HISTORY OF THE CONTINUUM IN THE 20^{th} CENTURY

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Abstract. This article provides a history of the continuum in the twentieth century. By "the continuum" is meant the cardinality of the real numbers as well as other cardinal numbers and combinatorial structures that are related to the continuum. Examples of the cardinals considered include the least number of meagre sets required to cover the real line, or the smallest cardinality of a non-measurable set. The combinatorial structures considered will mostly be associated with the family of subsets of the integers under almost inclusion or the family of sequences of integers under almost domination.

Contents

1. Introduction

The subject of this history is the cardinality of the continuum, as described by Georg Cantor, as well as the various other cardinal numbers defined by combinatorial structures associated with the continuum. Among these are the cardinals of various subsets of the ideals of meagre and Lebesgue null subsets of the reals. Typical examples of cardinals arising in this context are the least cardinal of a non-measurable set of reals or the least number of meagre sets whose union covers the reals. On the other hand, the combinatorial structures giving rise to cardinals are usually extremal objects embedded in simply definable equivalence relations or partial orders associated with the continuum. Typical examples include maximal families of subsets of $\mathbb{N}$ with all pairwise intersections being finite, or unbounded sets of sequences of natural numbers under the ordering of eventual dominance of minimal cardinality. Other examples of combinatorial structures associated with the continuum are sets of reals with unusual properties; *Lusin sets* — these are uncountable sets whose intersection with each meagre set is countable — provide typical examples of such sets. However, the theory of ultrafilters will play only a minor role in this discussion since these objects belong to a different realm than the continuum. As well, even though the descriptive set theory of the reals is very closely linked to the continuum, this topic would require its own chapter to be treated adequately.

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Instead, this article will trace the growing understanding of the continuum, as well as its related cardinals and associated combinatorial structures, starting at the beginning of the twentieth century and ending at the start of the twenty-first century. However, it would be more accurate to describe this as a history of the documentation of the subject as recorded in the published literature. No attempt has been made to interview the surviving contributors to the development of the subject, nor will this article attempt to settle controversial questions of provenance. Instead, the focus of this history is on the questions that stimulated research on the continuum and the often long chains of incremental progress that eventually lead to greater understanding and, in many cases, solutions to longstanding, open problems

Moreover, this is not intended to be a comprehensive history providing a catalogue of all significant results obtained in the last century. The reader looking for that much detail is advised to consult [25] or [13] as a starting point for further investigations into the research literature. The reader with no knowledge of the subject would also benefit from at least a superficial reading of [25] before turning to this history. For the benefit of such readers much of the standard terminology and definitions have been provided throughout the text; although, some of the technical definitions that are not central to the development of the subject are not provided. However, definitions are provided at that point in the text where the explanation contributes most to the development of the story rather than at their first appearance. An index has been provided, however, to help the reader searching for a definition or an explanation of some terminology. On the other hand, the full names of individuals mentioned in this history are used only at their first occurrence in the story and the index can be used to find these occurrences

2. Hilbert's Address

It is difficult to imagine that the story of the continuum in the twentieth century could start anywhere other than David Hilbert's 1900 address [81] in Paris. The very first of his celebrated list of problems presented at the International Congress of Mathematicians is the Continuum Hypothesis, which, by this point in time, had already been well studied. As Hilbert says, "The investigations of Cantor on such assemblages of points suggest a very plausible theorem, which nevertheless, in spite of the most strenuous efforts, no one has succeeded in proving."[1]. Hilbert goes on to emphasize a possible line of attack on the problem which, while not solving the problem, does serve as a suitable prelude for much of the developments in the study of the continuum that were to unfold throughout the coming century. In his address, Hilbert focusses his attention on "another very remarkable statement of Cantor's which stands in the closest connection with the theorem mentioned and which, perhaps, offers the key to its proof."[2] He is, of course, making reference here to the well ordering of the reals, a subject that would occupy the minds of mathematicians concerned with the foundations of their subject well into the coming century. However, Hilbert's interest here was not with foundations, but with understanding how the new concept of well ordering could be used to deal with existing problems such as that of whether $2^{\aleph_0} = \aleph_1$.

Both the novelty of the concept as well as the importance Hilbert attaches to it are underlined by the detailed description of what a well ordering of the reals would look like. This is to set up the question of "whether the totality of all numbers may not be arranged in another manner so that every partial assemblage may have a first element ... It appears to me most desirable to obtain a direct proof of this ... perhaps by actually giving an arrangement of numbers such that in every partial system a first

[1] Die Untersuchungen von Cantor über solche Punktmengen machen einen Satz sehr wahrscheinlich, dessen Beweis jedoch trotz eifrigster Bemühungen bisher noch Niemanden gelungen ist ... *The translation here, as well as later, is from* [82].

[2] Es sei noch eine andere sehr merkwürdige Behauptung Cantors erwähnt, die mit dem genannten Satze in engstem Zusammenhange steht und die vielleicht den Schlüssel zum Beweise dieses Satzes liefert.

number can be pointed out."[3] Later developments in descriptive set theory would have much to say on the extent to which Hilbert's plan could be carried out, but the importance from the point of view of the history of the continuum is Hilbert's clear vision of the potential connection between combinatorial structures, such as well ordered sets, and the topological or set theoretic continuum. His hope is that this will provide "a new bridge between the countable assemblage and the continuum."[4]

By the time of Hilbert's address it had already been more than 15 years since Cantor had concentrated his attention on perfect sets [41] and had shown them all to have the same cardinality as the continuum. It would be another 8 years after Hilbert's address before Felix Bernstein would make use [20] of the Axiom of Choice to construct what is now known as a Bernstein set, an uncountable set such that neither it nor its complement contains a perfect set. While Bernstein's construction put an end to any hope of establishing the Continuum Hypothesis using Cantor's theorem on the cardinality of perfect sets, it did provide another tool to be used in investigating the "new bridge" of Hilbert's address, between the countable and the continuum. This line of reasoning, looking for connections between infinite combinatorial objects and the continuum can be traced throughout the twentieth century. On one hand, advances in understanding the continuum followed from results such as Pavel Alexandroff and Felix Hausdorff's independent proofs [3, 75] that uncountable Borel sets contain perfect sets. On the other hand, important information about the combinatorial nature of ω_1 was obtained from constructions such as Hausdorff's (ω_1, ω_1^*)–gap [74] or Stevo Todorčević's counterexample [184] to $\aleph_1 \to [\aleph_1]^2_{\aleph_1}$. The motivation behind such results is often the hope of gaining better understanding of the connection between these two facets of the continuum, understanding such as that gained from Robert M. Solovay's coding by almost disjoint families. It is the purpose of this history of the continuum to examine this line of research and the critical developments along it.

While Hilbert's address might provide a dramatic starting point, the truth of the matter is that there is no convenient anchor for a history of twentieth century developments on the continuum. However, it is instructive to begin investigations into this history by considering those questions in analysis that had attracted the attention of mathematicians with a set theoretic interest at the end of the nineteenth and start of the twentieth centuries, questions which would turn out to have a critical influence on developments in set theory and, eventually, would lead to the detailed study of cardinal invariants[5] of the continuum. A focal point, if not a starting point, for these investigations is found a year before Hilbert's address in the 1899 thesis of René Baire [6]. Baire had set himself the task of analyzing pointwise limits of continuous functions. Among many other results, this thesis contains Baire's Theorem [6] that no interval is of the first category, a theorem which 70 years later could be interpreted as a weak form of Martin's Axiom. The developments leading from Baire's Theorem through Martin's Axiom to the Proper Forcing Axiom and its variants will be discussed in §8.

His continuation of this work six years later would introduce Baire's classification of functions as a hierarchy of classes, each the pointwise limit of functions from previous classes. His sufficient condition for a function to belong to this classification inspired Nikolai Lusin's [108] construction, assuming the Continuum Hypothesis, of what has become to be known as a Lusin set, even though Paul Mahlo's [115] article with a similar construction appeared a year earlier. Almost a century later this remains a critical example delineating the behaviour of cardinal invariants of the continuum. It is a happy

[3]Denn, wenn wir als Teilmenge die Punkte einer endlichen Strecke mit Ausnahme des Anfangspunktes der Strecke ins Auge fassen, so besitzt diese Teilmenge jedenfalls kein frühestes Element. Es erhebt sich nun die Frage, ob sich die Gesamtheit aller Zahlen nicht in anderer Weise so ordnen läßt, daß jede Teilmenge ein frühestes Element, hat, d.h. ob das Continuum auch als wohlgeordnete Menge aufgefaßt werden kann, was Cantor bejahen zu müssen glaubt.

[4]der Beweis dieses Satzes würde mithin eine neue Brücke schlagen zwischen der abzählbaren Menge und dem Continuum.

[5]The use of the term *invariant* to describe cardinals associated with various structures on the continuum is not universally accepted. For example, in [25] Andreas Blass uses the term *characteristic* instead, arguing that these cardinals are not at all invariant and can assume different values in different models of set theory. Nevertheless, since the term *invariant* has found widespread acceptance and usage in the literature, this survey will not deviate from this tradition.

circumstance for the development of mathematics that Lusin's construction seems to have been ignored by mathematicians, such as Abram Besicovitch and Fritz Rothberger, working on related questions much later. While a Lusin set would have been sufficient for their purposes, the concentrated sets they constructed, assuming the same hypothesis as Lusin, were revealed half a century later to be subtly distinct from Lusin sets. Moreover, their method of construction lead to important developments in the study of the structure of $\mathbb{N}^{\mathbb{N}}$ under the $\leq^*$ relation and the related cardinal invariants $\mathfrak{b}$ and $\mathfrak{d}$ which will be examined in §5.

The study of $\mathbb{N}^{\mathbb{N}}$ under the $\leq^*$ has its roots in the work of Paul du Bois-Reymond [51] and [52] on rates of convergence, referred to by du Bois-Reymond as pantachies. It would be more than thirty years before du Bois-Reymond's work would attract serious attention. G. H. (Godfrey Harold) Hardy [71] provided an English exposition and considered the implication for power series and L-functions. Hausdorff, on the other hand, recognized the critical role set theory would play in the study of pantachies. This will be the starting point for §5.

While du Bois-Reymond, his contemporary Jean-Gaston Darboux and, of course, Cantor were engaged in the first, tentative examinations of questions in analysis that would lead to interesting set theoretical developments it would be a generation later before the study of such questions found any real resonance among analysts. Writing in 1922, Henri Léon Lebesgue acknowledges[6] that Camille Jordan has established the usefulness of set theory and incorporated it into the mainstream of mathematics, even though he does refer to Darboux's counterexamples as "une sorte de musée de monstruosités". Indeed, Hélène Gispert argues [63] that the French tradition of interest in the theory of functions was the source of the increased interest in set theoretic questions, especially in the work of Émile Borel, Lebesgue and Baire. Given the influence of French mathematics, it is no surprise to see this interest reflected beyond France's borders. Indeed, the analysts in Moscow and Warsaw took up this direction so enthusiastically that it is fair to say that set theoretic developments in these two centres soon eclipsed those of Paris.

The year 1920 witnessed the founding of the journal *Fundamenta Mathematicae* in Warsaw and even a casual scanning of the early editions reveals the intense interest in set theoretic questions during those years, both in Warsaw and in Moscow. Alongside the Polish mathematicians Kazimierz Kuratowski, Stefan Banach, Hugo Steinhaus and, of course, Wacław Sierpiński one also see the names of Lusin, Pavel Urysohn and Mikhail Suslin although Russian contributions are seen to have declined with the rise of Stalin by the end of the 1920's. An enlightening indication of the set theoretic interests of these two groups can be found in the problems section at the end of each of the early issues. Perhaps the best known of these questions, due to Suslin, is recorded on page 223 of the section *Problèmes* of the first issue of *Fundamenta Mathematicae*:

Question 2.1. Can the real line be characterized as the unique complete, dense in itself linear order with the countable chain condition?[7]

The ramifications of the answer to Suslin's question are profound and lead to mathematics far beyond what could have been imagined by Suslin and his contemporaries at the time. These will be discussed in §8. As well, one finds Sierpiński asking:

Question 2.2. Is there a partition of the unit interval into $\aleph_1$ Borel sets?[8]

The key idea to the solution, Hausdorff's (ω_1, ω_1^*)–gap, had already appeared some ten years earlier, albeit in a rather obscure journal. In the same list of problems Sierpiński also asks:

[6]On page 92 of [107].

[7]Un ensemble ordonné (linéairement, sans sauts at sans lacunes, possédant cette propriété que tout ensemble d'intervalles contenant plus qu'un élément) n'empiétant pas les un sur les autres est toujours au plus dénombrable, est il nécessairement un continu linéaire ordinaire?

[8]Existe-il une décomposition d'un intervalle en $\aleph_1$ ensembles mesurables (B), non vides, sans point commun deux à deux?

Question 2.3. Is there an uncountable set of reals every homeomorphic image of which is Lebesgue null?[9]

Almost as an afterthought he asks whether assuming that $2^{\aleph_0} = \aleph_1$ helps. The answers to both of these questions would be known considerably before the answer to Suslin's question.

Other questions seem considerably ahead of their time, given the little that was known about the value of the continuum, but, nevertheless were asked by Sierpiński in the first issue of *Fundamenta Mathematicae*:

Question 2.4. Is it possible, without assuming that $2^{\aleph_0} = \aleph_1$, to show that there are $\aleph_1$ Lebesgue null sets whose union is not null? What of meagre sets? What of analytic sets?[10]

Two issues later, in a similar spirit, Stanisław Ruziewicz asks:

Question 2.5. Are sets of cardinality less than the continuum necessarily meagre?[11]

In the language of cardinal invariants, of course, Sierpiński is asking about the additivity of the null and meagre ideals, while Ruziewicz is asking about the uniformity invariant of the ideal of meagre sets. All these questions will be considered in §6.

Sierpiński's final question is about the additivity of analytic sets, while, in the same list Lusin asks what the cardinality of a co-analytic set is? This second question would soon [113] be answered by Lusin and Sierpiński himself, but this line of investigation leads away from the study of cardinal invariants. However, it is worthy of note that these sort of questions seemed natural long before they could be answered.

The following question from the first issue of *Fundamenta*, due to Steinhaus, deserves special mention:

Question 2.6. Is it possible, without assuming that $2^{\aleph_0} = \aleph_1$, to show that there is a subset of the plane which is null in all vertical sections, yet whose complement is null in all horizontal sections?[12]

Sierpiński would show that the answer is negative and this answer would play a central role in his book [164] on the Continuum Hypothesis. In the fourth issue Kuratowski asked:

Question 2.7. Given a set which is nowhere meagre, is it possible to partition it into two disjoint sets, both of which are nowhere meagre?[13]

He mentions that this can be done by assuming either that the set in question is analytic or by assuming the Continuum Hypothesis but, that the answer for an arbitrary subset of the reals is not known. Partial answers would invoke large cardinal assumptions, even though Lusin's final answer would have no need of these. This question will be revisited in §6.

These questions, and ones like them, will play an important role in this effort to trace the central developments leading to our current understanding of the continuum since they can be viewed as giving meaning to questions such as the Continuum Hypothesis. Given the broad range of independence results concerning the continuum and its various invariants, there may be a tendency to view any results in this area as merely a formalist exercise. Indeed, Paul Cohen himself admits to such a perspective in an essay

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[9]Existe-il un ensemble linéaire non dénombrable E, tel que tout ensemble linéaire homéomorphe avec E soit de mesure lebesguienne nulle? Peut-on démontrer l'existence d'un tel ensemble même en admettant que $2^{\aleph_0} = \aleph_1$?

[10]Peut-on démontrer sans l'hypthèse du continu ($2^{\aleph_0} = \aleph_1$) qu'une somme de $\aleph_1$ ensembles de mesure lebesguienne nulle n'est pas nćessairement de mesure (L) nulle? Qu'une somme de $\aleph_1$ ensembles de première catégorie n'est pas nécessairement de première catégorie? Qu'un produit de $\aleph_1$ ensembles (A) n'est pas nécessairement un ensemble (A)?

[11]Un ensemble (linéaire) de puissance inférieure à celle du continu, est-il nécessairement de la première catégorie de M. Baire?

[12]Peut-on démontrer sans l'hypothèse du continu l'existence d'un ensemble plan qui est de la mesure lebesguienne nulle sur toute parallèle à l'axe d'abscisses et dont le complémentaire est de mesure nulle sur toute parallèle à l'axe d'ordonnées?

[13]A étant un ensemble de nombres réels qui n'est de 1 catégorie dans aucun intervalle, existe-il une décomposition: $A = B + C$, $B \times C = 0$ telle que ni B ni C ne soient de 1 catégorie dans aucun intervalle?

written shortly after obtaining the independence of the Continuum Hypothesis — the impact of the forcing technique he developed to do this will be discussed in §7 — and he acknowledges that Abraham Robinson's views on the subject have convinced him to accept the formalist position. Commenting on this [45] he says,

> *It is a choice which carries with it certain heavy weights. Certainly one of the heaviest is the admission that CH, perhaps the first significant question about uncountable sets which can be asked, has no intrinsic meaning.*

If, contrary to Cohen's view, there is any meaning to be found in statements about the continuum, then, at least part of this meaning will be found in the applications to which these statements are put. The applications of interest to set theorists shortly after the First World War have been outlined in this section. Subsequent sections will survey the developments stemming from these.

3. The Weak Continuum Hypothesis

Lusin's 1914 construction [109] using transfinite induction and assuming $2^{\aleph_0} = \aleph_1$ to enumerate the reals and perfect sets is, of course, now found in most text books on set theory. But it is of interest to reflect on his conclusion and the use to which it was to put to construct what is now known as a Lusin set. More than a decade earlier, Baire had examined the classification of functions as a hierarchy of classes, each the pointwise limit of functions from previous classes. His necessary condition for a function f to belong to this classification scheme was that given any perfect set P there is a set M which is meagre relative to P such that the restriction of f to $P \setminus M$ is continuous. It is easy to see that the characteristic function of a Lusin set has this property; indeed, given any perfect set P there is a countable set M such that the restriction of the characteristic function to $P \setminus M$ is constant. But how did Lusin show that the characteristic function of the set he constructed does not belong to Baire's classification scheme? Had he known that all Borel sets have the Perfect Set Property, this would have been immediate. But this fact was to be established only two years later, in 1916, independently by Alexandroff [3] and Hausdorff [75]. Along with the later result that Baire functions are Borel, it is easy to show that the characteristic functions of Lusin sets are not Baire. However, this was not available to Lusin in 1914. He had to rely on a less constructive pigeonhole argument using that $2^{\aleph_0} < 2^{\aleph_1}$. Various results throughout the coming century would provide further illustrations of the usefulness of this assumption.

It is interesting to note that in the section *Problèmes* of Volume 4 of *Fundamenta Mathematicae* Sierpiński asks in Question 24 about the cardinality of all functions satisfying Baire's condition. It seems very likely that the motivation for this question was rooted in considerations of the hypothesis used in Lusin's argument for constructing a function satisfying Baire's condition but not belonging to the Baire classification. It is worth noting though that already in 1921 Lusin had shown [110] that there are uncountable non-Borel sets that are meagre in every perfect set. Nevertheless, more than decade later, it is clear that this sort of question continued to occupy both Lusin and Sierpiński. In [109] Sierpiński returns to the hypothesis $2^{\aleph_0} < 2^{\aleph_1}$ since its failure is required for the following hypothesis [112] of Lusin to hold: Every set of reals of cardinality $\aleph_1$ is co-analytic. Martin's Axiom is discussed in §8 and in the paper introducing this axiom [85] it is shown that it implies that if $2^{\aleph_0} > \aleph_1$ then Lusin's Hypothesis holds if and only there is a real t such that $\aleph_1 = \aleph_1^{L[t]}$ and, hence, there are models of Martin's Axiom in which Lusin's Hypothesis is true.

However, the weak form of the Continuum Hypothesis had already been examined by Hausdorff in 1907 and found to be useful. Hausdorff's work on maximal linear orderings of sequences of integers under eventual dominance is examined in more detail in §5. A key development in this early work occurred in [73] in which Hausdorff established that these maximal linear orders — which, following du Bois-Reymond, he called *pantachies* — must have cardinality $2^{\aleph_0}$. Moreover, if such a pantachie happens to have no (ω_1, ω_1^*)–gaps — these are defined in §5 — then the cardinality of the pantachie must be $2^{\aleph_1}$. In other words, if $2^{\aleph_0} < 2^{\aleph_1}$ then there must be (ω_1, ω_1^*)–gaps in each pantachie. As it happens, this early application of the weak Continuum Hypothesis was superseded two years later by Hausdorff's critically important discovery [74] that (ω_1, ω_1^*)–gaps actually exist without having to appeal to any hypothesis other than the Axiom of Choice.

Around the same time that Lusin and Sierpiński were considering the weak Continuum Hypothesis in the context of co-analytic sets of cardinality $\aleph_1$, across the ocean the same axiom was finding applications to the theory of metrizability of general topological spaces. In 1937 F. Burton Jones showed that if $2^{\aleph_0} < 2^{\aleph_1}$ then every separable, normal Moore space is metrizable. The point in the argument at which the weak Continuum Hypothesis plays a role is in showing that there are far more instances of normality — namely, disjoint closed sets that need to be separated by disjoint open sets — than can be handled by the $2^{\aleph_0}$ open sets that are available in a separable space.

About the same time as Jones was working on questions of metrizability, Fritz Rothberger was making some his initial advances in the study of the structure $\mathcal{P}(\mathbb{N})$ under inclusion modulo a finite set as well as $\mathbb{N}^{\mathbb{N}}$ under eventual dominance which are described in more detail in §5. In the spirit of Hausdorff's results on the weak Continuum Hypothesis and (ω_1, ω_1^*)–gaps, Rothberger showed that [147] this hypothesis implies that ω_1–limits exist. Recall that an ω_1–limit is a $\subseteq^*$ descending family $\{A_\xi\}_{\xi\in\omega_1}$ for which there does not exist an infinite X such that $X \subseteq^* A_\xi$ for each $\xi \in \omega_1$. Once again the key step is to find a family of $2^{\aleph_1}$ descending families any two of which are eventually almost disjoint. Assuming the weak Continuum Hypothesis, it impossible to have an infinite set below each of these.

After stating some open problems suggested by his results, Rothberger comments[14] "Many propositions are known that *imply* Lusin's "second continuum hypothesis" but very little is known about *consequences* of Lusin's hypothesis." He refers here explicitly to Sierpiński's 1935 paper [109] dealing with Lusin's hypothesis that all sets of reals of cardinality $\aleph_1$ are co-analytic, but he is actually referring to the hypothesis $2^{\aleph_0} = 2^{\aleph_1}$.

In the same 1948 paper [147] Rothberger considers the following two related notions:

- A Q–set is an uncountable set of reals every subset of which is a relative F_σ.
- A *denumerable base* for a family $\mathcal{G}$ of $\mathbb{R}$-valued functions on a set X is a countable family $\mathcal{F}$ of $\mathbb{R}$-valued functions on X such that every function in $\mathcal{G}$ is the pointwise limit of functions from $\mathcal{F}$.

He goes on to show that there is a Q–set if and only if there is an uncountable set $X \subseteq \mathbb{R}$ for which $\mathbb{R}^X$ has a denumerable base. Providing more evidence for his comment on page 43, he shows that both these assertions fail unless Lusin's Second Continuum Hypothesis holds. R. H. Bing uses this idea in 1951 [24] to show that if there is a Q–set then there is a separable, normal, non-metrizable Moore space.

In spite of the extensive use made of the hypothesis that $2^{\aleph_0} < 2^{\aleph_1}$ in various contexts and by geographically diverse groups, there is a clear lack of new ideas until the revived interest in such problems witnessed by the forcing era. The new impetus for applications of the inequality $2^{\aleph_0} < 2^{\aleph_1}$ came from two sources, one was Ronald Jensen's detailed analysis of Kurt Gödel's constructible universe and the other was rooted in the theory of abelian groups.

One of the outgrowths of considering inner models continues the line of attack followed by Jones, Rothberger, Bing and other researchers who took advantage of the fact that if $2^{\aleph_0} < 2^{\aleph_1}$ then there are more uncountable sets of reals than there are reals, or objects coded by reals. Jensen and Robert Solovay gained an even deeper insight into this phenomenon by devising [85] a coding by almost disjoint sets. Given an almost disjoint family $\{A_\xi\}_{\xi\in\omega_1}$ with the property that for any $X \subseteq \omega_1$ there is a set $A \subseteq \mathbb{N}$ such that $A_\xi \cap A$ is finite if and only if $\xi \in X$, it is natural to associate with each subset of ω_1 a subset of $\mathbb{N}$ which codes it in this way. Using this they are able to show that in a countable transitive model $\mathfrak{M}$ of set theory which shares the same $\aleph_1$ with L and $A \subseteq \omega_1$ there is a Cohen extension $\mathfrak{M}[x]$, with the same cardinals as $\mathfrak{M}$, such that x is a real from which A is constructible.

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[14] On page 43 of [147].

However, following a different line of enquiry, in studying Gödel's condensation arguments Jensen formulated the following principle known as $\diamondsuit$: There is an indexed family $\{X_\xi\}_{\xi\in\omega_1}$ such that

- $X_\xi \subseteq \xi$ for each countable limit ordinal ξ
- for any $X \subseteq \omega_1$ there is (a stationary set of) ξ such that $X \cap \xi = X_\xi$.

Jensen had shown that $\diamondsuit$ holds in the constructible universe and, even though it implies that $2^{\aleph_0} = \aleph_1$ the reverse implication fails. The fact that $\diamondsuit$ implies there is a Suslin tree will play a role in §8.

One of the first to realize the potential applications of $\diamondsuit$ was Saharon Shelah. who showed in [153] that $\diamondsuit$ implies that every Whitehead group of cardinality $\aleph_1$ is free. On the other hand, Shelah had also shown in [153] that assuming Martin's Axiom and the failure of the Continuum Hypothesis — which is discussed in §8 — there is a non-free Whitehead group of cardinality $\aleph_1$. In his proof Shelah isolated the following combinatorial consequence $C(S)$ of Martin's Axiom and the failure of the Continuum Hypothesis: S is a stationary subset of ω_1 consisting of limit ordinals and for each $\xi \in S$ there is an increasing sequence $\{\eta^i_\xi\}_{i\in\omega}$ converging to ξ and if $c_\xi : \omega \to 2$ is a colouring of the integers by two colours for each $\xi \in S$ then there is a single colouring $c : \omega_1 \to 2$ such that for each $\xi \in S$ the equality $c(\eta^i_\xi) = c_\xi(i)$ fails for only finitely many i. In their introduction to [49], Keith Devlin and Shelah recount the developments leading to a new consequence of the weak Continuum Hypothesis:

> *Shelah conjectured that $C(\omega_1)$ was consistent with ZFC + GCH. Devlin refuted this conjecture by showing that CH implies $\neg C(\omega_1)$... Devlin's original proof used metamathematical techniques (precisely, inner models of set theory). Devlin, Jensen and Shelah all independently observed that the proof could be modified to eliminate the use of inner models, and that the assumption of CH could be weakened to $2^{\aleph_0} < 2^{\aleph_1}$. Shelah took this a step further by "extracting" from the proof the principle Φ ...*

The principle Φ, which follows from the negation of Lusin's Second Continuum Hypothesis, is now known as weak $\diamondsuit$ and asserts the following: For any function F colouring the complete binary tree of height ω_1 with two colours, 0 and 1, there is a function $g : \omega_1 \to 2$ such that for any branch B of the binary tree the set of all ξ such that $F(B \restriction \xi) = g(\xi)$ is stationary. While the proof of Φ recorded in [49] is quite elementary and does not use concepts unavailable to Sierpiński, Rothberger or Lusin, it is clear from the description of the history leading to this proof that significant new ideas were needed to get to the elementary proof. As well, the formulation is very closely tied to the combinatorial problems considered by Shelah in dealing with Whitehead groups. It is no surprise then, that the weak $\diamondsuit$ implies there is a non-free Whitehead group. Applications to topology and model theory were soon also found and the principle continues to play a central role in set theory.

A more recent development along these lines is a family of parametrized versions of $\diamondsuit$ that are closely related to certain cardinal invariants of the continuum. One of these is introduced in [83] and is closely related to the cardinal $\mathfrak{d}$ examined in §5. Variations and generalizations of this to other cardinal invariants of the continuum are found in [129] and some of these have their roots in a much earlier weakening of $\diamondsuit$ referred to as $\clubsuit$ introduced by Adam Ostaszewski in [132]. Just as Φ does not require the Continuum Hypothesis, neither does $\clubsuit$. But to follow this line of development would lead considerably away from the continuum. The reader interested in recent developments is encouraged to consult the introduction of [36].

4. The Continuum Hypothesis

This history chose as its starting point Hilbert's address to the International Congress in 1900 because the first question on his famous list was Cantor's continuum problem. Cantor's unsuccessful search for a proof of his Continuum Hypothesis is well documented — for example see [127] or [46] — and Hilbert himself was to devote considerable energies to this problem some twenty-five years after his address, with no more success than Cantor. However, there were some minor successes, and even more notable failures, around this time as well.

Four years after after Hilbert's address in Paris the International Congress of Mathematicians met again in 1904 in Heidelberg. The Continuum Hypothesis was again the topic of at least one talk. Julius König presented a proof [94] in which he attempted to show that the Continuum Hypothesis is false on the grounds that $2^{\aleph_0} \neq \aleph_\xi$ for any ξ. The correct part of the proof is what is now known as König's Lemma: For any sequence of ordinals $\{\sigma_n\}_{n\in\omega}$ the inequality $\sum_{n\in\omega} \aleph_{\sigma_n} < \prod_{n\in\omega} \aleph_{\sigma_n}$ holds. It should be emphasized that it was the proof that was correct, not the statement. Indeed it was Hausdorff who was to make precise the notion of cofinality in his researches on order types which are discussed in §5. The error in König's argument was his use of an assertion of Bernstein that $\aleph_\alpha^{\aleph_0} = \aleph_\alpha 2^{\aleph_0}$ in which Bernstein had claimed, essentially, that if $\alpha > 0$ then the cofinality of $\aleph_\alpha$ is uncountable. Taking this into account one sees that what König's argument actually establishes is that the cofinality of $2^{\aleph_0}$ is uncountable.

Bernstein himself was also scheduled to give a talk on the continuum problem, resolving it in the opposite way from König. He did not give his talk though, but published a version a year later [19]. He argued that the continuum has cardinality $\aleph_1$ because each real number can be calculated according to some rule and counting the set of all such rules was to yield $\aleph_1$. The strategy employed by Hilbert a quarter century after his 1900 address followed similar lines. However, such arguments did not lead to any resolution of the problem until Gödel's insightful analysis of such ideas lead to the consistency of $2^{\aleph_0} = \aleph_1$ as exemplified by the constructible universe [66].

It should be mentioned that the continuum problem was considered to have two variants in the early years of the twentieth century. The strong form was whether what is now known as the continuum hypothesis is true: Does $2^{\aleph_0} = \aleph_1$? The alternate form of the continuum hypothesis merely asserted that there are only two possible values for the cardinalities of subsets of the reals. Assuming the Axiom of Choice, of course, these problems are equivalent but, since this axiom was still the subject of much controversy at the time, it made some sense to consider both forms of the continuum problem.

Interest in the alternate version of the continuum problem was stimulated by Cantor's result that the alternate Continuum Hypothesis is true for closed sets. Extending this to other classes of sets seemed a natural course to follow. Indeed, Gregory Moore claims Hausdorff's interest in order types was motivated by a desire to solve the Continuum Problem. According to Moore[15], Hausdorff "put forward a new way of posing the problem, in the hope that this would lead to a solution. Namely, CH is true if the following proposition holds ...: There is a dense order-type of power $\aleph_1$ having no (ω, ω^*)–gaps." However, Hausdorff's own words[16] do not add as much weight to the result quoted by Moore:

> *For simple reflection reveals that each everywhere dense type contains the countable everywhere dense type η; consequently,* each everywhere dense type without (ω, ω^*)–gaps contains the usual continuum as a subset. *As long as there is no success, for instance, in constructing an everywhere dense type of the second infinite cardinality without (ω, ω^*)–gaps and with that in verifying Cantor's hypothesis $2^{\aleph_0} = \aleph_1$, the questions of the existence of homogeneous types of the second infinite cardinality will be restricted to the 32 species with (ω, ω^*)–gaps.* (The translation is from [77].)

It seems from this passage that Hausdorff is primarily interested in the classification of linear order types, but that he nevertheless recognizes the significance of his work to the Continuum Problem.

However, after the first decade of the century the role of the Continuum Hypothesis as a construction tool began to predominate. Lusin's and Mahlo's [115] early construction of what is now known as a Lusin

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[15]On page 109 of [127].

[16](Page 156 of [72]) Denn eine einfache Überlegung lehrt, daßjeder überalldichte Typus den abzählbaren überalldichten Typus η, folglich *jeder überalldichte Typus ohne $\omega\omega^*$-Lücken das gewöhnliche Kontinuum als Teilmenge enthält.* Solange es also nicht etwa gelingt einen überalldichten Typus zweiter Mächtigkeit ohne $\omega\omega^*$-Lücken zu konstruieren und damit die Cantorsche Hypothese $2^{\aleph_0} = \aleph_1$ zu verifizieren, wird man die Frage nach der Existenz homogener Typen zweiter Mächtigkeit nur auf die 32 Spezies mit $\omega\omega^*$-Lücken beschränken.

set has already been mentioned in §2. As Akihiro Kanamori[17] points out, "{i}t was the mathematical investigation of CH that increasingly raised doubts about its truth and certainly its provability". Indeed, by 1934 such constructions had become sufficiently commonplace that Sierpiński was able to devote an entire book [164] to the equivalents and consequences of the Continuum Hypothesis, a hypothesis which he denoted "H".

Judging by its rank in the list of equivalents, the following proposition — answering Steinhaus' Question 2.6 — seemed to Sierpiński to be a critical equivalent of the Continuum Hypothesis:

P_1: The plane is the union of two sets, A and B, such that each vertical section of A and each horizontal section of B is countable. [18]

Sierpiński himself had shown this to be equivalent to the Continuum Hypothesis in 1919 [160]. From this follow various consequences which are also equivalent to the Continuum Hypothesis. For example, the plane is the union of countably many functions and their inverses if and only if the Continuum Hypothesis holds. As a remark, Sierpiński points out that a similar result holds for Euclidean space as well. Much later though [170], Sierpiński would show that the Continuum Hypothesis is equivalent to the following: There is a partition of Euclidean space $\mathbb{R}^3 = A_1 \cup A_2 \cup A_3$ such that if L is a line parallel to the x_i-co-ordinate axis then $A_i \cap L$ is not only countable, but actually finite. Moreover, there is a partition $\mathbb{R}^{k+2} = A_1 \cup A_2 \cup \ldots A_{k+2}$ such that if L is a line parallel to the x_i-co-ordinate axis then $A_i \cap L$ is finite if and only if $2^{\aleph_0} \leq \aleph_k$.

A related statement from 1988 [130] is even more striking: the Continuum Hypothesis is equivalent to the existence of a Peano-like function $P : \mathbb{R} \to \mathbb{R}^2$ such that at each $x \in \mathbb{R}$ at least one of the co-ordinate functions of P is differentiable, but even continuity can fail at the other co-ordinate. Moreover, this was improved by Jacek Cichoń and Michał Morayne to show that $2^{\aleph_0} \leq \aleph_k$ is equivalent to the existence of a function $P : \mathbb{R}^k \to \mathbb{R}^{k+1}$ such that at each $x \in \mathbb{R}$ at least k of the co-ordinate functions of P are differentiable, but even continuity may fail at the other co-ordinate. It is interesting to note though, that even though this result seems to be totally unrelated to P_1 it, nevertheless, justifies the prominent role of P_1 emphasized by Sierpiński. For example, one can prove that the Continuum Hypothesis implies the existence of a function $P : \mathbb{R} \to \mathbb{R}^2$ such that at each $x \in \mathbb{R}$ at least one of the co-ordinate functions of P is differentiable by starting with a decomposition of the plane as described by P_1. One then defines the two co-ordinate functions of the Peano curve in such a way that the first co-ordinate function guarantees that the image of $(\infty, 1)$ covers one half of the decomposition of the plane, regardless of the definition of the second co-ordinate function, while the second co-ordinate function guarantees that the image of $(-1, \infty)$ covers the other half of the decomposition of the plane, regardless of the definition of the first co-ordinate function.

Other similar results stemming from Sierpiński's P_1 can be found, for example, in [93]. However, it should be noted that considerably earlier, in 1967, Barbara Osofsky had discovered [131] algebraic results equivalent to the Continuum Hypothesis. For example, she showed that the homological dimension of the field of rational polynomials in 3 variables with real coefficients over the ring of such polynomials has dimension 2 if and only if the Continuum Hypothesis holds.

Even in situations in which the connection between combinatorics in Euclidean spaces of different dimension and cardinal arithmetic is not immediately apparent, other phenomena related to dimension manifest themselves. For example, a subset $X \subseteq \mathbb{R}^n$ is said to have *distinct distances* if any two distinct pairs of points from X determine distinct distances. Kenneth Kunen [98] generalized to $\mathbb{R}^n$ the following result of Erdös and Kakutani: Assuming the Continuum Hypothesis, there is a partition of $\mathbb{R}$ into countably many pieces such that each piece has distinct distances. The difficulty in generalizing results involving cardinal arithmetic from $\mathbb{R}$ to $\mathbb{R}^n$ appears in different contexts as well.

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[17] Page 25 of [90].

[18] L'ensemble du tous les points du plan est une somme de deux ensembles dont l'un est au plus dénombrable sur toute parralèle à l'axe d'ordonnées et l'autre est au plus dénombrable sur toute parralèle à l'axe d'abscisses.

The following is the last in Sierpiński's section on the equivalences of the Continuum Hypothesis and was studied first by Alfred Tarski [181] in 1928.

P_{11}: There do not exist more than $\aleph_1$ distinct infinite subsets of ω_1 any two of which have finite intersection.[19]

The restriction to infinite sets here is crucial for obtaining the equivalence with the Continuum Hypothesis, even though asking for the sets to be uncountable seems more natural. However, there may have been a good reason for dealing only with this case since it was shown much later, in 1976, that such questions lead to independence results and were beyond the techniques available to Sierpiński and his contemporaries. James Baumgartner showed [14] that the existence of a family of cardinality $\aleph_2$ of subsets of ω_1 all of whose pairwise intersections are finite is independent of the usual axioms of set theory.

The largest part of Sierpiński book deals with implications of the Continuum Hypothesis. The first of the properties listed by him that follow from the Continuum Hypothesis was the existence of a Lusin set and was denoted as C_1. The early work regarding the Lusin set is been described in §2 and later work will be described in §5. However, a key result of Rothberger [143] certainly bears mentioning here: the Continuum Hypothesis is equivalent to the existence of both Lusin and Sierpiński sets, at least of one of which has cardinality $2^{\aleph_0}$. While this may not be the most elegant of statements, it is significant in that Rothberger's proof isolates a key translational property that has turned out to be extremely useful in many situations. He shows that if there is a second category set of cardinality κ then $\mathbb{R}$ is the union of κ meagre sets; while, on the other hand if there is a non-measurable set of cardinality κ then $\mathbb{R}$ is the union of κ null sets. He argues by decomposing $\mathbb{R}$ into a null and a meagre set and translating these. It follows easily that both the Sierpiński and Lusin sets must be of size $\aleph_1$. The requirement that at least one of the Sierpiński and Lusin set have cardinality $2^{\aleph_0}$ seems awkward, but it is essential. Adding alternately Cohen and random reals to any model where $2^{\aleph_0} > \aleph_1$ produces a model with both Sierpiński and Lusin sets of size $\aleph_1$.

Also mentioned in §5 is the Property C whose central role Sierpiński acknowledges by introducing it right after the Lusin set existence hypothesis:

C_1: there is a set of cardinality the continuum whose intersection with each nowhere dense set is at most countable.[20]

He notes that the sets with Property C are invariant under continuous maps and then concludes that C_1 implies the following:

C_5: there is a set of cardinality the continuum no continuous image of which contains the unit interval.[21]

Sierpiński had established this from the Continuum Hypothesis in [161] in 1928. However, this is considerably weaker than C_1 as was shown by Arnold W. Miller [123] in 1983. In particular, he showed that in the iterated Sacks model every set of reals of cardinality $\aleph_2 = 2^{\aleph_0}$ can be mapped continuously onto the unit interval. This is discussed in §7. Moreover, he also remarks that it is consistent with any cardinal arithmetic that the real line can be partitioned into $\aleph_1$ pieces none of which can be mapped onto the unit interval.

An indication of the range of implications considered by Sierpiński is provided by the fact that the indexing of properties stops at C_{81}. All of these are now known to follow from Martin's Axiom and are discussed in the paper [116] by Anthony Martin and Solovay introducing this hypothesis. In [189] Todorčević singles out Sierpiński's Property P_3: There is an uncountable subset of the Hilbert cube

[19] Aucun ensemble de puissance $\aleph_1$ n'est une somme de plus que $\aleph_1$ ensembles infinis ayant deux à deux un nombre fini d'éléments communs.

[20] Il existe un ensemble linéaire de puissance du continu qui admet un ensemble au plus dénombrable de points communs avec tout ensemble (linéaire) parfait non-dense.

[21] Il existe un ensemble linéaire E de puissance du continu et tel que l'intervalle linéaire n'en est pas une image continue.

every uncountable subset of which is projected onto the interval $[0, 1]$ by all but finitely many of the co-ordinate projections. He says of this hypothesis:

> *Proving this statement from CH did cause a considerable difficulties ... and we conjecture that this is because of a too strong preoccupation with getting a statement equivalent to CH rather than just its consequence (like C_1). In fact, P_3 is so made (perhaps artificially) that the implication $P_3 \to CH$ is a triviality.*

Perhaps a similar "preoccupation" can be found in the result of Roy Davies who showed in [47] that the Continuum Hypothesis is equivalent to the assertion that each function from $\mathbb{R}^2$ to $\mathbb{R}$ can be represented as a sum of products of single variable functions. The last chapter of Sierpiński's book also contains several similar instances of this "preoccupation". The question of determining that an assertion is *not* a consequence of the Continuum Hypothesis however, is an even more delicate task.

The first result of this kind was obtained by Jensen and is explained in [48]. It had already been shown by Solovay and Stanley Tennenbaum [172] that it was consistent that no Suslin trees exists. Moreover, Jensen had also shown, see [48], that Suslin trees could be constructed by using his principle $\diamondsuit$ which holds in Gödel's constructible universe. This, however, did no preclude the possibility that the Continuum Hypothesis alone may be sufficient for the construction of a Suslin tree. Perhaps there was an argument involving some key idea that Sierpiński, Lusin, Duro Kurepa and the other early masters at using the Continuum Hypothesis had all missed. However, Jensen provided a sophisticated iterated forcing construction that established the consistency of the Continuum Hypothesis along with no Suslin trees. Later Uri Abraham, Devlin and Shelah modified this approach [4] to obtain that some of the consequences of Martin's Axiom and $2^{\aleph_0} > \aleph_1$ are consistent with the Continuum Hypothesis .

However, Shelah later provided a general method for establishing such results as those of Chapter V of [154] . The completeness systems introduced there together with techniques for iterating proper partial orders allowed Jensen's result to be established with methods amenable to other problems. For example in [2] Abraham and Todorčević study the following property of ideals on ω_1: every $\subseteq^*$-σ-directed ideal $\mathcal{I}$ on ω_1 is such that one of the following two alternatives holds:

- there exists an uncountable subset
- ω_1 can be decomposed into countably many subsets A_i such that $A_i \cap I$ is finite for all $i < \omega$ and $I \in \mathcal{I}$.

They show this to be consistent with the Continuum Hypothesis . While this statement is weaker than the Proper Forcing Axiom it suffices to imply that there are no Suslin trees and that all (ω_1, ω_1^*)–gaps are Hausdorff gaps.

The principle $\diamondsuit$ is discussed in §3 and it is mentioned that Shelah was one of the first to realize the potential applications of this principle and that he used it to great effect in studying the freeness of abelian groups. Another of the early applications of $\diamondsuit$ was to topology. In 1976 Ostaszewski used $\diamondsuit$ to construct [132] a perfectly normal, countably compact space that is not compact. In addition, the space constructed by Ostaszewski has the property that all closed sets are either countable or co-countable. The question of whether $\diamondsuit$ is necessary for establishing the existence of such a space, or whether the Continuum Hypothesis alone suffices had vexed a generation of topologists until a 1999 paper of Todd Eisworth and Judith Roitman showed [54] that the Continuum Hypothesis alone does not suffice. The key difficulty here is to devise a forcing that destroys Ostaszewski type spaces without adding reals and, more, such that the iteration also does not add reals. Shelah's general framework from [154] provides the starting point for Eisworth and Roitman's work.

An important result of Woodin delineates the possibilities for obtaining results independent of the Continuum Hypothesis. If there is a proper class of measurable Woodin cardinals, and Ψ is a σ^2_1 sentence — in other words, $\Psi = (\exists X)\Phi(X)$ where the quantifiers in Φ are all over reals, but X is allowed to be a set of reals — such that Ψ holds in some forcing extension of the set theoretic universe, then it holds in every forcing extension that satisfies also satisfies the Continuum Hypothesis. For most purposes, this is almost the same as saying that the assertion actually follows from the Continuum Hypothesis. In other words, in order to establish that a statement asserting the existence of a set of reals with a Borel definable property follows from the Continuum Hypothesis, it will quite surely suffice to show only that it is consistent with set theory. Note that this result does not apply to the existence of Ostaszewski's space since describing the this space requires a universal quantifier over sets of reals as well as an existential quantifier asserting the existence of the space. The original argument of Woodin was never published, but an accessible account of this can be found in [56].

5. Cardinal Invariants of the Continuum Associated with Convergence Rates

Investigations into the structure of sequences of natural numbers have their origins in the work of du Bois-Reymond [51], work which was later continued by mathematicians such as Jacques Hadamard [69], Hardy in [70] and [71], who used du Bois-Reymond's ideas as a starting point for investigation into the theory of L-functions and what are now known as Hardy fields. But, much more relevant from the point of view of this history is the work of Hausdorff and its continuation by Rothberger leading to the modern theory of the invariants $\mathfrak{b}$, $\mathfrak{d}$ and related notions.

The question on which du Bois-Reymond had fixed his attention was that of classifying rates of convergence occurring in analysis by attaching to them some notion of order of infinity in analogy with Cantor's. He approached this problem by defining an ordering on positive, real-valued functions by declaring $f < g$ if $\lim_{x\to\infty} f(x)/g(x) = 0$, $f > g$ if $\lim_{x\to\infty} f(x)/g(x) = \infty$ and $f \sim g$ if $\lim_{x\to\infty} f(x)/g(x)$ exists and is finite but non-zero. If $f < g$ then du Bois-Reymond considered that f represented a smaller infinity than g whereas if $f \sim g$ then the two functions were considered as representing the same infinity. This structure was referred to by du Bois-Reymond as a pantachie and it was du Bois-Reymond's intention to find in it the cut point separating convergence from divergence. There are several problems here, the existence of the cut point being one of them.

The even more serious problem however, is that, unlike Cantor's cardinalities under the Axiom of Choice, du Bois-Reymond's "infinitary calculus" of pantachies suffers from the existence of incomparable orders of infinity. Unlike Cantor's cardinals, this problem persists even assuming the Axiom of Choice and under any conceivable modification of the definition of the ordering of the types considered by du Bois-Reymond. Indeed Salvatore Pincherle, and later Hausdorff, had observed [73] that replacing the quotient by a difference in du Bois-Reymond's definition leads to an equally reasonable definition. Hausdorff explains[22]

> *In addition, starting out from the mode of expression that x^α is infinite of order α, people have repeatedly endeavored to associate to the elements of such restricted function classes magnitude-like symbols with corresponding laws of combination; among these are Stoltz's moments, Thomae's complex numbers, and the symbols of Pincherle, Borel, Bortolotti et al. These magnitudes do not satisfy the Archimedean axiom and are thus "actually" infinitely large or infinitely small relative to each other, which says nothing against their logical admissibility; their usefulness can be debated since, on the one hand, they break down with respect to the intermediate levels of scale, and, on the other hand, they are more complicated than the functions whose infinity they are supposed to express.* (Translation from [77].)

[22](Page108 of [73]) Nebenbei hat man sich, von der Ausdrucksweise ausgehend, daßx^α von der Ordnung α unendlich werde, mehrfach bemüht, den Elementen solcher beschränker Funktionsklassen größentartige Symbole mit entsprechenden Verknüpfungsgesetzen zuzuordnen; hierher gehören die Stolzschen Momente, die Thomaeschen komplexen Zahlen, die Symbole von Pincherle, Borel Bortolotti u. A. Diese Größen erfüllen das Archimedische Axiom nicht und sind also, relativ zu einander, "aktuell" unendlich groß oder unendlich klein, womit nichts gegen ihre logische Zulässigkeit gesagt ist; über ihre Zweckmäßigkeit läst sich streiten, denn die einen versagen gegenüber den Zwischenstufen der Skala und die andern sin komplizierter als die Funktionen, deren Unendlich sie ausdrüchken sollen.

One direction of research that has proved very productive is abstracting to the simple situation of comparing the growth rates of sequences term by term. This, of course, leads directly to the study of $\mathbb{N}^{\mathbb{N}}$ under $\leq^*$ which will be the subject of much of this history. Recall that if $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ then the relation $f \leq^* g$ is define to hold if and only if $f(n) \leq g(n)$ for all but finitely many n However, it should be remarked that this abstracted, discrete version of the problem is not entirely equivalent to the original problem concerning functions defined on $[0, \infty)$ considered by du Bois-Reymond. For example, Vladimir Kanovei and Peter Koepke examine various possibilities of the discrete orderings as well as the continuous structure $\mathcal{C}([0,\infty))$ and mention that, while the existence of (κ, λ^*)–gaps in $\mathbb{R}^{\mathbb{N}}$ under the co-ordinate-wise ordering implies the existence of similar gaps in $\mathcal{C}([0,\infty))$, the converse is an open question. In their discussion of Gödel's argument that $2^{\aleph_0} = \aleph_2$ Jörg Brendle, Paul Larson and Todorčević [37] examine this same structure in some detail, as does the survey article [151]. However, it is the discrete versions of problems on order types that have driven developments in the set theoretic study of the continuum and its associated structures.

It was on the basis of the existence of incomparable rates of convergence that Cantor [42] rejected du Bois-Reymond's work on this subject. On the other hand, Hausdorff was not at all discouraged by this realization and took up the challenge of analyzing the structure of rates of convergence in [73]. Given that there are incomparable rates of convergence, Hausdorff's starting point was to take an arbitrary maximal, linearly ordered family of rates of convergence which, extending du Bois-Reymond's terminology, he also called a pantachie. In Hausdorff's words[23], "if we designate it as our task to connect the infinitary rank ordering as a whole with Cantor's theory of order types, then nothing remains but to investigate the sets of pairwise comparable functions that are as comprehensive as possible." Hausdorff justifies the necessity of this approach by explaining[24] that, "all attempts to produce a simple (linearly) ordered set of elements in which each infinity occupies its specific place had to fail: the infinitary pantachie in the sense of du Bois-Reymond does not exist."

By the time he arrived at the study of pantachies, Hausdorff already had a wealth of experience in studying these abstract order types. Naturally, the same sort of questions which had appealed to his interest in studying abstract order types, figured prominently in forming his approach to pantachies. A significant part of Hausdorff's interest had to do with η–sets which, being generalizations of the rationals, had certain saturation properties. In contrast and conjunction to this study of saturated linear orders, it was reasonable for Hausdorff to consider gaps in linear orders and to attempt to use these as a classifying tool. Most of his 1906 paper [72] is devoted to this pursuit. Here he isolates 50 possible uncountable order types according to which gaps and order types appear and tackles the problem of realizing these types by homogeneous orders. The culmination of this work is his 1909 construction [74] of an (ω_1, ω_1^*)–gap. While the unresolved Continuum Hypothesis obviously thwarted Hausdorff's attempts at a complete solution to his classification problem, this seminal result must have provided a fair degree of satisfaction. Indeed, even in 1936, when Hausdorff republished it in a different form as [76], this argument still defined the state of the art.

However there is another path of enquiry, not entirely different, leading to the modern detailed studies of the structure $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. This has its roots in the investigations of certain sets of reals and problems of category and measure. In 1919 Borel [32] undertook a study of sets of Lebesgue measure zero and a classification scheme that ranked null sets $X \subseteq \mathbb{R}$ by assigning to them sequences $\sigma : \mathbb{N} \to \mathbb{R}^+$ according to whether there exists a sequence of intervals $\{I_n\}_{n\in\mathbb{N}}$ such that $\bigcup_n I_n \supseteq X$ and the length of I_n is less than $\sigma(n)$. This, of course, is reminiscent of du Bois-Reymond's scheme for classifying convergence and

[23](Page 110 of [73] and translation from [77]) Bezeichen wir es also als unsere Aufgabe, die infinitäre Rangordnung als Ganzes mit der Theorie der CANTORschen Ordnungstypen in Verbindung zu setzen, so bleibt nichts anders übrig als *möglichst umfassende Mengen paarweise vergleichbarer Funktionen* zu untersuchen . . .

[24](Page 107 of [73] and translation from [77]) . . . alle Versuche, eine einfach (linear) geordnete Menge von Elementen herzustellen, in der jedes Unendlich seine bestimmte Stelle inne hat, mußten aus diesem Grunde scheitern: *die infinitäre Pantachie* im Sinne DU BOIS-REYMOND *existiert nicht*.

Borel acknowledges the key fact established by du Bois-Reymond that countable families of sequences of natural numbers can be $\leq^*$-dominated in the limit. The genesis of strong measure zero sets can be traced to this paper and Borel's isolation of those sets which he describes by the phrase *"mesure asymptotique infèrieure a toute serie donné a l'avance"*. In particular, as set of reals X has *strong measure zero* if and only for any sequence of positive reals $\{\epsilon_n\}_{n=1}^{\infty}$ there is a corresponding sequence of intervals $\{I_n\}_{n=1}^{\infty}$ such that the length of I_n is less than ϵ_n and $X \subseteq \bigcup_n I_n$.

It was 15 years later that Besicovitch took up this study of the classification of null sets, apparently unaware of the earlier work of Borel. Much of Besicovitch's life's work was focused on questions of geometric measure theory and much of this theory examines measures obtained by assigning measure to certain open sets and then extending this to at least all Borel sets. The fractional Hausdorff measures are typical examples of this type of construction. It was therefore natural for Besicovitch to turn, at some point, to the simplest such example. This is obtained by taking a continuous, monotone function ϕ defined on the positive reals such that $\lim_{x\to 0^+} \phi(x) = 0$ and defining the measure of an interval (a, b) to be $\phi(b-a)$. This is then used to define an outer measure in the usual way. In [22, 23] Besicovitch states that his interest is centred on determining those subsets of the real line which are null with respect to any such measure. As well, he is interested in the related problem of determining those sets on which the variation of any continuous, monotone function is zero. Relying on the *Perfect Set Property* for Borel sets — namely, that every uncountable Borel set contains a copy of the Cantor set — Besicovitch observes that the only Borel sets with either of these properties are the countable ones. He then poses himself the question of whether this is true in general.

His answer is that a set of reals is null with respect to all measures obtained from continuous, monotone functions if and only if it has the property currently referred to as strong measure zero and denoted by Sierpiński as Property C. Besicovitch's main result in [22] is that, assuming the Continuum Hypothesis, there is an uncountable sets of reals which has strong measure zero; in other words, it is null with respect to all the measures Besicovitch considers and, furthermore, all continuous, monotone functions defined on the set have zero variation.

What Besicovitch actually constructs is what he calls a *concentrated* set, which he defines as an uncountable set $X \subseteq \mathbb{R}$ for which there is a countable set $H \subseteq \mathbb{R}$ such that any open set containing H contains all but countably many points of X. He provides the simple argument that concentrated sets have strong measure zero and then shows how to construct a concentrated set assuming the Continuum Hypothesis. From the point of view of the study of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ the notion of a concentrated set will turn out to be a critical notion. Seemingly unaware of Besicovitch's work, (although by 1942 he has met Besicovitch and the latter has introduced him to the Cambridge Philosophical Society) Rothberger will study the same family as well as the strong measure zero sets — although, using Sierpiński's [163] terminology of sets with Property C — establishing their relationship to $\mathfrak{b}$. An important realization, due to Sierpiński [169], established the correspondence between open sets containing the rationals and the sequences of natural numbers obtained by thinking of the sequences of naturals as functions $\Phi : \mathbb{Q} \to \mathbb{N}$ and then sending Φ to the open set $\bigcup_{q\in\mathbb{Q}}(q - 1/\Phi(q), q + 1/\Phi(q))$. This mapping establishes that these two structures are cofinal in each other and, so, for many purposes equivalent.

Before following these later developments though, it is worth returning twenty years to 1914 since it was in this year that Lusin's article [109] constructing a set with an even stronger property than that required by Besicovitch, but under the same assumption, appeared. These are now called Lusin sets and it is fortunate for the development of set theory that Besicovitch had ignored their existence, since the subtle nature of the differences between them and concentrated sets has lead to interesting mathematical developments. The central role of concentrated sets in the evolution of the study of $\mathfrak{b}$ has already been noted. There is no indication that finding a weaker form of a Lusin set was Besicovitch's motivation for his construction, although we now do know, with the hindsight provided by close to a century of study, that concentrated sets can exist even when Lusin sets do not. (More on this can be found in the discussion of the random real model in §7.) Even though he does not refer to Lusin's much earlier work, Besicovitch must have been aware of the possibility of Lusin's construction since he also establishes, using the Continuum Hypothesis, the existence of what he calls *rarified sets*, but which are now usually referred to as *Sierpiński sets*. Since these are the measure theoretic analogue of Lusin sets — namely, their intersection with every null set is countable — it is hard to imagine that Besicovitch would have overlooked the possibility of using Lusin's topological construction for the purposes of the first half of [22]. On the other hand, Besicovitch's interest in the second half of that paper is showing that it is possible to have planar sets of positive measure that contain no subset of finite, but non-zero, measure. Further evidence for the view that Besicovitch was not concerned with the distinction between concentrated and Lusin sets is provided by the fact that he does not ask any questions along these lines. He could, for example, have asked whether the family of strong measure zero sets is the same as the family of concentrated sets, but it was Sierpiński who asked this question in print in 1938 [168]. However, the fact that it was Besicovitch who finally did answer this question [21] in 1942 indicates his interest in this line of enquiry. The related question of whether the existence of a set with Property C implies the existence of a concentrated set, of course, would have to wait for methods capable of producing models with large continuum. The same remark applies to the question of whether the existence of a concentrated set implies the existence of a Lusin set. This would have to wait for the arrival of forcing constructions and, in particular, the random model of Solovay that is discussed in §7.

The first hint of the connection between strong measure zero sets and the cardinal invariant $\mathfrak{b}$ appears in Lusin's article [110]. His proof, in fact, shows that there is a set of reals of cardinality $\mathfrak{b}$ which is meagre in each perfect set, although Lusin was satisfied with simply producing an uncountable such set. This uncountable set has the same property that had motivated Lusin's earlier construction of a Lusin set using the Continuum Hypothesis; three years earlier he had used this to produce a counterexample to show that Baire's sufficient condition for belonging to the Baire classification was not necessary. In 1914, Lusin had not been able to use the 1916 result of Alexandroff and Hausdorff, independently, that any uncountable Borel set contains a perfect set and, so, he had had to rely on some trickery using cardinal arithmetic as described in §3. Now however, he had the same conclusion without having to rely on any hypothesis other than the Axiom of Choice.

Lusin's realization was that, by following Baire and associating with each sequence of integers $\sigma : \mathbb{N} \to \mathbb{N}$ the irrational $\nu(\sigma)$ obtained from the continued fraction

$$\nu(\sigma) = \frac{1}{\sigma(0) + \frac{1}{\sigma(1) + \frac{1}{\sigma(2) + \dots}}}$$

it is possible to take a scale $\{\sigma_\beta\}_{\beta\in\omega_1}$ and produce a set of irrationals $G = \{\nu(\sigma_\beta)\}_{\beta\in\omega_1}$. Lusin then uses the fact that, given any perfect set P one may as well assume that $G \cap P$ is dense in P and, hence, there is an ordinal $\gamma \in \mathfrak{p}$ such that $G_\gamma \cap P$ is dense in P where $G_\gamma = \{\nu(\sigma_\beta)\}_{\beta\in\gamma}$. Note that du Bois-Reymond's observation that countable families of sequences can be dominated plays a key role here in guaranteeing that $\gamma \in \mathfrak{p}$. Lusin then realizes that $G \setminus G_\gamma$ must be relatively meagre in P. The fact that G_γ itself is relatively meagre in P is immediate if one is only interested in producing an uncountable set since initial segments are countable. However, the tools were at hand for Lusin to have proved that this can also be done for a set of size $\mathfrak{p}$, the least cardinal of a family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ such that there is no infinite $X \subseteq \mathbb{N}$ such that $X \subseteq^* F$ for all $F \in \mathcal{F}$. He could have chosen γ to be the least ordinal such that G_γ is not meagre in P and obtained a contradiction using diagonalization and the Baire Category Theorem. However, at this early stage in the developments of set theory this sort of result, asserting the existence of sets of a cardinality not known to be different from $\aleph_1$, may have seemed unjustifiable.

In any case, this slightly more delicate argument would have to wait for Rothberger and his construction of what he called a λ–set which is not universally meagre. Kuratowski had introduced the notion of a set with property λ on page 269 of his monograph [100]. A λ–set is a subset of n-dimensional

Euclidean space with the property that everyone of its countable subsets is a relative G_δ. In [167] Sierpiński addresses the question of the additivity of the λ property and establishes that this family is countably additive if it is finitely additive and, moreover, to obtain additivity it suffices to show that the union of a λ–set with a countable set is always a λ–set. He defines a set $X \subseteq \mathbb{R}^n$ to have property λ' if $X \cup Y$ has property λ for every countable $Y \subseteq \mathbb{R}^n$ and asks whether every λ–set is also a λ'-set.

Rothberger[145] describes a construction of a λ–set which is not a λ'-set using only the Axiom of Choice, thus answering Sierpiński's question in the negative. In the same issue of *Fundamenta* Sierpiński shows [169] that Lusin's argument of 1917 — recorded in [110] — provides an alternate proof and, indeed, the key ideas were already in place at that time. From the point of view of the development of the study of cardinal invariants though, it is interesting to note that both Rothberger and Sierpiński had introduced notation for the cardinal now known as $\mathfrak{b}$ and had isolated it as a mathematical object of study in its own right. Sierpiński names it ϕ whereas Rothberger's argument relies on defining the property he calls $\mathbf{B}(\aleph_\xi)$ and which is defined to hold if every family of sequences of natural numbers of cardinality $\aleph_\xi$ is bounded. He then defines $\aleph_\eta$ to be the least cardinal for which $\mathbf{B}(\aleph_\eta)$ fails; in other words, in contemporary notation $\aleph_\eta = \mathfrak{b}$. Both arguments, of course, rely on constructing sequences of reals well-ordered by $\leq^*$ of length $\mathfrak{b}$ or, to be precise, such that the sequences of integers obtained by the continued fraction expansions of the corresponding reals are well ordered by $\leq^*$. However, Rothberger's interest in $\aleph_\eta$ is clear. He remarks, for example, that the cofinality of $\mathfrak{b}$ must be uncountable — but does not mention du Bois-Reymond — and, in a footnote, adds that this is close to all that is known about it. Do we know much more about $\mathfrak{b}$ today than Rothberger did in 1939? Answers to this question will be found when discussing Stephen Hechler's work on the subject in the seventies.

Both Rothberger's and Sierpiński's argument rely on showing that any G_δ containing the rationals must intersect any unbounded scale in the irrationals. Of course one can immediately conclude more: Any G_δ containing the rationals must contain all but a bounded subset of any scale in the irrationals. Recall that Besicovitch had defined a set X to be concentrated on the rationals if every G_δ containing the rationals contained all but countably many of the points of X. If an unbounded scale has length ω_1 then the result of Rothberger and Sierpiński shows that the scale must be concentrated on the rationals. Moreover, their result points to a useful generalization of Besicovitch's definition that can be phrased in terms of the cardinal invariant $\mathfrak{b}$. Recall that $\mathfrak{b}$ is the least cardinal of a family $\mathcal{B} \subseteq \mathbb{N}^{\mathbb{N}}$ such that for each $f \in \mathbb{N}^{\mathbb{N}}$ there is $b \in \mathcal{B}$ such that $b \not\leq^* f$.

This discussion has rushed ahead of many interesting and critical developments in the history of the $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ to which we will return shortly. But, the point to be emphasized here is the motivation for studying one particular version of the questions surrounding gaps and limits in the various discrete structures that might be viewed as abstractions of du Bois-Reymond's pantachies. Hausdorff himself had deemed worthy of examination many of these variants in 1907 and these are examined in detail in [91]. For example, two sequence of natural numbers a and b could be compared by asking whether $a(n) \lneq b(n)$ for all but finitely many n or whether it is simply the case that eventually $a(n) \leq b(n)$. Of course the gap structure of dyadic sequences was also of interest. However, the structure of sequences of natural numbers was the focus of the greatest attention of researchers such as Lusin, Sierpiński and Rothberger because of the natural map from the irrationals to $\mathbb{N}^{\mathbb{N}}$ using continued fractions already described by Baire in [5] and [7]. It has already been mentioned that the usefulness of this map was exploited by Lusin [109] and, later, Sierpiński and Rothberger. Indeed, many of the papers dealing with Lusin sets and λ–sets invoke the continued fraction map in an almost boiler plate fashion, the introductory remarks seeming very similar to each other. The point to realize though, is that it was the questions on the properties of subsets of the reals constructed using the Axiom of Choice, which drove developments on subsets of the irrationals and, this, in turn, focused attention on $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ because of Baire's continued fraction map.

Hausdorff however, was motivated by understanding the discrete structures themselves and devotes some energy in [73] to comparing the possible variants. He spends several pages[25] discussing whether continuous or arbitrary real-valued functions should be considered, whether or not they should be monotone, and eventually arrives at the conclusion that studying discrete sequences of numbers — *Zahlenreihen* or *Zahlenfolge* — is the correct level of generality.

Recall that du Bois-Reymond's initial investigations were about the structure of real valued functions defined on the positive reals ordered by limiting ratios. Hausdorff abstracted to the context of a discrete domain in 1909 [171] and his pantachies were maximal linearly ordered subsets of $\mathbb{R}^{\mathbb{N}}$ under the ordering $<^*$ defined by $f <^* g$ if and only if $f(n) < g(n)$ for all but finitely many $n \in \mathbb{N}$. For $\mathbb{N}^{\mathbb{N}}$ one can consider the two different orderings, $\leq^*$ and $<^*$, but, of course, for $2^{\mathbb{N}}$ only $\leq^*$ is interesting. In all of these structures it makes sense to examine the limit and gap structures considered by Hausdorff.

In Hausdorff's terminology, given a linearly ordered set $(L, \leq)$ and ordinals γ and λ the pair of subsets $\{a_\xi\}_{\xi\in\gamma}$ and $\{b_\xi\}_{\xi\in\lambda}$ will be said to form a (γ, λ^*)–gap if and only if

- $a_\xi \leq a_\eta \leq b_\alpha \leq b_\beta$ provided that $\xi \in \eta \in \gamma$ and $\beta \in \alpha \in \lambda$
- there is no $c \in L$ such that $a_\xi \leq c \leq b_\alpha$ and $a_\xi \neq c \neq b_\alpha$ for all $\xi \in \gamma$ and $\alpha \in \lambda$.

A γ–limit in $(L, \leq)$ is simply a $(\gamma, 1)$–gap. In this general setting of linear orders the notion of a *scale* corresponds to a cofinal, well-ordered subset of L. In a general partial order though, a scale is a cofinal, well ordered family — for our purposes a *scale* is a subset of $\mathbb{N}^{\mathbb{N}}$ cofinal and well-ordered by $\leq^*$. An important point to keep in mind is that, since Hausdorff was motivated by putting du Bois-Reymond's work on a solid foundation, his notion of pantachie, a maximal linearly ordered subset of $\mathbb{R}^{\mathbb{N}}$ with respect to $<^*$, was central to all his work, at least in 1909. For pantachies and, indeed, for all linearly ordered sets, the notions of unbounded and cofinal coincide; in particular, well-ordered cofinal or unbounded sets are both scales. It is not surprising, therefore, that these concepts do not figure prominently in Hausdorff's work. Rothberger on the other hand, who arrived at the same questions from a different direction, did formulate and analyze these different notions. In Rothberger's terminology a *tower* of length κ is simply a $(\kappa, 1)$–gap in $(2^{\mathbb{N}}, \leq^*)$. For Hausdorff's purposes this would be the same as a scale.

However, when dealing with problems about sets of reals, as in [76] where he considers the problem of partitioning the reals into $\aleph_1$ Borel sets, Hausdorff chose to view the reals as the Cantor set instead of $\mathbb{N}^{\mathbb{N}}$. In 1909 he had established the existence of (ω_1, ω_1^*)–gaps in the structure $\mathbb{R}^{\mathbb{N}}$ whereas, in 1936, by using the same methods he established the analogous result for $2^{\mathbb{N}}$. While the formulation for $\mathbb{R}^{\mathbb{N}}$ may have caused some slight awkwardness in establishing that the annuli defined from his gap are Borel, this would have been very similar for $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$. In the 1936 article [76], using the same argument Hausdorff establishes the very closely related result that there is an (ω_1, ω_1^*)–gap in the structure $(2^{\mathbb{N}}, \leq^*)$. Nevertheless the second paper quickly became widely cited and Lusin even published his own proof of the result in 1943 [114] and Rothberger was inspired to continue work on the subject. Why was the earlier paper ignored while the later one so enthusiastically accepted? In a footnote to the 1936 paper Hausdorff himself explains that he had the $\mathbb{R}^{\mathbb{N}}$ version of the result much earlier but that this paper is little known. Koepke and Kanovei offer the following explanation:

> *Clearly the gap construction was far ahead of the level of development and, perhaps, even motivation of set theory in the early years of the century. In addition, the paper was published in a rather provincial journal.*

The "level of development" seems unlikely — surely Lusin and Sierpiński would have understood the argument and embraced it. The "motivation" explanation is certainly more convincing. Keeping in mind the questions occupying Lusin and Sierpiński, questions leading to the discovery of Lusin sets and λ–sets, it seems more likely that Hausdorff's investigations into abstract order types did not seem germane at the time. As well, the motivation for Hausdorff's 1936 paper was Question 2.2 of Sierpiński.

[25] Pages 111 to 116.

This was something more likely to attract the attention of Lusin and Sierpiński who had been working on these questions for some time, and Rothberger, who was just starting in the area.

Nevertheless, the key results were in place by 1909. The (ω_1, ω_1^*)–gap has already been mentioned, but Hausdorff also built on du Bois-Reymonds work to show that there are no (ω, ω^*)–gaps in $(\mathbb{R}^{\mathbb{N}}, <^*)$, indeed, this is a key component in showing that there are (ω_1, ω_1^*)–gaps. He also established some important, but not as surprising, results assuming the Continuum Hypothesis; for example, there are ω_1–limits, (ω_1, ω^*)–gaps as well as towers and scales.

The study of these order types in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ was systematically taken up as a subject in its own right only some thirty years later in the 1941 work of Rothberger, who came to it through an interest in properties of sets of reals. Of course, some five years earlier, Hausdorff had dealt with Question 2.2 and Question 2.4 of Sierpiński by showing that (ω_1, ω_1^*)–gaps produce partitions of the reals into $\aleph_1$ Borel sets. Of course, the relevance of scales to λ–sets and their variants had been known for quite some time. But now, in 1936, Hausdorff showed that his much more sophisticated construction of 1909 also had applications to the study of sets of reals and, from the modern perspective, to cardinal invariants of the continuum. It is interesting that Hausdorff cites Question 2.4 as his motivation, referring to Sierpiński's questions as (A) and (B):

A: Can one show, without assuming the Continuum Hypothesis, that there is a union of $\aleph_1$ first category sets which is of second category?[26]

B: Can one show, without assuming the Continuum Hypothesis, that there is a union of $\aleph_1$ null sets which has positive outer measure?[27]

He is able to show, that given a gap $\{A_\xi\}_{\xi\in\omega_1} \cup \{B_\xi\}_{\xi\in\omega_1}$ the set of all $X \subseteq \mathbb{N}$ such that $A_\xi \subseteq^* X \subseteq^* B_\xi$ is an F_σ in the Cantor set. Consequently, the family of differences of these sets provides the required Borel partition. However, Hausdorff's analysis of the situation, while providing a solution to Question 2.2, does not answer Question 2.4. The fact that this is a hopeless task would not become evident until after Cohen's work of the early sixties.

In any case, these were the developments that lead to Rothberger's 1939 paper [145] in which he introduced his already mentioned concept $B(\aleph_\xi)$ to study λ–sets. By 1941, he had become aware of Besicovitch's work on concentrated sets and was able to more clearly outline the connection between these objects and $B(\aleph_\xi)$. In synthesizing earlier work of Sierpiński, as well as himself, Rothberger presents [146] the following picture of the knowledge about λ–sets and unbounded families in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. The following are all equivalent:

- Any subset of $\mathbb{N}^{\mathbb{N}}$ of size $\aleph_\xi$ is $\leq^*$-bounded.
- Any subset of the irrationals of size $\aleph_\xi$ has Property Θ, in other words, it is σ-compact.
- Given disjoint sets of reals A and B, with B an F_σ and A a set of cardinality no greater than $\aleph_\xi$, an F_σ can be found containing A and disjoint from B.
- The union of $\aleph_\xi$ sets with Property Θ has Property Θ.
- All sets of reals of cardinality $\aleph_\xi$ have property λ.
- All sets of reals of cardinality $\aleph_\xi$ have property λ'.
- All sets of reals of cardinality $\aleph_\xi$ have property σ, in other words, all relative G_δ-subsets are also F_σ.
- Every real-valued function on a set of reals of cardinality $\aleph_\xi$ is of Baire class no greater than 1.

Rothberger also points out the special role of $\aleph_1$ in these problems. The key result here is: The failure of $B(\aleph_1)$ is equivalent to the existence of an uncountable concentrated set of reals.

Recall that a set X is concentrated on the rationals if every G_δ containing the rationals contains all but a countable subset of X. The role of countability allows Rothberger's equivalence in the case of $\aleph_1$, but not for other cardinals.

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[26] Kann eine Summe von $\aleph_1$ Mengen 1. Kategorie von 2. Kategorie sein?

[27] Kann eine Summe von $\aleph_1$ Nullmengen von positivem äusserem Masse sein?

This special role of $\aleph_1$ would be more subtly elucidated in section 4 of Rothberger's paper. Here he returns to the question of gaps themselves. A preliminary key result here is that the failure of $B(\aleph_1)$ is equivalent to the existence of (ω_1, ω^*)–gaps in $(2^{\mathbb{N}}, \leq^*)$, even though his proof actually shows, in his notation, that there is an (ω_η, ω^*)–gap in $(2^{\mathbb{N}}, \leq^*)$.

However, in 1948 Rothberger shows that the existence of (ω_1, ω^*)–gaps in $(2^{\mathbb{N}}, \leq^*)$ implies the existence of ω_1–limits in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. He does this by setting up a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$ and then noting that if $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ and $f \leq^* g$ then $\{(n,m) \in \mathbb{N} \times \mathbb{N} \mid m \leq f(n)\} \subseteq \{(n,m) \in \mathbb{N} \times \mathbb{N} \mid m \leq g(n)\}$. Indeed, from the point of view of cardinal invariants, this correspondence takes gaps to gaps and can be used to establish an inequality between the associated cardinal invariants and need not be restricted to the case of $\aleph_1$.

In Lemma 7 of [147], however, Rothberger uncovers an essential use of $\aleph_1$ that stands besides Hausdorff's gap construction as a mile post in the understanding of the role of ω_1 in the combinatorics of the continuum. He begins by showing that the existence of an (ω_1, ω^*)–gap is equivalent to the following assertion: There are families $\{X_\alpha\}_{\alpha \in \omega_1}$ and $\{Y_m\}_{m \in \omega}$ of subsets of $\mathbb{N}$ such that $X_\alpha \subseteq^* Y_m$ for each $\alpha \in \omega_1$ and $m \in \omega$ yet there is no $Z \subseteq \mathbb{N}$ such that $X_\alpha \subseteq^* Z \subseteq^* Y_m$ for each $\alpha \in \omega_1$ and $m \in \omega$. In other words, Rothberger extends his interest to gaps that do not need to be linearly ordered, gaps that were not of interest to Hausdorff because his motivation was to make sense of the pantachies of du Bois-Reymond by considering maximal linear orders. He shows that, at least for the case of gaps which have $\aleph_1$ sets on one side and countably many on the other, there is no loss of generality in assuming linearity. He then uses this result to show that if there are no (ω_1, ω)–gaps then for every filter with base $\{Z_\alpha\}_{\alpha \in \omega_1}$ there is a $\subseteq^*$-descending family $\{B_\alpha\}_{\alpha \in \omega_1}$ such that $B_\gamma \subseteq^* Z_\gamma$. In the language of cardinal invariants the result can be stated as: If $\mathfrak{p} = \aleph_1$ then $\mathfrak{t} = \aleph_1$. But, from Rothberger's point of view, if $\mathfrak{p} = \aleph_1$ then there is an ω_1–limit in $(2^{\mathbb{N}}, \subseteq^*)$ and so, using his terminology, $\mathfrak{t}$ can be defined to be the least cardinal such that there is a $\mathfrak{t}$–limit.

Rothberger's idea here is that, given a filter base $\{Z_\alpha\}_{\alpha \in \omega_1}$ it is possible to first choose for each α a set A_α such that $A_\alpha \subseteq Z_\gamma$ for each $\gamma \in \alpha$. This is not difficult and was known already to du Bois-Reymond. However, there is no reason to believe that the family $\{A_\alpha\}_{\alpha \in \omega_1}$ is $\subseteq^*$-descending. To achieve this Rothberger chooses inductively $\subseteq^*$-descending $\{B_\alpha\}_{\alpha \in \omega_1}$ such that $Z_\gamma \supseteq^* B_\gamma \supseteq^* A_\beta$ for $\gamma \in \beta$. The reason he is able to do this is that failure at stage γ would guarantee that the family $\{B_\beta\}_{\beta \in \gamma} \cup \{A_\alpha\}_{\alpha \in \omega_1 \setminus \gamma}$ is an (ω_1, ω)–gap, contradicting his hypothesis. This is a central result which sheds considerable light on one of the major outstanding problems concerning cardinal invariants of the continuum: Is $\mathfrak{t} = \mathfrak{p}$?

Before leaving the work of Rothberger it is worthwhile to look at some of his other results that have to do with the structure of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, some of which we will return to later. Results such as that $\mathfrak{p} = \aleph_1$ implies the existence of ω_1–limits in $(2^{\mathbb{N}}, \subseteq^*)$ or the existence of Hausdorff's gaps, to say nothing of Gódel's consistency results, must surely have provided encouragement to researchers in the thirties and forties to continue trying to prove the Continuum Hypothesis. Some of Rothberger's results must have seemed tantalizingly close. For example, in [144] Rothberger shows that the Continuum Hypothesis is equivalent to the conjunction of two assertions of a cardinal invariant nature. The first is that there is a family $F \subseteq \mathbb{N}^{\mathbb{N}}$ of size $2^{\aleph_0}$ such that for each $g \in \mathbb{N}^{\mathbb{N}}$ the set $\{f \in F \mid f \leq^* g\}$ is countable. The second assertion is that there is a scale of cardinality $2^{\aleph_0}$ in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. The result is not at all difficult, but, in order to obtain it, it was crucial that Rothberger was looking at the whole structure $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ rather than a pantachie within it such as Hausdorff would have done in 1909. It is also noteworthy that Rothberger begins to consider cofinal families in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ rather than just unbounded families. The cardinal $\mathfrak{d}$ is certainly implicit in this work — recall that $\mathfrak{d}$ is defined to be the least cardinal of a cofinal family in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, namely a family $\mathcal{D} \subseteq \mathbb{N}^{\mathbb{N}}$ such that for each $f \in \mathbb{N}^{\mathbb{N}}$ there is $d \in \mathcal{D}$ such that $f \leq^* d$. However, it was first considered explicitly by Miroslav Katětov in [92] in the context of studying the character of points in topological spaces. Other cardinal invariants are also standing in the wings waiting to be introduced. In [147] he shows that the failure of the Lusin's weak Continuum

Hypothesis, namely $2^{\aleph_0} = 2^{\aleph_1}$, implies the existence of ω_1–limits. In the same paper he shows that, in modern terminology, if $\aleph_1$ is equal to the minimum of $\mathcal{N}$ and $\mathbf{non}\,\mathcal{M}$ then there are ω_1–limits.

However, a lengthy hiatus of close to twenty years would be experienced in the study of the structure $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ before it was known that the existence of ω_1–limits does not imply the Continuum Hypothesis, let alone what the nature of the relationship between these concepts is. So, before skipping these twenty years to begin looking at developments in the post-Cohen era, it is worthwhile reviewing the progress that had been made by 1952, the year that Rothberger published his last paper [148] on the subject. (The other paper Rothberger published that year, [149], was considered in §3.) Hausdorff's initial research on $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ was motivated by examining the pantachies of du Bois-Reymond. He knew that the axiom of choice produced these maximal linearly ordered subsets, but had more detailed information about them only by assuming the Continuum Hypothesis. In particular, Hausdorff knew that (ω_1, ω_1^*)–gaps existed, but he did not know whether they existed in all pantachies. Lusin, on the other hand, was brought to studying unbounded families by his interest in λ–sets. He was able to construct a λ–set without assuming the Continuum Hypothesis; but, the existence of Lusin sets or concentrated sets still requires some extra hypothesis. Rothberger continued both lines of investigation. The relationship between λ–sets and the structure of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ is detailed by him in his list of equivalences to sets of a fixed cardinality being $\leq^*$-bounded. He is also able to show that $\mathfrak{p} = \aleph_1$ implies the existence of ω_1–limits in $(2^{\mathbb{N}}, \subseteq^*)$ or, in other words, $\mathfrak{t} = \aleph_1$. Furthermore, Besicovitch had shown that there is a set with Property C which is not concentrated, but the question of whether the existence of a set with Property C implies the existence of a concentrated set remained unanswered in 1952.

Question 5.1. Does the existence of a concentrated set imply there is a Lusin set?

What is the role of $\mathfrak{b}$ in these questions? It was known by Rothberger that $\mathfrak{b} = \aleph_1$ is equivalent to the existence of a concentrated set and that concentrated sets have Property C. However, the following question could have been asked

Question 5.2. Is there a set with Property C even if $\mathfrak{b} > \aleph_1$?

Rothberger explicitly asks in [147] whether the implication in his Theorem 3 can be reversed, namely,

Question 5.3. Does $\mathfrak{b} = \aleph_1$ imply that $\mathfrak{t} = \aleph_1$?

The answers to all of these question would have to await the advent of forcing techniques.

The first of the results to address these unanswered questions about $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ dealt, unknowingly, with questions arising from Rothberger's article [145]. Recall that in a footnote Rothberger had remarked that the cofinality of $\mathfrak{b}$ is uncountable, but that no more is known about it. In [80] Hechler shows that Rothberger did indeed know all that it was possible to know at the time. However, before describing Hechler's result, it should be noted that history did not unfold as it should at this point. One might have expected that the newly available forcing techniques would have precipitated a re-examination of the intense work on set theoretic questions recorded in *Fundamenta Mathematicae* from the twenties to the end of the forties. However, the early applications of forcing techniques produced their own questions in some cases, and answered questions from entirely different sources in others.

Hechler's main theorem in [80] is the following: Given any partially ordered set $(P, \leq)$ such that:

- $|P| \leq 2^{\aleph_0}$
- for all countable subsets $X \subseteq P$ there is some $p \in P$ such that $x \leq p$ for all $x \in X$

there is a forcing extension in which there is $\Phi : P \to \mathbb{N}^{\mathbb{N}}$ such that $p \leq q$ if and only if $\Phi(p) \leq^* \Phi(q)$ and, moreover, the image of Φ is cofinal; in other words, for each $f \in \mathbb{N}^{\mathbb{N}}$ there is some $p \in P$ such that $\Phi(p) \geq^* f$.

The result can, in fact, be strengthened as has been noted by Max Burke [40] who provides a modern version of Hechler's argument. The strengthening Burke notes is that one does not need to assume that $|P| \leq 2^{\aleph_0}$ as long as one realizes that in the generic extension the size of the continuum will be $|P|^{\aleph_0}$.

As a particular corollary to Hechler's theorem it follows that any uncountable regular cardinal κ can play the role of $(P, \leq)$ in the theorem and, in this case, it follows that $\mathfrak{b} = \kappa$. So Rothberger's footnote is, indeed, accurate. Moreover, if $\kappa < \rho$ are two different uncountable regular cardinals then $\kappa \times \rho$ can play the role of $(P, \leq)$ in Hechler's theorem and, in this case, it follows that $\mathfrak{b} = \kappa$ even though there are also ρ–limits in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$.

Hechler's construction also plays a key role in solving Question 5.3. R. C. Solomon solves this [171] by first showing that the equalities $\mathfrak{b} = \aleph_2 = 2^{\aleph_0}$ and $2^{\aleph_1} = \aleph_3$ hold in a model obtained by Hechler's construction used to establish his consistency result for the case $(P, \leq) = (\omega_2, \in)$. Using Rothberger's result that the weak Continuum Hypothesis implies there are ω_1–limits — this was examined in §3 — it follows that $\mathfrak{t} = \aleph_1$ in this model. It is worthy of note that Solomon's discussion of sets with Property C mentions that they are the same as the sets which have μ^h-Hausdorff measure zero for each function h. This clear link to the work of Besicovitch may explain why Solomon was more familiar with the work of Rothberger than many North American set theorists, such as Hechler, were at the time.

In the same paper Solomon points to the significance of the cardinal arithmetic of this model and a necessary condition for solving Question 5.2. He notes that Rothberger had shown that $\mathfrak{b} = \aleph_1$ implies there is a set which is concentrated and hence has Property C. On the other hand, in [148] he also showed that if $X \subseteq \mathbb{R}$ has cardinality less than $\mathfrak{p}$ then X has Property C. So, in order to have a model of set theory without any uncountable set with Property C it is necessary that $\mathfrak{p} < \mathfrak{b}$. The fact that this can not be ruled out follows from the cardinal arithmetic of Hechler's model.

Not long afterwards, in 1976, Laver [105] was able to produce a model of set theory where all sets with Property C, or, as he referred to them, all sets with *strong measure zero*, are countable. This was a milestone result in several respects. The consistency result itself, of course, was of great interest. Laver's motivation here is rooted in considerations about strengthenings of the property of being Lebesgue null. Strong Measure Zero is one such strengthening, but so is being universally null. Laver reminds his reader that Hausdorff had established, as a corollary to the existence of (ω_1, ω_1^*)–gaps, that there are universally null sets of size $\aleph_1$. Laver points out that, *"Regarding universal measure zero sets, Hausdorff's theorem is best possible in the sense that there is a model of ZFC + $2^{\aleph_0} > \aleph_1$ in which there are no universal measure zero sets of power $\aleph_2$"*. He explains that adding ω_2 Sacks reals or Solovay reals to a model of the Continuum Hypothesis will provide the required counterexample and that this fact is an extension of unpublished results of Baumgartner establishing that there are no sets with Property C of cardinality $\aleph_2$ in these models.

The other milestone Laver's paper marks is a new level of sophistication in forcing methods. Countable support iterations had previously been used for partial orders which are either countably closed or, at least, not too far removed from the countably closed domain. In [105] (announced in [104]) Laver showed that this sort of support had much broader applicability. Efforts to generalize Laver's arguments would lead to the notion of Axiom A[28] forcing but, eventually Shelah was to show that countable support forcing could be used in contexts not even hinted at by Laver's result, although the Axiom A partial orders continue to provide important examples of Shelah's proper partial orders. However, this is leading us to topics that are discussed in greater detail in §7 since the Borel Conjecture paper was not Laver's only contribution to settling questions left unanswered by the efforts of Hausdorff, Rothberger, Sierpiński and their contemporaries.

In [106] Laver shows that it is possible to construct a model of set theory in which $2^{\aleph_0} > \aleph_1$ and the $2^{\aleph_0}$-saturated linear order — recall that Hausdorff had devoted considerable effort to analyzing precisely such orders some 70 years earlier — is embedded into the structure $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. But, just as Hechler's motivation for his embedding result was not in answering the questions that were beyond the techniques available to Rothberger, Laver's embedding result was used to answer a question raised by Solovay and Woodin concerning the automatic continuity of homomorphisms of Banach algebras. So, while Laver's result is directly related to the pantachies of Hausdorff, these are not even mentioned in [106]. Using

[28]These should not be confused with Axiom A from [116] which is now known as Martin's Axiom.

routine methods, Laver's embedding result can easily be modified so as to arrange that the linear order he embeds into $\mathbb{N}^{\mathbb{N}}$ is actually maximal, in other words, a pantachie. This then answers question α of Hausdorff's [73] which asks whether there is a pantachie without (ω_1, ω_1^*)–gaps.

Since Hechler has already been mentioned, it may be useful to pause briefly to explain why Laver's result is not implied by Hechler's. After all, one might argue, the $2^{\aleph_0}$-saturated linear order satisfies the weakened hypothesis of Hechler's theorem noticed by Burke, so it follows that it can be embedded, in fact embedded cofinally. The subtle error in this argument is that being $2^{\aleph_0}$-saturated is not an absolute property. So, while it is true that Hechler's argument will embed the ground model $2^{\aleph_0}$-saturated linear order into $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, the forcing which does this will destroy the saturation property. So this is the difficulty Laver had to overcome.

His approach was to embed the $2^{\aleph_0}$-saturated linear order by iteratively adding new generic elements to $\mathbb{N}^{\mathbb{N}}$ which would be the targets of the embedding. At the same time, it is possible to keep track of new gaps in the $2^{\aleph_0}$-saturated linear order which need to be mapped to $\mathbb{N}^{\mathbb{N}}$. This seems a natural strategy, but there is a critical obstacle to overcome and there is a degree of irony in the fact that, while Laver was not interested in Hausdorff's pantachies, the obstacle which he had to overcome was one discovered by Hausdorff in his study of pantachies, namely the (ω_1, ω_1^*)–gap. It may occur that the strategy used by Laver arrives at a situation where the embedding has already been defined on a subset of order type $\omega_1 + \omega_1^*$ in the saturated linear order and has been mapped to an (ω_1, ω_1^*)–gap in $\mathbb{N}^{\mathbb{N}}$. From the fact that the linear order is $2^{\aleph_0}$-saturated and $2^{\aleph_0} > \aleph_1$ in the final model, it follows that there must be an element A of the saturated linear order falling between both halves of the subset of order type $\omega_1 + \omega_1^*$. However, the entire strategy must fail at this stage since the (ω_1, ω_1^*)–gap can not, by definition, be filled by any image of A. Laver's key realization though, is that since the image of the embedding consists of generic elements of $\mathbb{N}^{\mathbb{N}}$, every potential (ω_1, ω_1^*)–gap in $\mathbb{N}^{\mathbb{N}}$ can be split by an element of $\mathbb{N}^{\mathbb{N}}$ obtained by forcing with an appropriate partial order.

Laver's realization that generic (ω_1, ω_1^*)–gaps can be split by suitable forcing unveils the fine structure of these objects which could not even have been formulated by Hausdorff. It was Kunen who provided [96, 16] an even clearer picture of the nature of Hausdorff's gaps as well as gaps of different cofinalities. To put Kunen's contribution in the proper context, it has to be noted that Lavers's embedded linear order, of course, has no (κ, λ^*)–gaps at all for κ and λ less than $2^{\aleph_0}$. On the other hand, there must be $(\mathfrak{c}, \mathfrak{c}^*)$–gaps in the embedded liner order since these can be constructed by a standard diagonalization argument using the $2^{\aleph_0}$-saturation property. However, Kunen was able to show that it is consistent with Martin's Axiom and $2^{\aleph_0} = \aleph_2$ that there is a linear order of cardinality $2^{\aleph_0}$ which does not embed into $(2^{\mathbb{N}}, \leq^*)$.

His key idea was to establish a relationship between the assertion that a subset of $(2^{\mathbb{N}}, \leq^*)$ of order type $\kappa + \lambda^*$ is a gap with the assertion that the natural order for forcing a set which splits the gap enjoys the countable chain condition. It will become apparent that this idea is very closely related to the construction of a Hausdorff gap. In fact, it shows that there is some justification in using the preposition "the" when referring to a Hausdorff gap. Given a family $\mathcal{Z} = \{Z_\gamma\}_{\gamma \in \kappa+\lambda^*}$ of subsets of $\mathbb{N}$ such that $Z_\gamma \subseteq Z_{\gamma'}$ if $\gamma < \gamma'$ let $\mathbb{P}(\mathcal{Z})$ be the partial order consisting of finite approximations to a set splitting $\{Z_\gamma\}_{\gamma\in\kappa}$ from $\{Z_\gamma\}_{\gamma\in\lambda^*}$. To be more precise, $p = (x, a, b) \in \mathbb{P}(\mathcal{Z})$ where x is a finite subset of $\mathbb{N}$, a is a finite subset of κ and x is a finite subset of λ^*. Moreover $\bigcap_{\gamma\in a} Z_\gamma \setminus \max(x) \supseteq \bigcup_{\gamma\in b} Z_\gamma \setminus \max(x)$. Kunen's key lemma[29] describes precisely when this partial order has the countable chain condition.

(1) If $\mathcal{Z}$ is filled — in other words, if $\mathcal{Z}$ is not a gap — then $\mathbb{P}(\mathcal{Z})$ has the countable chain condition.
(2) If the cofinality of either κ or λ is different from ω_1 then $\mathbb{P}(\mathcal{Z})$ has the countable chain condition.
(3) If $\kappa = \lambda = \omega_1$ then there is a countable chain condition partial order which forces an uncountable set of pairwise incomparable elements, known as an antichain, in $\mathbb{P}(\mathcal{Z})$.

[29]Theorem 4.2 of [16].

The key point to observe is that it is immediate that, assuming Martin's Axiom, the only possible (κ, λ^*)–gaps are

- $(\mathfrak{c}, \omega_1^*)$ and $(\omega_1, \mathfrak{c}^*)$–gaps
- $(\mathfrak{c}, 1)$ and $(1, \mathfrak{c}^*)$–gaps
- (ω_1, ω_1^*)–gaps
- $(\mathfrak{c}, \mathfrak{c}^*)$–gaps

and the $(\mathfrak{c}, 1)$ and $(1, \mathfrak{c}^*)$–gaps — or, limits in Hausdorff's and Rothberger's terminology — are easily constructed by induction under Martin's Axiom. However, the same argument does not work for $(\mathfrak{c}, \mathfrak{c}^*)$–gaps because an (ω_1, ω_1^*)–gap may arise at an intermediate stage bringing the induction to a halt. The same applies to $(\mathfrak{c}, \omega_1^*)$ and $(\omega_1, \mathfrak{c}^*)$–gaps.

Kunen was able to show that, in fact, it is consistent with Martin's Axiom that there are no $(\mathfrak{c}, \mathfrak{c}^*)$–gaps, nor are there $(\mathfrak{c}, \omega_1^*)$ or $(\omega_1, \mathfrak{c}^*)$–gaps. The strategy is to begin with a model of the Continuum Hypothesis and $\diamondsuit_{\omega_2}$ for the ordinals of cofinality ω_1. This version of $\diamondsuit_{\omega_2}$ is used to guess initial segments of the embedding, for example, of a $(\mathfrak{c}, \omega_1^*)$–gap. When an initial segment is trapped at stage δ, as

$$\mathcal{Z}_\delta = \{Z_\xi\}_{\xi \in \delta} \cup \{Z_\xi\}_{\xi \in \omega_1^*}$$

it follows that the cofinality of δ is ω_1. Hence the third clause of Kunen's lemma can be applied to it. In other words, there is a countable chain condition partial order which adds an uncountable antichain to $\mathbb{P}(\mathcal{Z}_\delta)$. Once there is an uncountable antichain in $\mathbb{P}(\mathcal{Z}_\delta)$ it will persist in all larger models and, hence, it is not possible in any later model to split $\mathcal{Z}_\delta$, by the first clause of Kunen's lemma. In other words, once a $(\mathfrak{c}, \omega_1^*)$–gap is trapped by the $\diamondsuit_{\omega_2}$-sequence, it is possible to force with a countable chain condition partial order so that the trapped gap can not be extended any further. It is easy to dovetail countable chain condition partial orders into this strategy to obtain a model of Martin's Axiom and no gaps other than those guaranteed by the construction of Hausdorff.

It is interesting to examine this proof in a bit more detail to see that it is even more closely connected to Hausdorff's construction than might be suggested by the statement of the result. Recall that the key idea of Hausdorff in the 1908 paper [74], and later in the 1936 paper [76], was to arrange that the gap $\mathcal{H} = \{Z_\xi\}_{\xi \in \omega_1} \cup \{W_\xi\}_{\xi \in \omega_1^*}$ constructed has the following property: Given any $\xi \in \omega_1$ and any $k \in \mathbb{N}$ there are only finitely many $\zeta \in \xi$ such that $Z_\zeta \subseteq k \cup W_\xi$. If there were a set $Y \subseteq \mathbb{N}$ such that $Z_\zeta \subseteq^* Y \subseteq^* W_\zeta$ for each $\zeta \in \omega_1$ then there would also be some $k \in \mathbb{N}$ and an uncountable set $\Gamma \subseteq \omega_1$ such that $Z_\zeta \subseteq Y \cup k \subseteq^* W_\zeta \cup k$ for each $\zeta \in \Gamma$. Any $\xi \in \Gamma$ with infinitely many predecessors in Γ then yields a contradiction. In particular, there are only finitely many $\zeta \in \xi \cap \Gamma$ such that $Z_\zeta \subseteq Y \cup k \subseteq^* W_\zeta \cup k$. In the terminology of the partial order $\mathbb{P}(\mathcal{H})$, the key property of the gap $\mathcal{H}$ constructed by Hausdorff is that for any uncountable set $\Gamma \subseteq \omega_1$ there are ζ and ξ in Γ such that $(\emptyset, \{\xi\}, \{\xi\})$ and $(\emptyset, \{\zeta\}, \{\zeta\})$ are incompatible. One may say that for the gap constructed by Hausdorff the partial order $\mathbb{P}(\mathcal{H})$ hereditarily fails to have the countable chain condition.

The success of Hausdorff's construction of an (ω_1, ω_1^*)–gap can be thought of as being the result of producing a family which is as chaotic as possible with respect to the finite defects allowed by the $\leq^*$ ordering. Todorčević has extended this approach by more carefully analyzing the defects in scales of length $\mathfrak{b}$. He defines an oscillation of two functions f and g in $\mathbb{N}^\mathbb{N}$, denoted by $\operatorname{osc}(f, g)$, to be the cardinality of the set $\{n \in \mathbb{N} \mid g(n) \leq f(n) \text{ and } g(n+1) > f(n+1)\}$. Observe that if $f \leq^* g$ then $\operatorname{osc}(f, g)$ is finite. In Corollary 1 of [185] (see also [188]) Todorčević establishes the following: If $B \subseteq \mathbb{N}^\mathbb{N}$ is any $\leq^*$-unbounded family of monotonically increasing functions such that all countable subsets of B have a $\leq^*$ upper bound in B then $\{\operatorname{osc}(f, g) \mid f, g \in B\} = \mathbb{N}$. This idea would later be used to great effect in the seminal paper [184] establishing the negative partition relation

$$\aleph_1 \not\to [\aleph_1]^2_{\aleph_1}$$

But the key realization contained in the earlier result is that the behaviour of initial segments of unbounded scales is as chaotic as possible. So, while Rothberger's remark that all that can be said about

$\mathfrak{b}$ is that it has uncountable cofinality is true, if one looks at the families of sequences exemplifying $\mathfrak{b}$ then one finds there a very complex structure.

Since Todorčević's results on unbounded sets in $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ do not use methods that were unavailable to Rothberger, or even to Sierpiński and Hausdorff before him, it is natural to ask why it took so long before this deeper structure was uncovered. A part of the explanation is certainly the difference in motivation for looking at $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Except for Hausdorff, all the researchers working on $(\mathbb{N}^{\mathbb{N}}, \leq^*)$, from Lusin to Rothberger, were primarily concerned with properties of subsets of the reals; sets with Property C, λ–sets and concentrated sets were the focus of greatest attention and the properties defining these sets are one dimensional in nature. It is an accident of terminology that these sets are often referred to as "linear" by the early researchers, but it is an accident worth keeping in mind. Hausdorff's construction of the gaps named in his honour are the only hint of an interaction between pairs of elements of $\mathbb{N}^{\mathbb{N}}$. Todorčević, however, was primarily interested in higher dimensional phenomena as can be seen by examining the generality of statements in [185]. Indeed, while the statement of Kunen's lemma has been stated in a forcing context, it is often more profitable to view Kunen's results in the context of partition relations as is done, for example, in [1].

Before leaving the history of $\mathbb{N}^{\mathbb{N}}$ there is still the question of the implications concerning concentrated sets, Lusin sets and sets with Property C. Since Lusin sets are concentrated and concentrated sets have Property C, the two questions to answer are:

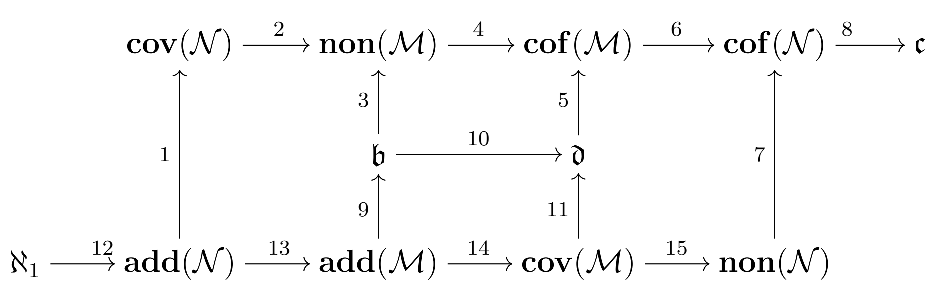
Question 5.4. Does the existence of an uncountable concentrated set imply the existence of a Lusin set?

Question 5.5. Does the existence of an uncountable set with Property C imply the existence of an uncountable concentrated set?

Since Rothberger showed that there is an uncountable concentrated set if and only if $\mathfrak{b} = \aleph_1$ the first question is equivalent to asking whether the equality $\mathfrak{b} = \aleph_1$ implies that there is a Lusin set. This is easily seen to be false in the model obtained by adding $\aleph_2$ random reals since all sets of size $\aleph_1$ are meagre in this model as was shown by Solovay. The second question is equivalent to asking whether $\mathfrak{b} > \aleph_1$ implies Borel's Conjecture. Martin's Axiom and $2^{\aleph_0} > \aleph_1$ easily shows this to be false too. However, these results may be better understood if considered at the same time as the historical developments surrounding the properties of measure and category.

6. The Cardinal Invariants of Measure and Category

The discussion of the cardinal $\mathfrak{b}$ in §5 was quite narrowly focused and readers familiar with the subject will have noticed, and perhaps even objected to, the lack of reference to the notions of measure and category. It is time to rectify this situation since, indeed, the cardinals $\mathfrak{b}$ and $\mathfrak{d}$ that are central to the study of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$ are just as central to the study of of measure and category. Coincidentally, these two invariants are also central in the diagram attributed by David Fremlin, [62] to Cichoń, as well as Anastasis Kamburelis and Janusz Pawlikowski:



in which $\mathcal{M}$ represents the ideal of meagre subsets of the reals and $\mathcal{N}$ represents the ideal of Lebesgue null subsets of the reals. Furthermore, if $\mathcal{I}$ is either $\mathcal{M}$ or $\mathcal{N}$ then

- $\mathbf{cov}(\mathcal{I})$ represents the least cardinality of a set $\mathcal{A} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{A} = \mathbb{R}$
- $\mathbf{add}(\mathcal{I})$ represents the least cardinality of a set $\mathcal{A} \subseteq \mathcal{I}$ such that $\bigcup \mathcal{A} \notin \mathcal{I}$
- $\mathbf{non}(\mathcal{I})$ represents the least cardinality of a set $A \subseteq \mathbb{R}$ such that $A \notin \mathcal{I}$
- $\mathbf{cof}(\mathcal{I})$ represents the least cardinality of a set $\mathcal{A} \subseteq \mathcal{I}$ such that for every $X \in \mathcal{I}$ there is $A \in \mathcal{A}$ such that $X \subseteq A$.

The definition of these four invariants can, of course, be applied to any ideal $\mathcal{I}$.

In fact, much of the development of the study of measure and category on the reals can be seen from the perspective of the cardinal invariants in Cichoń's diagram. Just as importantly, the diagram marks a watershed in the study of the continuum by attaching names and bringing to the forefront the cardinal invariants associated with the reals in its various guises. It seems a simple enough idea that one may wonder why it had not appeared earlier. A probable reason is the early preoccupation with the Continuum Hypothesis. Recall that one of Sierpiński's questions in the first volume of *Fundamenta Mathematicae* was Question 2.4 asking whether it can be proved, without appealing to the Continuum Hypothesis of course, that there are $\aleph_1$ null sets whose union is not null or $\aleph_1$ meagre sets whose union is not meagre. While it is true that in the same volume Ruziewicz asks Question 2.5, whether sets of cardinality less than $2^{\aleph_0}$ are necessarily meagre, the focus at that time is certainly in $\aleph_1$. This remains true even later in 1936 when Hausdorff tries to answer Question 2.4 using his (ω_1, ω_1)–gap. Given this approach, drawing something similar to the Cichoń diagram would certainly have seemed senseless since the entire diagram collapses to a single point in this case. With the range of possibilities for the continuum unveiled by Cohen though, it makes perfect sense and similar diagrams seem to have been used by many researchers in the early years after Cohen's introduction of forcing techniques. For example, Miller [120, 122] considers the same properties of measure and category as those that appear in the Cichoń diagram, but he is only interested in whether or not the cardinals are equal to the continuum. The organizational chart he uses is usually referred to as the Kunen–Miller chart.

So, the history of measure and category can begin by examining what could have been said about the Cichoń diagram in the years before the forcing era. It should be explained that an arrow from a cardinal invariant x to the invariant y means two things:

A: it is possible to prove that $x \leq y$ without appealing to extra axioms of set theory,
B: it is not possible to prove that $y \leq x$ without appealing to extra axioms of set theory.

In other words, each arrow corresponds to an inequality and none of the arrows can be replaced by an equality in the sense that for each of the 15 arrows there is a model of set theory where the cardinal invariant at the tail of the arrow is smaller than the one at its tip. This section will concentrate on the (A) component of the arrows in Cichoń's diagram, leaving the (B) component to §7. In fact, the (B) component is best described by the Kunen–Miller chart, a matrix whose entries correspond to possible combinations of cardinal invariants being equal or less than $\mathfrak{c}$. The final entries to this chart were added in [87].

The (A) part of Arrow 8 is nothing more than that each null sets is contained in a null G_δ and the number of such G_δ sets is no greater than $2^{\aleph_0}$ while the (A) part of Arrow 12 is just the additivity of Lebesgue measure. The (A) part of Arrows 1, 7, and 14 are immediate consequences of the definitions, as is the (A) part of Arrow 10. Recall that in [143] Rothberger showed that $\mathbf{non}(\mathcal{M}) \geq \mathbf{cov}(\mathcal{N})$ and $\mathbf{non}(\mathcal{N}) \geq \mathbf{cov}(\mathcal{M})$ or, in other words, he justified the (A) parts of Arrow 2 and Arrow 15. But, of course, this was not Rothberger's primary goal since, in fact, he was concerned with the Continuum Hypothesis. The result he was able to prove with the use these two inequalities is that the Continuum Hypothesis is equivalent to the existence of Lusin and Sierpiński sets at least one of which has cardinality $2^{\aleph_0}$. However, this is simply a convoluted way of expressing his key result which, from the modern perspective seems more elegant than the equivalence: If there are both Sierpiński and Lusin sets then they both have cardinality $\aleph_1$.

As well, as Miller[30] has observed, Rothberger's result has direct implications for forcing constructions. For example, adding κ Cohen reals will yield a Lusin set of size κ consisting of the Cohen reals themselves. The same is true of random reals. Iteratively adding a pair of reals, one Cohen and the other random, to a model of $2^{\aleph_0} > \aleph_1$ will yield both a Lusin and a Sierpiński set. Rothberger's argument shows that there is no way of modifying the support or iteration strategy to get one of the sets to have cardinality greater than $\aleph_1$.

What can be said of the other arrows? Part (A) of Arrow 3 is essentially due to Sierpiński who showed [169] that a subset of the irrationals is $\leq^*$-bounded if and only if it is σ-compact. It is important to recall that the representation of the irrationals as sequences of integers was already known to Lusin and Baire, but that the definition of the cardinal invariants associated with the meagre and null ideals are not specific to the metric space involved and, indeed, it can be shown that this is irrelevant for these invariants. For a result relating unboundedness to σ-compactness though, the role of the irrationals is, of course, crucial.

It has already been mentioned that in the same article Sierpiński established the cofinality preserving mappings between $\mathbb{N}^{\mathbb{N}}$ and the family of open sets containing the rationals. Using this, it is easy to establish the (A) part of Arrow 11. However, it is also worth observing that the nature of the map allows the definition of a Tukey function [191]. A Tukey function between partial orders is any function, not necessarily order preserving, which sends unbounded sets to unbounded sets. John Tukey defined these functions within a year or two of Sierpiński's results, but for quite different purposes. However it was to be only much later that the relevance of Tukey functions to cardinal invariants would become apparent, indeed, would become a guiding principle in their further study. Their relevance to the study of the continuum was realized by Peter Vojtas in [197] who called them Galois-Tukey functions, but some of the ideas are already apparent in the paper of Fremlin [62] which gave the Cichoń diagram its name. For example, a dual argument using the Tukey function established by Sierpiński yields the (A) part of Arrow 9.

Excluding the arrows involving the cofinality invariants, this leaves only Arrows 13 to discuss, the theorem of Tomek Bartoszynski. Certainly the notions involved were familiar long before Inequality 13 was proved; one need look no further than Question 2.4 to see this. In order to discuss Bartoszynski's Theorem in its historical context, it is worthwhile using the same notation he did, notation borrowed from Miller in [120]. $A(m)$ represents the statement that all families of less than $2^{\aleph_0}$ null sets have null union and similarly for $A(c)$ and category and $B(m)$ for covering. Earlier Miller had shown [120] that $\mathbf{add}(\mathcal{N}) \leq \mathfrak{d}$, an inequality that Cichoń's diagram reveals was later refined. Also, Kunen had shown that $\mathbf{add}(\mathcal{N}) \lneq \mathbf{add}(\mathcal{M})$) is possible.

Bartoszynski starts [11] by showing that $A(m)$ is equivalent to the statement that for any family of less than $2^{\aleph_0}$ convergent series there is a single convergent series dominating all of the members of the family; he calls this last property hD. It is interesting to recall, in this context, that Hausdorff had been interested in classifying the convergent series by finding a cut in a pantachie. The critical lemma then is one which establishes that hD is equivalent to a statement about what, in three years time, Bartoszynski [12] would be calling *slaloms*. Part of this lemma establishes that hD is equivalent to:

Statement 1. For every $F \subseteq \mathbb{N}^{\mathbb{N}}$ of size less than $2^{\aleph_0}$ there exist sets $I_n \subseteq \mathbb{N}$ such that $|I_n| < n^2$ and for every $f \in F$ there is some M such that $f(m) \in I_m$ for all $m \geq M$.

The key point here is that such statements are similar to combinatorial characterizations of the meagre ideal. The role of n^2 is simply that it produces a convenient convergent series $\sum_{n=1}^{\infty} n^{-2}$.

John Truss [190] and Miller [122] had shown that $\mathbf{add}(\mathcal{M}) = \mathfrak{c}$ if and only $\mathbf{cov}(\mathcal{M}) = \mathfrak{b} = \mathfrak{c}$. Indeed, the arguments show that $\mathbf{add}(\mathcal{M})$ is the minimum of $\mathbf{cov}(\mathcal{M})$ and $\mathfrak{b}$, as might be conjectured by considering Arrows 9 and 14 of the Cichoń diagram. Miller also shows in [122] that if $\mathfrak{b} = \mathfrak{c}$ then the equality $\mathbf{cov}(\mathcal{M}) = \mathfrak{c}$ can be characterized combinatorially by the statement:

[30] On page 205 of [125].

Statement 2. For every $F \subseteq \mathbb{N}^{\mathbb{N}}$ of cardinality less than $2^{\aleph_0}$ there is $g \in \mathbb{N}^{\mathbb{N}}$ such that for every $f \in F$ there are infinitely many $n \in \mathbb{N}$ such that $f(n) = g(n)$.

Slightly earlier, Michel Talagrand had found a similar combinatorial approach to category in his study [179] of measurable and non-measurable filters. A set $X \subseteq 2^{\mathbb{N}}$ is meagre if and only if there is $y : \mathbb{N} \to 2$ and an increasing sequence of natural numbers $\{n_i\}_{i=0}^{\infty}$ such that for every $x \in X$ there are only finitely many i such that $x \restriction [n_i, n_{i+1}) = y \restriction [n_i, n_{i+1})$.

Bartoszynski uses Statement 2 by adopting a different perspective on functions from $\mathbb{N}$ to $\mathbb{N}$ in order to show that Statement 1 implies Statement 2. Given a family of functions $F \subseteq \mathbb{N}^{\mathbb{N}}$ of size less than $2^{\aleph_0}$ he associates to $f \in F$ the function f' which maps n to the finite partial function $f \restriction [n^3, (n+1)^3)$. Statement 1 then allows the selection of a family I_n of size n^2 of functions defined on the interval $[n^3, (n+1)^3)$ such that for every $f \in F$ eventually $f \restriction [n^3, (n+1)^3)$ belongs to I_n. Since the domains of the functions in I_n are larger than $|I_n|$ it is easy to find a single $g_n : [n^3, (n+1)^3) \to \mathbb{N}$ which agrees with each member of I_n on at least one point in $[n^3, (n+1)^3)$. Stringing together the g_n produces the desired function g witnessing that Statement 2 holds. The argument for $\mathfrak{c}$ then converts without difficulty to a proof that Arrow 13 holds.

It is interesting that the same result along with strengthenings were obtained by Jean Raisonnier and Jacques Stern [137, 138] as well as Pawlikowski [134] but in a much different spirit. Whereas Bartoszynski's arguments would have been comprehensible to Sierpiński or Rothberger, even the statement of Pawlikowski's result would have required considerable explanation: *Let M be a standard model of ZFC. If the union of all Borel sets of measure (Lebesgue) zero coded in M has measure zero then the union of all Borel sets of first category coded in M is of first category.* What has become evident by the 1980's is that even combinatorial statements that could have been considered and even proved in the 1930's are often more profitably viewed from the perspective of adding a generic real over a certain model of set theory.

It is instructive to look a bit more closely at Pawlikowski's results. In [134] his arguments are typical of this approach. He first recasts Bartoszynski's Theorem in the following form: If M is a standard model of ZFC then Statement 1 holds for $F = \mathbb{N}^{\mathbb{N}} \cap M$ if and only if the union of all Borel sets of measure zero in M has measure zero. He then shows that if M is a standard model of ZFC such that Statement 1 holds for $F = \mathbb{N}^{\mathbb{N}} \cap M$ then the union of all meagre Borel sets in M is meagre. Bartoszynski's Theorem follows from this by realizing that if $\kappa < \mathbf{add}(\mathcal{N})$ and $\mathcal{X}$ is a family of κ meagre sets then one can take a model of sufficiently much set theory containing $\mathcal{X}$ yet still having cardinality κ. It follows from the recast version of Bartoszynski's Theorem that Statement 1 holds for the functions of this model and hence the union of all meagre sets in the model is meagre. In particular, the union of $\mathcal{X}$ is meagre.

Moreover, the techniques used in Pawlikowski's argument lend themselves to arguments providing a deeper explanation of some of the inequalities of Cichoń's diagram. For example, in Section 2.3 of [13] it is shown that there is a Tukey function from $(\mathcal{M}, \subseteq)$ to $(\mathcal{N}, \subseteq)$. It follows directly from the existence of this Tukey function that $\mathbf{add}(\mathcal{N}) \leq \mathbf{add}(\mathcal{M})$, namely the (A) part of Arrow 13 holds. Moreover, a duality argument using the Tukey function establishes that $\mathbf{cof}(\mathcal{N}) \leq \mathbf{cof}(\mathcal{M})$ or, in other words, that the (A) part of Arrow 6 holds.

It was mentioned that the cardinal invariants related to the cofinality of the ideals $\mathcal{N}$ and $\mathcal{M}$ did not receive much attention before the forcing era, but this is not entirely true. An important counterexample is the Sierpiński-Tarski duality theorem assuming the Continuum Hypothesis. This uses cofinality as a key part of an argument — although under the Continuum Hypothesis, of course, many of these invariants become difficult to distinguish — to construct a function from $\mathcal{M}$ to $\mathcal{N}$ somewhat in the spirit of a Tukey function. The idea behind the result is that measure and category are, in many senses, very similar so it would be very useful if information about one structure could be transferred to the other. Which direction is more useful depends on your point of view though. John C. Oxtoby takes the view that, since "*the theory of measure is more extensive and "important" than that of category, the service (of such a transfer) is mainly in the direction of measure theory*" [133]. On the other hand

Shelah comments that, "*Mathematicians who are not set theorists generally consider* "null" *as senior to* "meagre"*, that is, as a more important case; set theorists inversely, as set-theoretically Cohen reals are much more manageable than random reals ...*" [158]. The two points of view differ not only in bias, but also in perspective. For Shelah measure and category are the study of random and Cohen reals, whereas Oxtoby's perspective is more traditional.

The duality was first formulated and proved by Sierpiński in [165] and then later strengthened by Paul Erdös in [55]. The basic idea of the proof is that, assuming the Continuum Hypothesis , there are cofinal families of order type ω_1 under inclusion in both $\mathcal{N}$ and $\mathcal{M}$. Constructing a bijection which respects annuli of these cofinal families will be the required mapping. Sierpiński concludes that, assuming the Continuum Hypothesis, any property of sets mentioning only the notions of null and meagre can be dualized by interchanging the roles of "meagre" and "null" and the dual statement will have the same truth value.

The question naturally arises of how similar the properties, meagre and null, actually are. One indication that full duality does not hold has already been seen in the inequalities $\mathbf{add}(\mathcal{N}) \leq \mathbf{add}(\mathcal{M})$, although consistency arguments for the (B) part of Arrow 13 are needed to make this more credible. A slightly earlier hint at this phenomenon was Shelah's proof [155] that the consistency of having all sets of reals exhibit the Baire Property did not require a large cardinal, whereas the analogous result for measure, due to Solovay [174], does require an inaccessible.

But it would be very wrong to leave the impression that the role of inaccessible cardinals in the study of measure and category began to be appreciated only in the forcing era. This idea is quickly dispelled by considering Kuratowski's Question 2.7 about whether a set which is nowhere meagre can be partitioned into two disjoint sets, both of which are nowhere meagre. In 1933 Stanisław Ulam [194] had shown that if all weakly inaccessible cardinals are greater than $\mathfrak{c}$ then any second category set contains uncountably many disjoint second category sets. Assuming the same hypothesis Sierpiński was then [166] able to answer Question 2.7 in the positive. However, only a bit later, Lusin discovered [111] an even more ingenious argument that dispensed with the extra hypothesis entirely.

While there are a few results — such as Zbigniew Piotrowski and Andrzej Szymański's inequality establishing $\mathfrak{t} \leq \mathbf{add}(\mathcal{M})$ — that link the cardinal invariants of measure and category with others, it is worth taking a bit of a detour to consider the cofinalities of some of these invariants. Two results deserve special attention in this context. The first is Miller's proof that the cofinality of $\mathbf{cov}(\mathcal{M})$ is uncountable [121]. In stark contrast to this, Shelah showed that the cofinality of $\mathbf{cov}(\mathcal{N})$ can be countable [157] in some models. It is interesting to speculate whether such questions were ever considered by Baire or Lebesgue. When Baire showed that $\mathbf{cov}(\mathcal{M})$ is uncountable the notion of cofinality had not yet been properly formulated. Within five years though, it had been clarified in response to the Bernstein and König controversy after the 1904 International Congress of Mathematicians mentioned in §4 and so it may have occurred to Baire whether he could improve his result in the way Miller eventually did. The same may have occurred to Lebesgue but the answer would certainly have surprised him, as it does even those who first encounter it today.

Even though this will be the topic of the next section, it is not possible to end this section without mentioning the great body of work on finding models exhibiting the entire spectrum of behaviour allowed by the various cardinal invariants related to measure and category that are not ruled out by Cichoń's diagram. Indeed, it is certainly true that much of the advances in developing forcing techniques were spurred on by the goal of trying to find a complete understanding of the possible relations among the invariants of Cichoń's diagram. The reader interested in the details of this development is encouraged to consult [13] or [25] which devote considerable space to the this topic. Part of the story will be examined in §7.

7. What Forcing Arguments Reveal about the Continuum

The discussion in §6 focused entirely on condition (A) in the explanation of the meaning of the arrows in the Cichoń diagram. The use of forcing to obtain independence results, thereby establishing the (B) component of the arrows in the Cichoń diagram, is the goal of this section. The tremendous effect that Cohen's introduction of forcing techniques in the 1960's had on the development of the study of the continuum has been mentioned several times already. However, Cohen's motivation for his work seems to have been equally based on establishing the independence of the Axiom of Choice from the other axioms of set theory as it was for showing that the negation of the Continuum Hypothesis is consistent. As it turns out, the same reals used to establish the failure of the Axiom of Choice could be used to control the size of the continuum; indeed, the arguments in this case are somewhat simpler.

Cohen's original arguments on the general framework of forcing were soon discovered to be more cumbersome than necessary, but the combinatorial arguments on adding κ Cohen reals to a model of the Continuum Hypothesis are essentially the same as those used today. A difference that is inessential from the combinatorial point of view is that, instead of finite functions from κ to 2, Cohen's original argument uses finite collections of forcing statements. In order to show that new countable subsets of the ground model are contained in old countable subsets of the ground model Cohen shows that his partial order has what is now known as the *countable chain condition*, namely every family of pairwise incompatible elements is countable. Cohen's Lemma 2 on page 131 of [44] states exactly this and the proof can be recognized as a proof of what is now known as the Δ-system lemma. However, it would be Solovay who would realize in [174] that Cohen's partial order is equivalent, for the purposes of forcing, to the partial order of equivalence classes of Borel sets modulo the ideal of meagre sets. This is done as an afterthought in the context of proving the same result for random reals.

But before considering this it is worth noting that Cohen's interest in the Axiom of Choice resulted in him producing various models in [44] exhibiting different types of failures of the Axiom of Choice. One of these is a model where ω_1 is the union of countably many countably ordinals. This is not of great interest from the point of view of the continuum, but it is of interest to note that the method of proof relies on collapsing cardinals with finite conditions. Cohen clearly saw this as a generalization of the original Cohen partial order. Around the same time as Cohen's proof became available, Azriel Levy produced generalizations, one of them being to apply this collapsing partial order to inaccessible cardinals. The resulting partial order collapses all uncountable cardinals below the inaccessible, but not the inaccessible itself.

Solovay realized that this partial order has important homogeneity and embedding properties that can be used to control the appearance of reals in the generic extension. His key innovation, already appearing in [173], was the definition of what he called a "random real" over a model $\mathfrak{M}$ as a real which does not belong to any null set belonging to $\mathfrak{M}$. In order to have this definition make any sense at all, Solovay had had to develop the theory of codes for Borel sets. With the notion of a code for a Borel set it is possible to understand the notion of a real x not belonging to a null set in $\mathfrak{M}$ as meaning that there is no G_δ set with a code in $\mathfrak{M}$ which contains x. Establishing absoluteness for the concepts involved is a critical step in developing the theory of codes for Borel sets. In other words, Solovay had to show that if x does not belong to a Borel set with code c in some model containing both x and c then x does not belong to that Borel set in any other model containing both x and c.

He describes his intuition for arriving at the definition for a random real in [174] as follows:

> *Let x be a real random over $\mathfrak{M}$. An observer stationed in $\mathfrak{M}$ cannot have total knowledge about x (since x is not in $\mathfrak{M}$). However, he can have partial knowledge about x. For example, if B is a Borel set rational over $\mathfrak{M}$, then a natural question the observer can ask about x is "Is $x \in B$?" If $\mu(B) = 0$, then the answer is certainly no. On the other hand, if $\mu(B) > 0$, it is possible for x to be in B .... A similar discussion shows that if B_1 and B_2 are Borel sets rational over $\mathfrak{M}$, and $\mu((B_1 \Delta B_2) = 0$ (i.e., B_1 and B_2 are*

equal almost everywhere), then for x random over $\mathfrak{M}$, the questions "Is $x \in B_1$?" and "Is $x \in B_2$?" are equivalent.

With this discussion as motivation, Solovay then goes on to provide the now standard definition of random forcing as equivalence classes of Borel sets modulo the null ideal with the natural ordering. He notes that an analogue of a random real is obtained by replacing the null ideal by the ideal of meagre sets and he realizes that the associated partial order is precisely the forcing Cohen had used for his original consistency results on the failure of the Continuum Hypothesis. However, the random partial order plays a secondary role in Solovay's argument to the notion of a random real as one which avoids ground model null sets. It is this definition that plays a key role in establishing the consistency of all sets being Lebesgue measurable.

In [173] Solovay had shown that the measurability of all Σ^1_2 sets is equivalent to the statement that for any real a the set of reals not random over $L[a]$ is null and, similarly, that all Σ^1_2 sets having the Property of Baire is equivalent to the statement that for any real a the set of reals not Cohen over $L[a]$ is meagre. These ideas developed by Solovay eventually lead to the result of Raisonnier and Stern [137, 138] that if all Σ^1_2 sets are measurable then all Σ^1_2 sets have the Property of Baire. That the converse is false was established by Shelah and this will be discussed in a bit greater detail soon. This result is noteworthy in several ways, but it should be remarked here that it provided that first evidence that the duality hinted at by the results of Sierpiński [165] and Erdös fails. This has already been discussed in the section of Bartoszynski's work on additivity and this is closely related to the work of Raisonnier and Stern.

A precise discussion of Solovay's arguments establishing the measurability of all sets, thus contradicting the Axiom of Choice, would lead a bit too far away from the main theme of this article, but some key points must be mentioned. In exploiting Levy's inaccessible version of the collapsing functions considered by Cohen, Solovay observes that if the continuum is collapsed to be countable then the total number of codes for null sets in the ground model is also countable after the collapse and hence their union is also null. In other words, all but a null set of reals in the generic extension are random reals. (The same argument, of course, works for Cohen reals and the meagre ideal and this leads to the consistency of all sets having the Property of Baire.) Solovay had earlier [173] already established that this would yield the measurability of all Σ^1_2 if it was true with respect to all $L[a]$. The properties of the Levy collapse allowed precisely this.

The work of Solovay and Tennenbaum and Martin on iterations of forcing partial orders will be discussed in §8 but the same ideas were critical for the Levy collapse. In particular, Solovay showed that given any real a in the generic extension by the Levy collapse it is possible to find a small partial order completely embedded in the collapse such that a already belongs to the generic extension by this partial order. In a sense, this was the reverse of the problem considered in the work of Solovay and Tennenbaum and Martin. The second key observation is that the quotient of the Levy collapse by the the small algebra is again a Levy collapse in this intermediate model. In other words, the observation about almost all reals being random over this model is valid in this intermediate extension.

It is significant that Solovay's focus is on the measurability question and the parallel result for the Baire Property is treated almost as a footnote. It is interesting in this context to recall the views of Oxtoby and Shelah on the relative importance of the meagre and null ideals. Shelah's comment that the meagre ideal is more natural and tractable was likely based, in part, on its appearance in Cohen's original forcing arguments. But Solovay's introduction to his paper lists several consequences of the measurability of all sets to problems in analysis, such as Newtonian capacity, but none for having the Baire Property of all sets.

Given the very similar proofs for the two results, it came as a great surprise when Shelah showed [155] that the consistency of all sets having the Property of Baire could be established without assuming the existence of an inaccessible cardinal. The argument employs a similar strategy to Solovay's but the critical stumbling block is to obtain the embedding and homogeneity properties of the Levy collapse of an inaccessible without actually resorting to assuming that there is an inaccessible cardinal. The argument relies on a careful analysis of when countable partial orders can be completely embedded into a larger, homogeneous, partial order in such a way that all but a meagre set of later reals are Cohen over the countable partial order. As in Solovay's argument, the abundance of Cohen reals will guarantee the Baire Property for all sets of reals definable in a generic extension by the smaller order. The key point is the argument allowing the homogeneity arguments in the Levy collapse to be simulated. Shelah's ideas here allow various generalizations and lead to a large class of forcing partial orders known as "sweet". For example, the article of Andrzej Rosłanowski and Shelah [141] combines these ideas with norms on possibilities to provide a classification of countable chain condition partial orders according to whether they are Cohen-like or random-like.

While the notion of a random real played a central role in Solovay's work on the measurability question, the random algebra which generates random reals was of less significance to this argument. However, in work following on the heels of [174], Solovay showed [175] that, assuming the consistency of a measurable cardinal it is consistent that the continuum is real-valued measurable. This provided answers to questions with roots in the work of Ulam [193] in the 1930's. Ulam had shown that if κ is not a weakly inaccessible cardinal then there is no κ-complete measure that measures all subsets of κ. This did not exclude the possibility that there might be such a measure on the continuum though. On the other hand, Solovay also showed in [175] that if there is a real-valued measurable cardinal then there is an inner model with a measurable cardinal. This result, in combination with work of Dana Scott, showing [152] that measurable cardinals do not exist if $V = L$ points the direction in which to look for models where measurable cardinals do exist.

Solovay's consistency result relied on forcing with the measure algebra of size κ where κ is a 2-valued measurable cardinal. The continuum then becomes κ and the κ complete ultrafilter on κ becomes a κ-complete filter on $\mathbb{R}$. However, there are new subsets of κ not measured by the ground model ultrafilter. Solovay was able to exploit the measure on the forcing algebra by analyzing the names for sets of reals and assigning a measure to such names according the average probability that a real belongs to it. The probability is calculated in the forcing algebra, using its measure, while the average is calculated using the κ-complete measure on κ.

The generic model obtained by forcing with the measure algebra on 2^κ is usually known as the random real model. Since Solovay showed that the reals added are random, these are sometimes referred to as Solovay reals. Aside from producing a real-valued measure, the random real model is useful as an example exhibiting an interesting selection of values for various cardinal invariants. Moreover, the real-valued measurable cardinal imposes severe restrictions on the possible combinatorics at $\mathfrak{c}$, such as that $\Diamond_{\mathfrak{c}}$ holds. The article [61] can be consulted for more information. Even before Solovay's work, Petr Vopenka and Karel Hrbáček had shown [198] that if there is a Lusin set of size $2^{\aleph_0}$ then there can be no real-valued measure on the reals. The significance of this is that it points to the fact that Lusin sets can be considered as sets of Cohen reals. Indeed, it follows from the characterization of Cohen reals as those not belonging to any ground model meagre set — which was remarked on by Solovay, but even earlier also noticed by Vopenka and Hrbáček [198] — that an uncountable set of Cohen reals is a Lusin set. The Vopenka–Hrbáček result shows that something very much like the measure algebra needs to be used to get a real-valued measure since no Cohen reals can be added, at least not cofinally.

Before continuing to look at other methods of adding generic reals it is worth pausing to consider the two models, Cohen and random (sometime known as Solovay). These are already sufficient to display a contrasting range of possibilities for $\mathfrak{b}$ and $\mathfrak{d}$ and the other invariants of the Cichoń diagram. In the model obtained by adding κ Cohen reals to a model of the Continuum Hypothesis, Vopenka and Hrbáček showed that $\mathbf{cov}(\mathcal{M}) = \kappa$ while $\mathbf{non}(\mathcal{M}) = \aleph_1$. One therefore has that all the values of cardinals on the left half of the Cichoń diagram are $\aleph_1$ while those on the right hand side are all $\kappa = 2^{\aleph_0}$. In particular, this establishes the (B) component of Arrows 4, 10 and 14 in the Cichoń diagram. In the analogous model for random reals, Mathias showed that $\mathbf{cov}(\mathcal{N}) = \kappa$ while $\mathbf{non}(\mathcal{N}) = \aleph_1$. One therefore has that all the values of cardinals on the bottom row of the Cichoń diagram are $\aleph_1$ while those on the top row are all $\kappa = 2^{\aleph_0}$. Moreover $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = \aleph_1$. Hence, Arrows 1, 3, 5 and 7 in the Cichoń diagram are also fully justified..

A model where both $\mathfrak{b}$ and $\mathfrak{d}$ are the continuum but the Continuum Hypothesis fails can be obtained by Hechler's work, which has already been mentioned in the discussion of the structure of $(\mathbb{N}^{\mathbb{N}}, \leq^*)$. Indeed there is a simple version of Hechler's result, which corresponds to adding a κ-scale, that is very useful in distinguishing the behaviour of various cardinal invariants of the continuum. The partial order for adding a single real dominating all the ground model functions is usually referred to as the Hechler order or, simply, the partial order for adding a dominating real. In the generic extension obtained by iterating with finite support the Hechler partial with length κ, Hechler's results show that $\mathfrak{b} = \kappa$. Hechler did not have the general iteration techniques of Martin and Solovay [85] available to him and, in any case, the general results he obtained yielding a wide spectrum of possible structures on $\mathbb{N}^{\mathbb{N}}$ required methods beyond the linear iteration considered by Martin and Solovay. Results of Truss [190] establish that in the linearly iterated Hechler model $\mathbf{add}(\mathcal{M}) = \mathbf{cof}(\mathcal{M}) = \kappa$. It is interesting to note what Truss says of the motivation for his work, *"We first became interested in these properties on reading the paper by K. Kunen and F. D. Tall (Between Martin's axiom and Suslins's hypothesis, to appear), and in particular, in the case of* calibre*, on learning of R. M. Solovay's result that, assuming $A_{\aleph_1}$ (Martin's axiom), the measure algebra has calibre $\aleph_1$"* [31]. In other words, he was lead to studying properties of the continuum, in part, by work with its roots in set-theoretic topology. Other examples of this phenomenon are discussed in §9 and §8.

Further investigations revealed that arguments of Miller, Truss and Solovay combine to show that $\mathbf{add}(\mathcal{N}) \neq \mathbf{add}(\mathcal{M})$ in Hechler's model, thereby establishing that Bartoszynski's inequality, Arrow 13, cannot be reversed. It has already been mentioned that Truss and Miller had shown that $\mathbf{add}(\mathcal{M})$ is the minimum of $\mathbf{cov}(\mathcal{M})$ and $\mathfrak{b}$. Iterating Hechler reals and Cohen reals — this is not realy needed, since the finite support iteration will add Cohen reals itself — with finite support ω_2 times over a model of the Continuum Hypothesis then produces a model in which both $\mathfrak{b}$ and $\mathbf{cov}(\mathcal{M})$ have value $\aleph_2$. Hence $\mathbf{add}(\mathcal{M}) = \aleph_2$ also. Once it is shown that no random reals have been added, Solovay's characterization of these reals yields that $\mathbf{cov}(\mathcal{N}) = \aleph_1$ in this model and the (A) part of Arrow 1 then implies that $\mathbf{add}(\mathcal{N}) \neq \mathbf{add}(\mathcal{M})$. The details can be found in [122]. Duality of a Tukey function argument yields that Arrow 6 of the Cichoń diagram is also justified. Similar reasoning establishes that the (B) component for Arrows 2 and 15 is also justified by Hechler's model.

With the landscape of possibilities for values of the cardinal invariants having been revealed to be remarkably broad by the models of Cohen, Solovay and Hechler, the seventies witnessed a tremendous interest in exploring this landscape and determining its exact boundaries. A wealth of new forcing structures was developed in this period, most of them motivated as probes of the possibilities of cardinal invariants, but not all.

A notable exception is the Sacks real which was based on recursion theoretic arguments for constructing a minimal degree. Adding a Sacks real to Gödel's constructible universe produces a model that has exactly two degrees of constructibility. The behaviour of Sacks forcing[150] is at the opposite end of the cardinal invariant spectrum from Hechler forcing. While increasing the size of the continuum with Hechler forcing will increase most invariants along with the continuum, Sacks forcing increases the continuum by keeping most other invariants small. In particular, the side-by-side Sacks model establishes that Arrow 8 of the Cichoń diagram is also fully justified; indeed, Miller attributes the inequalities $\mathbf{non}(\mathcal{M}) < \mathfrak{c}$ and $\mathbf{non}(\mathcal{N}) < \mathfrak{c}$ to Gerald Sacks. While, the behaviour of most invariants is the same in both constructions, Baumgartner and Laver showed how to iterate Sacks reals in [17] based on Laver's earlier work [105] on the Borel conjecture. They point out that while many researchers had realized that Sacks reals could be added "side-by-side" or simultaneously, the iteration is a more delicate construction. It is shown in [177] that there are invariants distinguishing the two types of forcing.

[31]The article cited by Truss did appear later as [99].

An important realization of Miller is that the iterated Sacks model enjoys a certain type of homogeneity and he was able to exploit this in [123] to show that every set of reals of size $\aleph_2$ can be mapped continuously onto the reals. This shows that Sierpiński's Property C_5 mentioned in §4 is consistently false. Somewhat earlier, in 1980, Miller [119] had used a modification of Sacks forcing to get a partition of the reals into $\aleph_1$ closed sets together with the equality $\mathfrak{c} = \aleph_2$.

The notion of forcing now referred to as "Laver forcing" was introduced quite early as part of Laver's work on the Borel conjecture, even before the work with Baumgartner on iterating Sacks forcing. Recall from §5 that Borel had been concerned with an analysis of null sets based on the rates of convergence of the size of intervals and that Laver had provided a model where there are no uncountable sets with Property C, in the terminology of Rothberger. Noteworthy about Laver's argument is that it is the first instance of a countable support iteration used to iterate partial orders that add reals at each stage of the iteration. It became apparent that the strategy used by Laver was far more widely applicable and this was crystallized by Baumgartner into an axiomatic form. A class of partial orders satisfying a property Baumgartner called Axiom A [15] has turned out to be a very useful source of examples. As it turns out though, the iteration strategy devised by Laver was soon to be eclipsed by the far more general strategies for iterating proper partial orders, a class of partial orders including those satisfying Axiom A.

However, Laver reals continued to play a significant role in studying the continuum beyond the Borel Conjecture result. It was shown already in Laver's original article that $\mathfrak{b} = \aleph_2$ in the iterated Laver model. Haim Judah and Shelah showed that $\mathbf{non}(\mathcal{N}) = \aleph_1$ in this model as well, thus establishing [87] even more than is needed for (B) of Arrows 9 and 11. The crucial contribution of this work was to the theory of iteration of proper partial orders, since knowing that adding a single Laver real preserves the outer measure of the ground model reals does not, by itself, yield the desired result.

Woodin showed [88] that adding any number of random reals to Laver's model preserves the Borel conjecture. The significance here is that countable support iterations can only be used for iterations of length ω_2 without collapsing cardinals. This is the the key source of difficulties in trying to establish the consistency of $\mathfrak{p} < \mathfrak{t}$. Recall from §5 that Rothberger has shown that if $\mathfrak{p} = \aleph_1$ then, in fact, $\mathfrak{p} = \mathfrak{t}$. Hence a model where $\mathfrak{p} < \mathfrak{t}$ would require that $\aleph_3 \leq \mathfrak{t}$, putting this outside the realm of models constructed with countable support. Woodin's result provides at least some alternate method of creating models with $2^{\aleph_0} > \aleph_2$ in situations where countable support iterations seem to be essential, as is the case with the Borel Conjecture.

A variant of Laver forcing, due to Miller [124], has played a central role in furthering our understanding of forcing and the continuum. The partial order introduced by Miller is *"intermediate between Sacks perfect set forcing and Laver forcing"*. While defined originally as the partial order of perfect subsets of the real line in which the rationals are dense, it is equivalent to forcing with infinitely branching trees. Unlike Laver's partial order though, the conditions in Miller's partial order do not have to branch at every node, this is only required to happen cofinally often. In the paper introducing this notion, Miller shows that the generic real added is not dominated by the ground model and is of minimal degree. Moreover, any new real in the extension is either dominated by a ground model real or is itself generic with respect to Miller's partial order. A key property is shared with Sacks forcing: Every subset of $\mathbb{N}$ in the extension either contains or is disjoint from some infinite subset of $\mathbb{N}$ in the ground model. This last property implies that ultrafilters are generated by ground model sets and so, it allows much more control of these objects in forcing extensions.

This sort of property is extremely useful when dealing with problems such as destroying P–points. What was missing from Miller's construction though, was an iteration lemma, an iteration lemma which would soon be supplied by Shelah in [29]. In that paper Blass and Shelah introduce a proper forcing notion which has the property that P–points in the ground model generate P–points in the generic extension. The key point though is the iteration lemma asserting that in any countable support proper forcing iteration, if a ground model ultrafilter generates a P–point at every intermediate step of the iteration then it continues to do so at the limit. This then yields that iterating the partial order constructed by Blass and Shelah with countable support provides a model without P–points. However, not long after this construction was discovered, it was realized that Miller's rational perfect set forcing would also fit into the iteration scheme, considerably simplifying the argument.

This incident points to the critical role of iteration lemmas in sophisticated forcing arguments starting from the mid-1980's. A good illustration of this is provided in [117] by the forcing now known as the Mathias partial order. It is designed to add a generic ultrafilter and then diagonalize it — or at least, this is modern perspective — and it was introduced by Adrian Mathias to deal with questions on Ramsey theory with out the Axiom of Choice. Among the things he proves are that by collapsing a Mahlo cardinal one obtains a model where Dependent Choice holds yet there are no maximal almost disjoint families. (These are discussed in more detail in §9.) The questions Mathias considered did not require an elaborate iteration theory since he was able to extract a great deal of information by insightful arguments applied to a single stage forcing. However, Mathias forcing has become one of the standard partial orders in the theory of countable support iterations and the collection of cardinal invariant values in the iterated Mathias model are important for delineating the boundaries of the possible..

For example, recall from §5 that Rothberger had shown that $\mathfrak{t} \leq \mathbf{add}\,\mathcal{M}$ and that the inequality $\mathfrak{t} \leq \mathfrak{h}$ is discussed in §9. This raises the obvious question of whether these results can be improved to show that $\mathfrak{h} \leq \mathbf{add}\,\mathcal{M}$, but this fails in the Mathias model. In particular, $\mathfrak{t} = \mathbf{add}\,\mathcal{M} = \aleph_1$ in that model [50], yet $\mathfrak{h} = \aleph_2$.

The Mathias forcing is also significant in that it is a prototype for the single stage forcing in the 1982 construction by Shelah [156], of a model for $\mathfrak{s} > \mathfrak{b}$. This is the first instance of norms on possibilities and creature forcing which has had a profound influence on the theory of forcing. The memoir [140] describes the further developments along this line of research, while an excellent example of the strength of these methods can be found in the article [142] in which it is shown to be consistent that every real-valued function agrees with a continuous function on a non-measurable set. However, it is not the purpose of this section to provide a full account of the developments in forcing techniques since their introduction by Cohen in the early sixties. However, the far more modest goal of justifying the arrows in Cichoń's diagram has almost been achieved. All that remains to be shown is that condition (B) for Arrow 12 holds. This is the topic of §8 in which this and far more will be discussed.

8. The Baire Category Theorem and Martin's Axiom

Perhaps the longest and most remarkable chain of related ideas in the history of the continuum is that leading from Baire's work on continuous functions to Martin's Maximum. In his doctoral thesis [6] Baire provides a completely abstract definition of the notion of function, one based simply on existence rather than rules for calculating values. Moreover, he explains that he is interested in examining which properties of these abstract functions imply others. An example he provides foreshadows in a surprising way the uses to which the methods he developed in his thesis would be put. He reminds his readers of the existence of continuous functions without derivatives. In other words, continuous functions with derivatives are an exceptional subset of the family of all continuous functions and the one property does not imply the other.[32] Baire's work on classifying pointwise limits of continuous functions falls squarely into this quite modern, non-constructive view of functions.

His thesis contains what is now known as the Baire Category Theorem, that no interval in the real line is the union of countably many closed nowhere dense sets. While Baire makes very effective use

[32]Par example, on a reconnu, contrairement à ce qui avait été longtemps admis, qu'il existe des fonctions continues n'admettant pas de dérivée. Ce résultat doit être entenda de la manière suivante: le fait d'imposer à une fonction la continuité n'entraîne pas comme conséquence l'existence d'une dérivée; il en résulte que les fonctions continue qui admettent une dérivée ne forment qu'une classe particulière dans l'ensemble des fonctions continue; autrement dit, *c'est par exception qu'une fonction continue admet une dérivé.* (Page 2 of [6])

of this result and was clearly aware of its significance, it would take some thirty years before Banach would distill the constructive essence of Baire's Theorem into a method for proving the existence of mathematical objects. In 1931 Banach [10] and Stefan Mazurkiewicz [118] independently employed the Baire Category Theorem to show how to prove the existence of a continuous but nowhere differentiable function from the reals to the reals. The key idea is that Baire's Theorem is true in a much broader context than just the real numbers since his proof can easily be modified to apply to any separable, complete metric space. Banach then exploits the fact that the continuous functions on the unit interval form a separable, complete metric space with respect to the uniform norm.

Banach's idea has since been used countless times to construct examples of what are now known as generic objects. The term "generic" refers to the fact that the Baire Category Theorem provides more information than just existence, it actually shows that most objects have the desired property. For example, not only is there a continuous nowhere differentiable function, but most continuous functions are nowhere differentiable in the sense that all but a meagre subset of the space of continuous functions with respect to the uniform norm will be nowhere differentiable. This method of proof is often described as non-constructive, but it should be realized that any such proof can be transformed into a constructive argument by suitably enumerating the dense open sets used in the argument and tracing the steps to create an inductive construction. The usefulness of Banach's approach is that it allows one to ignore these bookkeeping details in favour of the conceptual. However, some forty years later Banach's ideas began to be used in the context of Martin's Axiom in a way that is essentially non-constructive.

Before examining these ideas though, it is necessary to take what may seem to be a digression and return to the year 1920 in which the first issue of *Fundamenta Mathematicae* appeared, the one in which Suslin asked Question 2.1. The motivation for the question is clear. Suslin was aware that the real line could be characterized as the unique separable, complete, linear order with neither end points nor isolated points. So Suslin is asking whether the separability condition can be replaced by the seemingly weaker one that all disjoint families of open sets are countable. The question of why Suslin was interested in this particular weakening can only be guessed at, but the future development of the subject was to reveal that it is a central concept and so it may well be correct to say that it was a natural question for him to ask.

However the methods available to Suslin and his contemporaries were far from adequate for making much progress on the problem. The one bit of early progress worth noting is a reformulation of the problem due to Kurepa. He realized [101] that by considering the set of intervals of a putative counterexample to Suslin's question one can create a tree structure capturing the relevant combinatorics. To be precise, Kurepa showed that the following are equivalent:

- The real line is not the only complete, linear order without endpoints or isolated points in which all disjoint families of intervals are countable.
- There is a tree of height ω_1 in which all nodes split, yet all incomparable sets of nodes are countable.

Trees satisfying the second alternative are called *Suslin trees*.

It is interesting to note that there is a tree structure similar to a Suslin tree that was studied for different reasons. It is easy to check that a Souslin tree can have no branches of length ω_1. However, trees satisfying just this weaker property are known as *Aronszajn trees* and the earliest recorded construction appears in [102]. The point to remark on is that this construction has been shown by Todorčević to be equivalent to Hausdorff's construction of an (ω_1, ω_1^*)–gap. This construction itself is similar to Lusin's construction of an almost disjoint family no two uncountable subsets of which can be separated as described in §9. So, even though Hausdorff had described his construction in 1908 it seems that 30 years later the subject had matured to the point that similar constructions were now being discovered by various researchers.

The status of Suslin trees, however, would not be resolved until forcing techniques became available. In 1967 Tomáš Jech proved [84] the consistency of the existence of a Suslin tree using an argument which could be rephrased as forcing with a countably closed partial order. The reviewer of the article for the AMS, Michael Morley makes the following comment in his review of this article:

> *The author notes that he has heard that S. Tennenbaum had obtained similar results in 1964. The facts known to the reviewer are as follows: (1) Tennenbaum announced the main result of this paper in a lecture at Harvard in January,1964; (2) his proof has been presented at numerous seminars; (3) it has never been published even as an abstract (excluding mimeographed notes); and (4) the author's proof is significantly different from Tennenbaum's.*

Indeed, Tennenbaum did subsequently publish [182] another proof of the consistency of the existence of a Suslin tree and, as Morley suggests, his proof is quite different from Jech's, relying on finite conditions rather than Jech's countable ones. About the same time, Ronald Jensen showed that $V = L$ also implies that there is a Suslin tree.

The key result in the opposite direction was Solovay and Tennenbaum's proof that it is consistent that there are no Souslin trees, thus showing that Suslin's question was not answerable by the methods available at the time he asked it. It is here that this seeming digression on Suslin's problem becomes relevant to the development stemming from the Baire Category Theorem. The important point to emphasize is that Solovay and Tennenbaum's proof relied on the new technique of iterated forcing. Tennenbaum had realized that thinking of the Suslin tree itself as a partial order allowed one to force with it. The result of this forcing is an ω_1 length path through the tree and it has already been noted that no Suslin tree can have such a path. In other words, forcing with a Suslin tree destroys that tree.

The critical property of a Suslin tree that allows it to be used as a forcing partial order without collapsing cardinals is its defining feature: There are no uncountable pairwise incompatible sets of elements of the tree considered as a partial order. This is the same condition Cohen had used to show that cardinals are not collapsed when increasing the size of the continuum. Solovay and Tennenbaum then developed the strategy of iteratively destroying each tree in turn. The new concept introduced here was that of creating a partial order which has the same effect as forcing to destroy one Souslin tree and then forcing over the resulting model to destroy the next Souslin tree and so on.

The next step was for Martin and Solovay to realize that the new iteration strategy allowed any partial order with the countable chain condition to be forced with in an iteration scheme. In its ultimate form, this meant that one could line up all partial orders with the countable chain condition and force with each in turn. Martin and Solovay explain in [85]:

> *Martin observed that the construction of $\mathcal{N}$ depended only on very general properties of the Cohen extensions $\mathcal{M} \to \mathcal{M}^T$. He and, independently Rowbottom, suggested an "axiom" which asserts that all Cohen extensions having these very general properties can be carried out inside the universe of sets: that the universe of sets is — so to speak —* closed *under a large class of Cohen extensions.*

Here $\mathcal{N}$ denotes the model constructed by Martin and Solovay.

The result is a strengthening of the Baire Category Theorem that applies to uncountably many dense sets and is valid in topological spaces satisfying the countable chain condition. Different variants of the countable chain condition define different classes of topological spaces to each of which correspond different versions of Martin's Axiom. The realization of the significance of the axiom to topological questions crystallized in the work of early researchers such as Franklin D. Tall [180] and István Juhász [89]. Of course, the work of Jensen on Souslin's Hypothesis [48] can also be considered to be topological in nature.

An important step in unravelling the connections between various versions of the Baire Category Theorem was obtained by Murray Bell, who showed [18] that the intersection of less that $\mathfrak{c}$ dense open sets in σ-centred spaces being non-empty is equivalent to $\mathfrak{p}$ being equal to the continuum. Once again the motivation for Bell was topological since the main application of his result is the existence of first countable Dowker spaces assuming that $\mathfrak{p} = \mathfrak{c}$. However, the combinatorial equivalent of his theorem has proved to be very useful: Martin's Axiom restricted to σ-centred partial orders holds if and only if $\mathfrak{p} = \mathfrak{c}$. The usefulness of this equivalence stems from the the fact that there is a broad range of σ-centred partial orders and these can be designed according to the application in which one is interested. Bell's Theorem allows the general techniques of Martin's Axiom to be applied in the situation where one knows only that $\mathfrak{p} = \mathfrak{c}$.

It was the set theoretic topologists who most vigorously applied Martin's Axiom in the early years after its appearance. Indeed, it can be said that, like the problems of analysis that drove set theoretic developments in the early years of the century, in the 1970's it was topological problems that provided the impetus for many new developments.

Central among these were the questions concerned with normality and metrizability. Recall from §3 that in 1937 Jones [86] had used the hypothesis that $2^{\aleph_0} < 2^{\aleph_1}$ to show that every separable, normal Moore space is metrizable. In 1951 Bing [24] had shown that a certain topological space constructed from an uncountable set of reals is a separable, normal, non-metrizable Moore space provided that the uncountable set of reals used is a Q–set. Recall also that Rothberger had shown that if there are no ω_1–limits, or equivalently, that $\mathfrak{p} > \aleph_1$ then there is a Q–set. It would then seem a natural progression to conclude that Martin's Axiom implies there is a separable, normal, non-metrizable Moore space. Indeed, this was announced by Jack Silver [180] and the proof was based on Bing's result, but with no reference to Rothberger's work. It was only later pointed out by David Booth [180] that Silver's result follows immediately from Rothberger's and Bing's work. However, it is clear that the set theoretic researchers of Europe and the general topologists of Texas had been unaware of each others work throughout the middle decades of the twentieth century. (A curious counterexample is provided by Émile Borel's address at the inauguration of Rice University in Houston, Texas in 1912 in which he developed some of the ideas leading to the Borel Conjecture.) The renewed interest in these question resulting from the availability of the new forcing techniques provided by Cohen and Solovay as well as Tall's thesis [180] on the normal Moore space problem finally rectified this situation and resulted in a lengthy period of intense activity.

The study of convergence properties as well as of $\beta\mathbb{N} \setminus \mathbb{N}$ proved to be particularly amenable to the newly available methods. Perhaps the most influential and far reaching development though, was the solution [186] of the S–space problem by Todorčević using innovative proper forcing techniques. The problem had been the object of intense study by various authors and several partial results had been available earlier. For example, Kunen showed in [97] that Martin's Axiom and the failure of the Continuum Hypothesis implied that there are no regular spaces all of whose finite products are S–spaces. However, the hypothesis used by Kunen is especially amenable to Martin's Axiom since it lends itself to countable chain condition partial orders. The case of an arbitrary S–space, however, required the use of Todorčević's techniques and this focused attention on the Proper Forcing Axiom described by Baumgartner in [16].

Cohen, in his concluding remarks in [44] says of the problem of determining the value of the continuum,

> *It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach $\mathfrak{c}$. Thus $\mathfrak{c}$ is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_\alpha$ where $\alpha = \aleph_\omega$ etc. This point of view regards $\mathfrak{c}$ as an incredibly rich set given to us by one bold new axiom, which can never be approximated by any piecemeal process of construction'.*

As it turns out, there are versions of Martin's Axiom sufficiently strong to establish the value of $\aleph_2$ for the continuum. However, as will be seen, while this does contradict Cohen's view that the continuum must be very large, it does not contradict his view that the continuum "can never be approximated by any piecemeal process of construction."

While many of the questions concerning cardinal invariants are settled by Martin's Axiom, indeed they are all equal to $2^{\aleph_0}$, other questions are not. For example, Martin's Axiom does not settle the question of whether there are $(\mathfrak{c}, \mathfrak{c}^*)$–gaps as was shown [96] by Kunen. Even less is determined by assuming the following consequence of Martin's Axiom and $2^{\aleph_0} > \aleph_1$ which is often referred to as $\mathrm{MA}_{\aleph_1}$: Given $\aleph_1$ dense open sets in a countable chain condition partial order, there is filter meeting each of them. In [99] Kunen and Tall consider variations on Martin's Axiom and remark on two different types of consequences of Martin's Axiom. They distinguish between what they call "combinatorial" consequences and "Souslin type" consequences. They point out that the combinatorial consequences "readily imply that $2^{\aleph_0} > \aleph_1$, while the Souslin type consequences do not." However, even the full $\mathrm{MA}_{\aleph_1}$ does not determine the value of the continuum as is immediate from the construction [172] of Solovay and Tennenbaum.

In [189] Todorčević explains the significance of this question.

> *The increasing sophistication in the area of iterated forcing resulted in a profusion of models of Set Theory built for the purpose of proving the independence of statements like Souslin's Hypothesis, Borel's Conjecture, and so on. It turns out that many such proofs, especially the deeper ones involving other than the finite support iterations, could not produce models where the continuum would have value different from $\aleph_2$ ... It is therefore natural to ask whether the statements in question do pose some restriction on the continuum such as $\mathfrak{c} = \aleph_2$. This kind of question is especially interesting when the considered statements are some kind of maximality principles, such as, for example, various forcing axioms asserting that we can have a sufficiently generic filter for every member of a certain class of posets.*

Therefore it was very surprising when Todorčević was able to show that the strengthening of $\mathrm{MA}_{\aleph_1}$ known as the Proper Forcing Axiom actually implies that $2^{\aleph_0} = \aleph_2$. It will be seen that the argument here does indeed have two distinct parts that might be seen as corresponding to Kunen and Tall's division of Martin's Axiom into "combinatorial" and "Souslin type" consequences. However, the story is not quite so simple and the two parts of Todorčević's argument involve several new ingredients that were not yet available at the writing of [99].

An early step in this direction was taken in [187]. Rado's Conjecture is a reflection type statement about linear orders: A family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies if and only if every subfamily of size $\aleph_1$ has this property. In [187] Todorčević studies this conjecture and shows that it implies that the size of continuum is at most $\aleph_2$.

However, further progress in this direction hinges on an axiom that might be viewed as being at the core of the "Souslin type" consequences mentioned by Kunen and Tall in [99], an axiom now commonly known as OCA. However, there is room for considerable confusion here since a related family of axioms was studied earlier by Uri Abraham, Matatyahu Rubin and Shelah in [1].

The semi-open colouring axiom (SOCA) states that for any second countable space X with $|X| = \aleph_1$ and any open set $U \subseteq [X]^2$ — this can be identified with a symmetric open subset of $X \times X$ — there is $A \subseteq X$ with $|A| = \aleph_1$ such that either $[A]^2 \subseteq U$ or $[A]^2 \cap U = \emptyset$. However, another axiom Abraham, Rubin and Shelah call OCA is also defined. An open colouring of a second countable Hausdorff space X with $|X| = \aleph_1$ is a finite cover U of $[X]^2$. A set $A \subseteq X$ is said to be U-homogeneous if $[A]^2$ is contained in one member of U. The open colouring axiom (OCA) states that for every such X and U, X can be partitioned into countably many U-homogeneous sets.

The axiom currently known as OCA was introduced by Todorčević in [186] and is actually a strengthening of SOCA: If X is a separable metric space and $G \subseteq [X]^2$ is open then either there is a decomposition of X into countably many pieces $\{X_i\}_{i\in\omega}$ such that $[X_i]^2 \cap G$ is empty for each i or there is an uncountable $H \subseteq X$ such that $[H]^2 \subseteq G$. Observe that the two alternatives are not symmetric as they are in SOCA.

The first component of Todorčević's argument was to show that OCA, as formulated by him, implies that $\mathfrak{b} = \aleph_2$. A crucial step in the argument in [186] is to show that every gap in $(\mathbb{N}^{\mathbb{N}}, <^*)$ is either of type (ω_1, ω_1^*) or of type (ω, ω_1^*) or of type (ω_1, ω^*). (As discussed in §5 the first type of gap was first constructed by Hausdorff, while the other two were analyzed by Rothberger.) On the other hand, in [185] it is shown that if $\mathfrak{b} > \aleph_2$ then there is a gap in $(\mathbb{N}^{\mathbb{N}}, <^*)$ which is neither of the Hausdorff nor Rothberger type. It is interesting to note that this argument depends on choosing a maximal linear order in $(\mathbb{N}^{\mathbb{N}}, <^*)$, in other words the pantachies as defined and studied by Hausdorff had not yet outlived their usefulness almost a century later.

The other essential step in the argument is to use the value of $\mathfrak{b}$ to code reals and put an upper bound on $\mathfrak{c}$. In this coding a well-ordered, unbounded subset of $\mathbb{N}^{\mathbb{N}}$ plays a central role and, of course, this set must have cardinality $\mathfrak{b}$. A scheme for having certain subsets of this well-ordered set code reals is described in [189] and this coding has the property that sets coding distinct reals have distinct suprema. Hence showing that each real is coded will establish that $\mathfrak{b}$ is a bound on the number of reals. Showing that each real actually is coded requires a relatively weak extension of Martin's Axiom, namely Martin's Axiom for partial orders of the form $P * Q$ where P is σ-closed and Q has the countable chain condition.

Before leaving the topic, it is interesting to note that Justin Moore [128] has shown that the conjunction of two versions of OCA — the original OCA as defined by Abraham, Rubin and Shelah and the later version of OCA defined by Todorčević and strengthening SOCA — also implies that $2^{\aleph_0} = \aleph_2$. It must be remarked though, that finding bounds on the continuum using variations of Martin's Axiom are not an isolated phenomenon, but should really be placed within the context of discoveries, many of them quite unexpected, about combinatorial implications of variations of Martin's Axiom whose consistency is obtained by assuming some large cardinal hypothesis. As examples, one can mention Todorčević's proof [183] that the Proper Forcing Axiom implies the failure of $\square_\kappa$ for all uncountable κ and the applications of Martin's Maximum in [58, 59, 60]. A very different approach to these axioms is provided [199] by Woodin's $P_{\max}$ which is a canonical model for the failure of the Continuum Hypothesis. However, providing more detail here would require too long a diversion for the scope of this article.

9. Cardinal Invariants of the Continuum Associated with $\beta\mathbb{N} \setminus \mathbb{N}$

The discussion of the structure of $\mathbb{N}^{\mathbb{N}}$, the invariants associated to measure and category and the development of Baire Category like theorems have brought us to consider $\mathfrak{b}$, $\mathfrak{d}$, the invariants of the ideals $\mathcal{N}$ and $\mathcal{M}$ as well as $\mathfrak{p}$ and $\mathfrak{t}$. Little mention so far has been made of the combinatorial invariants such as $\mathfrak{a}$, $\mathfrak{i}$, $\mathfrak{s}$ and $\mathfrak{r}$. Indeed, all of these seem to have escaped the attention of set theorists and topologists for considerably longer than the invariants discussed in §5. Certainly part of the explanation for this must be found in the fact that these later invariants are much less closely connected to the notion of rate of convergence which were of interest to du Bois-Reymond and Hausdorff after him. The interest in the invariants studied in §6 is motivated by very different considerations of course.

The cardinal $\mathfrak{a}$ is defined to be the least cardinality of a maximal, infinite family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ such that any two elements of $\mathcal{A}$ are almost disjoint, in the sense that their intersection is finite. The notion of an almost disjoint family was considered already in 1928 by both Sierpiński [162] and Tarski [181] in considerable generality and Sierpiński established the result — a result he viewed as paradoxical — that in any infinite set of cardinality κ it is possible to find more than κ pairwise almost disjoint sets. In the case that $\kappa = \aleph_0$ there are, in fact, $\mathfrak{c}$ almost disjoint sets. As a result, until the era of forcing and the ability to construct models where $\mathfrak{c}$ is large it would not have made much sense to consider the question of the cardinality of *maximal* almost disjoint families of subsets of the integers. It was only in 1972, just shortly after the Martin–Solovay paper introducing Martin's Axiom, that articles of Stephen Hechler [79] and Solomon [171] appeared considering this question.

It is worth noting that Hechler's paper appeared in a topology journal and that the cardinal $\mathfrak{a}$ was looked at as the least cardinal of a maximal, infinite family of disjoint clopen sets in $\beta\mathbb{N} \setminus \mathbb{N}$. Moreover, in Eric van Douwen 's very influential article [195] the discussion of $\mathfrak{a}$ highlights the Ψ space constructed from an almost disjoint family. Given an almost disjoint family $\mathcal{A}$, the Ψ–space has as points $\mathbb{N} \cup \mathcal{A}$ and the neighbourhoods of $A \in \mathcal{A}$ are of the form $A \setminus k$ while each $n \in \mathbb{N}$ is isolated. The study of the topological properties of this space has motivated much of the developments in studying $\mathfrak{a}$ and almost disjoint families in general.

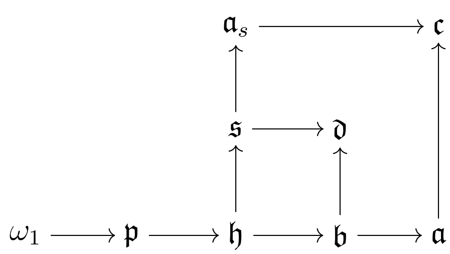
Hechler noted that Martin's Axiom implies that $\mathfrak{a} = \mathfrak{c}$ but, of course, the definition would not have been of much interest had he not been able to show that it is consistent that $\mathfrak{a} \neq \mathfrak{c}$. And, indeed, he was able to show that it is consistent that $\mathfrak{c} > \aleph_1$ and there are maximal almost disjoint families of each uncountable cardinality not greater than the continuum. By keeping the continuum smaller than the first singular, Hechler was able to avoid the question of singular cardinality for $\mathfrak{a}$. Much later, Jörg Brendle was able [34] to use a modification of a forcing technique due to Shelah to obtain the consistency of $\mathfrak{a} = \aleph_\omega$. The argument of Shelah which Brendle modified was originally used in 1999 to show that it is consistent that $\mathfrak{a} > \mathfrak{d}$, but the paper [159] appeared only five years later. The fact that $\mathfrak{a} \geq \mathfrak{b}$ was noted by Solomon in [171] and a model where $\mathfrak{a} \neq \mathfrak{b}$ was obtained by Shelah in 1982, along with several other inequalities, in [156], a paper whose influence on forcing constructions was discussed at the end of §7. However, obtaining a model where $\mathfrak{a} \neq \mathfrak{d}$ seemed to be beyond the capabilities of the techniques whose development was initiated in [156]. And, in fact, the methods used to finally obtain such a model where entirely novel, but their influence on future developments is yet to be determined. It is worth noting though that the following question [126] of Judith Roitman is still open: If there is a dominating family in $\mathbb{N}^\mathbb{N}$ of size $\aleph_1$ does it follow that $\mathfrak{a} = \aleph_1$? The appearance of $\aleph_1$ here is not as a place holder for $\mathfrak{c}$ as it might have been in the questions posed in the early issues of *Fundamenta Mathematicae*, but because of a fundamental problem in modifying Shelah's argument to handle this cardinal.

In discussing the cardinal $\mathfrak{a}$ it is worth remarking that in the same year that Hausdorff published his second paper on constructing Hausdorff gaps — the paper which finally got set theorists to take notice of a result Hausdorff had obtained close to thirty years earlier — Lusin produced a very similar argument [114] constructing an unusual almost disjoint family. What he did was to construct an almost disjoint family $\mathcal{A}$ such that no two disjoint, uncountable subfamilies can be separated: In other words, if $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{B} \cap \mathcal{A} = \emptyset$ then there is no $X \subseteq \mathbb{N}$ such that $B \subseteq^* X$ for all $B \in \mathcal{B}$ and $C \cap X$ is finite for all $C \in \mathcal{C}$. While it seems that Lusin was unaware of Hausdorff's work at the time — at least van Douwen expresses this opinion in [195] — it is at least clear that the work of du Bois-Reymond had motivated Lusin's interest in these questions, as it had Hausdorff's much earlier construction. Indeed, Lusin attributes the non-existence of non-separable countable families to du Bois-Reymond. In one sense, this is a stronger result than Hausdorff's gap construction because it yields $2^{\aleph_1}$ pairs that can not be separated. On the other hand, the pairs consist of almost disjoint sets rather than $\subseteq^*$-increasing towers as in Hausdorff's gaps. And, of course, the connection to the invariant $\mathfrak{a}$ is quite weak because there is no claim that Lusin's almost disjoint family is maximal. Moreover, using the Axiom of Choice to extend it to a maximal almost disjoint family is likely to destroy the key property of the Lusin family.

This very brittle behaviour of almost disjoint families is at the heart of some key questions still open in set theory. For example, in [78] Hechler raises the question of whether there is a *completely separable* maximal almost disjoint family. An almost disjoint family $\mathcal{A}$ is said to be completely separable if for every $X \subset \mathbb{N}$ either X is almost contained in the union of finitely many members of $\mathcal{A}$ or there is $A \in \mathcal{A}$ such that $X \supseteq A$. While this is still an open question, if the requirement of maximality is removed a solution was obtained by Petr Simon [9]. The argument used by Simon has its roots in methods developed by Bohuslav Balcar, Jan Pelant and Simon in [8] in which a cardinal invariant quite different in nature to those studied by Rothberger or Sierpiński is examined. The notation for the cardinal now known as $\mathfrak{h}$ was introduced in [9], but Balcar, Pelant and Simon designated it as κ — or, to be precise, $\kappa(\mathbb{N}^*)$ — in the earlier paper. It is defined to be the least cardinal of a *shattering matrix*, which is defined in [8] to be a collection of almost disjoint families $\{\mathcal{A}_\xi\}_{\xi\in\mathfrak{h}}$ such that for each infinite $X \subseteq \mathbb{N}$ there is some $\xi \in \mathfrak{h}$ such that there are at least two $A \in \mathcal{A}_\xi$ intersecting X on an infinite set.

The motivation behind the definition was firmly rooted in the study of $\beta\mathbb{N} \setminus \mathbb{N}$, the study of the number of nowhere dense sets required to cover $\beta\mathbb{N} \setminus \mathbb{N}$ to be precise. It did not take long, though, to realize that there is a close connection to distributivity properties of Boolean algebras. In the course of their analysis Balcar, Pelant and Simon show that $\mathfrak{h}$ is also the same as the minimal height of a tree π-base for $\beta\mathbb{N} \setminus \mathbb{N}$.

By the time of the survey article which revisited some of the results from [8] the article of van Douwen [195] had already made its mark on the community of set theoretic topologists and the position of $\mathfrak{h}$ among the cardinals considered in §5 seemed a natural question to answer. The following diagram describes the answer:



Here, as usual, an arrow signifies that the cardinal invariant at the tail of the arrow is less than or equal to the invariant at its tip and that it is consistent that inequality holds. The invariants $\mathfrak{b}$, $\mathfrak{d}$, $\mathfrak{h}$ and $\mathfrak{p}$ are treated in more detail in §5 but the invariants $\mathfrak{s}$ and $\mathfrak{a}_s$ have not yet been mentioned.

The invariant $\mathfrak{a}$ has a great many variants, some of which are known to be different from $\mathfrak{a}$ itself and others are not. The invariant $\mathfrak{a}_s$ is one of these and is defined to be the least cardinal of an almost disjoint family consisting of partial functions from $\mathbb{N}$ to $\mathbb{N}$. Therefore the inequality $\mathfrak{a} \leq \mathfrak{a}_s$ is immediate even though it is not indicated in the diagram. The cardinal $\mathfrak{s}$ is implicitly due to Booth in [31] where he studies the sequential compactness of products of the two point discrete space. He shows that 2^λ is sequentially compact if and only if for every family $\mathcal{A}$ of subsets of $\mathbb{N}$ of size λ there is some infinite $X \subseteq \mathbb{N}$ such that for each $A \in \mathcal{A}$ either $X \subseteq^* A$ or $X \cap A$ is finite. The cardinal invariant is now usually defined as the least cardinal for which this combinatorial equivalence of the sequential compactness of 2^λ fails; namely $\mathfrak{s}$ is the least cardinal of a family $\mathcal{A}$ of subsets of $\mathbb{N}$ such that for every infinite $X \subseteq \mathbb{N}$ there is some $A \in \mathcal{A}$ such that $X \setminus A$ and $X \cap A$ are both infinite. Moreover, Booth also showed that $\mathbf{non}(\mathcal{N}) \geq \mathfrak{s}$ which can be compared to the inequality $\mathfrak{t} \leq \mathbf{add}(\mathcal{M})$ established much earlier by Rothberger and discussed in §5.

The inequalities $\omega_1 \leq \mathfrak{p} \leq \mathfrak{h}$ are immediate, but more interesting are $\mathfrak{h} \leq \mathfrak{b}$ and $\mathfrak{s} \leq \mathfrak{a}_s$. Since it involves $\mathfrak{b}$, it should be no surprise that the first of the last two inequalities is established by appealing to arguments used by Rothberger, but a structure similar to a shattering matrix also plays a role. The argument for the second inequality is quite subtle and can be found in [9].

The problem of showing that none of these inequalities is actually an equality is the work of several authors. However, it is worth noting the simple fact that the invariant $\mathfrak{t}$ studied by Rothberger is bounded by $\mathfrak{h}$. The fact that $\mathfrak{p} < \mathfrak{t}$ is proved in [50] so this yields $\mathfrak{p} < \mathfrak{h}$. The consistency of $\mathfrak{s} < \mathfrak{b}$ is obtained by adding $\aleph_1$ of Solovay's random reals to a model of Martin's Axiom and $\mathfrak{c} = \aleph_2$ while arguments in a similar spirit (see [9]) yield all the other arrows except for $\mathfrak{d} < \mathfrak{a}$ and $\mathfrak{a} < \mathfrak{s}$. However, the consistency of $\mathfrak{d} < \mathfrak{a}$ has already been discussed as a result of Shelah [159] using techniques developed expressly for the purpose of obtaining this strict inequality. It had taken some twenty years to achieve this since the time of his seminal paper [156] in which he showed the consistency of $\mathfrak{a} \neq \mathfrak{b}$. In fact, in the same paper the consistency of $\mathfrak{a} < \mathfrak{s}$ was also proved. Noting that $\mathfrak{s} < \mathfrak{a}_s$, this also establishes the consistency of $\mathfrak{a} < \mathfrak{a}_s$, a quite unexpected result.

While it would not be correct to attribute the notion of an ultrafilter to Marshall Stone — indeed, Stanislaw Ulam showed [192] how the Axiom of Choice can be used to construct these objects in 1932 and even before him, in 1909, Frederic Riesz [139] had introduced the notion to little acclaim— it is certainly true that his work in [178] was a critical point in bringing to the attention of topologists the importance of this notion. While Stone's work on what is now known as the Čech–Stone compactification was motivated by his earlier research on the spectral theory of linear operators, Eduard Čech's seminal paper [43] on the subject grew out of earlier work of Andrei Nikolaevich Tychonov in this direction, even to the extent of continuing to use Tychonov's β notation for the compactification. While Čech's construction, unlike Stone's, was not based on prime ideals in Boolean algebras, he was certainly aware of the connections to set theory as the following remark from [43] makes clear:

> *Let I denote an infinite countable isolated space (e.g. the space of all natural numbers). It is an important open problem to determine the cardinal $\mathfrak{m}$ of $\beta(I)$. All I know about it is that $2^{\aleph_0} \leq \mathfrak{m} \leq 2^{2^{\aleph_0}}$*

It would not be long before this question was answered though. Published in the same year, an article by Bedřich Pospíšil showed [135] that in fact $\mathfrak{m} = 2^{2^{\aleph_0}}$.

However, had he used Stone's construction, Pospíšil's answer would have been immediately clear to Čech if he had also been aware of the following result [57] of Gregory Fichtenholz and Leonid Kanterovich: There is an independent family of subsets of $\mathbb{N}$ of cardinality $\mathfrak{c}$. It follows that each function from $\mathfrak{c}$ to 2 corresponds to a distinct ultrafilter. These very early results on $\beta\mathbb{N}$ were a clear signpost for set theorists to follow in the era of independence results. It has already been mentioned that Hechler's results on $\mathfrak{a}$ as well as the later work of Balcar, Pelant and Simon were clearly focused on $\beta\mathbb{N} \setminus \mathbb{N}$. It was perfectly natural, therefore, to consider some of the earliest constructions through the lens of independence results.

For example, given that there is an independent family of cardinality $\mathfrak{c}$, in analogy with the definition of $\mathfrak{a}$ one can ask for the least cardinal of a maximal independent family. This appears for the first time in an appendix to the survey article [196] which provides a proof due to Shelah that $\mathfrak{d} \leq \mathfrak{i}$ and that it is consistent that $\mathfrak{d} \neq \mathfrak{i}$.

Another cardinal invariant found in [196] that has shown itself to be quite closely connected to various notions is $\mathfrak{u}$, the least cardinal of a base for a non-principle ultrafilter on $\mathbb{N}$. Of course, the nature of points in $\beta\mathbb{N} \setminus \mathbb{N}$ was one of the first and most obvious questions to be considered by topologists, but, one might argue that the structure of this space is really not that closely related to the continuum since even the cardinalities of the two objects are different. While this is true to some degree, the cardinal $\mathfrak{u}$ has played an important role in the development of our understanding of the cardinal invariants associated with the continuum. Already in 1939 Pospíšil had considered [136] the question of showing that there is an ultrafilter on $\mathbb{N}$ with no base of cardinality smaller than $\mathfrak{c}$. However, the results of [17] from 1979 already yielded that it is consistent that $\mathfrak{u}$ is less than the continuum. This is quite surprising since $\mathfrak{u}$ seems as if it should be quite a large cardinal. Indeed there is no upper bound known for it other than $\mathfrak{c}$.

While a great deal of the interest in ultrafilters comes from topologists studying the Čech–Stone compactification there are other fields that have influenced development in this area as well. The study of abelian groups is one of them. Motivated by very different questions than du Bois-Reymond, Ernst Specker defined [176] growth types as ideals of non-decreasing functions from $\mathbb{N}$ to $\mathbb{N}$. By an ideal he meant a family of non-decreasing functions closed under pointwise sums and dominated functions. Rüdiger Göbel and Burkhard Wald defined [64, 65] an ordering on equivalence classes of these growth types and raised the question of their number. In providing answers to some of the questions raised by Göbel and Wald, Andreas Blass and Claude Laflamme [28] found it useful to introduce a new cardinal invariant known as $\mathfrak{g}$.

A family $\mathcal{S}$ of infinite subsets of $\mathbb{N}$ which is downwards closed under the $\subseteq^*$ relation is known as *groupwise dense* if for every infinite family of pairwise disjoint finite subsets of $\mathbb{N}$ there is an infinite subfamily whose union belongs to $\mathcal{S}$. The cardinal $\mathfrak{g}$ is defined to be the least cardinal of a family of groupwise dense families whose intersection is empty. Comparing this definition to the definition of $\mathfrak{h}$ using shattering matrices will reveal several similarities. Indeed, Blass and Laflamme remark in

[28] that the inequality $\mathfrak{h} \leq \mathfrak{g}$ is immediate. Not too much more difficult is the fact that $\mathfrak{g} \leq \mathfrak{d}$ which establishes a connection to growth types as considered more than a century earlier by du Bois-Reymond. In order to obtain the consistency of there being only four growth types, Blass and Laflamme employ the following hypothesis: $\mathfrak{u} \leq \mathfrak{g}$. Furthermore, they actually show that their hypothesis implies that for any two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ there is a finite-to-one function f such that $f(\mathcal{U}) = f(\mathcal{V})$, a hypothesis known as NCF, the Near Coherence of Filters.

NCF was introduced in [26] which dealt with models of arithmetic. However, it soon turned out to be applicable to a wide variety of problems. For example it is shown under this assumption that the ideal of compact operators on the separable Hilbert space is the sum of two smaller ideals. In a similar fashion, the Čech–Stone compactification of the half line is shown to have only one composant and applications to the same sort of questions about abelian groups — dealt with later [28] together with Laflamme — are addressed here as well.

The consistency of NCF, however, was only established in [29] by Blass and Shelah, although a simpler proof was provided in [30]. It is interesting to note that a key step on the road to this result was Shelah's construction [154] of a model where there are no P–points. This is yet another example of a forcing model devised to answer a topological question, but the implications of whose techniques have had far reaching consequences on the study of independence results about the continuum.

In [176] Specker studied the group $\mathbb{Z}^{\mathbb{N}}$, the direct product of countably many copies of the integers. Letting $\{e_n\}_{n=0}^{\infty}$ be the natural generators for this group, Specker showed that for any homomorphism $H : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ the value of $H(e_n)$ is 0 for all but finitely many of the generators e_n. Moreover, he also showed that this holds for many subgroups of $\mathbb{Z}^{\mathbb{N}}$, all of which have cardinality $\mathfrak{c}$. The question then arose of whether the cardinality of any such group must be maximal. However, Katsuya Eda showed in [53] that this question cannot be decided in the absence of some extra set theoretic axioms since he could show that the minimal cardinality of a subgroup of $\mathbb{Z}^{\mathbb{N}}$ satisfying Specker's condition can be no less than $\mathfrak{p}$ and no greater than $\mathfrak{d}$. Later, Blass [27] was able to improve this to show that the lower bound $\mathfrak{p}$ could be replaced by $\mathbf{add}(\mathcal{N})$ and the upper bound could be replaced by $\mathfrak{b}$. In the course of studying the cardinal invariants associated with the Specker phenomenon Blass introduced the notion of evasion and prediction.

A *predictor* consists of an infinite $D \subseteq \mathbb{N}$ and functions $\pi_d : \mathbb{N}^d \to \mathbb{N}$ for $d \in D$. Such a predictor is said to predict a function $f : \mathbb{N} \to \mathbb{N}$ if and only if for all but finitely many $d \in D$ the predictor guesses $f(d)$ in the sense that $\pi_d(f \restriction d) = f(d)$. Otherwise f has evaded the predictor. For any function $g : \mathbb{N} \to \mathbb{N}$ — to avoid trivialities we require that $g(i) > 2$ for all i — Blass defines $\mathfrak{e}_g$ to be the least cardinality of a family $\mathcal{E} \subseteq \prod_{i=0}^{\infty} g(i)$ such that for any predictor there is a function in $\mathcal{E}$ evading it. This cardinal has inspired a great deal of subsequent work by Blass as well as others such as Laflamme [103], Brendle [33, 35] and Brendle and Shelah [39, 38].

An interesting aspect of these evasion cardinals is that they are parametrized by the function g, which can be thought of as bounding the growth rates of the functions being predicted. This raises the possibility of infinite, indeed uncountable, families of cardinal invariants. The fact that this many invariants can actually be realized is a result [68] due to Martin Goldstern and Shelah who considered invariants defined in a spirit similar to the evasion numbers of Blass. As with the evasion numbers, the invariants of Goldstern and Shelah are parametrized by growth rates and they show that it is consistent that uncountably many of these parametrized invariants are all different. Of course, this requires the continuum to be at least $\aleph_{\omega_1}$. Moreover, it is possible to exercise some control over the values of the invariants providing a vast landscape of models of set theory which differ for very concrete reasons, namely the value of certain cardinal invariants of the continuum.

10. Epilogue

Since this history began with a list of questions occupying set theoretic researchers just after the first world war, it is fitting to provide a glimpse into the questions considered important at the end of the

20^{th} century. We are fortunate to have a record of those questions considered important in the eyes of one of the foremost contributors to the study of the continuum in the last half of the last century, Saharon Shelah. In an issue of *Fundamenta Mathematicae* devoted to his work, Shelah provided an introductory article [158] listing questions he would have liked to have seen solved at that time. Some have been solved since then, but others persist.

Among those is the question of the consistency of $\mathfrak{p} < \mathfrak{t}$. It is interesting, and quite revealing, to see why Shelah would consider this seemingly technical question sufficiently important to include on his millennium list. Indeed, he provides a detailed discussion of this matter but, before seeing what he has to say on this it is worth recalling that Rothberger had shown that if $\mathfrak{p} = \aleph_1$ then $\mathfrak{t} = \aleph_1$. Shelah explains further:

> *. . . the advances in proper forcing make us "rich in forcing" for $2^{\aleph_1} = \aleph_2$, making the higher values more mysterious. . . . So, because we know much more how to force to get $2^{\aleph_0} = \aleph_2$, the independence results on the problems of the interrelation of cardinal invariants of the continuum have mostly dealt with relationships of two cardinals, as their values are $\in \{\aleph_1, 2^{\aleph_0}\}$. Thus, having only two possible values $\{\aleph_1, \aleph_2\}$, among any three two are equal; the Pigeonhole Principle acts against us. As we are rich in our knowledge to force $2^{\aleph_0} = \aleph_2$, naturally we are quite poor concerning ZFC results. . . . We are not poor concerning forcing for the Continuum Hypothesis (and are rich in ZFC). But for $2^{\aleph_1} \geq \aleph_3$ we are totally lost: very poor in both directions. We would like to have iteration theory for length $\geq \omega_3$. I tend to think good test problems will be important in developing such iterations.*

So, by Rothberger's Theorem, the consistency of $\mathfrak{p} < \mathfrak{t}$ is a good test problem for iteration theory. Rephrased to avoid mention of consistency, the question of whether $\mathfrak{p} \geq \mathfrak{t}$ would have seemed natural and, very likely, have been an attractive problem to the early contributors to *Fundamenta*. The motivation of course, even if comprehensible, would have seemed entirely foreign. The questions these early set theoretic researchers dealt with could mostly be motivated by their connection to problems in analysis. Eighty years later, though, motivation comes from a very different source.

Shelah mentions several other test problems. Among them is the very general question of finding relationships between triples of the well studied cardinal invariants. Certainly the iteration problem will need to be solved to have a general tool for dealing with independence results along these lines. However, Shelah believes that there are outright theorems waiting to be discovered, but that they "are camouflaged by the independent statements". Emphasizing once more the source of his motivation, he goes on to say that, "cardinal invariants from this perspective are excellent excuses to find iteration theorems".

However, it is perhaps true that the major set theoretic question waiting to be resolved at the end of the twentieth century is the same one as at the start, the value of $2^{\aleph_0}$. The fact that this value can be calculated to be $\aleph_2$ under the Proper Forcing Axiom has already been discussed and there is a suggestion of Hugh Woodin [200, 201, 202] that related types of arguments might be used to establish a true value for the continuum. A further discussion along these lines would be far from the topic of this history, but it may be worth reflecting on an earlier failed attempt in this direction by Gödel [67].

His idea, discussed in much greater detail in [37], was that the conjunction of four statements would imply that $2^{\aleph_0} = \aleph_2$. The first two of these axioms have to do with a generalization of the cardinal invariant $\mathfrak{d}$ to the context of the cardinals $\aleph_n$. In particular he asserted the existence of a family of functions from ω_n to ω_n of cardinality $\aleph_{n+1}$ which dominates all such functions with respect to the partial order of domination on a cofinal subset of ω_n, yet all of whose initial segments form a set of cardinality $\aleph_n$. While this structure is reminiscent of that studied by Hausdorff, Lusin, Sierpiński, Rothberger, Hechler and all those that followed this line of enquiry, its behaviour, in fact, is quite different. A discussion of this difference would lead too far away from the continuum, but the interested reader might consult [95] as a starting point for further investigation.

The last axioms Gödel invoked are familiar from the early work of Hausdorff on pantachies mentioned in §5. Gödel hypothesized a complete scale in $\mathbb{R}^\omega$ with no increasing or decreasing ω_2 sequences with what Gödel called the "Hausdorff Continuity Axiom". The exact meaning of this is not clear, but Brendle, Larson and Todorčević provide three axioms, too technical to be included here, and show that these follow from Gödel's hypotheses under the correct interpretation. Along with the complete scales mentioned, sets of strong measure zero play a central role. So does the assertion, in the spirit of Baire's Category Theorem, that any set of reals of cardinality $\aleph_3$ which is the intersection of $\aleph_1$ open sets contains a perfect set. Brendle, Larson and Todorčević then show that the three axioms — axioms which would have been recognizable to Hausdorff, Lusin and Sierpiński, but perhaps not to Baire — imply that $2^{\aleph_0} = \aleph_2$.

While Gödel's plan for proving that $2^{\aleph_0} = \aleph_2$ must be considered a failure, it may well be that more sophisticated arguments eventually succeed with the more ambitious goal of establishing that some form of the Proper Forcing Axiom is actually true. The effect of this on the study of cardinal invariants of the continuum would be profound since all these invariants would take on the value $2^{\aleph_0} = \aleph_2$. Even most questions regarding Hausdorff gaps, special sets of reals and related structures would be settled if this were to be the case. Of course, it could not be denied that the study of the cardinal structures associated with the continuum would have been crucial in coming to this new understanding. Nevertheless, it is possible that their study would likely become no more than a footnote in the history of set theory — and perhaps this article will serve as that footnote.

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