

CT 2008

Injectivity, exponentiability,  
and completeness

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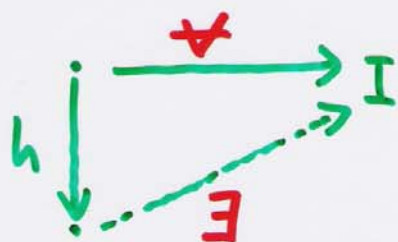
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## PLAN:

1. Generalities on injectivity and cogenerators
2. Case study: Nullstellensatz
3. Categorical versions of Birkhoff's and Banaschewski's Theorems
4. Injectivity = (co)completeness
5. Weak factorization systems and (weak) exponentiability, and an exercise in elementary linear algebra

The correspondence governing injectivity:

$\mathcal{C} : \quad \mathcal{H} \triangleright I \quad \mathcal{H} \triangleright J$



"I  $\mathcal{H}$ -injective"

"h  $J$ -extendable"

Maranda 1965

Examples:

$\mathcal{C} = \underline{\text{Set}}$      $\mathcal{H} = \text{Mono}$      $\mathcal{H}$ -inj. = non-empty

$\mathcal{C} = \underline{\text{Ord}}$      $\mathcal{H} = \text{Reg Mono}$     " = complete

$\mathcal{C} = \underline{\text{Fld}}$      $\mathcal{H} = \{\text{alg. exts}\}$     " = alg'ly closed

$\mathcal{C} = \underline{\text{Met}}$      $\mathcal{H} = \{\text{dense isom.}\}$     " = Gandy compl.

⋮

- 1) Characterization Problem  
2) Embeddability Problem

Being fixed under the correspondence:

Assume 1)  $\mathcal{H} \triangleright \mathcal{J}$

2)  $\forall X \exists \eta_X: X \rightarrow TX$  with  
 $\eta_X \in \mathcal{H}, TX \in \mathcal{J}$

Then: a)  $\mathcal{H} = \triangleright \mathcal{J} = \triangleright \{TX \mid X \in \mathcal{C}\}$

iff  $\mathcal{H}$  is left cancellable  
( $gh \in \mathcal{H} \Rightarrow h \in \mathcal{H}$ )

b)  $\mathcal{J} = \mathcal{H} \triangleright = \{\eta_X \mid X \in \mathcal{C}\} \triangleright$

iff  $\mathcal{J}$  is closed under retracts

( $I \xrightleftharpoons[r]{i} J \in \mathcal{J} \Rightarrow I \in \mathcal{J}$ )