Approximate Composition – Another Approach to Quantitative Concept Analysis?

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Searching Toronto to Rome: direct flights

YYZ

AC890

AZ651

FCO

.... and many more!
Searching Toronto to Rome: connecting flights

... and many more!

YYZ \rightarrow AC890 \rightarrow FCO

YYZ \rightarrow AZ651 \rightarrow NYC

NYC \rightarrow LHR

LHR \rightarrow FRA

FRA \rightarrow AMS

AMS \rightarrow FCO

... and many more!
Basic structure: metric(ally enriched) graph $\mathbb{X}$

- $\text{ob}\mathbb{X}$: vertices, objects $x, y, z, \ldots$ ("airports")
- $\mathbb{X}(x, y)$: edges, morphisms $f, f', \ldots : x \rightarrow y$ ("direct flights $x \rightarrow y$")
- $1_x : x \rightarrow x$: zero loop, identity morphism ("staying grounded at $x$")
- $d = d_{x,y}$ symmetric Lawvere metric ("price difference") on $\mathbb{X}(x, y)$:
  - $d(f, f) = 0$
  - $d(f, f') = d(f', f)$
  - $d(f, f'') \leq d(f, f') + d(f', f'')$

Note: for $f \neq f'$, $d(f, f') = 0$ or $d(f, f') = \infty$ are permitted.

Major shortcoming:
No comparison between direct and connecting flights!
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Richer structure: **metric approximate category** \( \mathbb{X} \)

- \( \text{ob} \mathbb{X}, \mathbb{X}(x, y), \ 1_x \in \mathbb{X}(x, x) \)
- “comparator” \( \delta = \delta_{x,y,z} : \mathbb{X}(x, y) \times \mathbb{X}(y, z) \times \mathbb{X}(x, z) \rightarrow [0, \infty] \)
  - \( \delta(f, 1_y, f) = 0 = \delta(1_x, f, f) \)
  - \( |\delta(a, h, c) - \delta(f, b, c)| \leq \delta(f, g, a) + \delta(g, h, b) \)

Geometrically: \( \delta \) represents area, volume, content, ...
Very brief tributes

- Karl Menger 1928: n-metrics, simplex inequality
- Siegfried Gähler 1963: 2-metrics, tetrahedral inequality
- Sammy Eilenberg and Max Kelly 1966: enriched categories
- Bill Lawvere 1973: distances as homs, triangle inequality as composition law, metrically enriched categories
- Abdelkrim Aliouche and Carlos Simpson 2017: approximate categorical structures, “directed tetrahedral” inequalities
- WT and Jiyu (Gates) Wang 2019: (quantalic generalization of) metagories, Yoneda embedding for metagories
- WT 2019 (hopefully): approximate 2-categories
Even richer structure: metric(ally enriched) category $\mathbb{X}$

- category $\mathbb{X}$ with underlying metric graph
- composition $\mathbb{X}(x, y) \times \mathbb{X}(y, z) \rightarrow \mathbb{X}(x, z)$ is contractive:

$$d(g \cdot f, g' \cdot f') \leq d(f, f') + d(g, g')$$

Met-Cat $\rightarrow$ Metag: $\delta(f, g, a) := d(g \cdot f, a)$

Metag $\rightarrow$ Met-Gph: $d(f, f') := \delta(f, 1_y, f') = \delta(1_x, f, f')$
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**Met-Cat \to Metag:** \( \delta(f, g, a) := d(g \cdot f, a) \)

**Metag \to Met-Gph:** \( d(f, f') := \delta(f, 1_y, f') = \delta(1_x, f, f') \)
Every (general) 2-metric space \((X, d)\) gives a chaotic metagory \(\underline{X}\):

\[
\text{ob}\underline{X} = X, \quad |\underline{X}(x, y)| = 1, \quad \delta(x \to y, y \to z, x \to z) := d(x, y, z).
\]

In particular: \(\mathbb{R}^n\) with its Euclidean 2-metric gives the metagory \(\mathbb{R}^n\).

We already saw: every metric category \(\underline{X}\) is a metagory.

In particular: \(\text{Met}, \text{Ban}_1, \ldots, \) are (large) metagories.

The first example describes a full embedding \(2\text{Met} \to \text{Metag}\), with reflector

\[
(X, \delta) \mapsto (\text{ob}X, d),
\]

\[
d(x, y, z) = \inf\{\delta(f, g, a) \mid f : x \to y, g : y \to z, a : x \to z\}.
\]
Examples

- Every (general) 2-metric space \((X, d)\) gives a chaotic metagory \(\mathbb{X}\):

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  \text{ob}\mathbb{X} = X, \quad |\mathbb{X}(x, y)| = 1, \quad \delta(x \to y, y \to z, x \to z) := d(x, y, z).
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\]
The quantalic view

\( \mathcal{V} = (\mathcal{V}, \leq, \otimes, k) \) commutative unital quantale, replacing Lawvere’s

\(([0, \infty], \geq, +, 0) \cong ([0, 1], \leq, \cdot, 1) : (\mathcal{V}, \leq) \) complete lattice, \((\mathcal{V}, \otimes, k) \) commutative monoid,

\[ u \otimes \bigvee_{i \in I} v_i = \bigvee_{i \in I} u \otimes v_i. \]

Examples of principal interest:

- Lawvere quantale
- \( 2 = (\{0, 1\}, \leq, \wedge, 1) \)
- \( \Delta = \{ \text{probability distribution functions} \ [0, \infty] \to [0, 1] \} \)
- \( = [0, \infty] \oplus [0, 1] \) in the cat. of comm. unital quantales
The quantalic view

\( \mathcal{V} = (\mathcal{V}, \leq, \otimes, k) \) commutative unital \textit{quantale}, replacing Lawvere’s

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Tensor, internal hom

- \mathcal{V} \text{ symmetric monoidal closed}
- \mathcal{V}\text{-Cat symmetric monoidal closed}
- \textbf{Met}_{\mathcal{V}} \text{ symmetric monoidal closed}
- \textbf{Met}_{\mathcal{V}}\text{-Gph symmetric monoidal closed}
- \textbf{Met}_{\mathcal{V}}\text{-Cat symmetric monoidal closed}

\[
d_{X \otimes Y}((f, g), (f', g')) = d_X(f, f') \otimes d_Y(g, g')
\]

\((f, f' : x \to y \text{ in } X \text{ and } g, g' : z \to w \text{ in } Y)\);

\[
d_{[X, Y]}(\alpha, \alpha') = \bigwedge_{x \in \text{ob} X} d_Y(\alpha_x, \alpha'_x)
\]

\((\alpha, \alpha' : F \to G \text{ nat. transf. of } \mathcal{V}\text{-contractive functors } F, G : X \to Y)\).
\( \mathcal{V} \) symmetric monoidal closed

\( \mathcal{V}\text{-Cat} \) symmetric monoidal closed

\( \text{Met}_\mathcal{V} \) symmetric monoidal closed

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d_{X \otimes Y}((f, g), (f', g')) = d_X(f, f') \otimes d_Y(g, g')
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\]

\((\alpha, \alpha': F \to G \text{ nat. transf. of } \mathcal{V}\text{-contractive functors } F, G: X \to Y)\).
What about $\text{Metag}_\mathcal{V}$?

- objects: (small) $\mathcal{V}$-metagories $\mathcal{X}, \mathcal{Y}, ...$
- morphisms: contractors $F : \mathcal{X} \to \mathcal{Y}$
- natural transformations $\alpha : F \to G$

\[
\begin{array}{ccc}
Fx & \xrightarrow{Ff} & Fy \\
\downarrow \alpha_x & & \downarrow \alpha_y \\
Gx & \xrightarrow{Gf} & Gy
\end{array}
\]

$k \leq \delta(Ff, \alpha_y, \alpha_f)$ and $k \leq \delta(\alpha_x, Gf, \alpha_f)$.

**Theorem:** $\text{Metag}_\mathcal{V}$ is symmetric monoidal closed!

\[1_F = (Ff)_{f: \mathcal{X} \to \mathcal{Y}}, \quad \delta_{\mathcal{X}, \mathcal{Y}}(\alpha, \beta, \gamma) = \bigwedge_{x \in \text{ob} \mathcal{X}} \delta_{\mathcal{Y}}(\alpha_x, \beta_x, \gamma_x)\]
What about $\text{Metag}_\mathcal{V}$?

- objects: (small) $\mathcal{V}$-metagories $X, Y, \ldots$
- morphisms: contractors $F : X \to Y$
- natural transformations $\alpha : F \Rightarrow G$

$$
\begin{array}{ccc}
Fx & \xrightarrow{Ff} & Fy \\
\downarrow{\alpha_x} & \alpha_f & \downarrow{\alpha_y} \\
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\end{array}
$$

$$k \leq \delta(Ff, \alpha_Y, \alpha_f) \quad \text{and} \quad k \leq \delta(\alpha_X, Gf, \alpha_f).$$

**Theorem:** $\text{Metag}_\mathcal{V}$ is symmetric monoidal closed!

$$1_F = (Ff)_{f : x \to y}, \quad \delta_{[X,Y]}(\alpha, \beta, \gamma) = \bigwedge_{x \in \text{ob } X} \delta_Y(\alpha_x, \beta_x, \gamma_x)$$
**Self-enrichment**

- **Cat** is a 2-category, *i.e.*, **Cat** is enriched in **Cat**.
- **Met\(_\mathcal{V}\)-Cat** is enriched in **Met\(_\mathcal{V}\)-Cat**:

\[
c_{x,y,z} : [x, y] \otimes [y, z] \to [x, z],
\]

\[
(\varphi : F \to G, \psi : H \to J) \mapsto (\psi \circ \varphi : HF \to JG)
\]

is \(\mathcal{V}\)-contractive!
Again: what about \textbf{Metag}_\mathcal{V}? 

\[
\begin{array}{c}
\begin{tikzpicture}
    \node (HF) at (0, 1) {$HF$};
    \node (HG) at (1, 1) {$HG$};
    \node (JF) at (0, 0) {$JF$};
    \node (JG) at (1, 0) {$JG$};
    \node (HFx) at (0, -1) {$HFx$};
    \node (HGy) at (1, -1) {$HGy$};
    \node (JFx) at (0, -2) {$JFx$};
    \node (JGy) at (1, -2) {$JGy$};
    \draw[->,double] (HF) to node [above] {$H\varphi$} (HG);
    \draw[->] (HF) to node [left] {$\psi F$} (JF);
    \draw[->] (HF) to node [right] {$\psi G$} (HG);
    \draw[->] (JF) to node [below] {$J\varphi$} (JG);
    \draw[->,double] (HFx) to node [above] {$H\varphi_f$} (HGy);
    \draw[->] (HFx) to node [left] {$\psi_{Fx}$} (JFx);
    \draw[->] (HFx) to node [right] {$\psi_{Gy}$} (HGy);
    \draw[->] (JFx) to node [below] {$J\varphi_f$} (JGy);
\end{tikzpicture}
\end{array}
\]

If $F, G, H, J$ are just contractors of $\mathcal{V}$-metagories, put

\[
(\psi \circ \varphi)_f := \psi_{\varphi_f} : HFx \longrightarrow JGy,
\]

for all $f : x \rightarrow y$ in $\mathcal{X}$.

**Theorem:** \textbf{Metag}_\mathcal{V} is enriched in \textbf{Metag}_\mathcal{V}!
\( \mathbb{V} \) as an internal monoid in \( \text{Met}_{\mathbb{V}} \)-\text{Cat} 

Internal hom:

\[
z \leq u \rightarrow v \iff z \otimes u \leq v
\]

\( \mathbb{V} \)-metric:

\[
d_{\mathbb{V}}(u, v) = (u \rightarrow v) \land (v \rightarrow u)
\]

\( \otimes : \mathbb{V} \otimes \mathbb{V} \rightarrow \mathbb{V} \) is \( \mathbb{V} \)-contractive:

\[
d_{\mathbb{V}}(u, v) \otimes d_{\mathbb{V}}(w, z) \leq d_{\mathbb{V}}(u \otimes w, v \otimes z)
\]

Strategy:

Expand these facts to \( \mathbb{V} \)-distributors (= \( \mathbb{V} \)-(bi)modules, \( \mathbb{V} \)-profunctors)!

Goal:

Embed a \( \mathbb{V} \)-metagory into a \( \mathbb{V} \)-metric category!
\( \mathcal{V} \) as an internal monoid in \( \text{Met}_{\mathcal{V}}\text{-Cat} \)

**Internal hom:**

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z \leq u \rightsquigarrow v \quad \iff \quad z \otimes u \leq v
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**\( \mathcal{V} \)-metric:**

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Expand these facts to \( \mathcal{V} \)-distributors (= \( \mathcal{V} \)-(bi)modules, \( \mathcal{V} \)-profunctors)!

**Goal:**

Embed a \( \mathcal{V} \)-metagory into a \( \mathcal{V} \)-metric category!
\( \mathcal{V}\)-distributors of \( \mathcal{V}\)-metric spaces

\( X, Y \) \( \mathcal{V}\)-metric spaces

\[ \varphi = (\varphi(x, y))_{x \in X, y \in Y} \quad \text{\( \mathcal{V}\)-distributor} \iff \varphi(x, y) \in \mathcal{V} \text{ with} \]

\[ \varphi(x, y) \otimes d_Y(y, y') \leq \varphi(x, y'), \quad d_X(x', x) \otimes \varphi(x, y) \leq \varphi(x', y) \]

Composition of \( \varphi : X \to Y \) followed by \( \psi : Y \to Z \):

\[ (\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \varphi(x, y) \otimes \psi(y, z) \]

The one-object category \( \mathcal{V} \) embeds fully into \( \text{Dist}_{\mathcal{V}} \):

\[ \mathcal{V} \to \text{Dist}_{\mathcal{V}}, \; \nu \mapsto (\nu : 1 \to 1) \]

What about its \( \mathcal{V}\)-metric structure?
$\mathcal{V}$-distributors of $\mathcal{V}$-metric spaces

$X, Y \quad \mathcal{V}$-metric spaces

$\varphi = (\varphi(x, y))_{x \in X, y \in Y} \quad \mathbf{\varphi}$-distributor $\iff \varphi(x, y) \in \mathcal{V}$ with

$\varphi(x, y) \otimes d_Y(y, y') \leq \varphi(x, y'), \quad d_X(x', x) \otimes \varphi(x, y) \leq \varphi(x', y)$

Composition of $\varphi : X \rightarrow Y$ followed by $\psi : Y \rightarrow Z$:

$(\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \varphi(x, y) \otimes \psi(y, z)$

The one-object category $\mathcal{V}$ embeds fully into $\mathbf{Dist}_\mathcal{V}$:

$\mathcal{V} \rightarrow \mathbf{Dist}_\mathcal{V}, \quad \nu \mapsto (\nu : I \rightarrow I)$

What about its $\mathcal{V}$-metric structure?
**Dist**\(\mathcal{V}\) as a **Met**\(\mathcal{V}\)-enriched category

\(\varphi, \varphi' : X \rightarrow Y:\)

\[
d(\varphi, \varphi') := \bigwedge_{x \in X, y \in Y} d_{\mathcal{V}}(\varphi(x, y), \varphi'(x, y))
\]

makes every \(\text{Dist}^\mathcal{V}(X, Y)\) a *separated* \(\mathcal{V}\)-metric space, that is

\[
d(\varphi, \varphi') \geq k \text{ only if } \varphi = \varphi'.
\]

**Theorem:**

For every (small) \(\mathcal{V}\)-metagory \(\mathbb{X}\), the \(\mathcal{V}\)-metagory \([\mathbb{X}^{\text{op}}, \text{Dist}^\mathcal{V}]\) is (induced by) a separated \(\mathcal{V}\)-metric category.

Now we got to embed \(\mathbb{X}\) into this \(\mathcal{V}\)-metric category! But:
Not every metagory may be isometrically embedded into a metric category! (Aliouche and Simpson)
Every $\mathcal{V}$-metric category is a transitive $\mathcal{V}$-metagory.

Transitivity is not hereditary (under isometric embeddings).

\[
\bigvee_{a:x \to z} \delta(f, g, a) \otimes \delta(a, h, c) = \bigvee_{b:y \to w} \delta(f, b, c) \otimes \delta(g, h, b),
\]
Transitive $\mathcal{V}$-metagories

Every $\mathcal{V}$-metric category is a transitive $\mathcal{V}$-metagory.

Transitivity is not hereditary (under isometric embeddings).
**Theorem**

Let \( \mathcal{X} \) be a transitive \( \mathcal{V} \)-metagory. Then there is an isometric \( \mathcal{V} \)-contractor

\[
y : \mathcal{X} \rightarrow \mathcal{[X^{\text{op}}, \text{Dist}_\mathcal{V}]}, \quad w \mapsto (y_w : \mathcal{X}^{\text{op}} \rightarrow \text{Dist}_\mathcal{V}, x \mapsto \mathcal{X}(x, w))
\]

mapping \( \mathcal{X} \) into a separated \( \mathcal{V} \)-metric category, as follows:

For every object \( w \) and \( f : x \rightarrow y \) in \( \mathcal{X} \), one has the \( \mathcal{V} \)-distributor

\[
y_w(f) : \mathcal{X}(y, w) \rightarrow \mathcal{X}(x, w), \quad y_w(f)(b, c) = \delta(f, b, c),
\]

for all \( b : y \rightarrow w, c : x \rightarrow w \); this gives the contractor \( y_w \);

for \( m : w \rightarrow v \) in \( \mathcal{X} \) one has the natural transf. \( y_m : y_w \rightarrow y_v \) with

\[
(y_m)_x : \mathcal{X}(x, w) \rightarrow \mathcal{X}(x, v), \quad (y_m)_x(c, e) = \delta(c, m, e)
\]

for all \( c : x \rightarrow w, e : x \rightarrow v \) in \( \mathcal{X} \).

\( y \) maps objects injectively; same for morphisms if \( \mathcal{X} \) is separated.
Theorem

$\mathbf{X}$ transitive $\mathcal{V}$-metagory. Then there is an isometric $\mathcal{V}$-contractor

$$y : \mathbf{X} \longrightarrow [\mathbf{X}^{\text{op}}, \textbf{Dist}_{\mathcal{V}}], \quad w \mapsto (y_w : \mathbf{X}^{\text{op}} \longrightarrow \textbf{Dist}_{\mathcal{V}}, x \mapsto \mathbb{X}(x, w))$$

mapping $\mathbf{X}$ into a separated $\mathcal{V}$-metric category, as follows:

For every object $w$ and $f : x \longrightarrow y$ in $\mathbf{X}$, one has the $\mathcal{V}$-distributor

$$y_w(f) : \mathbb{X}(y, w) \longrightarrow \mathbb{X}(x, w), \quad y_w(f)(b, c) = \delta(f, b, c),$$

for all $b : y \longrightarrow w$, $c : x \longrightarrow w$; this gives the contractor $y_w$;

for $m : w \longrightarrow v$ in $\mathbf{X}$ one has the natural transf. $y_m : y_w \longrightarrow y_v$ with

$$(y_m)_x : \mathbb{X}(x, w) \longrightarrow \mathbb{X}(x, v), \quad (y_m)_x(c, e) = \delta(c, m, e)$$

for all $c : x \longrightarrow w$, $e : x \longrightarrow v$ in $\mathbf{X}$.

$y$ maps objects injectively; same for morphisms if $\mathbf{X}$ is separated.
Comparison with the standard $\mathcal{V}$-Yoneda

If $\mathcal{X}$ is a $\mathcal{V}$-metric category:

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\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\tilde{y}} & [\mathcal{X}^{\text{op}}, \text{Met}_\mathcal{V}] \\
\mathcal{X} & \xrightarrow{y} & [\mathcal{X}^{\text{op}}, \text{Dist}_\mathcal{V}]
\end{array}
\]

Note: generally, $y$ is not full!
Corollary

The unit $X \rightarrow \text{Path}X$ of the right-adjoint functor $\text{Met}_\mathcal{V}\cdot\text{Cat} \rightarrow \text{Metag}_\mathcal{V}$ at the transitive $\mathcal{V}$-metagory $X$ is an isometry.

$$\begin{array}{c}
\text{Path}X \\
\downarrow \\
X \\
y \\
\downarrow \\
[X^{\text{op}}, \text{Dist}_\mathcal{V}].
\end{array}$$
Warning:
A 2-metric space, seen as a metagory, is generally not transitive!
In particular: $\mathbb{R}^n$ is not transitive!

Nevertheless:
The metagory Path $\mathbb{R}^n$ is metric!
For a \( \mathcal{V} \)-metagory \( \mathbb{X} \), consider:

(i) \( \mathbb{X} \) is (induced by) a \( \mathcal{V} \)-metric category;

(ii) for all \( f : x \to y, g : y \to z \) in \( \mathbb{X} \), there is \( a : x \to z \) in \( \mathbb{X} \) with \( k \leq \delta(f, g, a) \);

(iii) for all \( f : x \to y, g : y \to z \) in \( \mathbb{X} \), \( k \leq \bigvee_{a : x \to z} \delta(f, g, a) \otimes \delta(f, g, a) \);

(iv) \( \mathbb{X} \) is transitive.

Then (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv), and (i) \( \Leftrightarrow \) (ii) if \( \mathbb{X} \) is separated.

Furthermore, if \( \mathcal{V} \) satisfies \( \bigvee \{ \varepsilon \mid \varepsilon \ll k \} = k \), then (iii) is equivalent to

(iii') for all \( \varepsilon \ll k \) in \( \mathcal{V} \), \( f : x \to y, g : y \to z \) in \( \mathbb{X} \), there is \( a : x \to z \) in \( \mathbb{X} \) with \( \varepsilon \leq \delta(f, g, a) \otimes \delta(f, g, a) \).
Sufficient conditions for transitivity

For a $\mathcal{V}$-metagory $\mathbb{X}$, consider:

(i) $\mathbb{X}$ is (induced by) a $\mathcal{V}$-metric category;

(ii) for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $k \leq \delta(f, g, a)$;

(iii) for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, $k \leq \bigvee_{a : x \to z} \delta(f, g, a) \otimes \delta(f, g, a)$;

(iv) $\mathbb{X}$ is transitive.

Then (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv), and (i) $\iff$ (ii) if $\mathbb{X}$ is separated.

Furthermore, if $\mathcal{V}$ satisfies $\bigvee \{\varepsilon \mid \varepsilon \ll k\} = k$, then (iii) is equivalent to

(iii') for all $\varepsilon \ll k$ in $\mathcal{V}$, $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\varepsilon \leq \delta(f, g, a) \otimes \delta(f, g, a)$.
Let $\varepsilon \in \mathcal{V}$.

A $\mathcal{V}$-metagory $\mathbb{X}$ is $\varepsilon$-categorical if for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\varepsilon \leq \delta(f, g, a)$.

**Theorem:**
Assume that the quantale $\mathcal{V}$ satisfies $\bigvee \{ \varepsilon \otimes \varepsilon \mid \varepsilon \ll k \} = k$. Then

$$k\text{-}\text{Metag}_\mathcal{V} \subseteq \bigcap_{\varepsilon \ll k} \varepsilon\text{-}\text{Metag}_\mathcal{V} \subseteq \text{TransMetag}_\mathcal{V}.$$  

For $\mathcal{V}$-metagories $\mathbb{X}$ and $\mathbb{Y}$, if $\mathbb{Y}$ is $k$-categorical, so is $[\mathbb{X}, \mathbb{Y}]$. If both $\mathbb{X}$ and $\mathbb{Y}$ are $k$-categorical or transitive, $\mathbb{X} \otimes \mathbb{Y}$ has the respective property; likewise for the property of being $\varepsilon$-categorical for all $\varepsilon \ll k$. 
Let $\varepsilon \in \mathcal{V}$.

A $\mathcal{V}$-metagory $\mathcal{X}$ is $\varepsilon$-categorical if for all $f: x \to y$, $g: y \to z$ in $\mathcal{X}$, there is $a: x \to z$ in $\mathcal{X}$ with $\varepsilon \leq \delta(f, g, a)$.

**Theorem:**
Assume that the quantale $\mathcal{V}$ satisfies $\bigvee \{\varepsilon \otimes \varepsilon | \varepsilon \ll k\} = k$. Then

$$k\text{-Metag}_\mathcal{V} \subseteq \bigcap_{\varepsilon \ll k} \varepsilon\text{-Metag}_\mathcal{V} \subseteq \text{TransMetag}_\mathcal{V}.$$  

For $\mathcal{V}$-metagories $\mathcal{X}$ and $\mathcal{Y}$, if $\mathcal{Y}$ is $k$-categorical, so is $[\mathcal{X}, \mathcal{Y}]$. If both $\mathcal{X}$ and $\mathcal{Y}$ are $k$-categorical or transitive, $\mathcal{X} \otimes \mathcal{Y}$ has the respective property; likewise for the property of being $\varepsilon$-categorical for all $\varepsilon \ll k$. 

Walter Tholen (York University)

Approximate Composition

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Let $\varepsilon \in \mathcal{V}$.

A $\mathcal{V}$-metagory $\mathbb{X}$ is $\varepsilon$-categorical if for all $f : x \to y$, $g : y \to z$ in $\mathbb{X}$, there is $a : x \to z$ in $\mathbb{X}$ with $\varepsilon \leq \delta(f, g, a)$.

**Theorem:**
Assume that the quantale $\mathcal{V}$ satisfies $\bigvee \{ \varepsilon \otimes \varepsilon \mid \varepsilon \ll k \} = k$. Then

$$k\text{-Metag}_{\mathcal{V}} \subseteq \bigcap_{\varepsilon \ll k} \varepsilon\text{-Metag}_{\mathcal{V}} \subseteq \text{TransMetag}_{\mathcal{V}}.$$

For $\mathcal{V}$-metagories $\mathbb{X}$ and $\mathbb{Y}$, if $\mathbb{Y}$ is $k$-categorical, so is $[\mathbb{X}, \mathbb{Y}]$. If both $\mathbb{X}$ and $\mathbb{Y}$ are $k$-categorical or transitive, $\mathbb{X} \otimes \mathbb{Y}$ has the respective property; likewise for the property of being $\varepsilon$-categorical for all $\varepsilon \ll k$. 
Let $\mathcal{X}$ be a metagory given by airports, "sufficiently many" flights between them, and a comparator function for direct versus connecting flights.

Then $\text{Path}\mathcal{X}$ carries a \textit{metric} structure into which $\mathcal{X}$ is isometrically embedded as a submetagory.
Thanks!