Injectivity by slice = Projectivity by coslice

Assume: $H \sqsubset Q$ & $\forall q \in H \quad \exists q = f \cdot z$

Then:

$H^0 = Q \iff Q$ closed under retracts by slice:

$q \cdot r \in Q, r \cdot i = 1 \implies q \in Q$

$H = Q \iff H$ closed under contractions by coslice

$i \cdot h \in H, r \cdot i = 1 \implies h \in H$
"Economy class" (T-Rosicky 2002):

Have: \[
\begin{tikzcd}
A \xrightarrow{f} B
\end{tikzcd}
\]
with \( \{ h \}_{i}^{f} = \{ \eta_{i} \}_{i}^{f} \subset C / B \)

Put: \( \mathcal{H} = \{ h \mid \exists i: \eta_{i} = 1, i \cdot h = \eta_{h} \} \)

\( Q = \{ q \mid \exists r: r \cdot q = 1, q \cdot f = \eta_{q} \} \)

Then: \( \mathcal{H} = a Q \) and \( Q = \mathcal{H}^{0} \).

Theorem: CHNB, small comp. & cocomp., wellpw, residually small. Then:

\( \exists (\text{Mono}, Q) \) weak fact. system (factorial)

\( \iff \) Mono is stable under pushout.
(Weakly) $h^*$-communiversal arrows

$C$ with pbs, $h: C \to B$, hence

$h^*: C/B \to C/C$

If $(h \cdot -)^* h = h^*$, every $h^*$-communiversal arrow is iso precisely when $h$ is monic.

Prop. Let $h$ be monic. Then:

1. $\omega: h^*(p) \to \overline{p}$ weakly $\implies$ $\omega$ split epi in $C/B$
2. $\omega$ $h^*$-communiversal $\implies$ $\omega$ iso
"Exponentiability $\Rightarrow$ Injectivity" (T-CT00)

Let $f = (A \xrightarrow{F} C \xrightarrow{h} B)$ such that

- $\exists$ $h^*$-couniversal arrow $w: h^*(p) \to p$, $h$ mono
- or $\exists$ weakly $h^*$-couniv. arrow $w$ for $p$, $H$ Split mono in $\mathcal{E}$

Then:

$\exists f = (A \xrightarrow{\overline{h}} D \xrightarrow{\overline{F}} B)$

Furthermore:

This factorization is essential if $w$ is iso, in particular, if $h$ is monic.