

## A natural closure for $\underline{\text{U-Cat}}$

$$l_X^* \leq f^* \cdot f_* \quad , \quad f_* \cdot f^* \leq l_Y^*$$

$$f \text{ fully faithful} \Leftrightarrow l_X^* = f^* \cdot f_* \quad a(x, x') = b(f(x), f(x'))$$

$$f \text{ L-dense} \Leftrightarrow f_* \cdot f^* = l_Y^*$$

$$\Leftrightarrow (g \cdot f = h \cdot f \Rightarrow g \cong h)$$

"epi up to iso"

$$M \subseteq X$$

$\bar{M}$  = least subobject of  $X$  in which  $M$  is L-dense

$$= \{ y \in X \mid a \leq \bigvee_{x \in M} b(y, x) \oplus b(x, y) \}$$

L-separation, L-completion

$X$  L-separated  $\Leftrightarrow \Delta_X \subseteq X \times X$  L-closed

$\Leftrightarrow \forall g, h: P \rightarrow X$   
 $(g \approx h \Rightarrow g = h)$

$\Leftrightarrow \gamma: X \rightarrow \hat{X} = U^{X^{op}}$  is 1-1

Lemma:  $U = (U, \dashv)$  is L-separated  
and {fully faithful} - injective,  
in particular: {fff, L-dense} - injective  
pseudo  
"L-injective"

Naturally:

$\gamma: X \rightarrow \tilde{X} := \overline{\gamma(X)} \subseteq \hat{X} = U^{X^{op}}$

should be a "completion"

Theorem (Lawvere 1973,  
Clementino, Hofmann, Stubbe, T04-08)

Equivalent are for a  $\mathcal{U}$ -category  $X$ :

(i)  $X$   $L$ -complete (every  $\varphi + \psi: X \rightarrow Y$   
in  $\mathcal{U}\text{-Mod}$  is representable  
as  $f_* + f^*$  for some  $f$ ).

(ii)  $X$   $L$ -injective.

(iii)  $X$  is a (pseudo-)retract of  $\tilde{X}$ .

(iv)  $\gamma: X \rightarrow \tilde{X}$  is onto.

For a considerable\* generalization  
of this theorem  $\rightarrow$  Hofmann's talk

- \*)
- $\mathcal{U} \rightsquigarrow \mathcal{J}$  "topological theory"
  - relativized injectivity

Reflection, inj. hull, cogenerator:

- $U\text{-Cat}_{\text{cpl}} \hookrightarrow U\text{-Cat}_{\text{sep}}$  reflective, reflection:

$$\gamma: X \longrightarrow \tilde{X} \quad L\text{-dense full embedding}$$

- $\gamma$  is actually  $\{L\text{-dense full emb.}\}$ -inj. hull

Note: Functorial  $\mathcal{H}$ -inj. hulls are rare since:

$$\eta: T \rightarrow T \text{ ptw. } \mathcal{H}\text{-inj. hull} \left\{ \begin{array}{l} \text{and ptw. monic} \end{array} \right\} \Rightarrow \eta \text{ ptw. epic}$$

- $X \in U\text{-Cat}_{\text{cpl}} \iff \exists X \xrightarrow{\quad} \prod_I U$   
full  $L$ -closed emb.