A SMALL DOWKER SPACE FROM A CLUB-GUESSING PRINCIPLE.

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Abstract. We present the construction of a new Dowker space from a special type of club-guessing ladder system. These types of guessing principle have previously been used to construct spaces consistent with MA+\neg CH. Thus, this construction may shed light on the open problem whether MA+\neg CH is consistent with the existence of a Dowker spaces of size \aleph_1.

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1. Introduction

Dowker proved that a product X \times [0, 1] is normal if and only if X is normal and countable paracompact [Dow51]. Subsequently, any normal space X that has non-normal product with the closed unit interval has come to be called a Dowker space. Whether ZFC implies there is a Dowker space of cardinality \omega_1 is a particular and important instance of the general “small Dowker space” question. Indeed, it is not known whether MA+\neg CH or PFA implies there are no Dowker spaces of size \aleph_1 (problem 10 of [Sze08]).

Although MA+\neg CH decides a great deal about structures of size \omega_1, there are a number of counterexamples to this general principle. E.g., the existence of first countable S-space ([AT84]), or perfectly normal not realcompact spaces of size \aleph_1 ([Her04], [HI04]) are both independent of MA+\neg CH. The latter result used a club-guessing ladder system to construct a perfectly normal not realcompact spaces followed by a consistency result to show that the club guessing principle was consistent with Martin’s Axiom. The main motivation of this paper was to try to use similar techniques to obtain a Dowker space of size

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\(\aleph_1\) in a model where MA+¬CH holds. While we have not been able to obtain such a space, we do present a construction of an example using a club-guessing principle not previously seen in the literature, and we conjecture that similar techniques should lead to an example consistent with MA+¬CH or even PFA.

It should be remarked that it would even be interesting to show that this example is consistent with some weak form of Martin’s Axiom. Yorioka has recently shown that generalizations of one of Rudin’s constructions cannot be Dowker assuming \(\mathcal{K}_2(\text{rec})\) [Yor08] and it would be interesting in general to understand what kind of small Dowker spaces there are in the model for Katetov’s problem (where \(\mathcal{K}_2(\text{rec})\) holds) [LT02].

In Section 2 we introduce the guessing principle, explain how to construct our space, and introduce the properties of the guessing sequence that imply the space is Dowker. In Section 3 we present the construction of the required guessing sequence assuming \(V = L\). In section 4 we show how to modify the space to make it locally compact and first countable.

2. Building the space from the guessing sequence

A ladder system on \(\omega_1\) is a sequence \(\vec{E} = \langle E_\alpha : \alpha \in \text{Lim} (\omega_1) \rangle\) such that each \(E_\alpha \subseteq \alpha\) is unbounded in \(\alpha\) of order type \(\omega\).

If \(\vec{E}\) is a ladder system on \(\omega_1\), then we define a topology \(\tau\) on \(\omega_1\) associated with \(\vec{E}\) by declaring final segments of its elements \(E_\alpha\) as weak neighbourhoods of \(\alpha\); that is, a set \(U \subseteq \omega_1\) is defined to be \(\tau\)-open if for each \(\alpha \in U\) there exists some \(\beta < \alpha\) such that \(E_\alpha \setminus \beta \subseteq U\). One can easily check that this is a topology, and that with this topology, \(\omega_1\) is a regular space. Of course, we cannot expect to always get a normal topology. However, if \(\vec{E}\) has good guessing properties, then \((\omega_1, \tau)\) can be normal. In order to state the necessary lemma we need some definitions: we say that \(\vec{E}\) is strong club guessing if for every club \(C \subseteq \omega_1\) there is a club \(K \subseteq \omega_1\) so that \(\alpha \in K \Rightarrow (\exists \beta < \alpha)(E_\alpha \setminus \beta \subseteq C)\) and \(\vec{E}\) is 2-stationary hitting if for every pair of stationary sets \(S, T \subseteq \omega_1\) there is some \(\gamma \in \omega_1\) such that both \(E_\gamma \cap S\) and \(E_\gamma \cap T\) are cofinal in \(\gamma\). This type of guessing principle was used in [HI04] to construct a perfectly normal not realcompact space, please refer to this paper for a proof of the following lemma.

**Lemma 1.** If \(\vec{E}\) is strong club guessing and 2-stationary hitting, then \((\omega_1, \tau)\) is normal.
Recall that Dowker proved in [Dow51] that a normal space $X$ is countably paracompact if whenever \( \{D_n\}_{n \in \omega} \) is a decreasing family of closed subsets of $X$ whose intersection is empty, there exists a family \( \{U_n\}_{n \in \omega} \) of open sets which has empty intersection and, for each $n \in \omega$, $U_n \supseteq D_n$. We start by considering a countable partition of $\omega_1$ into stationary subsets \( \{S_n : n \in \omega\} \), and we will make the union of the first $n$ many stationary sets open. The complement of that union will be the closed set $D_n$ witnessing that countable paracompactness fails.

So, we need that the elements of the ladder system $E$ “look back”; that is, $\alpha \in S_n$ must give us $E_\alpha \subseteq \bigcup_{k=0}^n S_k$.

**Proposition 2.** Suppose \( \{S_n : n \in \omega\} \) is a partition of $\omega_1$ into stationary subsets and $E$ is a strong club guessing ladder system such that $\alpha \in S_n \Rightarrow E_\alpha \subseteq \bigcup_{k=0}^n S_k$. Moreover assume that for each $n$ and for each pair of stationary sets $A, B \subseteq \bigcup_{i < n} S_i$ there is $\alpha \in S_n$ such that $E_\alpha \cap A$ and $E_\alpha \cap B$ are infinite. Then, letting $\tau$ be the topology associated with $E$, the space $(\omega_1, \tau)$ is a Dowker space.

**Proof.** It is easy to see that the hypotheses imply that the ladder system is 2-stationary hitting, so by lemma 1 the space is normal. To see that the space is not countably paracompact, let $D_n = \omega_1 \setminus \bigcup_{i < n} S_i$. Then \( \{D_n : n \in \omega\} \) is a decreasing sequence of closed sets with empty intersection. Now suppose that \( \{W_n : n \in \omega\} \) is a sequence of $\tau$-open subsets of $\omega_1$ such that for each $n \in \omega$, $W_n \supseteq D_n$. Suppose further that $\bigcap_{n \in \omega} W_n = \emptyset$. Then $(\forall \alpha \in \omega_1) (\exists n \in \omega) (\alpha \notin W_n)$, and thus for some fixed $m \in \omega$ there is a stationary set $X \subseteq \omega_1 \setminus W_m$. As $W_m$ is open, $\text{cl}_\tau(X) \subseteq \omega_1 \setminus W_m$. However, there must be $\delta \in S_{m+1}$ such that

$$\delta \in \text{cl}_\tau(X) \subseteq \omega_1 \setminus W_m,$$

which contradicts that $S_{m+1} \subseteq W_m$. Hence the space is not countably paracompact. \qed

Before we present the construction of a ladder system satisfying the hypotheses of the previous proposition, we make a few observations about spaces constructed from ladder systems. Recall that a space $X$ is a scattered if and only if every non-empty subset contains an isolated point. One can also define $X^{(0)} = X$, $X^{(\alpha+1)} = (X^{(\alpha)})'$ — the set of limit points, and $X^{(\alpha)} = \bigcap_{\beta < \alpha} X^{(\beta)}$ in case $\alpha$ is a limit ordinal. Then $X$ is scattered space iff there is some ordinal $\alpha$ such that $X^{(\alpha)} = \emptyset$. If such an $\alpha$ exists, then we say that the scattered height of $X$ is the least $\alpha$ for which $X^{(\alpha)} = \emptyset$. 

Lemma 3. If \( \vec{E} \) is strong club guessing ladder system and \( \tau \) is the topology on \( \omega_1 \) described above, then \((\omega_1, \tau)\) has uncountable scattered height.

Proof. To see that \((\omega_1, \tau)\) has uncountable scattered height, it will suffice to show, by induction for every \( \alpha < \omega_1 \), that \((\omega_1)^{(\alpha)}\) contains a club subset of \( \omega_1 \).

Suppose \( \alpha < \omega_1 \) is a limit ordinal. Choose \( \beta_n \not\in (\omega_1)^{(\alpha)} \). Then \((\omega_1)^{(\alpha)} = \bigcap_{n=0}^{\infty} (\omega_1)^{(\beta_n)} \). Thus we only need to make sure that for successor stages at least a club of limit points is preserved. But that is easy as \( \vec{E} \) is strong club guessing sequence. \( \square \)

By a result in [HI04] we know that using ladder systems \( \vec{E} \) we cannot hope for an example consistent with MA+\( \neg \)CH unless we use “longer” ladders. That is, we will need that for at least a stationary set of \( \alpha \)'s the \( E_\alpha \)'s have order type \( \alpha \). We refer to this kind of sequence as a guessing sequences. The following result established the possibility of getting normal topologies with club guessing sequences even in the presence of Martin’s Axiom.

Theorem 4 ([Her04]). If ZFC is consistent, then it is also consistent with ZFC that MA+\( \neg \)CH holds and there exists a strong club guessing and 2-stationary hitting guessing sequence \( \vec{E} \). Moreover, the resulting topology \( \tau \) is normal.

However, the guessing sequence given by this theorem is not built around a required partition of \( \omega_1 \) into stationary sets. The difficulty in obtaining a Dowker topology using a guessing sequence is to guarantee not only the \( E_\alpha \subseteq \bigcup_{k=0}^{n} S_k \) but that the \( \tau \)-closure of \( E_\alpha \) stays inside the first \( n \)-levels of the space, \( \bigcup_{k=0}^{n} S_k \). This we were not able to obtain.

3. Construction of the guessing sequence

In this section we construct a ladder system \( \vec{E} = \langle E_\gamma : \gamma \in \text{Lim} (\omega_1) \rangle \) with enough properties and then we define a topology on \( \omega_1 \) assigning compact open neighbourhoods to the points in \( E_\gamma \), for selected \( \gamma \). The method to construct this ladder system is different to the one used in [HI04]. There a forcing iteration of non-proper posets was used, here we use Gödel’s Constructibility Axiom, \( V = L \).

Proposition 5. Assume \( V = L \). If \( \{S_n : n \in \omega\} \) is a partition of \( \omega_1 \) into stationary subsets, then there is a strong club guessing ladder system \( \vec{E} \) such that \( \alpha \in S_n \Rightarrow E_\alpha \subseteq \bigcup_{k=0}^{n} S_k \), and moreover \( \vec{E} \) has the
following property: If \( A, B \subseteq \bigcup_{k=0}^{n} S_k \) are stationary, then there exists a stationary \( T \subseteq S_{n+1} \) such that
\[
\gamma \in T \Rightarrow \sup (A \cap E_{\gamma}) = \gamma = \sup (B \cap E_{\gamma}).
\]

Proof. Fix the partition \( \{S_n : n \in \omega \} \) of \( \omega \) into stationary sets. To aid in the notation let \( f : \omega \rightarrow \omega \) be such that \( S_n = f^{-1}(\{n\}) \) for each \( n \in \omega \). For each limit ordinal \( \gamma \in \text{Lim}(\omega_1) \) let \( A_{\gamma} \) be defined by \( \alpha \in A_{\gamma} \) if and only if
\[
\begin{align*}
(\text{i}) & \quad L_\alpha \models \text{ZF}^- \text{ (i.e. ZF without the Powerset Axiom),} \\
(\text{ii}) & \quad \gamma = \omega_1^{L_\alpha}, \text{ and} \\
(\text{iii}) & \quad f \upharpoonright \gamma \in L_\alpha \text{ and } L_\alpha \models "f \upharpoonright \gamma \text{ codes a partition into stationary sets."}
\end{align*}
\]

Then \( |A_{\gamma}| \leq \aleph_0 \), for each \( \gamma \in \text{Lim}(\omega_1) \), since \( \{\rho \in \omega_1 : L_\rho \prec L_{\omega_1}\} \) is unbounded in \( \omega_1 \) and \( L_{\omega_1} \models \gamma < \omega_1. \) Note that \( \{\gamma < \omega_1 : A_{\gamma} \neq \emptyset\} \) contains a club set. Now let
\[
G_\gamma = \{C \subseteq \gamma : C \text{ is club in } \gamma \& (\exists \alpha \in A_{\gamma}) (C \in L_\alpha)\}.
\]

Then \( G_\gamma \) is countable and closed under finite intersections. Finally, let \( H_\gamma \) be defined as: \( \langle A^0, A^1 \rangle \in H_\gamma \) if and only if
\[
\begin{align*}
(\text{i}) & \quad (\forall i \in 2) (A^i \subseteq \gamma), \\
(\text{ii}) & \quad (\exists \alpha \in A_{\gamma}) (\forall i \in 2) (A^i \in L_\alpha), \\
(\text{iii}) & \quad (\forall C \in G_\gamma) (\forall i \in 2) (C \cap A^i \neq \emptyset), \\
(\text{iv}) & \quad (\forall i \in 2) (\forall \xi \in A^i) (f(\xi) < f(\gamma)).
\end{align*}
\]

Note that given \( \langle A^0, A^1 \rangle \in H_\gamma \) we have that \( L_\alpha \models "A^i \text{ is stationary in } \gamma", \) for a suitable \( \alpha \in A_{\gamma} \). \( H_\gamma \) is also countable.

We will use induction to construct each term of the guessing ladder. For instance, the member \( E_\gamma \) will be constructed as follows. Let us start by assuming \( \gamma \in S_m \) and enumerating \( G_\gamma \) as \( \{C_n\}_{n \in \omega} \) and \( H_\gamma \) as \( \{\langle A_n^0, A_n^1 \rangle\}_{n \in \omega} \), where in the list of elements of \( H_\gamma \) we mention each \( \aleph_0 \) times; and choose a cofinal sequence \( \{\alpha_n : n \in \omega \} \) of \( A_\gamma \) so that for all \( n \in \omega \) \( \langle A^0_n, A^1_n \rangle, C_m \in L_{\alpha_n}, \) for all \( m \leq n \).

Choose an increasing sequence \( \langle \delta_n : n \in \omega \rangle \) cofinal in \( \gamma \). We know that, for each \( i \in 2 \), \( L_{\alpha_0} \models "A^i_0 \cap C_0 \text{ is stationary in } \gamma"; \) working inside \( L_{\alpha_0} \) we can choose \( \gamma_0 \in A^0_0 \cap C_0 \setminus \delta_0 \) and \( \gamma_1 \in A^1_0 \cap C_0 \setminus \gamma_0; \) then apply the same now inside \( L_{\alpha_1} \) and so on; in general we select \( \gamma_{2n} \in A^0_n \cap \bigcap_{m \leq n} C_m \setminus \max \{\delta_n, \gamma_{2n-1}\} \) and \( \gamma_{2n+1} \in A^1_n \cap \bigcap_{m \leq n} C_m \setminus \gamma_{2n}. \) Finally we let
\[
E_\gamma = \{\gamma_n : n \in \omega\}.
\]

Clearly \( E_\gamma \) defined this way will be a cofinal in \( \gamma \) and for each \( \langle A^0, A^1 \rangle \in H_\gamma \) we have that \( \sup (E_\gamma \cap A^i) = \gamma, \) for \( i \in 2. \) Observe that if \( \gamma \in S_m, \)
then \( E_\gamma \subseteq \bigcup_{k=0}^{m} S_k \) as \( E_\gamma \) contains only points from the \( A_n^i \) forming the pairs elements of \( \mathcal{H}_\gamma \).

**Claim 6.** If \( C \subseteq \omega_1 \) is closed unbounded and \( \langle A^0, A^1 \rangle \) is a pair of stationary subsets such that \( f(A^0 \cup A^1) \subseteq m+1 \), then there exist a stationary \( T \subseteq S_{m+1} \) and club \( K \subseteq \omega_1 \) such that \( \langle A^0 \cap \gamma, A^1 \cap \gamma \rangle \in \mathcal{H}_\gamma \), for all \( \gamma \in T \), and \( C \cap \gamma \in \mathcal{G}_\gamma \), for all \( \gamma \in K \).

**Proof of Claim 6.** Let \( \langle A^0, A^1 \rangle \) and \( C \) be given as above; we will find \( T \) and \( K \). By recursion, define a sequence of elementary submodels \( M_\nu \prec L_{\omega_2}, \ \nu < \omega_1 \), as follows:

- \( M_0 \) is the smallest \( M \prec L_{\omega_2} \) such that \( \langle A^0, A^1 \rangle, C \in M \),
- \( M_{\nu+1} \) is the smallest \( M \prec L_{\omega_2} \) such that \( M_\nu \cup \{M_\nu\} \subseteq M \),
- \( M_\xi = \bigcup_{\nu<\xi} M_\nu \) if \( \xi \) is a limit ordinal.

By the Condensation Lemma, \( M_\nu \cap L_{\omega_1} \) is transitive. Let \( \alpha_\nu = M_\nu \cap \omega_1 \). Then \( \{ \alpha_\nu : \nu < \omega_1 \} \) is a normal sequence in \( \omega_1 \). Let

\[
\pi_\nu : M_\nu \cong L_{\beta_\nu}.
\]

Clearly then

\[
\pi_\nu \upharpoonright L_{\alpha_\nu} = \text{id} \upharpoonright L_{\alpha_\nu}, \quad \pi_\nu(\omega_1) = \alpha_\nu,
\]

\[
\pi_\nu(C) = C \cap \alpha_\nu, \quad \pi_\nu(\langle A^0, A^1 \rangle) = \langle A^0 \cap \alpha_\nu, A^1 \cap \alpha_\nu \rangle.
\]

Consider \( K \) the set of all limit points of the set \( \{ \alpha_\nu : \nu < \omega_1 \} \). Then \( K \) is club in \( \omega_1 \). Let \( \gamma \in K \) be given. For some limit ordinal \( \lambda < \omega_1 \),

\[
\gamma = \sup_{\nu < \lambda} \alpha_\nu = \sup_{\nu < \lambda} \beta_\nu
\]

and hence \( \gamma = \alpha_\lambda \). To see this, it suffices to prove that for all \( \nu < \omega_1 \), \( \alpha_\nu < \beta_\nu < \alpha_{\nu+1} \). Clearly \( \alpha_\nu < \beta_\nu \). But \( \beta_\nu \) is definable from \( M_\nu \) since \( L_{\beta_\nu} \) is the transitive collapse of \( M_\nu \), and moreover this definition relativises to \( L_{\omega_2} \). So as \( M_\nu \in M_{\nu+1} \prec L_{\omega_2} \) we have \( \beta_\nu \in M_{\nu+1} \). Hence \( \beta_\nu \in \alpha_{\nu+1} \), and the affirmation is confirmed.

Note that \( \beta_\lambda \in \mathcal{A}_\gamma \) since we have that \( L_{\beta_\lambda} \models \gamma = \omega_1 \) and \( L_{\beta_\lambda} \models \text{ZF}^- \), hence \( C \cap \gamma = \pi_\lambda(C) \in L_{\beta_\lambda} \) and of course \( L_{\beta_\lambda} \models \pi_\lambda(C) \) is club set in \( \gamma \). Thus \( C \cap \gamma \in \mathcal{G}_\gamma \).

Also, \( \langle A^0 \cap \gamma, A^1 \cap \gamma \rangle \in L_{\beta_\lambda} \). To obtain the rest of our Claim 6, we need to find the stationary set \( T \). For this let \( \mu_\gamma = \sup \mathcal{A}_\gamma \), for each \( \gamma < \omega_1 \). Let us define

\[
E = \left\{ \gamma \in S_{m+1} : (\forall i \in 2)(L_{\mu_\gamma} \models "A^i \cap \gamma \text{ is stationary in } \gamma = \omega_1^{L_{\mu_\gamma}}") \right\}.
\]

**Claim 7.** The set \( E \) is stationary in \( \omega_1 \).
Proof of Claim 7. Let $F$ be a club subset of $\omega_1$. To find an ordinal in the intersection of $F$ and $E$, let $\xi < \omega_2$ be the least ordinal such that $\{S_k\}_{k \in \omega}, F, A^0, A^1 \in L_\xi$, and for $n < \omega$ let $\kappa_n$ be the $(n + 1)$-th ordinal greater than $\xi$ such that $L_{\kappa_n} \models \text{ZF}^-$. For each $n \in \omega$ define countable submodels $M^n_\nu \prec L_{\kappa_n}$, for $\nu < \omega_1$, as follows:

- $M^n_\nu$ is the smallest $M \prec L_{\kappa_n}$ such that $\{S_k\}_{k \in \omega}, F, A^0, A^1 \in M$,
- $M^{n+1}_\nu$ is the smallest $M \prec L_{\kappa_n}$ such that $M^n_\nu \cup \{M^n_\nu\} \subseteq M$, and
- $M^n_\nu = \bigcup_{i < \nu} M^i_\nu$, when $\nu$ is a limit ordinal.

Now, set $\alpha^n_\nu = M^n_\nu \cap \omega_1$. Then the sequence $\langle \alpha^n_\nu : \nu < \omega_1 \rangle$ is normal for every $n \in \omega$. Put $G_n = \{\nu < \omega_1 : \nu = \alpha^n_\nu\}$ and let $G = \bigcap \{G_n : n \in \omega\}$. Then

$$\nu \in G \Rightarrow \nu = \alpha^n_\nu,$$

for all $n \in \omega$.

Fix $\nu = \min(G \cap S_{m+1})$, and consider transitive collapsing maps $\pi_n : M^n_\nu \cong L^n_\nu$, for $n \in \omega$. Then $\pi_n \upharpoonright \nu = \text{id} \upharpoonright \nu$, $\pi_n(\omega_1) = \nu$, $\pi_n(S_k) = S_k \cap \nu$ for all $k \in \omega$, $\pi_n(F) = F \cap \nu$, $\pi_n(A^i) = A^i \cap \nu$ and $\gamma^n_\nu \in A_\nu$. Let $\gamma_\nu = \sup_{n < \omega} \gamma^n_\nu$. Then $L_\gamma_\nu \models \text{“} \nu = \omega_1 \text{“}$, $L_\gamma_\nu \models \text{“} S_k \cap \nu \text{ is stationary} \text{“}$ for every $k \in \omega$, $L_\gamma_\nu \models \text{“} A^i \cap \nu \text{ is stationary} \text{“}$ for every $i \in 2$ and $L_\gamma_\nu \models \text{“} F \cap \nu \text{ is cofinal in } \nu \text{“}$. If $\mu \geq \gamma_\nu$, we want to prove $\mu \notin A_\nu$ to have that $\mu_\nu = \gamma_\nu$. Suppose $L_\mu \models \text{“} \nu = \omega_1 \text{“}$. Let $\check{\xi}_\nu$ be the least ordinal such that

$$\{S_k \cap \nu\}_{k \in \omega}, \{A^i \cap \nu\}_{i \in 2}, F \cap \nu \subseteq L_{\check{\xi}_\nu}.$$

Then we have $\pi_n(\xi) = \check{\xi}_\nu$, for all $n \in \omega$. It follows that for $n > 0$,

$$(\pi_n)^{-1}(\gamma^n_\nu) = \kappa_0, (\pi_n)^{-1}(\gamma^0_\nu) = \kappa_0, \ldots , (\pi_n)^{-1}(\gamma^{n-1}_\nu) = \kappa_{n-1}$$

and $\gamma^n_\nu$ is the $(n + 1)$-th ordinal greater than $\check{\xi}_\nu$ such that $L_{\gamma^n_\nu} \models \text{ZF}^-$. Thus $\langle \gamma^n_\nu : n \in \omega \rangle$ is definable in $L_{\gamma_\nu}$ and hence $L_{\gamma_\nu}$ cannot be a model of $\text{ZF}^-$, so $\gamma_\nu \notin A_\nu$. If $\mu > \gamma_\nu$, then $\langle \gamma^n_\nu : n \in \omega \rangle \subseteq L_\mu$. Working inside $L_\mu$ we can define the $L_\mu$-versions, $\overline{G}_n$, of the club’s $G_n \subseteq \omega_1$ defined earlier (with $\check{\xi}_\nu$ in place of $\xi$, $\gamma^n_\nu$ in place of $\kappa_n$, etc.). Then $\overline{G}_n = G_n \cap \nu$, for all $n \in \omega$. Thus

$$L_\mu \models \text{“} \bigcap_{n \in \omega} G_n \cap \nu \text{ is club in } \nu \text{“},$$

and hence $L_\mu \models (\bigcap_{n \in \omega} G_n \cap \nu) \cap (S_{m+1} \cap \nu) \neq \emptyset$. Nevertheless, this is impossible since $\nu = \min(G \cap S_{m+1})$. Thus, it is not true that $L_\mu \models \text{ZF}^-$,
and hence $\mu_{\nu} = \gamma_{\nu}$. In summary,

$$L_{\gamma_{\nu}} \models "\nu = \omega_1",$$

$$L_{\gamma_{\nu}} \models "A^i \cap \nu \text{ is stationary}",$$

for every $i \in 2$, and

$$L_{\gamma_{\nu}} \models "F \cap \nu \text{ is cofinal in } \nu".$$

The first two parts together with the choice of $\nu$ imply that $\nu \in E$ and the third one implies that $\nu \in F$ since $F$ is closed. This completes the proof of Claim 7.

□

So, the set $E$ is stationary and put $T = E \cap K$. For if $\gamma \in T$ then as we have seen $\beta_\lambda \in A_\gamma$, $L_{\mu_\gamma} \models "\gamma = \omega_1"$ and $L_{\mu_\gamma} \models "A^i \cap \gamma \text{ is stationary in } \gamma"$ for every $i \in 2$, although $L_{\mu_\gamma}$ is not a model of $\text{ZF}^-$. However, if $C \in L_\beta$ is a club subset of $\gamma$ for some $\beta \in A_\gamma$, then $C \in L_{\mu_\gamma}$ and $C$ is still a club since this notion is absolute. Therefore, for all $i \in 2$, $L_{\mu_\gamma} \models C \cap (A^i \cap \gamma) \neq \emptyset$ as we needed to prove. So, $((A^0 \cap \gamma, A^1 \cap \gamma) \in \mathcal{H}_\gamma)$.

□

This claim finishes the proof of the proposition.

□

Take a ladder system $\overrightarrow{E}$ given by Proposition 5. Applying Proposition 2 and considering the topology $\tau$ associated to $\overrightarrow{E}$, we obtain a Dowker topology on $\omega_1$.

4. A LOCALLY COMPACT FIRST COUNTABLE MODIFICATION OF THE SPACE

The space in previous section is locally countable but it cannot be first countable and hence it cannot be locally compact either. We want to present a modification of the construction obtaining a locally compact example. We shall use again the ladder system $\overrightarrow{E}$ constructed in the previous section. But instead of using final segments of its elements as weak neighbourhoods we will choose suitable compact neighbourhoods for the points using the ladder system $\overrightarrow{E}$ in the usual way (e.g., see [dC76]). The resulting topology will be coarser than the topology $\tau$ but neighbourhoods will still “look to the left.” Hence the resulting topology will again fail to be countably paracompact. To guarantee that the normality will not be destroyed we need to build separations for possible disjoint closed subsets. For this we need to assume $\diamondsuit^+$. We are going to use a $\diamondsuit^+$ sequence that captures quintuples of subsets of $\omega_1$. I.e., we will assume
there is a sequence $\langle D_\alpha : \alpha \in \text{Lim} (\omega_1) \rangle$ so that for any subsets $A, B, C$ and $K$ of $\omega_1$ we have that there is a club $D \subseteq \omega_1$ such that

$$\left( \forall \gamma \in D \right) \left( \langle A \cap \gamma, B \cap \gamma, C \cap \gamma, K \cap \gamma, D \cap \gamma \rangle \in D_\gamma \right).$$

For a quintuplet $q = \langle A, B, C, K, D \rangle$ we are going to denote by $q \upharpoonright \gamma$ the quintuplet $\langle A \cap \gamma, B \cap \gamma, C \cap \gamma, K \cap \gamma, D \cap \gamma \rangle$. We are also going to use the collections $G_\alpha$ defined in the first part of the proof of Proposition 5.

We will define, by recursion on $\beta \in \omega_1$, topologies $\rho_\beta$ on $[0, \beta)$. Having defined $\rho_\beta$, we will say that a quintuplet $q = \langle A, B, C, K, D \rangle \in D_\beta$ is important if

1. $A$ and $B$ are $\rho_\beta$-closed,
2. $C \in G_\beta$,
3. $(A \cup C) \cap B = \emptyset$,
4. $K \subseteq C$ and
5. $K$ and $D$ are clubs in $\beta$.

We then also define $\rho_\beta$-open sets $U_0^q$ and $U_1^q$ for all important quintuplets $q \in D_\beta$. The topologies and pairs of open sets satisfy the following inductive hypotheses:

1. For each $\gamma < \beta$, $\rho_\beta$ is a “conservative extension” of $\rho_\gamma$. I.e., the $\rho_\beta$ subspace topology on $[0, \gamma)$ and $\rho_\gamma$ coincide,
2. $\rho_\beta$ is finer than the order topology on $\omega_1$ and coarser than $\tau \upharpoonright [0, \beta)$, where $\tau$ is the topology associated with the ladder system $E$,
3. For each important $q \in D_\beta$, $U_0^q$ and $U_1^q$ form an open separation for $A \cup C$ and $B$ in $\beta$ with the topology $\rho_\beta$,
4. these open separations are coherent along $K \cap D$; that is, $U_i^q \upharpoonright \gamma = U_i^q \cap \gamma$, for each $i \in 2$ and each $\gamma < \beta$ such that $\gamma \in K \cap D$,
5. if $q \in D_\beta$ is an important quintuplet and if $\sup(D \cap K) = \beta$, then $U_0^q \cup \{ \beta \}$ is a neighbourhood of $\beta$.

We assume we have topologies $\rho_\beta$ defined on $[0, \beta)$, for all $\beta < \alpha$, satisfying the above five clauses. We want to extend this topology to a topology $\rho_{\alpha+1}$ on $[0, \alpha]$ defining neighbourhoods of $\alpha$ and defining separations for all possible $A$ and $B$ that appear in some $q_n$, where $\{ q_n : n \in \omega \}$ is an enumeration of all important quintuplets in $D_\alpha$. We do this extension of $\rho_\alpha$ also by recursion assuming that we are done choosing $U_{q_k}^i$ for $k < n$. We have two exclusive cases to complete the definition of $U_{q_n}^i$, being $q_n = \langle A_n, B_n, C_n, K_n, D_n \rangle$:
Lemma 8. The space $(\omega_1, \rho)$ is normal.

Proof. Let $A_0, A_1$ be stationary subsets of $\omega_1$. Then there must exist $\gamma \in \text{Lim} (\omega_1)$ such that $\gamma \in \text{cl}_\rho (A_i)$, for each $i \in \{0, 1\}$. Therefore no two stationary sets have disjoint closures. Thus if we consider two disjoint $\rho$-closed subsets $A \subseteq \omega_1$ and $B \subseteq \omega_1$, without loss of generality we may assume that $B$ is not stationary, and hence there is a club $C \subseteq \omega_1$ such that $C \cap B = \emptyset$. For $C$, we know there is a club $K$ such that $\gamma \in K \Rightarrow E_\gamma \subseteq^* C$ as $\overrightarrow{E}$ is strong club guessing ladder system. Using our $\Diamond^+$-sequence we know that there is a club $D \subseteq \omega_1$ such that $q \upharpoonright \gamma \in D_\gamma$, for $\gamma \in D$, where $q = \langle A, B, C, K, D \rangle$. And moreover, for $\gamma \in D \cap K$, we have built $\rho$-open subsets $U_{q, \gamma}^0$ and $U_{q, \gamma}^1$ so that

\begin{enumerate}
    \item $(A \cup C) \cap [0, \gamma) \subseteq U_{q, \gamma}^0 \subseteq [0, \gamma],$
    \item $(A \cup C) \cap [\gamma, \omega_1) \subseteq U_{q, \gamma}^1 \subseteq [\gamma, \omega_1],$
    \item $[0, \gamma) \cap W_{\rho} = [0, \gamma) \cap \text{cl}_\rho [0, \gamma)$
\end{enumerate}
A SMALL DOWKER SPACE FROM A CLUB-GUESSING PRINCIPLE.

(2) $B \cap [0, \gamma) \subseteq U_{q|\gamma}^1 \subseteq [0, \gamma)$;
(3) $\xi < \eta$ \& $\xi \in D \cap K \cap \eta \Rightarrow U_{q|\xi}^i = U_{q|\eta}^i \cap [0, \xi)$ and
(4) $U_{q|\gamma}^0 \cap U_{q|\gamma}^1 = \emptyset$.

So we let $U_q^0 = \bigcup \{ U_{q|\xi}^0 : \xi \in D \cap K \}$ and $U_q^1 = \bigcup \{ U_{q|\xi}^1 : \xi \in D \cap K \}$
to obtain a separation for $A$ and $B$. 

REFERENCES


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