+1}} B_i = \check{W}_{t^\frown Y}\text{”}$$

(8) $p_n \Vdash$ “$\displaystyle\sum_{i=N_n+1}^{\infty} r_i^k < J^*(K_n)$”

(9) $\displaystyle\sum_{i=0}^{n} \frac{|A_i|}{K_i+1} < 1/2$.

Given that this can be done, define $p_\omega = \bigcap_{n=0}^{\infty} p_n$ and note that each A_n is a maximal antichain in $p_\omega^{<\infty}$. By Induction Hypotheses 2, 4 and 5 it follows that $p_\omega \in \mathbb{P}$.

For each n and $t \in A_n$ and $Y \in \Sigma_{p_\omega}(t)$ define $F_t(Y) = W_{t^\frown Y}$ and note that $\lambda(F_t(Y)) < J^*(K_n)$. Since $\nu(\Sigma_{p_\omega}(t)) = \nu(\Sigma_{p_n}(t)) \geq K_n + 1$ for $t \in A_n$ it follows that if S_t is defined to be

$$\left\{(x,v) \in [0,1]^k \times P \mid \nu(\langle x, v, F_t\rangle_{K_n}) < K_n\right\} \tag{2.13}$$

then $\lambda \times \sigma(S_t) \leq 1/(K_n+1)$. It follows that

$$\lambda \times \sigma\left(\bigcup_{n=0}^{\infty} \bigcup_{t\in A_n} S_t\right) \leq \sum_{n=0}^{\infty} \frac{|A_n|}{K_n+1} \leq 1/2$$

by Induction Hypothesis 9. Therefore it is possible to choose $x^* \in [0,1]^k$ such that

$$\sigma\left(\left\{v \in P \;\middle|\; (x^*, v) \notin \bigcup_{n=0}^{\infty} \bigcup_{t\in A_n} S_t\right\}\right) > 1/3 \tag{2.14}$$

Now refer to Definition 2.1 to choose $v^* \in P$ such that

$$(g^*, v^*) \in G \times P \setminus \left(\bigcup_{n=0}^{\infty} \bigcup_{t\in A_n} S_t\right) \tag{2.15}$$

and, using the inequality $\lambda(W) < J(1/2, 1/3)$ of Induction Hypothesis 6 and the fact that the action $*$ is measure preserving, such that

$$\lambda^s(W \cap (g^* * \Lambda(v^*))) \leq 1/2. \tag{2.16}$$

Now let $p^* \leq p_\omega$ be defined by

$$p^* = \{c \in p_\omega \mid (\forall n)(\forall m) \text{ if } c \restriction m \in A_n \text{ then } c(m) \in \langle x^*, v^*, F_{c\restriction m}\rangle_{K_n}\} \tag{2.17}$$

and note that $p^* \in \mathbb{P}$ since (x^*, v^*) satisfies Condition 2.15. Letting G be the canonical name for the generic real obtained from $\mathbb{P}$, let g_n be a name for the unique element of A_n such that $g_n \subseteq G$ and note that

$$p^* \Vdash \text{“}\bigcup_{i=0}^{\infty} B_i \subseteq W \cup \bigcup_{n=0}^{\infty} W_{g_n^\frown G(|g_n|)}\text{”}$$

by Condition 7 and so

$$p^* \Vdash \text{“}\lambda^s\left((\Lambda(v^*) + x^*) \cap \left(\bigcup_{i=0}^{\infty} B_i\right)\right) \leq \sum_{n=0}^{\infty} \lambda^s\left((\Lambda(v^*) + x^*) \cap (F_{g_n}(G(|g_n|)))\right) + \lambda^s((\Lambda(v^*) + x^*) \cap W)\text{”}. \tag{2.18}$$

Observe that by Inequality 2.16 it follows that the last summand in Inequality 2.18 is less than 1/2. From Definition 2.17 of p^* it follows that $G(|g_n|) \in \langle x^*, v^*, F_{g_n}\rangle_{K_n}$ for all n and hence

$$\lambda^s\left((\Lambda(v^*) + x^*) \cap (F_{g_n}(G(|g_n|)))\right) = \Lambda^{F_{g_n}(G(|g_n|))}(v^*, x^*) \leq 1/K_n.$$

Therefore

$$p^* \Vdash \text{“}\lambda^s\left((\Lambda(v^*) + x^*) \cap \left(\bigcup_{i=0}^{\infty} B_i\right)\right) \leq \sum_{i=0}^{\infty} 1/K_i + 1/2 < 1\text{”}$$

by Induction Hypothesis 9. It follows from the hypothesis of Definition 2.1 that $\lambda^s(\Lambda(v^*)) \geq 1$ and therefore the fact that $p^* \Vdash$ “$\Lambda(v^*) + x^* \not\subseteq \bigcup_{i=0}^{\infty} B_i$” contradicts that (x^*, v^*) belongs to V.

In order to see that the induction can be carried out, begin by letting $K_0 = 2$ so that Induction Hypothesis 9 is satisfied. Next, choose $p_0^1 \leq p$ and N_0 such that

$$p_0^1 \Vdash \text{“}\sum_{i=N_0+1}^{\infty} r_i^k < J^*(K_0)\text{”}.$$

Then choose $p_0^2 \leq p_0^1$ such that $p_0^2 \Vdash$ “$\bigcup_{i=0}^{N_0} B_i = \check{W}$” for some open set W. Let $t \in (p_0^2)^{<\infty}$ be such that $\nu(\Sigma_{p_0^2}(t)) > K_0 + 1$ and let $p_0 = p_0^2 \cap [t]$ and let $A_0 = \{t\}$. Induction Hypothesis 6 is then satisfied. Induction Hypotheses 4, 7 and 8 are not relevant at this stage.

Now suppose that N_n, p_n, K_n and $\{A_n\}_{i=0}^{n}$ have been constructed satisfying the induction hypotheses and select $K_{n+1} \geq n + 1$ so large that

$$\sum_{i=0}^{n} \frac{|A_i|}{K_i + 1} + \frac{\sum_{t \in A_n} |\Sigma_{p_n}(t)|}{K_{n+1} + 1} < 1/2. \tag{2.19}$$

Choose $p_n^1 \leq p_n$ such that for each $t \in A_n$ and $Y \in \Sigma_{p_n}(t)$ there is some $N_{t^\frown Y}$ such that

$$p_n^1 \cap [t^\frown Y] \Vdash \text{“}\sum_{i=N_{t^\frown Y}}^{\infty} r_i^k < J^*(K_{n+1})\text{”} \tag{2.20}$$

and let $N_{n+1} = \max\{N_{t^\frown Y} \mid t \in A_n \text{ and } Y \in \Sigma_{p_n}(t)\}$. Then choose $p_n^2 \leq p_n^1$ such that for each $t \in A_n$ and $Y \in \Sigma_{p_n}(t)$ there is an open set $W_{t^\frown Y}$ such that

$$p_n^2 \cap [t^\frown Y] \Vdash \text{“}\bigcup_{i=N_n}^{N_{n+1}} B_i = \check{W}_{t^\frown Y}\text{”}.$$

For each $t \in A_n$ and $Y \in \Sigma_{p_n}(t)$ select $\tau(t, Y) \in (p_n^2)^{<\infty}$ such that $t^\frown Y \subseteq \tau(t, Y)$ and $\nu(\Sigma_{p_n^2}(\tau(t, Y))) \geq K_{n+1} + 1$ and such that $|\tau(t, Y)| > n + 1$ so that Induction Hypothesis 3 will hold if A_{n+1} is defined to be $\{\tau(t, Y) \mid t \in A_n \text{ and } Y \in \Sigma_{p_n}(t)\}$. Then let

$$p_{n+1} = \bigcup \{p_n^2 \cap [\tau(t, Y)] \mid t \in A_n \text{ and } Y \in \Sigma_{p_n}(t)\}$$

guaranteeing that Induction Hypothesis 5 holds.

To see that Induction Hypothesis 8 holds use Condition 2.20. Condition 2.19 guarantees that Induction Hypothesis 9 holds and everything else is immediate from the construction.

Finally, no restriction will be imposed on κ. It suffices to assume that $\kappa = \omega$. For $p \in \mathbb{P}_\omega$, $n \in \omega$ and $\theta : n \to T^{<\infty}$ define

$$p[\theta] = \begin{cases} p(j) & \text{if } j \geq n \\ p(j) \cap [\theta(j)] & \text{if } j \in n \end{cases}$$

noting, of course, that $p[\theta]$ may not belong to $\mathbb{P}_\omega$ in general.

If the theorem is false then, as in the previous cases, there is $p \in \mathbb{P}_\omega$ such that

$$p \Vdash_{\mathbb{P}_\omega} \text{“}\bigcup_{x \in V \cap [0,1]^k} \bigcup_{v \in V \cap P} (\Lambda(v) + x) \subseteq \bigcup_{i=0}^{\infty} B_i \text{ and } \sum_{i=0}^{\infty} r_i^k < J(1/2, 1/3)\text{”}$$

where, as before, B_i is the cube $\prod_{j=1}^{k} (q_{j,i}, q_{j,i} + r_i)$.

Fix a function $e : \mathbb{N} \to \mathbb{N}$ such that the pre-image of each integer under e is infinite and such that $e(i) \leq i$ for each $i \in \mathbb{N}$. Now construct by induction on n conditions $p_n \in \mathbb{P}_\omega$, integers N_n and $K_n \geq 1$ and $A_n^{e(i)} \subseteq T^{<\infty}$ for $i \leq n$ such that

1. $p_0 \leq p$ and $p_{n+1} \leq p_n$ for each n
2. $\{[t]\}_{t \in A_n^i}$ is a finite open cover of $p_n(i)$ by disjoint non-empty sets if A_n^i is defined
3. if $0 \leq i \leq n$ and $i \neq e(n+1)$ then $A_n^i = A_{n+1}^i$ if A_n^i is defined
4. if $t \in A_n^{e(n)}$ then $|t| > n$
5. $\Sigma_{p_n(i)}(t) = \Sigma_{p_m(i)}(t)$ for each $m \geq n$ and for each $t \in A_n^i$ — as a consequence of this it is possible to define $\Sigma(A_n^i) = \left\{ t^\frown X \;\middle|\; t \in A_n^i \text{ and } X \in \Sigma_{p_n(i)}(t) \right\}$ and have that $\Sigma(A_n^i) = \left\{ t^\frown X \;\middle|\; t \in A_n^i \text{ and } X \in \Sigma_{p_m(i)}(t) \right\}$ for any $m \geq n$
6. $\nu(\Sigma_{p_n(e(n))}(t)) \geq K_n + 1 \geq n$ for each $t \in A_n^{e(n)}$
7. there is an open set W such that $p_0 \Vdash$ "$\bigcup_{i=0}^{N_0} B_i = \check{W}$"
8. for each $\theta \in \prod_{i=0}^{n} \Sigma(A_n^i)$ there is an open set W_θ such that $p_{n+1}[\theta] \Vdash$ "$\bigcup_{i=N_n+1}^{N_{n+1}} B_i = \check{W}_\theta$" using the convention that $\Sigma(A_n^i) = \{\emptyset\}$ in case A_n^i is not defined
9. $p_n \Vdash$ "$\displaystyle\prod_{i=0}^{e(n)-1} \left|\Sigma(A_n^i)\right| \prod_{i=e(n)+1}^{n} \left|\Sigma(A_n^i)\right| \sum_{i=N_n+1}^{\infty} r_i^k < J^*(K_n)$"
10. $\displaystyle\sum_{i=0}^{n} \frac{2|A_i^{e(i)}|}{K_i} < 1/2.$

Given that this can be done, define p_ω by setting $p_\omega(i) = \bigcap_{n=0}^{\infty} p_n(i)$ and note that each A_n^i is a maximal antichain in $p_\omega(i)$. As before, the induction hypotheses and the choice of the enumerating function e yields that $p_\omega(i) \in \mathbb{P}$ for each i and so $p_\omega \in \mathbb{P}_\omega$.

For each n and $t \in A_n^{e(n)}$ and $Y \in \Sigma_{p_\omega(e(n))}(t)$ and

$$\theta \in \prod_{i=0}^{e(n)-1} \Sigma(A_n^i) \times \prod_{i=e(n)+1}^{n} \Sigma(A_n^i)$$

define $\theta * (t, Y) \in \prod_{i=0}^{n} \Sigma(A_n^i)$ by

$$\theta * (t, Y)(i) = \begin{cases} \theta(i) & \text{if } i \neq e(n+1) \\ t^\frown Y & \text{if } i = e(n+1) \end{cases}$$

and then define

$$F_t(Y) = \bigcup \left\{ W_{\theta * (t,Y)} \;\middle|\; \theta \in \prod_{i=0}^{e(n)-1} \Sigma(A_n^i) \times \prod_{i=e(n)+1}^{n} \Sigma(A_n^i) \right\} \tag{2.21}$$

and note that $\lambda(F_t(Y)) < J^*(K_n)$.

Since $\nu(\Sigma_{p_\omega(e(n))}(t)) = \nu(\Sigma_{p_n(e(n))}(t)) \geq K_n + 1$ for $t \in A_n^{e(n)}$ it follows that if S_t is defined by Expression 2.13 then $\lambda \times \sigma(S_t) \leq 1/(K_n + 1)$ and the proof then proceeds as in the previous cases. □

3. AN INTERMEDIATE NORM

In order to establish the validity of Hypothesis 2.12 an intermediate family of norms will be introduced and an inequality will be proved between these new norms and those of §2. It will then be shown in §4 that the analogue of Hypothesis 2.12 holds for these new norms. This will be enough to conclude that Hypothesis 2.12 itself holds. For the next definition recall Notation 2.1.

Definition 3.1. Let (Q_1, μ_1) and (Q_2, μ_2) be probability spaces and $\gamma > 0$. For A and B measurable subsets of $(Q_1 \times Q_2)^k$ define $A \Subset_\gamma B$ by induction on the integer k. If $k = 1$ then $A \Subset_\gamma B$ if and only if $A \subseteq B$ and either

$$(\exists p \in Q_1) B\langle p\rangle \neq \emptyset \text{ and } \mu_2(B\langle p\rangle \setminus A\langle p\rangle) < \gamma \tag{3.1}$$

or

$$\mu_1(\{p \in Q_1 \mid A\langle p\rangle = \emptyset \neq B\langle p\rangle\}) < \gamma. \tag{3.2}$$

If $k > 1$ then $A \Subset_\gamma B$ if and only if $A \subseteq B$ and

$$\{x \in Q_1 \times Q_2 \mid A\langle x\rangle \Subset_\gamma B\langle x\rangle \neq \emptyset\} \Subset_\gamma \pi(B). \tag{3.3}$$

where $\pi(B) = \{x \in Q_1 \times Q_2 \mid B\langle x\rangle \neq \emptyset\}$. Note that the first use of $\Subset_\gamma$ uses the inductive hypothesis for the $(k-1)$-fold product.

In the case $k = 1$ it is useful to think of the relation $A \Subset_\gamma B$ as saying that either A contains all but a set of measure γ of a vertical section of B or there is a function contained in A whose domain is all but a set of measure γ of the domain of B. If $k > 1$ then the definition of $A \Subset_\gamma B$ is more complicated because of the Fubini type product used in the inductive definition.

Definition 3.2. Given $M \in \mathbb{N}$, $\gamma > 0$, $\xi > 0$ and $\mathcal{Y} \subseteq \mathcal{P}_\xi(M)$ define a set $\Theta_\gamma(\mathcal{Y}) \subseteq \mathbb{N}$ by defining which $j \in \mathbb{N}$ belong to $\Theta_\gamma(\mathcal{Y})$.

- $0 \in \Theta_\gamma(\mathcal{Y})$ if and only if $\mathcal{Y} \neq \emptyset$.
- $1 \in \Theta_\gamma(\mathcal{Y})$ if and only if $\cup\mathcal{Y} = M$.
- If $j \geq 1$ then $j+1 \in \Theta_\gamma(\mathcal{Y})$ if and only if for every pair of finite probability spaces (Q_1, μ_1) and (Q_2, μ_2) and any $A \subseteq (Q_1 \times Q_2)^j$ and $\Psi : A \to M$ there is $Y \in \mathcal{Y}$ such that $\Psi^{-1}Y \Subset_\gamma A$.

Define $\rho_\gamma(\mathcal{Y}) = j$ if $j+1$ is the least integer that does not belong to $\Theta_\gamma(\mathcal{Y})$. Observe that if $\mathcal{Y}_1 \subseteq \mathcal{Y}_2$ then $\Theta_\gamma(\mathcal{Y}_1) \leq \Theta_\gamma(\mathcal{Y}_2)$ and if $j+1 \in \Theta_\gamma(\mathcal{Y})$ then $j \in \Theta_\gamma(\mathcal{Y})$.

For technical reasons, a superset of $\Theta_\gamma(\mathcal{Y})$ will be needed. If $\delta > 0$ and $L \in \mathbb{N}$ then a probability space will be said to be (δ, L)-fine if and only if every set of measure greater than δ has cardinality at least L. $\Theta_\gamma^{\delta,L}(\mathcal{Y})$ is defined the same way as $\Theta_\gamma(\mathcal{Y})$ except that if $j \geq 1$ then $j+1 \in \Theta_\gamma^{\delta,L}(\mathcal{Y})$ if and only if for every pair of finite (δ, L)-fine probability spaces (Q_1, μ_1) and (Q_2, μ_2) and any $A \subseteq (Q_1 \times Q_2)^j$ and $\Psi : A \to M$ there is $Y \in \mathcal{Y}$ such that $\Psi^{-1}Y \Subset_\gamma A$.

Observe that, while there are similarities to Θ of Definition 2.4 there is a crucial difference: The definition of $\Theta(\mathcal{Y})$ is inductive while the definition of $\Theta_\gamma(\mathcal{Y})$ is not. (Of course, this relies on the inductive definition of $\Subset_\gamma$.) Nevertheless, the two sets are linked, as will be seen in the next result, Lemma 3.3. Keep in mind that Λ, J, K and $\{\epsilon(i)\}_{i=0}^\infty$ are still fixed as in §2. Before proceeding to the proof of Lemma 3.3 the observation that Definitions 3.1 and 3.2 have equivalent martingale versions will turn out to be useful.

Definition 3.3. For a fixed integer k recall that a martingale is a family of finite probability spaces indexed by sequences of length less than or equal to k and denoted $\mathfrak{M}(t) = (M_t, \sigma_t)$ for t a sequence of length less than or equal to k. The integer k will be called the length of the martingale. (For the purposes of this section $\mathfrak{M}(t)$ will always be a product of two spaces, but this can be suppressed for the moment.) The probability space $S(\mathfrak{M})$ is defined by induction on k.

If $k = 0$ the space is $S(\mathfrak{M}) = \mathfrak{M}(\emptyset)$. If $k > 0$ then define $\mathfrak{M}_x(t) = \mathfrak{M}(x^\frown t)$ for t of length less than k. Then $S(\mathfrak{M}) = (M_\mathfrak{M}, \sigma_\mathfrak{M})$ where $M_\mathfrak{M}$ is the set of pairs (x, y) such $x \in M_\emptyset$ and $y \in M_{\mathfrak{M}_x}$ with the measure $\sigma_\mathfrak{M}$ defined by

$$\sigma_\mathfrak{M}(X) = \sum_{x \in \mathfrak{M}(\emptyset)} \sigma_\emptyset(\{x\}) \sigma_{\mathfrak{M}_x}(X\langle x\rangle)$$

for $X \subseteq M_\mathfrak{M}$.

Definition 3.4. Let a martingale of length k of products of probability spaces and denoted by $\mathfrak{M}(t) = (M_t, \sigma_t)$ be given. For A and B measurable subsets of $M_\mathfrak{M}$ define $A \Subset_\gamma^\mathfrak{M} B$ by induction on the integer k. If $k = 0$ then $A \Subset_\gamma^\mathfrak{M} B$ is defined exactly as in Definition 3.1. If $k \geq 1$ then $A \Subset_\gamma^\mathfrak{M} B$ if and only if $A \subseteq B$ and

$$\left\{x \in M_\emptyset \;\middle|\; A\langle x\rangle \Subset_\gamma^{\mathfrak{M}_x} B\langle x\rangle \neq \emptyset\right\} \Subset_\gamma \pi(B) \tag{3.4}$$

where $\pi(B)$ is defined in analogy with Definition 3.1 and the notation $A\langle x\rangle$ is now used to denote all $t \in M_{\mathfrak{M}}$ such that $t(0) = x$.

Definition 3.5. Just as in Definition 3.2 let $\bar{M} \in \mathbb{N}$, $\gamma > 0$, $\xi > 0$ and $\mathcal{Y} \subseteq \mathcal{P}_\xi(\bar{M})$ be given and define a set $\Theta^+_\gamma(\mathcal{Y}) \subseteq \mathbb{N}$ by defining which $j \in \mathbb{N}$ belong to $\Theta^+_\gamma(\mathcal{Y})$. The definition for 0 and 1 is exactly the same as in Definition 3.2. If $j \geq 1$ then $j + 1 \in \Theta_\gamma(\mathcal{Y})$ if and only if for every martingale $\mathfrak{M}$ of length j of products of pairs of probability spaces and any $A \subseteq M_{\mathfrak{M}}$ and $\Psi : A \to \bar{M}$ there is $Y \in \mathcal{Y}$ such that $\Psi^{-1}Y \Subset^{\mathfrak{M}}_\gamma A$. A version of this definition restricted to martingales consisting of (δ, L)-fine probability spaces will also be needed.

Lemma 3.1. *For any $\bar{M} \in \mathbb{N}$, $\gamma > 0$, $\xi > 0$ and $\mathcal{Y} \subseteq \mathcal{P}_\xi(\bar{M})$ the sets $\Theta_\gamma(\mathcal{Y})$ and $\Theta^+_\gamma(\mathcal{Y})$ are the same.*

Proof. Since products are simple martingales, it suffices to show that $\Theta_\gamma(\mathcal{Y}) \supseteq \Theta^+_\gamma(\mathcal{Y})$. Given a martingale $\mathfrak{M}$ of length $j - 1$ and $A \subseteq M_{\mathfrak{M}}$ and $\Psi : A \to \bar{M}$ let $\mathfrak{M}(t)$ be the product of the two probability spaces (Q^t_0, σ^t_0) and (Q^t_1, σ^t_1). Let D consist of all $t \restriction n$ for some $t \in M_{\mathfrak{M}}$ and $n < j$. For $i \in 2$ let

$$(Q^*_i, \sigma^*_i) = \prod_{t\in D}(Q^t_i, \sigma^t_i)$$

and then define a mapping $R : \prod_{n\in j} Q^*_0 \times Q^*_1 \to M_{\mathfrak{M}}$ by $R(x)(n) = x(n)(R(x) \restriction n)$ noting that this is an inductive definition and that $x(n)$ is a function with domain D and $R(x) \restriction n \in D$.

Let $A^* = R^{-1}A$ and defined $\Psi^* : A^* \to \bar{M}$ by $\Psi^*(x) = \Psi(R(x))$. A direct verification establishes that $(\Psi^*)^{-1}Y \Subset_\gamma A^*$ if and only $\Psi^{-1}Y \Subset^{\mathfrak{M}}_\gamma A$ for $Y \in \mathcal{Y}$. □

The following technical fact can be established by routine arguments.

Lemma 3.2. *If (Q_0, σ_0) and (Q_1, σ_1) are Polish probability spaces, $\epsilon > 0$ and $A \subseteq Q_0 \times Q_1$ is measurable of positive measure and $\zeta < 1$ then there are $\{A^0_j \times A^1_j\}^k_{j=1}$ such that*

1. *each A^i_j is a measurable subset of Q_i*
2. *if $j \neq i$ then $A^0_j \cap A^0_i = \emptyset$*
3. $\sigma_0 \times \sigma_1 \left(A \setminus \bigcup^k_{j=1} A^0_j \times A^1_j\right) < \epsilon$
4. $\dfrac{\sigma_1(A\langle x\rangle \cap A^1_j)}{\sigma_1(A^1_j)} > \zeta$ *for each j and $x \in A^0_j$.*

Proof. Regularity of the measures and Fubini's Theorem yield a compact set $C \subseteq A$ and an open set $U \supseteq A$ such that $\sigma_0 \times \sigma_1(U \setminus C)$ is so small that

$$\sigma_0(\{q \in Q_0 \mid \sigma_1(C\langle q\rangle)/\sigma_1(U\langle q\rangle) < \zeta\}) < \epsilon/2$$

Let $\{B_n\}_{n\in\omega}$ enumerate a base for $Q_0 \times Q_1$ consisting of rectangular sets and let $U_n = \bigcup\{B_j \mid B_j \subseteq U \text{ and } j \leq n\}$ and choose k large enough that $C \subseteq U_k$ still holds. Since there are only finitely many sets that can be formed by the first k basic open sets, the result follows. □

Lemma 3.3. *If $j \in \mathbb{N}$ and $\gamma < \min(\epsilon(j), K^{q'} j^{q'} J^*(j)^{q'/q})$ and $\mathcal{Y} \subseteq \mathcal{P}(M)$ then $\Theta_\gamma(\mathcal{Y}) \subseteq \Theta(\mathcal{Y})$. Indeed, $\Theta^{\delta,L}_\gamma(\mathcal{Y}) \subseteq \Theta(\mathcal{Y})$ for any $\delta > 0$ and L.*

Proof. Proceed by induction on j to show that if $j \in \Theta_\gamma(\mathcal{Y})$ then $j \in \Theta(\mathcal{Y})$, the cases $j = 0$ and $j = 1$ being trivial.

Now suppose that $2 \notin \Theta(\mathcal{Y})$; in other words, there is a function F from $\mathcal{Y}$ to the open subsets of $\mathbb{R}^k$ of measure less than $J^*(1)$ such that, letting

$$A = \left\{(x, p) \in [0, 1]^k \times P \mid 1 \notin \Theta(\langle x, p, F\rangle_1)\right\}$$

it follows that $\lambda \times \sigma(A) > 1/2$. Recall from Lemma 2.1 that the mapping $\Lambda^{F(Y)}$ is measurable for each $Y \in \mathcal{Y}$. Hence it is possible to find a measurable $A' \subseteq A$ such that $\lambda \times \sigma(A') > 1/2$ and each of these mappings is continuous on A'.

For each $(x, p) \in A'$ it follows that there is some $a \in M \setminus \cup\langle x, p, F\rangle_1$. Note that $Y \notin \langle x, p, F\rangle_1$ if and only if $\Lambda^{F(Y)}(p, x) > 1$ and so there is an open rectangle $U_1(Y) \times U_0(Y)$ which is a neighbourhood of (x, p) such that $\Lambda^{F(Y)}(p',x') > 1$ for all $(x',p') \in U_1(Y) \times U_0(Y) \cap A'$. Hence, letting $U_i = \bigcap\{U_i(Y) \mid a \in Y \in \mathcal{Y}\}$, there is a neighbourhood $U_1 \times U_0$ of (x,p) such that $a \notin \cup\langle x',p',F\rangle_1$ for all $(x',p') \in U_1 \times U_0 \cap A'$. It follows from applying Lemma 3.2 to sufficiently many of these neighbourhoods that there are pairwise disjoint measurable rectangles $\{U_1^j \times U_0^j\}_{j=1}^m$ such that:

(1) for each j there is $a_j \in M$ such that $a_j \notin \cup\langle x,p,F\rangle_1$ for all $(x,p) \in U_1^j \times U_0^j \cap A'$

(2) $\lambda \times \sigma\left(A' \cap \bigcup_{j=1}^m U_1^j \times U_0^j\right) > 1/2$

(3) $\sigma_0(A'\langle x\rangle \cap U_j^0) > \sigma_0(U_j^0)\zeta$ for each j and $x \in U_j^1$ where $\zeta < 1$ is sufficiently large that $(2\epsilon(1)-\gamma)\zeta > \epsilon(1)$.

For the last assertion use the assumption on γ that $\gamma < \epsilon(1)$.

Now let $\mathcal{M}_1$ be the finite Boolean algebra of subsets of $[0,1]^k$ generated by $\{U_1^j\}_{j=1}^m$ and let $\mathcal{M}_0$ be the finite Boolean algebra of subsets of P generated by $\{U_0^j\}_{j=1}^m$. Let Q_i be the set of all atoms of $\mathcal{M}_i$. Let μ_0 be the measure defined on the subsets of Q_0 by assigning $\mu_0(X) = \sum_{a \in X}\sigma(a)$ and let μ_1 be the measure defined on the subsets of Q_1 by assigning $\mu_1(X) = \sum_{a\in X}\lambda(a)$ so that both (Q_0,μ_0) and (Q_1,μ_1) are finite probability spaces. If they are not (δ,L)-fine, use the fact that P is atomless to refine the atoms so that (Q_0,μ_0) and (Q_1,μ_1) are (δ,L)-fine. Let

$$A^* = \left\{q_1 \in Q_1 \;\middle|\; \mu_0\left(\left\{q_0 \in Q_0 \;\middle|\; (\exists j \leq m)\ q_1 \times q_0 \subseteq U_1^j \times U_0^j\right\}\right) > 2\epsilon(1)\right\}$$

and note that $\mu_1(A^*) > 2K^{q'}J^*(1)^{q'/q}$ by Definition 2.3 since (Q_0,μ_0) and (Q_1,μ_1) are probability spaces. Let $\bar{A} = \left\{(q_1,q_0) \;\middle|\; q_1 \in A^* \text{ and } (\exists j \leq m)\ q_1 \times q_0 \subseteq U_1^j \times U_0^j\right\}$. It is then possible to define $\Psi : \bar{A} \to M$ such that for each $(q_1,q_0) \in \bar{A}$ there is some j such that $q_1 \times q_0 \subseteq U_1^j \times U_0^j$ and $\Psi(q_1,q_0) = a_j$. Since $\rho_\gamma(\mathcal{Y}) \geq 2$ it is possible to find $Y \in \mathcal{Y}$ such that $\Psi^{-1}Y \Subset_\gamma \bar{A}$. There are now two cases to consider.

Case One. There is some $q_1 \in A^*$ such that $\mu_0((\bar{A} \setminus (\Psi^{-1}Y))\langle q_1\rangle) < \gamma$.

Since $\mu_0(\bar{A}\langle q\rangle) > 2\epsilon(1)$ for each $q \in A^*$ it follows that $\mu_0((\Psi^{-1}Y)\langle q_1\rangle) > 2\epsilon(1) - \gamma$. If $q_0 \in (\Psi^{-1}Y)\langle q_1\rangle$ then $\Psi(q_1,q_0) \in Y$ and so Y does not belong to $\langle x,p,F\rangle_1$ for any $(x,p) \in A'$ such that $x \in q_1$ and $p \in q_0$. Fix some $x^* \in q_1$. Therefore $\Lambda^{F(Y)}(p,x^*) > 1$ for every $p \in \cup(\Psi^{-1}Y)\langle q_1\rangle \cap A'\langle x^*\rangle$. Using the fact that the $U_1^j \times U_0^j$ are pairwise disjoint, it follows that $\sigma(\cup(\Psi^{-1}Y)\langle x^*\rangle \cap A'\langle x^*\rangle) > (2\epsilon(1) - \gamma)\zeta > \epsilon(1)$ and so $\lambda(F(Y)) > J^*(1)$ by Definition 2.1. This contradicts the hypothesis on F that the measure of $F(Y)$ is less than $J^*(1)$.

Case Two. $\mu_1(\{q_1 \in Q_1 \mid (\Psi^{-1}Y)\langle q_1\rangle = \emptyset \neq \bar{A}\langle q_1\rangle\}) < \gamma$.

Observe that $q_1 \in A^*$ if and only if $\emptyset \neq \bar{A}\langle q_1\rangle$. Therefore

$$\mu_1\left(\{q_1 \in A^* \mid (\Psi^{-1}Y)\langle q_1\rangle = \emptyset\}\right) < \gamma < K^pJ^*(1)^{p/q}$$

and so $\mu_1(\{q_1 \in A^* \mid (\Psi^{-1}Y)\langle q_1\rangle \neq \emptyset\}) > \mu_1(A^*) - K^pJ^*(1)^{p/q} > K^pJ^*(1)^{p/q}$. From Inequality 2.7 with $\epsilon = 1$ and the fact that the measure of $F(Y)$ is less than $J^*(1)$ it follows that it is possible to choose $x^* \in q_1^* \in A^*$ so that $(\Psi^{-1}Y)\langle q_1^*\rangle \neq \emptyset$ and such that $\Lambda^{F(Y)}(p,x^*) < 1$ for all $p \in P$. Choose any $q_0^* \in (\Psi^{-1}Y)\langle q_1^*\rangle$ and $p^* \in q_0^*$ such that $(x^*,p^*) \in A'$ and note that then $\Psi(q_1^*,q_0^*) \in Y$. On the other hand, it follows that $\Lambda^{F(Y)}(p^*,x^*) < 1$ by the choice of x^* and so $Y \in \langle x^*,p^*,F\rangle_1$ and hence, since $(x^*,p^*) \in A'$, $\Psi(q_1^*,q_0^*) \notin Y$. This contradiction finishes the argument for the case $j = 2$.

Now assume that $j \geq 2$ and that $j+1 \in \Theta_\gamma(\mathcal{Y})$. Let F be a function from $\mathcal{Y}$ to the open subsets of $\mathbb{R}^k$ of measure less than $J^*(j)$. As in the initial case, let

$$A = \left\{(x,p) \in [0,1]^k \times P \mid j \notin \Theta(\langle x,p,F\rangle_j)\right\}$$

and, aiming for a contradiction, suppose that $\lambda \times \sigma(A) > 1/(j+1)$. Furthermore, it may be assumed that that the mapping $\Lambda^{F(Y)}$ is continuous on A for each $Y \in \mathcal{Y}$. From the induction hypothesis it follows from applying Lemma 3.2 that there are pairwise disjoint measurable rectangles $\{U_1^n \times U_0^n\}_{n=1}^m$ such that:

(1) for each n there is there are finite probability spaces (Q_0,μ_0) and (Q_1,μ_1) and $B \subseteq (Q_0 \times Q_1)^{j-1}$ and $\Psi : B \to M$ witnessing that $j \notin \Theta_\gamma(\langle x,p,F\rangle_j)$ if $(x,p) \in A \cap U_1^n \times U_0^n$ in that $\Psi^{-1}Y \not\Subset_\gamma B$ for each $Y \in \langle x,p,F\rangle_j$

(2) $\lambda \times \sigma \left(A \cap \bigcup_{j=1}^{m} U_1^j \times U_0^j \right) > 1/2$

(3) $\sigma_0(A\langle x\rangle \cap U_0^j) > \sigma_0(U_0^j)\zeta$ for each j and $x \in U_1^j$ where $\zeta < 1$ is sufficiently large that $(2\epsilon(j) - \gamma)\zeta > \epsilon(j)$.

As in the initial case, let $\mathcal{M}_1$ be the finite Boolean algebra of measurable subsets of $[0,1]^k$ generated by $\{U_1^n\}_{n=1}^m$ and $\mathcal{M}_0$ be the finite Boolean algebra of measurable subsets of P generated by $\{U_0^n\}_{n=1}^m$. Letting Q_i be the atoms of $\mathcal{M}_i$, if $\tilde{A}$ is defined to be the set of all $(q_1, q_0) \in Q_1 \times Q_0$ such that there exist finite probability spaces $(Q_0^{q_1,q_0}, \mu_0^{q_1,q_0})$ and $(Q_1^{q_1,q_0}, \mu_1^{q_1,q_0})$ and $B^{q_1,q_0} \subseteq (Q_0^{q_1,q_0} \times Q_1^{q_1,q_0})^{j-1}$ as well as a function $\Psi^{q_1,q_0} : B^{q_1,q_0} \to M$ such that

$$(\forall (x,q) \in (q_1,q_0) \cap A)(\forall Y \in \langle x, p, F\rangle_j)(\Psi^{q_0,q_1})^{-1} Y \not\Subset_\gamma B^{q_0,q_1}$$

then $\mu_1 \times \mu_0(\tilde{A}) > 1/(j+1)$ where each μ_i is the measure defined on Q_i as in the initial case. Let

$$A^* = \left\{ q_1 \in Q_1 \;\middle|\; \mu_0(\tilde{A}\langle q_1\rangle) > 2\epsilon(j) \right\}$$

and note that $\mu_1(A^*) > 2j^{q'}K^{q'}J^*(j)^{q'/q}$. Let $\bar{A} = \left\{ (q_1,q_0) \in \tilde{A} \mid q_1 \in A^* \right\}$.

Note that defining the martingale $\mathfrak{M}$ by setting $\mathfrak{M}(\emptyset) = (Q_1 \times Q_0, \mu_1 \times \mu_0)$ and, if $t \neq \emptyset$, setting

$$\mathfrak{M}(t) = (Q_1^{q_1,q_0} \times Q_0^{q_1,q_0}, \mu_1^{q_1,q_0} \times \mu_0^{q_1,q_0})$$

where $t(0) = (q_1, q_0)$ satisfies Definition 3.4 and has length j. Let B consist of all $(q_0,q_1)^\frown t \in M_{\mathfrak{M}}$ such that $t \in B^{q_0,q_1}$. Let Ψ be defined on B by $\Psi((q_0,q_1)^\frown t) = \Psi^{q_0,q_1}(t)$.

As $j+1 \notin \Theta_\gamma(\mathcal{Y})$ it is possible to use Lemma 3.1 find $Y \in \mathcal{Y}$ such that $\Psi^{-1}Y \Subset_\gamma^{\mathfrak{M}} B$. In other words,

$$\left\{ (q_1,q_0) \in Q_1 \times Q_0 \;\middle|\; (\Psi^{-1}Y)\langle (q_1,q_0)\rangle \Subset_\gamma^{\mathfrak{M}_{(q_1,q_0)}} B\langle (q_1,q_0)\rangle \right\} \Subset_\gamma \pi(B) = \bar{A}.$$

Once again there are two cases to consider.

Case One. There is some $q_1 \in A^*$ such that

$$\mu_0 \left(\left\{ q_0 \in Q_0 \;\middle|\; (q_1,q_0) \in \bar{A} \text{ and } (\Psi^{-1}Y)\langle (q_1,q_0)\rangle \not\Subset_\gamma^{\mathfrak{M}_{q_1,q_0}} B\langle (q_1,q_0)\rangle \right\} \right) < \gamma.$$

Observe that $\mu_0(\bar{A}\langle q_1\rangle) > 2\epsilon(j)$ since $q_1 \in A^*$. Furthermore, $B\langle (q_1,q_0))\rangle = B^{q_1,q_0}$ for any $q_0 \in Q_0$. Moreover, $\Psi((q_1,q_0)^\frown x) = \Psi^{q_1,q_0}(x)$ for any $x \in B^{q_1,q_0}$. Hence if

$$E = \left\{ q_0 \in Q_0 \;\middle|\; (q_1,q_0) \in \bar{A} \text{ and } (\Psi^{q_1,q_0})^{-1} Y \Subset_\gamma B^{q_1,q_0} \right\}$$

then $\mu_0(E) \geq 2\epsilon(j) - \gamma > \epsilon(j)$ by the assumption on γ. Choose $x^* \in q_1$. Then if $p \in q_0 \in E$ and $(x,p) \in A$ it cannot be the case that $Y \in \langle x^*, p, F\rangle_j$ since Ψ^{q_1,q_0} was chosen to be a counterexample to $j \in \Theta_\gamma(\langle x, p, F\rangle_j)$. Therefore $\Lambda^{F(Y)}(p, x^*) > 1/j$ whenever $p \in \cup E$. Since $\sigma(\cup E \cap A\langle x^*\rangle) > (2\epsilon(j) - \gamma)\zeta > \epsilon(j)$ it follows that $\lambda(F(Y)) > J^*(j)$ contradicting that F was chosen so that $\lambda(F(Y)) < J^*(j)$.

Case Two. If E is defined to be

$$\left\{ q_1 \in A^* \;\middle|\; (\forall q_0 \in Q_0)(\Psi^{-1}Y)\langle (q_1,q_0)\rangle \not\Subset_\gamma^{\mathfrak{M}_{q_1,q_0}} B\langle (q_1,q_0)\rangle \text{ or } (q_1,q_0) \notin \bar{A} \right\}$$

then $\lambda(E) < \gamma$.

Notice that $B\langle (q_1,q_0)\rangle \neq \emptyset$ if and only if $(q_1,q_0) \in \bar{A}$. Also,

$$A^* \setminus E = \left\{ q_1 \in A^* \;\middle|\; (\exists q_0 \in Q_0)\,(\Psi^{q_1,q_0})^{-1}\, Y \Subset_\gamma^{\mathfrak{M}_{q_1,q_0}} B^{q_1,q_0} \right\}$$

and, moreover, $\mu_1(A^* \setminus E) > 2j^{q'}K^{q'}J^*(j)^{q'/q} - \gamma > j^{q'}K^{q'}J^*(j)^{q'/q}$. Furthermore, since the measure of $F(Y)$ is less than $J^*(j)$ it follows that

$$\lambda \left(\left\{ x \in [0,1]^k \;\middle|\; (\exists p \in P)\Lambda^{F(Y)}(p,x) \geq 1/j \right\} \right) \leq j^{q'}K^{q'}J^*(j)^{q'/q}$$

and so it is possible to choose $q_1^* \in A^*$ and $x^* \in q_1^*$ such that $\Lambda^{F(Y)}(p^*, x^*) < 1/j$ for all $p \in P$ and such that $\left(\Psi^{q_1^*,q_0^*}\right)^{-1} Y \Subset_\gamma^{\mathfrak{M}_{q_1,q_0}} B^{q_1^*,q_0^*}$ for some $q_0^* \in Q_0$. If $p^* \in q_0^* \cap A\langle x^*\rangle$ then $\Lambda^{F(Y)}(p^*, x^*) < 1/j$ implies that

$Y \in \langle x^*, p^*, F\rangle_j$. However, the fact that $\left(\Psi^{q_1^*,q_0^*}\right)^{-1} Y \Subset_\gamma^{\mathfrak{M}_{q_1,q_0}} B^{q_1^*,q_0^*}$ contradicts the choice of $\Psi^{q_1^*,q_0^*}$ and $B^{q_1^*,q_0^*}$ and the fact that $(x^*, p^*) \in A$. □

4. The Pigeonhole Principle

The key step in justifying Hypothesis 2.12 will be establishing a measure theoretic pigeonhole principle. The description of the argument involves considerable notation, so an informal introduction to the reasoning behind this pigeonhole principle is likely to be helpful. Roughly speaking, it will be shown that for any $\gamma, \zeta > 0$ and $j \in \mathbb{N}$ there is a sufficiently large integer M such that for probability spaces (P_1, σ_1) and (P_2, σ_2) and any $\Psi : B \to M$ where $B \subseteq (P_1 \times P_2)^j$ there is a $W \subseteq M$ such that $|W| < \zeta M$ and $\Psi^{-1}W \Subset_\gamma B$. For the purposes of this sketch it can be assumed that $B = (P_1 \times P_2)^j$.

If $j = 1$ then let $E_x = \{\Psi(x, y) \mid y \in P_2\}$ for each $x \in P_1$. Fix $k < \zeta M$ where M has been chosen to be so large that k can be assumed to have the following property: If k elements of M are painted white or black at random with black having probability ζ then the chances of having at least one element painted black are greater than $1 - \gamma$. Given that $|E_x| > k$ for all x, choosing a random subset W of M by giving each element of M a chance ζ of belonging to W will yield a set W such that $\Psi^{-1}W$ is likely to hit all but γ of the E_x. In other words, $\Psi^{-1}W \Subset_\gamma P_1 \times P_2$.

On the other hand, if there is even one x such that $|E_x| \leq k$ then letting $W = E_x$ works. However, extending this strategy to even the case $j = 2$ encounters some problems caused by the fact that W is chosen probabilistically in one case, and according to a definite rule in the other. For example, if $j = 2$ then for some $(x, y) \in P_1 \times P_2$ the W that works for the restriction of Ψ to $(P_1 \times P_2)^2\langle(x, y)\rangle$ may be chosen probabilistically for some (x, y) while for others it might be E_z for some z such that $E_{x,y,z}$ is small, where $E_{x,y,z} = \{\Psi(x, y, z, w) \mid w \in P_2\}$. However, the probabilistic choice actually yields many W that work; indeed, it will be argued that almost all W will work.

But what happens if all of the $E_{x,y,z}$ are small? (This will be the assumption under which Sublemma 4.1 is proved.) If M is sufficiently large than the argument for the case $j = 1$ can be carried out with k much smaller than ζM. Indeed, if k is sufficiently small and there are sufficiently many z, choosing elements of M at random will ensure that at least one of the z will be lucky enough to have all the elements of $E_{x,y,z}$ chosen, provided the $\{E_{x,y,z}\}_{z \in P_2}$ are disjoint. But there is no reason to expect them to be disjoint. However, the following measure theoretic version of the Δ-system lemma, Lemma 4.1, shows that one can expect to have a large family of z that form a Δ-system with root $E_{x,y}$.

Lemma 4.1. *Recall from §3 that a probability space is said to be (δ, L)-fine if and only if every set of measure greater than δ has cardinality at least L. If L and k are integers and τ a positive real then there is an integer $\Xi(k, L, \tau) \geq k$ such that for any $(\tau/2^k, L)$-fine probability space $(P, \sigma, \mathcal{M})$ and any measurable $Q : P \to [\mathbb{N}]^{\leq k}$ there is $E \in [\mathbb{N}]^{\leq \Xi(k,L,\tau)}$ such that for all $U \in \mathcal{M}$, if $\sigma(U) \geq \tau$ and $|U| \geq L$ then there is $X \in [U]^L$ such that the family $\{Q(x) \setminus E\}_{x \in X}$ is pairwise disjoint.*

Now the $\{E_{x,y,z} \setminus E_{x,y}\}_z$ are pairwise disjoint for many z, guaranteeing that at least one of the z will be lucky enough to have all the elements of $E_{x,y,z} \setminus E_{x,y}$ chosen. However, if this strategy is to be used to get a column witnessing that $\Psi^{-1}W \Subset_\gamma (P_1 \times P_2)^2$ — in other words, to get $x \in P_1$ such that for most $y \in P_2$ there is z such that all the elements of $E_{x,y,z} \setminus E_{x,y}$ are contained in W — then the problem of getting W to contain all of the remaining $E_{x,y}$ still exists.

Two alternatives have to be considered. In order to state these alternatives, use Lemma 4.1 to find, loosely speaking, a Δ-system with root E_x for the family $\{E_{x,y}\}_{y \in P_2}$ and a Δ-system with root E for the family $\{E_x\}_{x \in P_1}$. Either there is some $x \in P_1$ such that for most $y \in P_2$, for most $z \in P_1$, $(E_x \setminus E) \cap E_{x,y,z} \neq \emptyset$ or there is not. If there is such an x then let $W = E_x \setminus E$ and observe that $(\Psi^{-1}W)\langle(x, y)\rangle$ contain the graph of a function whose domain is almost all of P_1 for almost all $y \in P_2$. In other words, $\Psi^{-1}W \Subset_\gamma (P_1 \times P_2)^2$ in this case.

On the other hand, if there is no such x then for every $x \in P_1$ there are many $y \in P_2$ such that there are many $z \in P_1$ so that $(E_x \setminus E) \cap E_{x,y,z} = \emptyset$. The existence of many y and z allows Lemma 4.1 to be used to conclude that for any x, a randomly chosen W is likely to have at least one y_x such that there is at least one z so that:

(1) $(E_x \setminus E) \cap E_{x,y_x,z} = \emptyset$
(2) $E_{x,y_x,z} \setminus E_{x,y_x} \subseteq W$
(3) $E_{x,y_x} \setminus E_x \subseteq W$.

This means that one can expect y_x to exist for all but γ of the $x \in P_1$. In order for $\Psi^{-1}W \Subset_\gamma (P_1 \times P_2)^2$ it suffices to establish that for each such x there is $z \in P_1$ such that $E_{x,y_x,z} \subseteq W$. But it is immediate that

$$(E_{x,y_x,z} \setminus E_{x,y_x}) \cup (E_{x,y_x} \setminus E_x) \subseteq W$$

and, by choosing M large enough, it can be guaranteed that E is small enough that it can be included in W as well. Hence, the only way that $E_{x,y_x,z}$ could fail to be a subset of W is that $(E_x \setminus E) \cap E_{x,y_x,z} \not\subseteq W$. But, since z can be chosen so that $(E_x \setminus E) \cap E_{x,y_x,z} = \emptyset$ this is no problem.

Arguments continuing this line of reasoning require a slight modification when $j > 2$, but they will still yield Lemma 4.2. However, it should be apparent at this point that it is time to end the introductory sketch and to provide the necessary details and calculations. The first order of business is to establish the validity of Lemma 4.1.

Proof of Lemma 4.1. Proceed by induction on k the case $k = 0$ being trivial since setting $\Xi(0, L, \tau) = 0$ works for all L and τ. For $k \geq 1$ let $\Xi(k, L, \tau)$ be greater than $2kL/\tau + \Xi(k-1, L, \tau/2)$ and k. Given Q let $\mathcal{A} \subseteq \mathcal{M}$ be a maximal pairwise disjoint family such that for each $A \in \mathcal{A}$

- $\sigma(A) \geq \tau/2$
- there does not exist $X \in [A]^L$ such that $\{Q(x)\}_{x \in X}$ are pairwise disjoint.

It follows that for each $A \in \mathcal{A}$ there is $E_A \in [\mathbb{N}]^{<kL}$ such that $Q(a) \cap E_A \neq \emptyset$ for each $a \in A$. Since $|\mathcal{A}| \leq 2/\tau$ it follows that if E^1 is defined to be $\bigcup_{A \in \mathcal{A}} E_A$ then $|E^1| \leq 2kL/\tau$.

Now define

$$Q^*(p) = \begin{cases} Q(p) \setminus E^1 & \text{if } p \in \cup\mathcal{A} \\ \emptyset & \text{if } p \notin \cup\mathcal{A} \end{cases}$$

and note that $Q^* : P \to [\mathbb{N}]^{\leq k-1}$ is measurable and the probability space (P, σ) is $((\tau/2)/2^{k-1}, L)$-fine By the induction hypothesis there is $E^2 \in [\mathbb{N}]^{\Xi(k-1,L,\tau/2)}$ such that for every $U \in \mathcal{M}$ such that $\sigma(U) \geq \tau/2$ there is $X \in [U]^L$ such that the family $\{Q^*(x) \setminus E^2\}_{x \in X}$ is pairwise disjoint. Now let $E = E^1 \cup E^2$ and note that $|E| \leq \Xi(k, L, \tau)$.

Moreover, if $U \in \mathcal{M}$ and $\sigma(U) \geq \tau$ then if $\sigma(U \setminus \cup\mathcal{A}) > \tau/2$ it follows from the maximality of $\mathcal{A}$ that there is some $X \in [U \setminus \cup\mathcal{A}]^L$ such that $\{Q(x)\}_{x \in X}$ are pairwise disjoint. Hence it may be assumed that $\sigma(U \cap (\cup\mathcal{A})) \geq \tau/2$. Then there is $X \in [U \cap (\cup\mathcal{A})]^L$ such that $\{Q^*(x) \setminus E^2\}_{x \in X}$ are pairwise disjoint. In other words, $\{Q(x) \setminus E\}_{x \in X}$ are pairwise disjoint. □

The next lemma is stated for finite probability spaces but the proof shows that it also applies to Borel functions on Polish measure spaces all of whose projective subsets are measurable. Of course, for finite measure spaces or for finitely additive total measures on $\mathbb{N}$ this hypothesis is trivially satisfied but, if one assumes the measurability of all projective sets or the existence of real valued measurable cardinals then the argument of Lemma 4.2 has wider applicability. Whether these very strong hypotheses are actually required is not known.

Lemma 4.2. *Suppose that $\xi \in (0, 1)$, $\gamma \in (0, 1)$ and that j is a non-zero integer. There is an integer $M(\xi, \gamma, j)$ and $\epsilon^* > 0$ and $L^* \in \mathbb{N}$ such that for any two finite, (ϵ^*, L^*)-fine, probability spaces (P_1, σ_1) and (P_2, σ_2) and any $B \subseteq (P_1 \times P_2)^j$ and any function $\Psi : B \to M(\xi, \gamma, j)$ there is $W \in \mathcal{P}_\xi(M(\xi, \gamma, j))$ such that $\Psi^{-1}W \Subset_\gamma B$.*

Proof. If ξ, γ and j have been given, define $\bar{\gamma} = \gamma/2$ and let $\bar{\xi} = \xi/2(j+1)$. Define $\beta(0, \delta) = 1 - \delta$ and then inductively define

$$\beta(n+1, \delta) = 1 - \frac{\max(1 - \beta(n, \delta)(1 - \delta) + \delta, 1 - \beta(n, \delta), \delta)}{\bar{\gamma}} \tag{4.1}$$

for $\delta > 0$. It follows by induction that $\lim_{\delta \to 0} \beta(n, \delta) = 1$ for each n so choose $\bar{\delta} \in (0, \bar{\gamma})$ so small that $\beta(j, \bar{\delta}) > 1/2$. Let $\beta(i) = \beta(i, \bar{\delta})$. Observe that $\bar{\gamma} \leq 1/2$ implies that $\beta(i) \geq \beta(i+1)$ for all i.

Begin by choosing $k_1 \in \mathbb{N}$ so large that

$$\left(1-\bar{\xi}\right)^{k_1} < \bar{\delta} \tag{4.2}$$

and, given k_n, choose L_n so large that

$$\left(1-\bar{\xi}^{k_n}\right)^{L_n} < \bar{\delta} \tag{4.3}$$

and, given L_n, let

$$k_{n+1} = \Xi(k_n, L_n, \bar{\delta}) \tag{4.4}$$

where Ξ is the function described in Lemma 4.1. Since the function Ξ satisfies that $\Xi(k, L, \tau) \geq k$ the sequence $\{k_n\}_{n=1}^{2j+1}$ is monotone. Let $\epsilon^* = \bar{\delta}/2^{k_{2j+1}}$ and $L^* = L_{2j+1}$.

Then let $M = M(\xi, \gamma, j) \in \mathbb{N}$ be sufficiently large that

$$\frac{k_{2j+1}}{M} < \bar{\xi} \tag{4.5}$$

and, furthermore, such that the probability that $\sum_{i=1}^{M} X_i < 2\bar{\xi}M$ is greater than $2^{-1/j}$ where $\{X_i\}_{i=1}^{M}$ are independent $\{0,1\}$-valued random variables with mean $\bar{\xi}$. In order to avoid dealing with probabilities, however, let $\bar{\mu}$ be the probability measure on $\{0,1\}$ defined by $\bar{\mu}(\{1\}) = \bar{\xi}$ and $\bar{\mu}(\{0\}) = 1-\bar{\xi}$ and let $\mu = \bar{\mu}^M$ be the M-fold product of $\bar{\mu}$ on 2^M. Elements of 2^M will be identified with subsets of M by their supports; in other words, μ will be thought of as a measure on $\mathcal{P}(M)$ and hence the probability inequality following Inequality 4.5 is equivalent to

$$\mu\left(\left\{W \subseteq M \;\middle|\; |W| < 2\bar{\xi}M\right\}\right) > \frac{1}{2^{1/j}}. \tag{4.6}$$

In preparation for the proof some conventions for denoting coordinates will be introduced. If $z \in (P_1 \times P_2)^m$ then

$$z = ((z_1(1), z_2(1)), (z_1(2), z_2(2)), \ldots, (z_1(m), z_2(m)))$$

while if $z \in (P_1 \times P_2)^m \times P_1$ then

$$z = ((z_1(1), z_2(1)), (z_1(2), z_2(2)), \ldots, (z_1(m), z_2(m)), z(m+1)).$$

For $z \in (P_1 \times P_2)^j$ and, for any $\ell \leq j$ let

$$z_1^*(\ell) = ((z_1(1), z_2(1)), (z_1(2), z_2(2)), \ldots, (z_1(\ell-1), z_2(\ell-1)), z_1(\ell))$$
$$z^*(\ell) = ((z_1(1), z_2(1)), (z_1(2), z_2(2)), \ldots, (z_1(\ell), z_2(\ell)))$$

and $z^*(0)$ will be defined to be $\emptyset$. If $z \in (P_1 \times P_2)^m$ and $x \in P_1$ then $z^\frown x$ will denote

$$((z_1(1), z_2(1)), (z_1(2), z_2(2)), \ldots, (z_1(m), z_2(m)), x)$$

with the convention that $\emptyset^\frown x = x$ in the case $m = 0$. If $z \in (P_1 \times P_2)^m \times P_1$ and $x \in P_2$ then $z^\frown x$ will denote

$$((z_1(1), z_2(1)), (z_1(2), z_2(2)), \ldots, (z_1(m), z_2(m)), (z(m+1), x)).$$

Now suppose that (P_1, σ_1) and (P_2, σ_2) are finite, (ϵ^*, L^*)-fine, probability spaces and that $B \subseteq (P_1 \times P_2)^j$ and $\Psi : B \to M$. For $x \in (P_1 \times P_2)^{j-1} \times P_1$ define $E_x = \{\Psi(x^\frown y) \mid y \in B\langle x\rangle\}$. Before considering the general case, assume that $|E_x| \leq k_1$ for each $x \in (P_1 \times P_2)^{j-1} \times P_1$. It will first be shown that in this case the following sublemma holds. Most of the argument to follow will be devoted to this special case; the general result will then follow with only a little more effort.

Sublemma 4.1. *There is $W \subseteq \mathcal{P}_{(2j+1)\bar{\xi}}(M)$ such that $\Psi^{-1}W \Subset_{\bar{\gamma}} B$.*

Proof. If $j = 1$ then choosing any $x \in P_1$ such that $B\langle x\rangle \neq \emptyset$ and letting $W = E_x$ will satisfy that $\Psi^{-1}W \Subset_{\gamma} B$ and $|W| \leq k_1 \leq k_3 \leq \bar{\xi}M$. Therefore assume that $j \geq 2$.

Now suppose that $1 \leq \ell \leq j$ and that $E_x \in [M]^{\leq k_{2\ell-1}}$ has been defined for each $x \in (P_1 \times P_2)^{j-\ell} \times P_1$. For $x \in (P_1 \times P_2)^{j-\ell}$ let $E_x \subseteq M$ be such that

$$|E_x| \leq \Xi(k_{2\ell-1}, L_{2\ell-1}, \bar{\delta}) = k_{2\ell} \tag{4.7}$$

$$(\forall U \subseteq P_1) \text{ if } \sigma_1(U) \geq \bar{\delta} \text{ then } (\exists Z \in [U]^{L_{2\ell-1}}) \ \{E_{x^\frown z} \setminus E_x\}_{z\in Z} \text{ is pairwise disjoint} \tag{4.8}$$

using the convention that $(P_1 \times P_2)^0 = \{\emptyset\}$ and $(P_1 \times P_2)^0 \times P_1 = P_1$ in the case that $j = \ell$. On the other hand, suppose that $1 \leq \ell < j$ and that $E_x \in [M]^{\leq k_{2\ell}}$ has been defined for each $x \in (P_1 \times P_2)^{j-\ell}$. For $x \in (P_1 \times P_2)^{j-\ell-1} \times P_1$ let $E_x \subseteq M$ be such that

$$|E_x| \leq \Xi(k_{2\ell}, L_{2\ell}, \bar{\delta}) = k_{2\ell+1} \tag{4.9}$$

$$(\forall U \subseteq P_2) \text{ if } \sigma_2(U) \geq \bar{\delta} \text{ then } (\exists Z \in [U]^{L_{2\ell}}) \ \{E_{x^\frown z} \setminus E_x\}_{z\in Z} \text{ is pairwise disjoint.} \tag{4.10}$$

Now begin by defining

$$C_1 = \left\{ z \in (P_1 \times P_2)^{j-1} \times P_1 \;\middle|\; B\langle z\rangle \neq \emptyset \text{ and } \bigcup_{\ell=1}^{j-1} \left(E_{z_1^*(\ell)} \setminus E_{z^*(\ell-1)}\right) \cap E_z = \emptyset \right\}$$

and, given that $C_\ell \subseteq (P_1 \times P_2)^{j-\ell} \times P_1$ has been defined, for each $z \in (P_1 \times P_2)^{j-\ell}$ let $\Theta(z) = C_\ell\langle z\rangle$. For $z \in (P_1 \times P_2)^{j-(\ell+1)} \times P_1$ let

$$\Omega(z) = \left\{ y \in P_2 \;\middle|\; \sigma_1(\Theta(z^\frown y)) > \bar{\delta} \right\}$$

if $\ell = 1$ while if $1 < \ell < j$ then let

$$\Omega(z) = \left\{ y \in P_2 \;\middle|\; \sigma_1(\Theta(z^\frown y)) < \bar{\delta} \right\}$$

noting the crucial difference in the two definitions. Then let

$$C_{\ell+1} = \left\{ z \in (P_1 \times P_2)^{j-(\ell+1)} \times P_1 \;\middle|\; B\langle z\rangle \neq \emptyset \text{ and } \sigma_2(\Omega(z)) < \bar{\delta} \right\}.$$

There are now two cases to consider.

Case One. $C_j \neq \emptyset$.

Define H_ℓ by induction on $\ell \geq 2$ to be an $(\ell - 2)$-ary function on $\mathcal{P}(M)$. If $\ell = 2$ then let $H_2 = C_2$ and if $\ell \geq 3$ define $H_\ell(W_1, W_2, \ldots, W_{\ell-2})$ to be the set of all $x \in C_\ell$ such that

$$\sigma_2(H^*_\ell(x, W_1, W_2, \ldots, W_{\ell-2})) < \bar{\gamma} \tag{4.11}$$

where $H^*_\ell(x, W_1, W_2, \ldots, W_{\ell-2})$ is defined to be

$$\{y \in P_2 \mid (\forall z \in P_1)\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \not\subseteq W_1 \text{ or } x^\frown y^\frown z \notin H_{\ell-1}(W_2, W_3, \ldots, W_{\ell-2})\}.$$

Claim 1. If $\ell \geq 3$ and $x \in C_\ell$ then $\mu^{\ell-2}(D(x)) \geq \beta(\ell)$ where $D(x) = \left\{ \vec{W} \in \mathcal{P}(M)^{\ell-2} \;\middle|\; x \in H_\ell(\vec{W}) \right\}$.

Proof. The argument will proceed by induction on ℓ. In preparation for the main argument however, observe that since $\ell \geq 3$ then for any $y \in P_2 \setminus \Omega(x)$ it must be the case that $\sigma_1(\Theta(x^\frown y)) \geq \bar{\delta}$. Hence there is $Z(y) \in [\Theta(x^\frown y)]^{L_{2\ell-3}}$ such that $\{E_{x^\frown y^\frown z} \setminus E_{x^\frown y}\}_{z\in Z(y)}$ is a pairwise disjoint family. Therefore

$$\mu(D_0(x^\frown y)) = 1 - \prod_{z\in Z(y)} \left(1 - \bar{\xi}^{|E_{x^\frown y^\frown z} \setminus E_{x^\frown y}|}\right) \geq 1 - \left(1 - \bar{\xi}^{k_{2\ell-3}}\right)^{L_{2\ell-3}} > 1 - \bar{\delta}$$

where $D_0(x^\frown y)$ is defined to be $\{W \subseteq M \mid (\exists z \in Z(y))\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W\}$. Let $\{D_1(x^\frown y^\frown z)\}_{z\in Z(y)}$ be a partition of $D_0(x^\frown y)$ such that if $W \in D_1(x^\frown y^\frown z)$ then $E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W$. Moreover, since $Z(y) \subseteq \Theta(x^\frown y)$ it follows that $x^\frown y^\frown z \in C_{\ell-1}$ if $z \in Z(y)$.

The initial case of the induction is $\ell = 3$. In this case, if $y \in P_2 \setminus \Omega(x)$ then

$$\mu\left(\{W \mid (\exists z)\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W \text{ and } x^\frown y^\frown z \in H_2 = C_2\}\right) \geq \tag{4.12}$$

$$\mu\left(\{W \mid (\exists z \in Z(y))\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W\}\right) \geq \mu\left(\bigcup_{z\in Z(y)} D_1(x^\frown y^\frown z)\right) = \mu(D_0(x^\frown y)) > 1 - \bar{\delta} \geq \beta(2)(1 - \bar{\delta})$$

by Equality 4.1. On the other hand, if $\ell > 3$ and $y \in P_2 \setminus \Omega(x)$ then the induction hypothesis implies that $\mu^{\ell-3}(D(x^\frown y^\frown z)) \geq \beta(\ell-1)$ for any $z \in Z(y)$ and therefore if $\ell > 3$ then

$$\mu^{\ell-2}\left(\{(W_1, W_2, \ldots, W_{\ell-2}) \mid (\exists z)\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W_1 \text{ and } x^\frown y^\frown z \in H_{\ell-1}(W_2, W_3, \ldots, W_{\ell-2})\}\right) \geq \tag{4.13}$$

$$\mu^{\ell-2}\left(\bigcup_{z \in Z(y)} D_1(x^\frown y^\frown z) \times D(x^\frown y^\frown z)\right) \geq \sum_{z \in Z(y)} \mu(D_1(x^\frown y^\frown z))\beta(\ell-1)$$
$$= \mu(D_0(x^\frown y))\beta(\ell-1) > (1-\bar{\delta})\beta(\ell-1).$$

In either case — either Inequality 4.12 or Inequality 4.13 — it follows that for any $y \in P_2 \setminus \Omega(x)$

$$\mu^{\ell-2}\left(\left\{\vec{W} \;\middle|\; y \in H^*_\ell(x, \vec{W})\right\}\right) < 1 - (1-\bar{\delta})\beta(\ell-1)$$

and so

$$\int_{P_2 \setminus \Omega(x)} \mu^{\ell-2}\left(\left\{\vec{W} \;\middle|\; y \in H^*_\ell(x, \vec{W})\right\}\right) d\sigma_2(y) < 1 - (1-\bar{\delta})\beta(\ell-1).$$

Hence

$$\int_{P_2} \mu^{\ell-2}\left(\left\{\vec{W} \;\middle|\; y \in H^*_\ell(x, \vec{W})\right\}\right) d\sigma_2(y) < 1 - (1-\bar{\delta})\beta(\ell-1) + \bar{\delta}$$

and therefore by Fubini's Theorem

$$\int_{\mathcal{P}(M)^{\ell-2}} \sigma_2(H^*_\ell(x, \vec{W})) d\mu^{\ell-2}(\vec{W}) < 1 - \beta(\ell-1)(1-\bar{\delta}) + \bar{\delta}.$$

Therefore, using Inequality 4.11 and Markov's Inequality,

$$\mu^{\ell-2}\left(\left\{\vec{W} \in \mathcal{P}(M)^{\ell-2} \;\middle|\; x \in H_\ell(\vec{W})\right\}\right) = \mu^{\ell-2}\left(\left\{\vec{W} \in \mathcal{P}(M)^{\ell-2} \;\middle|\; \sigma_2(H^*_\ell(x, \vec{W})) < \bar{\gamma}\right\}\right)$$
$$\geq 1 - \frac{1 - \beta(\ell-1)(1-\bar{\delta}) + \bar{\delta}}{\bar{\gamma}}$$

and comparing this with the definition of $\beta(\ell)$ establishes that the claim is proved. □

Claim 2. Let $\ell \geq 2$. If $x \in H_\ell(W_2, W_3, \ldots, W_{\ell-1})$ and $W_1 \supseteq E_x \setminus E_{x^*(j-\ell)}$ then

$$\Psi^{-1}\left(\left(\bigcup_{i=1}^{j-\ell} E_{x^*_1(i)} \setminus E_{x^*(i-1)}\right) \cup \bigcup_{i=1}^{\ell-1} W_i\right)\langle x^*(j-\ell)\rangle \Subset_{\bar{\gamma}} B\langle x^*(j-\ell)\rangle$$

using the convention that $\bigcup_{i=1}^{0} E_{x^*_1(i)} \setminus E_{x^*(i-1)} = \emptyset$ in the case that $\ell = j$.

Proof. This is established by induction on ℓ. If $\ell = 2$ then define $E' = \bigcup_{i=1}^{j-1} E_{x^*_1(i)} \setminus E_{x^*(i-1)}$ and note that

$$\left(\bigcup_{i=1}^{j-2} E_{x^*_1(i)} \setminus E_{x^*(i-1)}\right) \cup W_1 \supseteq E'. \tag{4.14}$$

Since $x \in C_2$ it follows that $\sigma_2(\Omega(x)) < \bar{\delta}$ and $B\langle x\rangle \neq \emptyset$. Furthermore, if $y \in \Omega(x)$ then $\sigma_1(\Theta(x^\frown y)) > \bar{\delta}$ because of the special definition in the case $x \in (P_1 \times P_2)^{j-2} \times P_1$. In other words, noting that $x^\frown y \in (P_1 \times P_2)^{j-1}$ and paying attention to the definition of C_1

$$\sigma_2\left(\left\{y \in P_2 \;\middle|\; \sigma_1\left(\left\{z \in P_1 \;\middle|\; E' \cap E_{x^\frown y^\frown z} = \emptyset\right\}\right) > \bar{\delta}\right\}\right) < \bar{\delta}$$

and so, recalling that $E_{x^\frown y^\frown z}$ is the image of $\{x^\frown y^\frown z\} \times B\langle x^\frown y^\frown z\rangle$ under Ψ,

$$\sigma_2\left(\left\{y \in P_2 \;\middle|\; \sigma_1\left(\left\{z \in P_1 \;\middle|\; \Psi^{-1}E'\langle x^\frown y^\frown z\rangle = \emptyset\right\}\right) > \bar{\delta}\right\}\right) < \bar{\delta}$$

and so $x^*_1(j-1)$ witnesses that $\Psi^{-1}(E')\langle x^*(j-2)\rangle \Subset_{\bar{\delta}} B\langle x^*(j-2)\rangle$ according to Alternative 3.1 of Definition 3.1. Because $\bar{\delta} < \bar{\gamma}$ and Inclusion 4.14 holds, the result now follows.

For the induction step suppose the claim has been proved for $\ell - 1$ and that $W_1 \supseteq E_x \setminus E_{x^*(j-\ell)}$ and $x \in H_\ell(W_2, W_3, \ldots, W_{\ell-1})$. From the induction hypothesis it follows that if $E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W_2$ and $x^\frown y^\frown z \in H_{\ell-1}(W_3, W_4, \ldots, W_{\ell-1})$ then

$$\Psi^{-1}\left(\left(\bigcup_{i=1}^{j-(\ell-1)} E_{(x^\frown y^\frown z)^*_1(i)} \setminus E_{(x^\frown y^\frown z)^*(i-1)}\right) \cup \bigcup_{i=2}^{\ell-1} W_i\right) \langle x^\frown y\rangle \Subset_{\bar{\gamma}} B\langle x^\frown y\rangle.$$

Note that $(x^\frown y^\frown z)^*_1(j-(\ell-1)) = x$ and, since $W_1 \supseteq E_x \setminus E_{x^*(j-\ell)}$, it follows that

$$\left(\bigcup_{i=1}^{j-(\ell-1)} E_{(x^\frown y^\frown z)^*_1(i)} \setminus E_{(x^\frown y^\frown z)^*(i-1)}\right) \cup \bigcup_{i=2}^{\ell-1} W_i \subseteq \left(\bigcup_{i=1}^{j-\ell} E_{x^*_1(i)} \setminus E_{x^*(i-1)}\right) \cup \bigcup_{i=1}^{\ell-1} W_i.$$

Hence,

$$\sigma_2\left(\left\{y \in P_2 \;\middle|\; \Psi^{-1}\left(\left(\bigcup_{i=1}^{j-\ell} E_{x^*_1(i)} \setminus E_{x^*(i-1)}\right) \cup \bigcup_{i=1}^{\ell-1} W_i\right) \langle x^\frown y\rangle \not\Subset_{\bar{\gamma}} B\langle x^\frown y\rangle\right\}\right) \le \sigma_2(H^*_\ell(x, W_2, W_3, \ldots, W_{\ell-1})) < \bar{\gamma}. \tag{4.15}$$

But note that $x^\frown y$ in Inequality 4.15 is the same as $x^*(j-\ell)^\frown x_1(j-(\ell-1))^\frown y$ and so $x_1(j-(\ell-1))$ is a witness to the fact that

$$\Psi^{-1}\left(\left(\bigcup_{i=1}^{j-\ell} E_{x^*_1(i)} \setminus E_{x^*(i-1)}\right) \cup \bigcup_{i=1}^{\ell-1} W_i\right) \langle x^*(j-\ell)\rangle \Subset_{\bar{\gamma}} B\langle x^*(j-\ell)\rangle$$

satisfies Alternative 3.1 of Definition 3.1. Moreover, $x \in H_\ell(W_2, W_3, \ldots, W_{\ell-1}) \subseteq C_\ell$ implies $B\langle x\rangle \neq \emptyset$. Hence $B\langle x^*(j-\ell)\langle x_1(j-(\ell-1))\rangle\rangle \neq \emptyset$ as required. $\square$

Now choose $x \in C_j$. If $j = 2$ then $x \in H_2$ and so Claim 2 and Inequality 4.5 can be directly applied to find W such that $E_x \subseteq W \subseteq M$ and $|W| \le k_3 < \bar{\xi}M$. If $j \ge 3$ then from Claim 1 and the fact that $\beta(j) > 1/2$ and Inequality 4.6 it follows that there are $\{W_i\}_{i=2}^{j-1}$ such that $|W_i| < 2\bar{\xi}M$ for each i and such that $x \in H_j(W_2, W_3, \ldots, W_{j-1})$. Let $W_1 = E_x$ and define $W = \bigcup_{i=1}^{j-1} W_i$. It follows from Inequality 4.5 that

$$|W| < 2(j-2)\bar{\xi}M + k_{2j+1} < (2j+1)\bar{\xi}M$$

and, moreover, the hypotheses of Claim 2 are satisfied by $W_1, W_2, \ldots W_{\ell-1}$ and x. Therefore

$$\Psi^{-1}\left(\left(\bigcup_{i=1}^{j-j} E_{x^*_1(i)} \setminus E_{x^*(i-1)}\right) \cup W\right) \langle x^*(j-j)\rangle \Subset_{\bar{\gamma}} B\langle x^*(j-j)\rangle$$

and so $\Psi^{-1}W \Subset_{\bar{\gamma}} B$.

Case Two. $C_j = \emptyset$.

The general scheme of the proof is the same as in the first case. However, the lack of symmetry between P_1 and P_2 requires some slight modifications of the argument. Define an ℓ-ary function H_ℓ on $\mathcal{P}(M)$ by induction on ℓ. If $\ell = 1$ then

$$H_1(W) = \left\{x \in (P_1 \times P_2)^{j-1} \;\middle|\; B\langle x\rangle = \emptyset \text{ or } (\exists z \in \Theta(x))\ E_{x^\frown z} \setminus E_x \subseteq W\right\}$$

and if $\ell \ge 2$ define $H_\ell(W_1, W_2, \ldots, W_\ell)$ to be the set of all $x \in (P_1 \times P_2)^{j-\ell}$ such that

$$\sigma_1\left(H^*_\ell(x, W_1, W_2, \ldots, W_\ell)\right) < \bar{\gamma} \tag{4.16}$$

where $H^*_\ell(x, W_1, W_2, \ldots, W_\ell)$ is defined to be

$$\{y \in P_1 \mid (\forall z \in P_2)\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \not\subseteq W_1 \text{ or } x^\frown y^\frown z \notin H_{\ell-1}(W_2, \ldots, W_\ell)\}. \tag{4.17}$$

Claim 3. If $x \in (P_1 \times P_2)^{j-\ell}$ and either

- $B\langle x\rangle = \emptyset$

- $\ell = 1$ and $\sigma_1(\Theta(x)) \geq \bar{\delta}$
- or $\ell > 1$ and $\sigma_1(\Theta(x)) < \bar{\delta}$

then $\mu^\ell(D(x)) > \beta(\ell)$ where $D(x) = \left\{\vec{W} \in \mathcal{P}(M)^\ell \;\middle|\; x \in H_\ell(\vec{W})\right\}$.

Proof. Begin by considering the case that $\ell = 1$ in which there are two possibilities. The first is that $B\langle x\rangle = \emptyset$ and in this case it suffices to observe that $D(x) = \mathcal{P}(M)$. In the second case $\sigma_1(\Theta(x)) \geq \bar{\delta}$ and so there is then $Z \in [\Theta(x)]^{L_1}$ such that $\{E_{x^\frown z} \setminus E_x\}_{z\in Z}$ is a pairwise disjoint family and hence

$$\mu\left(\{W \subseteq M \mid (\forall z \in Z)\ E_{x^\frown z} \setminus E_x \not\subseteq W\}\right) < \bar{\delta}$$

by Inequality 4.3. Hence $\mu(\{W \subseteq M \mid x \in H_1(W)\}) > 1 - \bar{\delta} \geq \beta(1)$.

On the other hand, if $\ell > 1$ and $\sigma_1(\Theta(x)) < \bar{\delta}$ then note that if $y \in P_1 \setminus \Theta(x)$ then $x^\frown y \notin C_\ell$ and so there are again two possibilities: either $B\langle x^\frown y\rangle = \emptyset$ or $\sigma_2(\Omega(x^\frown y)) \geq \bar{\delta}$. In the second case there is $Z \in [\Omega(x^\frown y)]^{L_{2(\ell-1)}}$ such that $\{E_{x^\frown y^\frown z} \setminus E_{x^\frown y}\}_{z\in Z}$ is a pairwise disjoint family. Therefore

$$\mu(D_0(x^\frown y)) \geq 1 - \left(1 - \bar{\xi}^{k_{2(\ell-1)}}\right)^{L_{2(\ell-1)}} > 1 - \bar{\delta}$$

where $D_0(x^\frown y)$ is defined to be $\{W \subseteq M \mid (\exists z \in Z)\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W\}$. Let $\{D_1(x^\frown y^\frown z)\}_{z\in Z}$ be a partition of $D_0(x^\frown y)$ such that if $W \in D_1(x^\frown y^\frown z)$ then $E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W$. Moreover, since $Z \subseteq \Omega(x^\frown y)$ it follows that either $\ell = 2$ and $\sigma_1(\Theta(x^\frown y^\frown z)) \geq \bar{\delta}$ for all $z \in Z$ or, $\ell > 2$ and $\sigma_1(\Theta(x^\frown y^\frown z)) < \bar{\delta}$ for all $z \in Z$. In either case the hypothesis of the claim holds for $x^\frown y^\frown z \in (P_1 \times P_2)^{j-(\ell-1)}$ and so the induction hypothesis implies that $\mu^{\ell-1}(D(x^\frown y^\frown z)) > \beta(\ell - 1)$ for any $z \in Z$. Therefore

$$\begin{aligned}\mu^\ell\left(\{(W_1, W_2, \ldots, W_\ell) \mid (\exists z)\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W_1 \text{ and } x^\frown y^\frown z \in H_{\ell-1}(W_2, W_3, \ldots, W_\ell)\}\right) \geq \\ \mu^\ell\left(\bigcup_{z\in Z} D_1(x^\frown y^\frown z) \times D(x^\frown y^\frown z)\right) \geq \sum_{z\in Z} \mu(D_1(x^\frown y^\frown z))\beta(\ell - 1) \\ = \mu(D_0(x^\frown y))\beta(\ell - 1) > (1 - \bar{\delta})\beta(\ell - 1)\end{aligned} \tag{4.18}$$

for all $y \in P_1 \setminus \Theta(x)$.

If $B\langle x^\frown y\rangle = \emptyset$ then $E_z = \emptyset$ for any $z \supseteq x^\frown y$. It then follows from the induction hypothesis that

$$\begin{aligned}\mu^\ell\left(\{(W_1, W_2, \ldots, W_\ell) \mid (\exists z)\ E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W_1 \text{ and } x^\frown y^\frown z \in H_{\ell-1}(W_2, W_3, \ldots, W_\ell)\}\right) \geq \\ \mu^\ell\left(\mathcal{P}(M) \times D(x^\frown y^\frown z)\right) \geq \beta(\ell - 1)\end{aligned} \tag{4.19}$$

The rest of the proof of Claim 3 is the same as the proof of Claim 1 with the probability space (P_1, σ_1) playing the role of (P_2, σ_2) and ℓ in place of $\ell - 2$. □

Claim 4. If $x \in H_\ell(W_1, W_2, \ldots, W_\ell)$ then

$$\Psi^{-1}\left(E_\emptyset \cup \bigcup_{i=1}^{j-\ell}\left(E_{x^*(i)} \setminus E_{x_1^*(i)}\right) \cup \bigcup_{i=1}^{\ell} W_i\right)\langle x\rangle \Subset_{\bar{\gamma}} B\langle x\rangle.$$

Proof. This is also proved by induction on ℓ. If $x \in H_1(W)$ and $B\langle x\rangle = \emptyset$ the result is trivial. Otherwise there is some $z \in \Theta(x)$ such that $E_{x^\frown z} \setminus E_x \subseteq W$. Since $z \in \Theta(x)$ it follows that $x^\frown z \in C_1$ and so

$$\bigcup_{i=1}^{j-1}\left(E_{x_1^*(i)} \setminus E_{x^*(i-1)}\right) \cap E_{x^\frown z} = \emptyset \tag{4.20}$$

and since

$$E_{x^\frown z} \subseteq (E_{x^\frown z} \setminus E_x) \cup E_\emptyset \cup \left(\bigcup_{i=1}^{j-1}\left(E_{x^*(i)} \setminus E_{x_1^*(i)}\right) \cup \left(E_{x_1^*(i)} \setminus E_{x^*(i-1)}\right)\right)$$

it follows from Identity 4.20 that

$$E_{x^\frown z} \subseteq (E_{x^\frown z} \setminus E_x) \cup E_\emptyset \cup \bigcup_{i=1}^{j-1} \left(E_{x^*(i)} \setminus E_{x_1^*(i)}\right) \subseteq W \cup E_\emptyset \cup \bigcup_{i=1}^{j-1} \left(E_{x^*(i)} \setminus E_{x_1^*(i)}\right)$$

or, in other words,

$$\emptyset \neq B\langle x^\frown z\rangle \subseteq \Psi^{-1}\left(W \cup E_\emptyset \cup \bigcup_{i=1}^{j-1} \left(E_{x^*(i)} \setminus E_{x_1^*(i)}\right)\right)\langle x^\frown z\rangle.$$

Therefore z is a witness to the fact that

$$\Psi^{-1}\left(E_\emptyset \cup \bigcup_{i=1}^{j-1} \left(E_{x^*(i)} \setminus E_{x_1^*(i)}\right) \cup W\right)\langle x\rangle \Subset_{\bar\gamma} B\langle x\rangle.$$

Now assume that $\ell > 1$ and $x \in H_\ell(W_1, W_2, \ldots, W_\ell)$. In other words, since $\ell > 1$ it must be the case that Inequality 4.16 holds. From the induction hypothesis and Definition 4.17 it follows that if $y \in P_1 \setminus H_\ell^*(x, W_1, W_2, \ldots, W_\ell)$ then there exists $z \in P_2$ such that $E_{x^\frown y^\frown z} \setminus E_{x^\frown y} \subseteq W_1$ and

$$\Psi^{-1}\left(E_\emptyset \cup (E_{x^\frown y^\frown z} \setminus E_{x^\frown y}) \cup \bigcup_{i=1}^{j-\ell}(E_{x^*(i)} \setminus E_{x_1^*(i)}) \cup \bigcup_{i=2}^{\ell} W_i\right)\langle x^\frown y^\frown z\rangle \Subset_{\bar\gamma} B\langle x^\frown y^\frown z\rangle$$

and therefore $P_1 \setminus H_\ell^*(x, W_1, W_2, \ldots, W_\ell)$ is a subset of

$$\left\{y \in P_1 \;\middle|\; (\exists z \in P_2)\ \Psi^{-1}\left(E_\emptyset \cup \bigcup_{i=1}^{j-\ell}(E_{x^*(i)} \setminus E_{x_1^*(i)}) \cup \bigcup_{i=1}^{\ell} W_i\right)\langle x^\frown y^\frown z\rangle \Subset_{\bar\gamma} B\langle x^\frown y^\frown z\rangle\right\}$$

and Claim 4 now follows from Definition 3.1 and Inequality 4.16. □

Note that the second alternative of the hypothesis of Claim 3 includes the possibility that $j = \ell$ and $x = \emptyset$. Indeed, in the case that $j = \ell$ the hypothesis is automatically satisfied since $\sigma_1(\Theta(\emptyset)) = \sigma_1(C_j) = \sigma_1(\emptyset) = 0$. (Recall that the proof of Sublemma4.1 began by eliminating the case $j = 1$.) Hence, by an argument using Inequalities 4.6 and 4.5 as in the first case, there is $(W_1, W_2, \ldots, W_j) \in \mathcal{P}(M)^j$ such that $\emptyset \in H_j(W_1, W_2, \ldots, W_j)$ and $|W| < (2j+1)\bar\xi M$ where $W = E_\emptyset \cup \bigcup_{i=1}^{j} W_i$. From Claim 4 it follows that

$$\Psi^{-1}\left(\bigcup_{i=1}^{j-j}(E_{x^*(i)} \setminus E_{x_1^*(i)}) \cup W\right)\langle\emptyset\rangle \Subset_{\bar\gamma} B\langle\emptyset\rangle$$

and hence $\Psi^{-1}W \Subset_{\bar\gamma} B$. This establishes that Sublemma 4.1 holds. □

With Sublemma 4.1 in place it is now possible to consider the general case. Given $B \subseteq (P_1 \times P_2)^j$ let $B_0 = \left\{z \in B \;\middle|\; k_1 \geq |E_{z_1^*(j)}|\right\}$ and let $B_1 = B \setminus B_0$. It has already been shown that there is $W \subseteq M$ such that $|W| < (2j+1)\bar\xi M$ and $(\Psi \restriction B_0)^{-1}W \Subset_{\bar\gamma} B_0$. It will be shown by induction on j that if $(\Psi \restriction B_0)^{-1}(W) \Subset_{\bar\gamma} B_0$ then

$$\mu(\{U \subseteq M \mid \Psi^{-1}(W \cup U) \Subset_\gamma B\}) > \beta(j) \tag{4.21}$$

noticing the role γ rather than $\bar\gamma$. Observe that the hypothesis includes the case that $B_0 = \emptyset$.

If $j = 1$ let $B_k^* = \{x \in P_1 \mid \emptyset \neq B\langle x\rangle \subseteq B_k\}$ for $k < 2$. Then by analyzing the meaning of Definition 3.1 for $(\Psi \restriction B_0)^{-1}(W) \Subset_{\bar\gamma} B_0$ there are two cases to consider corresponding to Conditions 3.1 and 3.2. The first is that there is some $x \in B_0^*$ such that $\sigma_2(\{y \in P_2 \mid \Psi(x^\frown y) \notin W\}) < \bar\gamma$. Then, since $B_0\langle x\rangle = B\langle x\rangle$, it is immediate that $\Psi^{-1}(W) \Subset_{\bar\gamma} B$ and hence $\Psi^{-1}(W \cup U) \Subset_\gamma B$ for every $U \subseteq M$. (Note that the assertion that $B_0\langle x\rangle = B\langle x\rangle$ is only true because $j = 1$.) The second possibility is that $\sigma_1(\{x \in P_1 \mid B_0\langle x\rangle \neq \emptyset \text{ and } (\forall y \in B_0\langle x\rangle)\ \Psi(x^\frown y) \notin W\}) < \bar\gamma$. In this case for each $x \in B_1^*$

$$\mu(\{U \subseteq M \mid U \cap E_x = \emptyset\}) = (1-\bar\xi)^{|E_x|} \leq (1-\bar\xi)^{k_1} < \bar\delta$$

and so

$$\int_{\mathcal{P}(M)} \sigma_1\left(\{x \in B_1^* \mid U \cap E_x = \emptyset\}\right) d\mu(U) = \int_{B_1^*} \mu(\{U \subseteq M \mid U \cap E_x = \emptyset\}) d\sigma_1(x) < \bar{\delta}$$

and so

$$\mu\left(\{U \subseteq M \mid \sigma_1(\{x \in B_1^* \mid U \cap E_x = \emptyset\}) < \bar{\gamma}\}\right) > 1 - \frac{\bar{\delta}}{\bar{\gamma}} \geq \beta(1).$$

This immediately implies that $\mu(\{U \subseteq M \mid \Psi^{-1}(U) \Subset_{\bar{\gamma}} B_1\}) > \beta(1)$ and hence

$$\mu\left(\{U \subseteq M \mid \Psi^{-1}(U \cup W) \Subset_{\gamma} B\}\right) > \beta(1)$$

as required.

If $j > 1$ and $(\Psi \restriction B_0)^{-1} W \Subset_{\bar{\gamma}} B_0$ then according to Definition 3.1 there are two cases to consider corresponding to Conditions 3.1 and 3.2. The first, corresponding to Condition 3.1, is that there is some $x \in P_1$ such that $B_0\langle x\rangle \neq \emptyset$ and $\sigma_2(P_2 \setminus G) < \bar{\gamma}$ where

$$G = \left\{y \in P_2 \mid (\Psi \restriction B_0)^{-1}(W)\langle x^\frown y\rangle \Subset_{\bar{\gamma}} B_0\langle x^\frown y\rangle \text{ or } B_0\langle x^\frown y\rangle = \emptyset\right\}.$$

Observe that if $B_0\langle x^\frown y\rangle = \emptyset$ then trivially $(\Psi \restriction B_0\langle x^\frown y\rangle)^{-1}(W) \Subset_{\bar{\gamma}} B_0\langle x^\frown y\rangle$ and so, by the induction hypothesis, for each $y \in G$

$$\mu\left(\{U \subseteq M \mid \Psi^{-1}(W \cup U)\langle x^\frown y\rangle \Subset_{\gamma} B\langle x^\frown y\rangle\}\right) \geq \beta(j-1).$$

Integrating and applying Fubini's Theorem yields that

$$\int_{\mathcal{P}(M)} \sigma_2\left(\{y \in G \mid \Psi^{-1}(W \cup U)\langle x^\frown y\rangle \not\Subset_{\gamma} B\langle x^\frown y\rangle\}\right) d\mu(U) =$$

$$\int_{G} \mu\left(\{U \subseteq M \mid \Psi^{-1}(W \cup U)\langle x^\frown y\rangle \not\Subset_{\gamma} B\langle x^\frown y\rangle\}\right) d\sigma_2(y) \leq 1 - \beta(j-1)$$

and applying Markov's Inequality yields that

$$\mu\left(\{U \subseteq M \mid \sigma_2\left(\{y \in G \mid \Psi^{-1}(W \cup U)\langle x^\frown y\rangle \not\Subset_{\gamma} B\langle x^\frown y\rangle\}\right) < \bar{\gamma}\}\right) > 1 - \frac{1 - \beta(j-1)}{\bar{\gamma}}$$

and the right hand side of this inequality is at least $\beta(j)$. This suffices because the inequality $\sigma_2(P_2 \setminus G) < \bar{\gamma}$ implies that

$$\mu\left(\{U \subseteq M \mid \sigma_2(\{y \in P_2 \mid \Psi^{-1}(W \cup U)\langle x^\frown y\rangle \not\Subset_{\gamma} B\langle x^\frown y\rangle\}) < \gamma\}\right) > \beta(j)$$

and hence $\mu\left(\{U \subseteq M \mid \Psi^{-1}(W \cup U) \Subset_{\gamma} B\}\right) > \beta(j)$.

The other possibility, corresponding to Condition 3.2, is that if G is defined to be

$$\left\{x \in P_1 \mid (\exists y \in P_2)\ \Psi^{-1}(W)\langle x^\frown y\rangle \Subset_{\bar{\gamma}} B_0\langle x^\frown y\rangle \text{ or } B_0\langle x\rangle = \emptyset\right\}$$

then $\sigma_1(P_1 \setminus G) < \bar{\gamma}$. Choose any function $y : G \to P_2$ such that if $B_0\langle x\rangle \neq \emptyset$ then $\Psi^{-1}(W)\langle x^\frown y(x)\rangle \Subset_{\bar{\gamma}} B_0\langle x^\frown y(x)\rangle$ and, if $B\langle x\rangle \neq \emptyset = B_0\langle x\rangle$ then $B\langle x^\frown y(x)\rangle \neq \emptyset$. For each $x \in G$ applying the induction hypothesis to either $\Psi \restriction B_0\langle x^\frown y(x)\rangle$ or $\Psi \restriction B\langle x^\frown y(x)\rangle$ yields that

$$\mu\left(\{U \subseteq M \mid \Psi^{-1}(W \cup U)\langle x^\frown y(x)\rangle \Subset_{\gamma} B\langle x^\frown y(x)\rangle\}\right) \geq \beta(j-1).$$

Integrating over x and applying Fubini's Theorem and Markov's Inequality as before yields that

$$\mu\left(\{U \subseteq M \mid \sigma_1\left(\{x \in G \mid \Psi^{-1}(W \cup U)\langle x^\frown y(x)\rangle \not\Subset_{\gamma} B\langle x^\frown y(x)\rangle\}\right) < \bar{\gamma}\}\right) > 1 - \frac{1 - \beta(j-1)}{\bar{\gamma}}$$

and the argument can now be completed exactly as in the previous case.

Finally apply Inequality 4.6 and Sublemma 4.1 to find W and U such that $\Psi^{-1}(W \cup U) \Subset_{\gamma} B$ and $|W| < (2j+1)\bar{\xi}M$ and $|U| < 2\bar{\xi}M$. Hence $|W \cup U| < 2(j+1)\bar{\xi}M = \xi M$ as required. □

Corollary 4.1. *For any $n \in \mathbb{N}$ there is $M(n) \in \mathbb{N}$ such that $\nu(\mathcal{P}_{2^{-n}}(M(n))) > n$.*

Proof. Let $M(n) = M(2^{-n}, \gamma, n)$ be as defined in Lemma 4.2 with $\gamma = \min(\epsilon(j), K^p j^p J^*(j)^{p/q})/2$ and apply Lemma 4.2 and the version of Lemma 3.3 for (ϵ^*, L^*)-fine probability spaces. □

5. Applications and Questions

In order to apply Theorem 2.1 it is necessary to look for families satisfying the hypotheses of Definition 2.1 and Inequality 2.2. Several such exist in the literature.

Theorem 5.1 (Stein [12])**.** *If Λ is as defined in Example 2.1 and if $k \geq 3$ then the maximal operator M_Λ is bounded*

Theorem 5.2 (Bourgain [1], But see also Marstrand [8])**.** *If Λ is as defined in Example 2.1 and if $k = 2$ then the maximal operator M_Λ is bounded*

Theorem 5.3 (Falconer [3] and Marstrand [7])**.** *If Λ is as defined in Example 2.2 then the maximal operator M_Λ is bounded.*

Corollary 5.1. *For $k \geq 2$ it is consistent that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ spheres in $\mathbb{R}^k$ whose union is not null in $\mathbb{R}^k$.*

Proof. Use Theorem 2.1 and Theorem 5.1 if $k \geq 3$ and Theorem 5.2 if $k = 2$. □

Corollary 5.2. *It is consistent that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ planes in the Euclidean 3-space whose union is not null.*

Proof. Use Theorem 2.1 and Theorem 5.3. □

Since Corollaries 5.1 and 5.2 can both be obtained by appealing to Besicovitch duality arguments the key point here is that the theorems quoted actually apply to much broader classes of sets. For example, Theorem 5.2 actually yields that the maximal operator defined for the boundary of a smooth centrally symmetric convex body in $\mathbb{R}^2$ is bounded and hence one gets as a corollary that for any such set B it is consistent that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ homothetic copies of B whose union is not null.

It is curious that, while the analogue of Corollary 5.2 for lines in the plane — in other words Komjath's original question — remains true by the results of [11], Theorem 2.1 can not be applied because the associated maximal operator is not bounded. The non-boundedness follows from the existence of a Besicovitch set, a null set in the plane containing line segments in all directions (see §7.1 of [4]). In this context it is worth mentioning a result of Talagrand [13] which shows the existence of an analogue of the Besicovitch set for circles. In particular, he shows that there exists a family of circles of the plane whose union is of measure zero, but such that the set of the centres is of non-zero linear measure. This provides the same sort of obstacle as the Besicovitch set to proving the consistency of: Every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ circles in the plane *with centres on the x-axis* whose union is not null. One might hope to obtain this by applying the dualty betweeen lines and circles via complex inversion to the answer to Komjath's question. However, this would require that in the model answering Komjath's question the set of lines of positive measure would have to be of a restricted type in order that the circles obtained by complex inversion all had their centres on a straight line. This remains an open question with the possibility that Talagrand's construction can be used to provide a positive result.

The introduction remarked that a broad class of problems concerns determining which homemorphic images W of the unit interval in $\mathbb{R}^n$ have the property that $\aleph_1$ isometric copies of W can have non-null union while $\aleph_1$ sized sets of reals are all null. Even very natural examples of W provide open problems.

Question 5.1. Is it consistent that every set of reals of size $\aleph_1$ is null yet there are $\aleph_1$ spirals in $\mathbb{R}^3$ whose union is not null?

This is actually a family of questions depending on precisely what is meant by a spiral. If

$$\Lambda(t) = \{(t\cos(x), t\sin(x), tx) \mid 0 \leq x \leq 1\}$$

for $t \in [1, 2]$ then Pramanik and Seeger have shown [9] that the associated maximal function is bounded. However, Condition 3 of Definition 3 does not hold in this case. On the other hand, if

$$\Lambda(t, s) = \{(t\cos(x), t\sin(x), sx) \mid 0 \leq x \leq 1\}$$

for $(t, s) \in [1, 2]^2$ then it can be shown that Condition 3 of Definition 2.1 does hold, but the associated maximal function is not known to bounded. However [9] does have results in this direction.

Finally, there is the question of establishing consistent inequalities such as $\mathrm{add}(\mathcal{P}^3) \neq \mathrm{add}(\mathcal{C}^2)$, in other words inequalities involving cardinal invariants for both of which the associated maximal operator is bounded. Several difficulties would have to be overcome to make progress in this direction. First, substantial changes to the forcing partial order would be required since the objective now would be to cover, for example, ground model circles with a null set while keeping the measure of the ground model lines positive. Assuming this could be done, a more serious and potentially more interesting challenge, a challenge to harmonic analysis, would arise in finding the correct boundedness theorems for maximal operators.

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