Monoidal Topology

Walter Tholen

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Tychonoff’s Theorem

\[ \prod_{i \in I} X_i \text{ compact if all } X_i \text{ compact} \]

Proof:

Geometric Argument

Convergence Argument:

Most Books: Engelking, ...

Few Books: Willard, ...

Involved

Trivial

Why?
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Why?
Initial topology

\[ f_i : X \rightarrow Y_i, \; Y_i \in \text{Top} \; (i \in I) \]

Geometric Description:

- Collect all \( f_i^{-1}(V), \; V \subseteq Y_i \) open
- **Generate** a topology from these!

Convergence Description:

- \( x \rightarrow y : \iff \forall i \in I : f_i[x] \rightarrow f_i(y) \)
- This is (the conv. of) a topology!

By contrast:
Initial topology

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Geometric Description:

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- **Generate** a topology from these!

By contrast:

Convergence Description:

- \( x \rightarrow y :\Leftrightarrow \forall i \in I : f_i[x] \rightarrow f_i(y) \)

This is (the conv. of) a topology!
Final topology

\[ f_i : X_i \to Y, X_i \in \text{Top} \quad (i \in I) \]

Geometric Description:

- \( V \subseteq Y \) open: \( \iff \)
- \( \forall i \in I : f_i^{-1}(V) \subseteq X_i \) open

This is a topology!

Convergence Description:

- Collect all \( f_i[x] \to f_i(y) \) for \( x \to y \) in \( X_i \)

- Generate (the conv. of) a topology!
$f_i : X_i \rightarrow Y$, $X_i \in \text{Top}$ ($i \in I$)

**Geometric Description:**
- $V \subseteq Y$ open $\iff \forall i \in I : f_i^{-1}(V) \subseteq X_i$ open

This **is** a topology!

**Convergence Description:**
- Collect all $f_i[x] \rightarrow f_i(y)$ for $x \rightarrow y$ in $X_i$

- **Generate** (the conv. of) a topology!
First conclusions by a categorical topologist

- Appreciate the importance of topological functors, such as $\text{Top} \to \text{Set}$, $\text{Unif} \to \text{Set}$, $\text{TopGrp} \to \text{Grp}$, ...

- While it is beautiful to have self-duality of topological functors: all “initials” (infs) exist $\iff$ all “finals” (sups) exist, ...

- ... it may not always be convenient to express infs in terms of sups, or conversely.

- Treat opens/closeds/neighbourhoods and convergence side by side!

This talk is about a categorical formalization of convergence that has many predecessors:

- 1968: Manes, Wyler, Gähler, Möbus, Höhle, Flagg, Kopperman, ...
- 2002: Clementino, Hofmann, Seal, T, ....

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The “Two-Axiom Miracle” in Algebra

**Example:** $M$-sets ($M$ a monoid) $\xrightarrow{a} M \times X$, $a(\alpha, x) = \alpha \cdot x$

- $\forall \alpha, x \in M \times X$, we have:
  - $e_M \cdot x = x$
  - $(\alpha \beta) \cdot x = \alpha \cdot (\beta \cdot x)$
  - $f(\alpha \cdot x) = \alpha \cdot f(x)$

- $M \times X$ (with the obvious action) is the free $M$-set over a set $X$.
- Eilenberg-Moore: One may replace $M \times X$ by the free group, free ring, free Lie algebra, or any free algebra in a variety, to see that...

- ... the Two-Axiom Miracle continues throughout Algebra.
**Example:** $M$-sets ($M$ a monoid) \[ M \times X \xrightarrow{a} X, \quad a(\alpha, x) = \alpha \cdot x \]

\[
\begin{align*}
  e_M \cdot x &= x \\
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$X \xrightarrow{(e_M, 1_X)} M \times X \quad M \times M \times X \xrightarrow{1_M \times a} M \times X \quad M \times X \xrightarrow{1_M \times f} M \times Y$

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- ... the Two-Axiom Miracle continues throughout Algebra.
Manes 1968: compact Hausdorff spaces

Replace $M \times X$ by $\beta X$ = set of ultrafilters on $X$:

\[
e_X(x) = \dot{x} \quad m_X(\mathcal{F}) = \Sigma \mathcal{F} \text{ ("Kowalsky sum")}: \quad \begin{align*}
A \in \Sigma \mathcal{F} &\iff \{ \mathcal{F} \in \beta X \mid A \in \mathcal{F} \} \in \mathcal{F} \\
A \in \beta a(\mathcal{F}) = a[\mathcal{F}] \text{ ("image" of $\mathcal{F}$)} &\iff \{ \mathcal{F} \in \beta X \mid a(\mathcal{F}) \in A \} \in \mathcal{F}
\end{align*}
\]

\[
\lim \dot{x} = x \quad \lim(\lim \mathcal{F}) = \lim \Sigma \mathcal{F} \quad \begin{align*}
f(\lim \mathcal{F}) &= \lim(f[\mathcal{F}]) \\
\mathcal{F} \to y &\implies f[\mathcal{F}] \to f(y)
\end{align*}
\]
Replace $M \times X$ by $\beta X = \text{set of ultrafilters on } X$:

$$ e_X(x) = x, \quad m_X(\mathcal{X}) = \Sigma \mathcal{X} \text{ ("Kowalsky sum")}: $$

$$ A \in \Sigma \mathcal{X} \iff \{ x \in \beta X \mid A \in x \} \in \mathcal{X} $$

$$ A \in \beta a(\mathcal{X}) = a[\mathcal{X}] \text{ ("image" of } \mathcal{X}) $$

$$ \iff \{ x \in \beta X \mid a(x) \in A \} \in \mathcal{X} $$

$$ \lim \dot{x} = x, \quad \lim(\lim \mathcal{X}) = \lim \Sigma \mathcal{X} $$

$$ \dot{x} \to x, \quad \mathcal{X} \to \mathcal{Y} \text{ and } \mathcal{Y} \to \mathcal{Z} \Rightarrow \Sigma \mathcal{X} \to \mathcal{Z} $$

$$ \beta f(x) = f[\mathcal{x}] \text{ ("image") } $$

$$ B \in f[\mathcal{x}] \iff f^{-1}(B) \in \mathcal{x} $$

$$ \lim(\lim \mathcal{X}) = \lim f[\mathcal{x}] $$

$$ \mathcal{x} \to \mathcal{y} \Rightarrow f[\mathcal{x}] \to f(\mathcal{y}) $$
Barr 1970: arbitrary topological spaces?

Replace the map $\beta X \xrightarrow{a} X$ by a relation $\beta X \xrightarrow{a} X$.

Recall that a relation $a$ is a map precisely when

- defined everywhere: existence of convergence points: compactness;
- defined uniquely: uniqueness of convergence points: Hausdorffness.

What are the axioms on $a$ characterizing it as a topological convergence relation?
Barr 1970: arbitrary topological spaces?

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The “Two-Axiom Miracle” continues in Topology!

A conv. rel. of a top. sp. $X \iff$
- $\dot{x} \rightarrow x$
- $x \rightarrow \eta$ and $\eta \rightarrow z \Rightarrow \Sigma x \rightarrow z$

$f : X \rightarrow Y$ continuous $\iff$
$(x \rightarrow y \Rightarrow f[x] \rightarrow f(y))$

\[ \begin{array}{ccc}
X & \xrightarrow{e_X} & \beta X \\
\downarrow 1_X & \leq & a \\
X & \downarrow m_X & \geq \beta \\
X & \downarrow \beta X & \xrightarrow{\beta a} \beta X \\
\beta X & \xrightarrow{\beta a} \beta X & \xrightarrow{\beta a} X \\
\beta X & \xrightarrow{\beta a} X & \xrightarrow{\beta f} \beta Y \\
X & \xrightarrow{f} Y \\
b & \leq & Y
\end{array} \]
The “Two-Axiom Miracle” continues in Topology!

A convex relation of a top. sp. $X \iff$
- $x \to x$
- $x \to y$ and $y \to z \Rightarrow \Sigma x \to z$

$f : X \to Y$ continuous $\iff$
$(x \to y \Rightarrow f[x] \to f(y))$

$$
\begin{align*}
\text{Diagram:}
\end{align*}
$$
What does $\beta a$ mean when $a$ is just a relation?

More generally:
For a relation $r : X \rightarrow Y$, what does $\beta r : \beta X \rightarrow Y$ mean?

Present $r$ as a span $r = \left( \begin{array}{ccc} r_1 & \rightarrow & r_2 \\ X & \downarrow R & \rightarrow & Y \end{array} \right)$

The Barr extension of $\beta$ to a relation $r$ is given by:

$$r = r_2 \cdot r_1^\circ$$

$$x r y \iff \exists p \in R : \begin{cases} r_1(p) = x \\ r_2(p) = y \end{cases}$$

$$\beta r := (\beta r_2) \cdot (\beta r_1)^\circ$$

$$x(\beta r)\eta \iff \exists p \in \beta R : \begin{cases} r_1[p] = x \\ r_2[p] = \eta \end{cases}$$
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$$x r y \iff \exists p \in R : \begin{array}{l} r_1(p) = x \\ r_2(p) = y \end{array} \quad \quad \forall p \in \beta R : \begin{array}{l} r_1[p] = x \\ r_2[p] = y \end{array}$$
YES  One may replace $\beta X$ by $\gamma X = \text{set of filters on } X$
and describe topological spaces with the same two axioms, but:

NO  It is \textit{not} sufficient to just mimic Barr’s extension to relations!

More significantly:
One loses the ability to do meaningfully topology in this environment

See:  Seal 2005, “\textit{Monoidal Topology}” 2014
Filters instead of ultrafilters?

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and describe topological spaces with the same two axioms, but:

NO It is not sufficient to just mimic Barr’s extension to relations!

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From $\beta$ to any Set-monad $T$

$e_X : X \to TX$ nat. $m_X : TTX \to TX$ nat. $Tf : TX \to TY$ functorial

+ Two axioms making $(T, m, e)$ look like a monoid: “monad”
+ Provision for “extending” $T$ from maps to relations

$T$-relational spaces $(X, a)$ and continuous maps $f : (X, a) \to (Y, b)$:

```
<table>
<thead>
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<th>X</th>
<th>TX</th>
<th>TTX</th>
<th>TX</th>
<th>TY</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_X$</td>
<td>$m_X$</td>
<td>$T_a$</td>
<td>$a$</td>
<td>$b$</td>
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<tr>
<td>$1_X$</td>
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<td>$a$</td>
<td>$a$</td>
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<td>$\leq$</td>
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```

$e_X(x) \ a \ x$ (¿ $(Ta) \eta$ and $\eta \ a \ z \Rightarrow m_X(\bar{x}) \ a \ z$) ($\chi \ a \ y \Rightarrow Tf(\chi) \ b \ f(y)$)
From $\beta$ to any $\textbf{Set}$-monad $T$

\[ e_X : X \to TX \text{ nat.} \quad m_X : TTX \to TX \text{ nat.} \quad Tf : TX \to TY \text{ functorial} \]

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$T$-relational spaces $(X, a)$ and continuous maps $f : (X, a) \to (Y, b)$:

\begin{align*}
X & \xrightarrow{e_X} TX & TTX & \xrightarrow{Ta} TX & TX & \xrightarrow{Tf} TY \\
\downarrow 1_X & \leq & \downarrow m_X & \geq & \downarrow a & \downarrow a & \leq \downarrow b \\
X & & TX & & X & & Y \\
\downarrow a & & \downarrow a & & \downarrow f & & \\
X & & X & & Y
\end{align*}

\[ e_X(x) a x \quad (\exists (Ta) \eta \text{ and } \eta a z \Rightarrow m_X(\exists) a z) \quad (\forall a y \Rightarrow Tf(x) b f(y)) \]
\((T, 2)\text{-Cat}\)

\[ T = \text{Id}: \quad \text{Ord} = \text{(pre)ordered sets} \]
\[ x \leq x, \quad (x \leq y \text{ and } y \leq z \Rightarrow x \leq z) \]

\[ T = M \times (-): \quad M\text{-Ord} = \text{“}M\text{-ordered sets”} \]
\[ x \leq e_M x, \quad (x \leq_\alpha y \text{ and } y \leq_\beta z \Rightarrow x \leq_\beta_\alpha z) \]

\[ T = \beta: \quad \text{Top} = \text{topological spaces} \]
From Boolean relations to quantale-valued relations

\( r : X \times Y \to 2 \) to become \( r : X \times Y \to V \) for

\( V \) unital (commutative) quantale

= complete lattice with monoid structure \( V = (V, \otimes, k) \) s.th.

\[
\begin{align*}
  u \otimes \bigvee_{i \in I} v_i &= \bigvee_{i \in I} u \otimes v_i, \\
  (\bigvee_{i \in I} v_i) \otimes u &= \bigvee_{i \in I} v_i \otimes u
\end{align*}
\]

\( V \)-relational composition of \( r : X \to Y \) followed by \( s : Y \to Z \):

\[
(s \circ r)(x, z) = \bigvee_{y \in Y} s(y, z) \otimes r(x, y)
\]

- \( V = 2 = \{0 < 1\} \) with \( u \otimes v = u \land v, k = 1 \)
- \( V = ([0, \infty], \geq) \) with \( u \otimes v = u + v, k = 0 \) (Lawvere 1973)
- \( V = (2^M, \subseteq) \) with \( A \otimes B = \{\alpha\beta \mid \alpha \in A, \beta \in B\}, k = \{e_M\} \)
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\bigvee_{i \in I} u \otimes v_i = u \otimes \bigvee_{i \in I} v_i, \quad \bigvee_{i \in I} v_i \otimes u = \bigvee_{i \in I} v_i \otimes u
\]

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\[ u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i, \quad (\bigvee_{i \in I} v_i) \otimes u = \bigvee_{i \in I} v_i \otimes u \]

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- \( V = (2^M, \subseteq) \) with \( A \otimes B = \{\alpha \beta | \alpha \in A, \beta \in B\}, k = \{e_M\} \)
\( V\text{-Cat} = (\text{Id}, V)\text{-Cat} \quad (\mathcal{T} = \text{Id}) \)

\[
\begin{align*}
X & \xrightarrow{e_X = 1_X} X \\
\downarrow 1_X & \quad \downarrow a & \downarrow m_X = 1_X \\
X & \quad X & \quad X
\end{align*}
\]

\[
\begin{align*}
X & \xrightarrow{a} X \\
\downarrow & \quad \downarrow \geq & \downarrow \\
X & \quad X & \quad X
\end{align*}
\]

\[
\begin{align*}
X & \xrightarrow{f} Y \\
\downarrow a & \quad \downarrow \leq & \downarrow b \\
X & \quad X & \quad Y
\end{align*}
\]

\[
\begin{align*}
1_X & \leq a \\
1_X & \leq a(x, x) \\
a \cdot a & \leq a \\
a(y, z) \otimes a(x, y) & \leq a(x, z) \\
f \cdot a & \leq b \cdot f \\
a(x, y) & \leq b(f(x), f(y))
\end{align*}
\]

- \( V = 2: \quad V\text{-Cat} = \text{Ord} = (\text{pre})\text{ordered sets} \)
- \( V = 2^M: \quad V\text{-Cat} = (M \times (\_), 2)\text{-Cat} = M\text{-ordered sets} \)
- \( V = [0, \infty]^{\text{op}}: \quad V\text{-Cat} = \text{Met} = (\text{generalized}) \text{ metric spaces} \)
- \( 0 \geq a(x, x) \)
- \( a(y, z) + a(x, y) \geq a(x, z) \)
- \( a(x, y) \geq b(f(x), f(y)) \)

Walter Tholen (York University)

Monoidal Topology

Zhang Zhou, 25–30 Nov. 2015 15 / 32
\( \mathbb{V}\text{-Cat} = (\text{Id}, \mathbb{V})\text{-Cat} \quad (T=\text{Id}) \)

\[
\begin{array}{c}
\begin{array}{ccc}
X & \xrightarrow{e_X=1_X} & X \\
\downarrow & \leq & \downarrow \\
1_X & \downarrow & a \\
\downarrow & \downarrow & \downarrow \\
X & \xrightarrow{m_X=1_X} & X \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
1_X \leq a \\
k \leq a(x, x)
\end{array}
\]

\[
\begin{array}{cccc}
a \cdot a \leq a \\
a(y, z) \otimes a(x, y) \leq a(x, z) \\
a(x, y) \leq b(f(x), f(y))
\end{array}
\]

\[
\begin{array}{cccc}
f \cdot a \leq b \cdot f
\end{array}
\]

- \( \mathbb{V} = 2 \): \( \mathbb{V}\text{-Cat} = \text{Ord} = (\text{pre})\text{ordered sets} \)
- \( \mathbb{V} = 2^M \): \( \mathbb{V}\text{-Cat} = (M \times (-), 2)\text{-Cat} = M\text{-ordered sets} \)
- \( \mathbb{V} = [0, \infty]^{\text{op}} \): \( \mathbb{V}\text{-Cat} = \text{Met} = (\text{generalized}) \text{ metric spaces} \)
**V-Cat = (Id, V)-Cat (T=Id)**

- \(V = 2\): \(V\)-Cat = **Ord** = (pre)ordered sets
- \(V = 2^M\): \(V\)-Cat = \((M \times (\_ , 2))\)-Cat = \(M\)-ordered sets
- \(V = [0, \infty]^{op}\): \(V\)-Cat = **Met** = (generalized) metric spaces

\[
\begin{align*}
1_X \leq a & \quad a \cdot a \leq a \\
\kappa \leq a(x, x) & \quad a(y, z) \otimes a(x, y) \leq a(x, z) \\
0 \geq a(x, x) & \quad a(x, y) \geq b(f(x), f(y))
\end{align*}
\]
General case: \((V, \otimes, k)\) (symmetric) monoidal-closed category

A \(V\)-category \((X, a)\) has a set \(X\) of objects with “hom-objects”
\[ a(x, y) = \hom_X(x, y) \in V \] and \(V\)-arrows
\[ k \to a(x, x) \quad a(y, z) \otimes a(x, y) \to a(x, z) \]
subject to natural “monoidal” conditions

\(V\)-functor \(f : (X, a) \to (Y, b)\) is an “object map” \(f : X \to Y\)
equipped with \(V\)-arrows
\[ a(x, y) \to b(f(x), f(y)) \]
subject to natural conditions
\(V = \text{Set}: V\text{-Cat} = \text{Cat} = \) the category of small ordinary categories
Why “$V$-$\text{Cat}$”? Eilenberg and Kelly 1966

General case: $(V, \otimes, k)$ (symmetric) monoidal-closed category

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$$a(x, y) = \text{hom}_X(x, y) \in V$$

and $V$-arrows

$$k \rightarrow a(x, x) \quad a(y, z) \otimes a(x, y) \rightarrow a(x, z)$$

subject to natural “monoidal” conditions

$V$-functor $f : (X, a) \rightarrow (Y, b)$ is an “object map” $f : X \rightarrow Y$
equipped with $V$-arrows

$$a(x, y) \rightarrow b(f(x), f(y))$$

subject to natural conditions

$V = \text{Set}$: $V$-$\text{Cat} = \text{Cat} =$ the category of small ordinary categories
Why “$V$-Cat”? Eilenberg and Kelly 1966

General case: $(V, \otimes, k)$ (symmetric) monoidal-closed category

A $V$-category $(X, a)$ has a set $X$ of objects with “hom-objects”
$a(x, y) = \text{hom}_X(x, y) \in V$ and $V$-arrows

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$V = \text{Set}: \quad V\text{-Cat} = \text{Cat} = \text{the category of small ordinary categories}$
\((T, V)\)-spaces \((X, a)\) and continuous maps \(f : (X, a) \to (Y, b)\):

- \(k \leq a(e_X(x), x)\)
- \(Ta(\bar{x}, \eta) \otimes a(\eta, z) \leq a(m_X(\bar{x}), z)\)
- \(a(x, y) \leq b(Tf(x), f(y))\)
\((T, V)\)-spaces \((X, a)\) and \textit{continuous} maps \(f : (X, a) \to (Y, b)\):

\[ 
\begin{array}{cccc}
X & \xrightarrow{e_X} & TX & \\
\downarrow^{1_X} & \leq & \downarrow^{a} & \\
X & & TX & \\
\end{array} \quad \quad \begin{array}{cccc}
TTX & \xrightarrow{Ta} & TX & \\
m_X & \geq & a & \\
TX & \xrightarrow{a} & X & \\
\end{array} \quad \quad \begin{array}{cccc}
TX & \xrightarrow{Tf} & TY & \\
a & \leq & b & \\
X & \xrightarrow{f} & Y & \\
\end{array}
\]

- \(k \leq a(e_X(x), x)\)
- \(Ta(x, \eta) \otimes a(\eta, z) \leq a(m_X(x), z)\)

\[ a(x, y) \leq b(Tf(x), f(y)) \]
Topologicity

Basic Theorem
• \( (T, V)\text{-Cat} \) is topological over \( \text{Set} \), hence complete, cocomplete, \( \text{etc} \).
• The forgetful functor has both a left- and a right adjoint (discrete and indiscrete structures);
• its fibres are complete lattices.

Initial structure \( a \) on \( X \) with respect to \( f_i : X \to (Y_i, b_i) \):

\[
a(x, y) = \bigwedge_{i \in I} b_i(Tf_i(x), f_i(y))
\]

Principal Examples
\[
\begin{array}{ccc}
T & V & 2 \quad [0, \infty]^\text{op} \\
\text{Id} & \text{Ord} & \text{Met} \\
\beta & \text{Top} & \text{App} = \text{approach spaces: Lowen 1997} \\
\end{array}
\]

\( a(x, y) \) = measure of convergence of \( x \) to \( y \), two axioms

Alternative axiomatization by point-set distance
Topologicity

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Alternative axiomatization by point-set distance
Basic Theorem

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Principal Examples

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Alternative axiomatization by point-set distance

- $a(x, y) =$ measure of convergence of $x$ to $y$, two axioms
Let’s do Topology!

\((X, a)\) Hausdorff: \(a \cdot a^\circ \leq 1_X\) \((\perp < a(\overline{a}, x) \otimes a(\overline{a}, y) \Rightarrow x = y)\)

\((X, a)\) compact: \(a^\circ \cdot a \geq 1_{TX}\) \(\forall \overline{a} \in TX\) \(k \leq \bigvee_{x \in X} a(\overline{a}, x)\)

Silent hypotheses on \(V\):

- \(V\) commutative
- \(k = \top > \perp\) (\(V\) is “integral” and non-trivial)
- \(k \leq \bigvee_{i \in I} u_i \iff k \leq \bigvee_{i \in I} u_i \otimes u_i\) (\(V\) is “superior”)
- \((u \lor v = \top\) and \(u \otimes v = \perp \Rightarrow u = \top\) or \(v = \top\)) (\(V\) is “lean”)

Okay for \(V = 2, [0, \infty]^{op}\), or any linearly ordered frame, but not for \(V = 2^M\)
Let’s do Topology!

\((X, a)\) Hausdorff: \(a \cdot a^\circ \leq 1_X\) \((\bot < a(\delta, x) \otimes a(\delta, y) \Rightarrow x = y)\)

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Compact + Hausdorff is algebraic

\[
\begin{array}{cccc}
T & V & (T, V)\text{-Cat}_{\text{Comp}} & (T, V)\text{-Cat}_{\text{Haus}} \\
\text{Id} & 2 & \text{Ord} & \text{discrete ordered sets} \\
\text{Id} & [0, \infty]^{\text{op}} & \text{Met} & \text{discrete (generalized) metric spaces} \\
\beta & 2 & \text{Comp} & \text{Haus} \\
\beta & [0, \infty]^{\text{op}} & \text{App}_{0}\text{-Comp} & \text{approach spaces whose induced pseudotopology is Hausdorff}
\end{array}
\]

Manes’ Theorem generalized:
\[
(T, V)\text{-Cat}_{\text{CompHaus}} = (T, V)\text{-Cat}_{\text{Comp}} \cap (T, V)\text{-Cat}_{\text{Haus}} = \text{Set}^T \\
= \text{Eilenberg-Moore algebras w.r.t. } T
\]

Proof (Lawvere, Clementino-Hofmann)
\[
(a \cdot a^\circ \leq 1_X \text{ and } 1_{TX} \leq a^\circ \cdot a) \iff a \dashv a^\circ \iff a \text{ is (induced by) a map.}
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Compact + Hausdorff is algebraic

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**Tychonoff’s Theorem**

\(V \text{ completely distributive}
(\forall i \in I: X_i = (X_i, a_i) \text{ compact}) \Rightarrow (X, a) = \prod_{i \in I} X_i \text{ compact}

**Proof**  (Schubert 2005)  For all \(z \in TX:

\[
\bigvee_{x \in X} a(z, x) = \bigvee_{x \in X} \bigwedge_{i \in I} a_i(Tp_i(z), p_i(x)) = \bigwedge_{i \in I} \bigvee_{x_i \in X_i} a_i(Tp_i(z), x_i) \geq k
\]
Equationally-def'd properties cont’d: T1, core-compact

$(X, a : TX \to X)$

- $1_X \leq a \cdot e_X$

**T1:**

$(T = \beta, V = 2 :)$

$1_X \geq a \cdot e_X$

$(\dot{x} \to y \Rightarrow x = y)$

- $a \cdot Ta \leq a \cdot m_X$

**core compact:**

$a \cdot Ta \geq a \cdot m_X$

$(T = \beta, V = 2 :)$

$(\Sigma X \to z \Rightarrow \exists \eta : X \to \eta \to z)$

$\Leftrightarrow \forall x \in B \subseteq X \text{ open}$

$\exists A \subseteq X \text{ open (} x \in A << B)$

$\Leftrightarrow X \text{ exponentiable in } \text{Top}$

$\Leftrightarrow \forall Y \exists Y^X \forall Z \exists \text{ nat. bij. corr.}$

$(Z \to Y^X \Leftrightarrow Z \times X \to Y)$
Equationally-def'd properties cont’d: T1, core-compact

\[(X, a : TX \leftrightarrow X)\]

- \(1_X \leq a \cdot e_X\)

**T1:**

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\[a \cdot Ta \geq a \cdot m_X\]

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\(\exists A \subseteq X \text{ open} \quad (x \in A \ll B)\)

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Equationally-def’d properties cont’d: T1, core-compact

\[(X, a : TX \leftrightarrow X)\]

- \(1_X \leq a \cdot e_X\)
- \(a \cdot Ta \leq a \cdot m_X\)

\[\text{T1:} \quad 1_X \geq a \cdot e_X\]

\((T = \beta, V = 2 :)\)

\((\dot{x} \rightarrow y \Rightarrow x = y)\)

\[\text{core compact:} \quad a \cdot Ta \geq a \cdot m_X\]

\((T = \beta, V = 2 :)\)

\((\Sigma x \rightarrow z \Rightarrow \exists \eta : X \rightarrow \eta \rightarrow z)\)

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\[\Leftrightarrow \forall Y \exists Y^X \forall Z \exists \text{ nat. bij. corr.} \quad (z \rightarrow Y^X \Leftrightarrow Z \times X \rightarrow Y)\]
Equationally-def’d properties cont’d: $T_1$, core-compact

$$(X, a : TX \leftrightarrow X)$$

- $1_X \leq a \cdot e_X$

**$T_1$:**

$$1_X \geq a \cdot e_X$$

($T = \beta, V = 2$ :)

$$(x \rightarrow y \Rightarrow x = y)$$

- $a \cdot Ta \leq a \cdot m_X$

**core compact:**

$$a \cdot Ta \geq a \cdot m_X$$

($T = \beta, V = 2$ :)

$$(\Sigma X \rightarrow z \Rightarrow \exists \eta : X \rightarrow \eta \rightarrow z)$$

$\Leftrightarrow \forall x \in B \subseteq X \text{ open}$

$$\exists A \subseteq X \text{ open} (x \in A \ll B)$$

$\Leftrightarrow X \text{ exponentiable in } \textbf{Top}$

$\Leftrightarrow \forall Y \exists Y^X \forall Z \exists \text{ nat. bij. corr.}$

$$(Z \rightarrow Y^X \Leftrightarrow Z \times X \rightarrow Y)$$
Normal, extremally disconnected

Preparation: Induced “order”

on $\beta X$:

$$\begin{align*}
\text{Top} & \to \text{Ord} \\
X & \mapsto (\beta X, \leq)
\end{align*}$$

$\tau \leq \eta :\iff \forall A \subseteq X \text{ closed} \\
(A \in \tau \Rightarrow A \in \eta)$

on $TX$:

$$\begin{align*}
(\text{T, V)}\text{-Cat} & \to \text{V-Cat} \\
(X, a) & \mapsto (TX, \hat{a})
\end{align*}$$

$\hat{a} = (TX \xrightarrow{m^\circ X} TTX \xrightarrow{Ta} TX)$

“Adjoint significance” of $\leq$:

$$\begin{align*}
\text{Top} & \to \text{OrdCompHaus} \\
(T, V)\text{-Cat} & \to \text{V-Cat}^T \\
(X, a) & \mapsto (TX, \hat{a}, m^X)
\end{align*}$$
Normal, extremally disconnected

Preparation: Induced “order”

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\[ X \mapsto (\beta X, \leq) \]
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“Adjoint significance” of \( \leq \):
\[ \text{Top} \to \text{OrdCompHaus} \]
\[ (T, V)\text{-Cat} \to V\text{-Cat} \]
\[ (X, a) \mapsto (TX, \hat{a}) \]
\[ \hat{a} = (TX \xrightarrow{m_X^{\circ}} TTX \xrightarrow{T^a} TX) \]

\[ \text{X} \leq \text{Y} : \iff \forall A \subseteq X \text{ closed} \]
\[ (A \in \tau \Rightarrow A \in \eta) \]
Normal, extremely disconnected

\( X \in \text{Top} \) normal \( \iff \) \( (X, a) \in (T, V) \)-\textbf{Cat} normal

\[ \forall x \exists y \exists z \exists w : \iff \hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a} \]

\( X \) extrem’ly disconnected \( \iff \) \( (X, a) \) extremally disconnected

\[ \forall x \exists y \exists z \exists w : \iff \hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ \]

\[ \iff (TX, \hat{a}) \text{ ext. disc. } V\text{-space} \]
\[ \iff (TX, \hat{a}^\circ) \text{ normal } V\text{-space} \]

Note: \((X, a)\) compact Hausdorff \( \Rightarrow \) \((X, a)\) normal
Normal, extremally disconnected

\[ X \in \text{Top} \text{ normal} \iff (X, a) \in (T, V)\text{-Cat} \text{ normal} \]

\[ \forall \, r \in \mathcal{R} \, \exists \, r' \in \mathcal{R} \, \eta \quad :\iff \hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a} \]

\[ X \text{ extrem'ly disconnected} \iff (X, a) \text{ extremally disconnected} \]

\[ \forall \, r \in \mathcal{R} \, \exists \, r' \in \mathcal{R} \, \eta \quad :\iff \hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ \]

\[ \iff (TX, \hat{a}) \text{ ext. disc. } V\text{-space} \]
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Note: \((X, a) \text{ compact Hausdorff} \Rightarrow (X, a) \text{ normal}\)
Normal, extremally disconnected

\[ X \in \mathbf{Top} \text{ normal} \iff (X, a) \in (T, V)\text{-Cat} \text{ normal} \]

\[ \forall x \in X \exists y \in Y : \hat{a} \cdot \hat{a}^\circ \leq \hat{a}^\circ \cdot \hat{a} \]

\[ X \text{ extrem'ly disconnected} \iff (X, a) \text{ extremally disconnected} \]

\[ \forall x \in X \exists y \in Y : \hat{a}^\circ \cdot \hat{a} \leq \hat{a} \cdot \hat{a}^\circ \]

\[ \iff (TX, \hat{a}) \text{ ext. disc. } V\text{-space} \]
\[ \iff (TX, \hat{a}^\circ) \text{ normal } V\text{-space} \]

Note: \((X, a)\) compact Hausdorff \(\Rightarrow (X, a)\) normal
The categorical imperative: What about morphisms?

\[ f : (X, a) \rightarrow (Y, b) \]

- \( f \cdot a \leq b \cdot Tf \quad \text{f proper} : \iff \quad f \cdot a \geq b \cdot Tf \)
- \( a \cdot (Tf)^\circ \leq f^\circ \cdot b \quad \text{f open} : \iff \quad a \cdot (Tf)^\circ \geq f^\circ \cdot b \)

\[ f : X \rightarrow Y \]

\[ \begin{array}{ccc}
\text{proper} & \text{open} \\
x \leq z & z \leq x \\
f(x) \leq y & y \leq f(x)
\end{array} \]

\[ \text{Ord} = 2\text{-Cat} \]

\[ \text{Top (\( \beta, 2 \))-Cat} \]

\[ \begin{array}{ccc}
\bar{x} \rightarrow Z & \bar{z} \rightarrow X \\
f[\bar{x}] \rightarrow y & \eta \rightarrow f(x)
\end{array} \]
### The Categorical Imperative: What about morphisms?

The categorical imperative: What about morphisms?

\[ f : (X, a) \rightarrow (Y, b) \]

- \( f \cdot a \leq b \cdot Tf \) \( f \) proper \( \iff \) \( f \cdot a \geq b \cdot Tf \)
- \( a \cdot (Tf)^\circ \leq f^\circ \cdot b \) \( f \) open \( \iff \) \( a \cdot (Tf)^\circ \geq f^\circ \cdot b \)

### Proper Functionality

\[ f : X \rightarrow Y \]

- Proper: \( x \leq z \)
- Open: \( z \leq x \)

\[ \text{Ord} = 2\text{-Cat} \]

- \( f(x) \leq y \)
- \( y \leq f(x) \)

### Topological Relations

- \[ \text{Top} \; (\beta, 2)\text{-Cat} \]
  - \( f[\xi] \rightarrow y \)
  - \( f(\eta) \rightarrow f(x) \)
Basic Stability Properties for proper/open maps

- Isomorphisms are proper/open
- proper/open maps are closed under composition
- \( g \cdot f \) proper/open, \( g \) injective if and only if \( f \) proper/open
- \( g \cdot f \) proper open, \( f \) surjective if and only if \( g \) proper/open

In addition: Proper/open is stable under pullback:

\[
\begin{array}{ccc}
X \times_Y Z & \xrightarrow{p_2} & Z \\
\downarrow p_1 & & \downarrow g \\
X & \xrightarrow{f} & Y
\end{array}
\]

\( f \) proper/open if and only if \( p_2 \) proper/open
Kuratowski-Mrówka Theorem

Under mild hypotheses on $T$, $V$:
$$(X, a) \rightarrow 1 \text{ proper } \iff (X, a) \text{ compact}$$

Theorem (Clementino-T 2007)
$$f : (X, a) \rightarrow (Y, b) \text{ proper } \iff$$
- $f$ has compact fibres
- $Tf : (X, \hat{a}) \rightarrow (Y, \hat{b}) \text{ proper}$

(in Top, App, ...)
$$\iff$$
- $f$ has compact fibres
- $f$ is closed
$$\iff f \text{ is stably closed}$$

Corollary
- $X$ compact
  $$\iff \forall Z : X \times Z \rightarrow Z \text{ closed (equ’ly: proper)}$$
- $(X \xrightarrow{f} Y)$ proper
  $$\iff \forall (Z \rightarrow Y) : (X \times_Y Z \rightarrow Z) \text{ closed (proper)}$$
Kuratowski-Mrówka Theorem

Under mild hypotheses on $T$, $V$:

$(X, a) \rightarrow 1$ proper $\iff (X, a)$ compact

Theorem (Clementino-T 2007)

$f : (X, a) \rightarrow (Y, b)$ proper $\iff$

- $f$ has compact fibres
- $Tf : (X, \hat{a}) \rightarrow (Y, \hat{b})$ proper

(in Top, App, ...)

$\iff$

- $f$ has compact fibres
- $f$ is closed

$\iff f$ is stably closed

Corollary

- $X$ compact $\iff \forall Z : X \times Z \rightarrow Z$ closed (equ’ly: proper)
- $(X \xrightarrow{f} Y)$ proper $\iff \forall (Z \rightarrow Y) : (X \times_Y Z \rightarrow Z)$ closed (proper)
Kuratowski-Mrówka Theorem

Under mild hypotheses on $T, V$:

$$(X, a) \to 1 \text{ proper} \iff (X, a) \text{ compact}$$

Theorem (Clementino-T 2007)

\[ f : (X, a) \to (Y, b) \text{ proper} \iff \]

\[ \begin{align*}
& \bullet \ f \text{ has compact fibres} \\
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(in Top, App, ...)

\[ \iff \]

\[ \begin{align*}
& \bullet \ f \text{ has compact fibres} \\
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& \iff f \text{ is stably closed} \\
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Corollary

\[ \begin{align*}
& \bullet \ X \text{ compact} \iff \forall Z : X \times Z \to Z \text{ closed (equ’ly: proper)} \\
& \bullet \ (X \overset{f}{\to} Y) \text{ proper} \iff \forall (Z \to Y) : (X \times_Y Z \to Z) \text{ closed (proper)} \\
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**Kuratowski-Mrówka Theorem**

Under mild hypotheses on $T$, $V$:

$$(X, a) \to 1 \text{ proper} \iff (X, a) \text{ compact}$$

**Theorem (Clementino-T 2007)**

$f : (X, a) \to (Y, b) \text{ proper} \iff$

- $f$ has compact fibres
- $Tf : (X, \hat{a}) \to (Y, \hat{b}) \text{ proper}$

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$\iff$

- $f$ has compact fibres
- $f$ is closed

$\iff$ $f$ is *stably* closed

**Corollary**

- $X$ compact $\iff \forall Z : X \times Z \to Z \text{ closed (equ’ly: proper)}$
- $(X \overset{f}{\to} Y) \text{ proper} \iff \forall (Z \to Y) : (X \times_Y Z \to Z) \text{ closed (proper)}$
Tychonoff-Frolík-Bourbaki Theorem

Conclusion: proper = fibred version of compact
Consequently: categorically proven statements for compact objects
transfer to proper morphisms, and conversely.

Theorem: $V$ completely distributive. Then:

$$f_i : X_i \rightarrow Y_i \text{ proper } (i \in I) \Rightarrow \prod_{i \in I} f_i : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i \text{ proper}$$

Note, by contrast (not by categorical dualization!):

$$f_i : X_i \rightarrow Y_i \text{ open } (i \in I) \Rightarrow \bigsqcup_{i \in I} f_i : \bigsqcup_{i \in I} X_i \rightarrow \bigsqcup_{i \in I} Y_i \text{ open}$$
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Some Remarks

Starting with an axiomatically given class of “closed morphisms” one establishes a categorical theory of compactness and Hausdorff separation:

Pénon 1972, T 1999, Clementino-Giuli-T 2004, Clementino-Colebunders-T 2014, ...

Recently this approach has been exploited for the category $\text{TopGrp}$ of topological groups by He-T, extending the Dikranjan-Uspenskij product theorem for categorically compact groups to *categorically proper* homomorphisms of topological groups.
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Q1: Do we really need the $V$ of $(T, V)$-$\text{Cat}$?

YES:
All topological notions presented depend on $T, V$, not just on $(T, V)$-$\text{Cat}$. This is so already for $\textbf{Top}$ when presented via filter convergence instead of ultrafilter convergence! But

NO:
It is possible to always replace $V$ by 2 (i.e., have no “fuzziness”!) if

• you are only interested in the category itself and
• you accept a more complicated $T$:

Theorem (Hofmann-Lowen 2014)
Given $T, V$, there is a monad $\Pi = \Pi(T, V)$ such that

$$(T, V)$-$\text{Cat} \cong (\Pi, 2)$-$\text{Cat}$$

Special case: $T = \text{Id} \Rightarrow \Pi = P_V = V$-presheaf monad:

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Q2: Should one consider a quantaloid $Q$ instead of $V$?

YES

First indication:
Take $Q = DV$ (Stubbe, Zhang, ...) and obtain:

$$(T, Q)\text{-Cat} = \{\text{partial } (T, V)\text{-spaces}\}$$

In particular:

$$D[0, \infty]^\text{op} \text{-Cat} = \{\text{partial metric spaces}\}$$

(T 2015: AMS-Portugal Meeting, Porto, 2015)
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Your questions?

THANK YOU!