A natural closure for U-$\underline{\mathrm{Cat}}$

$$1_X^* \le f^* \cdot f_* \quad , \quad f_* \cdot f^* \le 1_Y^*$$

f fully faithful $\iff 1_X^* = f^* \cdot f_*$ $\qquad a(x,x') = b(f(x), f(x'))$

f L-dense $\qquad \iff f_* \cdot f^* = 1_Y^*$

$\qquad\qquad\qquad \iff (g \cdot f = h \cdot f \Rightarrow g \cong h)$

"epi up to iso"

$M \subseteq X$

$\overline{M}$ = least subobject of X in which M is L-dense

$$= \{ y \in X \mid k \le \bigvee_{x \in M} b(y,x) \otimes b(x,y) \}$$

L-separation, L-completion

X L-separated $\Leftrightarrow \Delta_X \subseteq X \times X$ L-closed

$\Leftrightarrow \forall g, h : P \to X$
$(g \cong h \Rightarrow g = h)$

$\Leftrightarrow y : X \to \hat{X} = \mathcal{V}^{X^{op}}$ is 1-1

Lemma: $\mathcal{V} = (V, \multimap)$ is L-separated
and {fully faithful}-injective,
in particular: $\underbrace{\{\text{fff}, \text{L-dense}\}\text{-}\overset{\text{pseudo}}{\wedge}\text{injective}}_{\text{"L-injective"}}$

Naturally:

$$y : X \longrightarrow \tilde{X} := \overline{y(X)} \subseteq \hat{X} = \mathcal{V}^{X^{op}}$$

should be a "completion"

Theorem (Lawvere 1973,
Clementino, Hofmann, Stubbe, T 04-08)

Equivalent are for a $\mathcal{V}$-category X:

(i) X L-complete (every $\varphi \dashv \psi : X \nrightarrow Y$ in $\mathcal{V}$-Mod is representable as $f_* \dashv f^*$ for some f).

(ii) X L-injective.

(iii) X is a (pseudo-) retract of $\tilde{X}$.

(iv) $y : X \to \tilde{X}$ is onto.

For a considerable* generalization of this theorem $\to$ Hofmann's talk

*) • $\mathcal{V} \rightsquigarrow \mathcal{T}$ "topological theory"
• relativized injectivity

Reflection, inj. hull, cogenerator:

- $\mathcal{V}\text{-}\underline{\mathrm{Cat}}_{cpl} \hookrightarrow \mathcal{V}\text{-}\underline{\mathrm{Cat}}_{sep}$ reflective, reflection:

  $y: X \longrightarrow \tilde{X}$ L-dense full embedding

- y is actually {L-dense full emb.}-inj. hull

  Note: Functorial $\mathcal{H}$-inj. hulls are rare since:

  $$\left.\begin{array}{l} \eta: 1 \to T \text{ ptw. } \mathcal{H}\text{-inj. hull} \\ \text{and ptw. monic} \end{array}\right\} \Rightarrow \eta \text{ ptw. epic}$$

- $X \in \mathcal{V}\text{-}\underline{\mathrm{Cat}}_{cpl} \Leftrightarrow \exists\, X \underset{\text{full L-closed emb.}}{\rightarrowtail} \prod_I \mathcal{V}$