CT 2008

Injectivity, exponentiability, and completeness

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PLAN:

1. Generalities on injectivity and cogenerators
2. Case study: Nullstellensatz
3. Categorical versions of Birkhoff's and Banaschewski's Theorems
4. Injectivity = (co)completeness
5. Weak factorization systems and (weak) exponentiability, and an exercise in elementary linear algebra

The correspondence governing injectivity:

$\mathcal{C}$: $\quad h \triangleright I \qquad \mathcal{H} \triangleright \mathcal{J}$

$$\begin{tikzcd} \bullet \arrow[r, "\forall"] \arrow[d, "h"'] & I \\ \bullet \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

"I $\mathcal{H}$-injective"

"h $\mathcal{J}$-extendable"

Maranda 1965

Examples:

$\mathcal{C} = \underline{\text{Set}}$ $\quad \mathcal{H} = \text{Mono}$ $\quad \mathcal{H}$-inj. = non-empty

$\mathcal{C} = \underline{\text{Ord}}$ $\quad \mathcal{H} = \text{RegMono}$ $\quad$ " = complete

$\mathcal{C} = \underline{\text{Fld}}$ $\quad \mathcal{H} = \{\text{alg. exts}\}$ $\quad$ " = alg'ly closed

$\mathcal{C} = \underline{\text{Met}}$ $\quad \mathcal{H} = \{\text{dense isom.}\}$ $\quad$ " = Cauchy compl.

$\vdots$

$\rightsquigarrow$ 1) Characterization Problem

2) Embeddability Problem

Being fixed under the correspondence:

Assume 1) $\mathcal{H} \boxslash \mathcal{J}$

2) $\forall X \; \exists \eta_X : X \to TX$ with

$\eta_X \in \mathcal{H}, TX \in \mathcal{J}$

Then: a) $\mathcal{H} = {}^{\boxslash}\mathcal{J} = {}^{\boxslash}\{TX \mid X \in \mathcal{C}\}$

iff $\mathcal{H}$ is left cancellable

$(gh \in \mathcal{H} \Rightarrow h \in \mathcal{H})$

b) $\mathcal{J} = \mathcal{H}^{\boxslash} = \{\eta_X \mid X \in \mathcal{C}\}^{\boxslash}$

iff $\mathcal{J}$ is closed under retracts

$(I \underset{r}{\overset{i}{\rightleftarrows}} J \in \mathcal{J} \Rightarrow I \in \mathcal{J})$