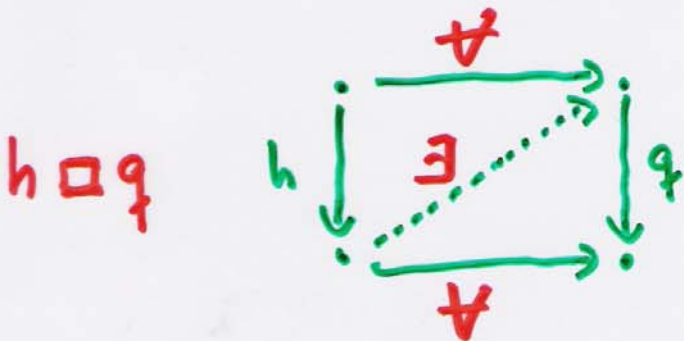


Injectivity by slice = Projectivity by coslice



$h \square q$
 $q \square h$

"left/right lifting prop."

Assume: $\mathcal{H} \square Q$ & $\forall f \in \mathcal{H} \exists \pi_f \in Q$

Then:

$\mathcal{H} \square Q \Leftrightarrow Q$ closed under retracts by slice:
 $q \cdot r \in Q, r \cdot i = 1 \Rightarrow q \in Q$

$\mathcal{H} = \square Q \Leftrightarrow \mathcal{H}$ closed under coretracts by coslice:
 $i \cdot h \in \mathcal{H}, r \cdot i = 1 \Rightarrow h \in \mathcal{H}$

"Economy class" (T-Rosidy' 2002):

Have: $\left(\begin{array}{ccc} & \xrightarrow{\gamma_f} & \\ A & \xrightarrow{f} & B \\ & & \downarrow \pi_f \end{array} \right)_f$ with $\{\gamma_f\}_f \square \{\pi_f\}_f \in \mathcal{C}/\mathcal{B}$

Put: $\mathcal{H} = \{h \mid \exists i: \pi_h \cdot i = 1, i \cdot h = \gamma_h\}$

$\mathcal{Q} = \{g \mid \exists r: r \cdot \gamma_g = 1, g \cdot r = \pi_g\}$

Then $\mathcal{H} = \square \mathcal{Q}$ and $\mathcal{Q} = \mathcal{H} \square$.

Theorem $\mathcal{C} \text{ HNB}$, small compl. & cocompl, wellpwd, residually small. Then:

\exists (Mono, \mathcal{Q}) weak fact. system (functorial)

\Leftrightarrow Mono is stable under pursuit.

(Weakly) h^* -couniversal arrows

\mathcal{C} with pbs, $h: C \rightarrow B$, hence

$$h^*: \mathcal{C}/B \rightarrow \mathcal{C}/C$$

If $(h_! \dashv) h^* \dashv h_*$, every h^* -couniversal arrow is iso precisely when h is monic.

Prop. Let h be monic. Then:

1. $w: h^*(\bar{p}) \rightarrow p$ weakly h^* -couniversal $\left. \vphantom{w: h^*(\bar{p}) \rightarrow p} \right\} \Rightarrow w$ split epi in \mathcal{C}/B
2. w h^* -couniversal $\Rightarrow w$ iso

"Exponentiability \Rightarrow Injectivity" (T-CT00)

\mathcal{C} with pbs, \mathcal{H} pb stable.

Let $f = (A \xrightarrow[p]{\mathcal{H}^\square} C \xrightarrow[\mathcal{H}]{h} B)$ s.th.

- \exists h^* -cominversal arrow $w: h^*(\bar{p}) \rightarrow p$, h mono
- or • \exists wkly h^* -cominv. arrow w for p , \mathcal{H} -Splitting $\subseteq \mathcal{H}$

Then:

$$\exists f = (A \xrightarrow[\mathcal{H}]{\bar{h}} D \xrightarrow[\mathcal{H}^\square]{\bar{p}} B)$$

\uparrow
 t

Furthermore:

This factorization is **essential** if w is iso, in particular, if h is monic.