Injectivity by slice = Projectivity by coslice

$$\begin{tikzcd} \bullet \arrow[r, "\forall"] \arrow[d, "h"'] & I \\ \bullet \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

$$\begin{tikzcd} & \bullet \arrow[d, "q"] \\ P \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & \bullet \end{tikzcd}$$

$h \,\square\, q$
$$\begin{tikzcd} \bullet \arrow[r, "\forall"] \arrow[d, "h"'] & \bullet \arrow[d, "q"] \\ \bullet \arrow[ur, dashed, "\exists"] \arrow[r, "\forall"'] & \bullet \end{tikzcd}$$
"left/right lifting prop."

Assume: $\mathcal{H} \,\square\, Q$ & $\forall f \; \exists$
$$\begin{tikzcd} & \bullet \arrow[dr, "\pi_f \in Q"] & \\ \bullet \arrow[ur, "\mathcal{H} \ni \iota_f"] \arrow[rr, "f"'] & & \bullet \end{tikzcd}$$

Then:

$\mathcal{H}^{\square} = Q \iff Q$ closed under retracts by slice:

$q \cdot r \in Q, \; r \cdot i = 1 \implies q \in Q$

$\mathcal{H} = {}^{\square}Q \iff \mathcal{H}$ closed under coretracts by coslice

$i \cdot h \in \mathcal{H}, \; r \cdot i = 1 \implies h \in \mathcal{H}$

"Economy class" (T-Rosický 2002):

Have: $\left(\begin{tikzcd} & \cdot \arrow[dr, "\pi_f"] & \\ A \arrow[ur, "\eta_f"] \arrow[rr, "f"'] & & B \end{tikzcd}\right)_f$ with $\{\eta_f\}_f \,\square\, \{\pi_f\}_{f \in \mathcal{C}/B}$

Put: $\mathcal{H} = \{ h \mid \exists i : \pi_h \cdot i = 1,\ i \cdot h = \eta_h \}$

$Q = \{ q \mid \exists r : r \cdot \eta_q = 1,\ q \cdot r = \pi_q \}$

Then $\mathcal{H} = {}^{\square}Q$ and $Q = \mathcal{H}^{\square}$.

Theorem $\mathcal{C}$ HNB, small compl. & cocompl., wellpwd, residually small. Then:

$\exists$ (Mono, Q) weak fact. system (functorial) $\Leftrightarrow$ Mono is stable under pushout.

(Weakly) h^*-couniversal arrows

$\mathcal{C}$ with pbs, $h : C \to B$, hence

$$h^* : \mathcal{C}/B \longrightarrow \mathcal{C}/C$$

If $(h_! \dashv) h^* \dashv h_*$, every h^*-couniversal arrow is iso precisely when h is monic.

Prop. Let h be monic. Then:

1. $\left.\begin{array}{l} w : h^*(\bar{p}) \to p \text{ weakly} \\ h^*\text{-couniversal} \end{array}\right\} \Rightarrow w$ split epi in $\mathcal{C}/B$
2. w h^*-couniversal $\Rightarrow w$ iso

"Exponentiability $\Rightarrow$ Injectivity" (T-CT00)

$\mathcal{C}$ with pbs, $\mathcal{H}$ pb stable.

Let $f = (A \xrightarrow[\mathcal{H}^{\square}]{p} C \xrightarrow[\mathcal{H}]{h} B)$ s.th.

- $\exists$ h^*-couniversal arrow $w : h^*(\bar{p}) \to p$, h mono

or
- $\exists$ wkly h^*-couniv. arrow w for p, $\mathcal{H}\cdot\mathrm{SplitMono} \subseteq \mathcal{H}$

Then:

$$\exists \quad \begin{tikzcd} f = (A \arrow[r, "\bar{h}", "\mathcal{H}"'] & D \arrow[loop below, "t"] \arrow[r, "\bar{p}", "\mathcal{H}^{\square}"'] & B) \end{tikzcd}$$

Furthermore:

This factorization is essential if w is iso, in particular, if h is monic.